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Variational resummation for ϵ -expansions of critical exponents of nonlinear $O(n)$ -symmetric σ -model in $2 + \epsilon$ dimensions

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Abstract

We develop a method for extracting accurate critical exponents from perturbation expansions of the $O(n)$ -symmetric nonlinear σ -model in $D = 2 + \epsilon$ dimensions. This is possible by considering the ϵ -expansions in this model as strong-coupling expansions of functions of the variable $\tilde{\epsilon} \equiv 2(4 - D)/(D - 2)$, whose first five weak-coupling expansion coefficients of powers of $\tilde{\epsilon}$ are known from ϵ -expansions of critical exponents in $O(n)$ -symmetric ϕ^4 -theory in $D = 4 - \epsilon$ dimensions. © 2000 Published by Elsevier Science B.V. All rights reserved.

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1. Critical exponents of the $O(n)$ -universality class can be calculated with high accuracy from standard resummation procedures of renormalization group functions of ϕ^4 -field theory [1,2]. For the classical Heisenberg model, where $n = 3$, the critical exponent ν governing the divergence of the coherence length as $\xi \propto |T - T_c|^{-\nu}$ has been calculated from seven-loop perturbation expansions in three dimensions [2] as $\nu = 0.7073 \pm 0.0030$ [1], whereas five-loop expansions in $D = 4 - \epsilon$ dimensions [3,4] extrapolated to $\epsilon = 1$ give $\nu = 0.7050 \pm 0.0055$ [1]. Apart from the initial expansion coefficients, the

resummation procedures incorporate information on the large-order growth of the coefficients obtained from semiclassical considerations [5,6]. Results very close to the above numbers were recently obtained from a novel strong-coupling ϕ^4 -theory [7–10] in $D = 3$ [9,10] as well as $4 - \epsilon$ dimensions [12–14].

It is generally accepted that, as a consequence of the *universality hypothesis* of critical phenomena of all systems with equal Goldstone bosons, the same critical exponents should be obtainable from renormalization group studies of $O(n)$ -symmetric nonlinear σ -models in $D = 2 + \epsilon$ dimensions at $\epsilon = 1$, if the second-order character of the transition is not destroyed by fluctuations. These conditions restrict the comparison to $n > 2$. For $n = 1$ (Ising case), there are no Goldstone bosons, and for $n = 2$ (XY-model), the transition is of infinite order, for which the divergence of the correlation length with temper-

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ature cannot be parametrized like $\xi \propto |T - T_c|^{-\nu}$, as shown by Kosterlitz and Thouless.

Unfortunately, the ϵ -expansions of the nonlinear σ -models have, up to now, remained rather useless for any practical calculation, due to their non-Borel character [15]. This has led some authors to doubt the use of such expansions around the lower critical dimension altogether [16]. Basis of these doubts is the increasing relevance of ignored higher powers of the derivative term in the calculations [17]. Such a situation would be quite unfortunate, since it would jeopardize other interesting theories which depend on similar relationships, such as Anderson's theory of localization [18].

Fortunately, the counter-arguments are not completely convincing since they involve an interchange of limits in the analytic continuation in ϵ and the increase of the number of derivatives [19], so that hope remains. The purpose of this note is to confirm this hope and to lend further support to the intimate relationship of ϵ - and ε -expansions. This is done by determining from a combination of the two expansions an accurate critical exponent ν for the classical Heisenberg model for all dimensions $2 \leq D \leq 4$.

2. So far, the ϵ -expansions of ν^{-1} and the anomalous dimension η have been calculated up to the powers ϵ^4 [15,20–23]:

$$\begin{aligned} \nu^{-1}(\epsilon) = & \epsilon + \frac{\epsilon^2}{n-2} + \frac{\epsilon^3}{2(n-2)} \\ & - [30 - 14 + n^2 + (54 - 18n)\zeta(3)] \\ & \times \frac{\epsilon^4}{4(n-2)^3} + \dots, \quad (1) \\ \eta(\epsilon) = & \frac{\epsilon}{n-2} - \frac{(n-1)\epsilon^2}{(n-2)^2} + \frac{n(n-1)\epsilon^3}{2(n-2)^3} \\ & - (n-1) [-6 + 2n + n^2 \\ & + (-12 + n + n^2)\zeta(3)] \frac{\epsilon^4}{4(n-2)^4} \\ & + \dots \quad (2) \end{aligned}$$

The singularity at $n = 2$ reflects the above-discussed restriction of the upcoming considerations to $n > 2$. When evaluated at $\epsilon = 1$, the first series yields for

the three-dimensional $O(3)$ -model the diverging successive values $\nu^{-1} = (1, 2, 2.5, 3.25)$. The often-employed Padé approximations do not help, with the best of them, the $[1,2]$ -approximation, giving the too large value $\nu = 2$. So far, the only result which is not too far from the true value has been obtained via the Padé–Borel transform [15]

$$P^{[1,2]}(\epsilon, t) = \frac{\epsilon t}{1 - \epsilon t/2 + \epsilon^2 t^2/6}, \quad (3)$$

from which one obtains the ϵ -dependent inverse critical exponent

$$\nu^{-1}(\epsilon) = \int_0^\infty dt e^{-t} P^{[1,2]}(\epsilon, t). \quad (4)$$

Its value at $\epsilon = 1$ is $\nu^{-1} \approx 1.252$, corresponding to $\nu \approx 0.799$, which is still considerably larger than the accurate value 0.705. The other Padé–Borel approximants are singular and thus of no use. See Fig. 1 for plots of the integrands.

A direct evaluation of the series for the other critical exponent, the anomalous dimension η , yields the successive values $(2, -2, 4, -5)$, which are completely useless. Here the nonsingular Borel–Padé approximations $[2,1]$, $[1,2]$, and $[1,1]$ yield 0.147, 0.150, and 0.139, rather than the correct value 0.032.

3. The remedy for these problems comes from a combination of the theory developed in Refs. [7–10] with a procedure developed in Ref. [11]. The theory allows us to extract the strong-coupling properties of a ϕ^4 -theory from perturbation expansions. In particular, it renders the power behavior of the renormalization constants for large bare couplings g_0 , and from this all critical exponents of the system. By using the known expansion coefficients of the renormalization constants in three dimensions up to six loops, we were able to derive extremely accurate values for the critical exponents. The method is a systematic extension to arbitrary orders [24] of the Feynman–Kleinert variational approximation to path integrals [25]¹. For an anharmonic oscillator, this so-called *variational perturbation theory* [28] yields

¹ A similar approach has been pursued independently by Refs. [26,27].

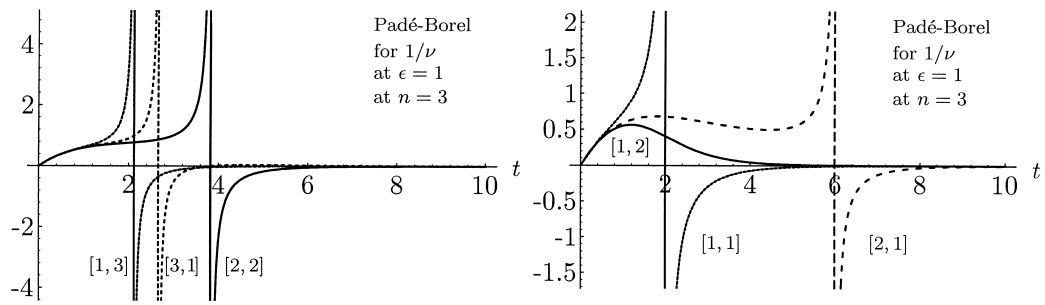


Fig. 1. Integrands of the Padé–Borel transform (4) for the Padé approximants [1,3], [3,1], [2,2] and for [1,1], [2,1], [1,2] at $\epsilon = 1$. Only the last is integrable, yielding $\nu^{-1} \approx 1.25183 \approx 1/.79883$.

expansions which converge uniformly in the coupling strength and exponentially fast, like $\exp(-\text{const} \times N^{1/3})$ in the order N of the approximation, as was observed in [29,30,28] and proved in [31]². The extension to field theory was achieved in Refs. [7,8], and showed to same type of convergence, but with the fractional power $1/3$ replaced by the irrational power $1 - \omega$, where ω is the critical exponent governing the approach to scaling.

This theory is combined with the procedure of Ref. [11] which allows us to interpolate variationally functions for which we know strong- and weak-coupling expansions. The resummation to be performed will be based on rewriting the above ϵ -expansions in such a way that they may be considered as strong-coupling expansion of functions, whose weak-coupling expansions are provided by power series expansion in powers of $\epsilon = 4 - D$, which are known from ϕ^4 -theory in $D = 4 - \epsilon$ dimensions. In this way we shall be able to derive accurate critical exponents ν^{-1} from the non-Borel expansion (1).

4. Let us briefly recall the interpolation procedure [11] by which a divergent weak-coupling expansion

in some variable g_0 of the type

$$E_N(g_0) = \sum_{n=0}^N a_n g_0^n$$

can be combined with a strong-coupling expansion of the type

$$E_M(g_0) = g_0^{p/q} \sum_{m=0}^M b_m (g_0^{-2/q})^m.$$

Previously treated examples [11] were the anharmonic oscillator with parameters $p = 1/3, q = 3$ for the energy eigenvalues, and the Fröhlich polaron with $p = 1, q = 1$ for the ground-state energy and $p = 4, q = 1$ for the mass. As described in detail in [28], the first step is to rewrite the weak-coupling expansion with the help of an auxiliary scale parameter κ as

$$E_N(g_0) = \kappa^p \sum_{n=0}^N a_n \left(\frac{g_0}{\kappa^q} \right)^n \tag{5}$$

where κ is eventually set equal to 1. We shall see below that the quotient p/q parametrizes the *leading power behavior* in g_0 of the strong-coupling expansion, whereas $2/q$ characterizes the *approach* to the leading power behavior. In a second step we replace κ by the identical expression

$$\kappa \rightarrow \sqrt{K^2 + \kappa^2 - K^2} \tag{6}$$

containing a dummy scaling parameter K . The series (5) is then reexpanded in powers of g_0 up to the order N , thereby treating $\kappa^2 - K^2$ as a quantity of order g_0 . The result is most conveniently expressed

² A convergence proof for the anharmonic oscillator which is completely equivalent to our results in [31] was given by [32]. Predecessors of these works which failed to explain the exponentially fast convergence in the strong-coupling limit observed in Ref. [30] are Refs. [33–37].

in terms of dimensionless parameters $\hat{g}_0 \equiv g_0/K^q$ and $\sigma \equiv (1 - \hat{\kappa}^2)/\hat{g}_0$, where $\hat{\kappa} \equiv \kappa/K$. Then the replacement (6) amounts to

$$\kappa \rightarrow K(1 - \sigma \hat{g}_0)^{1/2}, \quad (7)$$

so that the reexpanded series reads explicitly

$$W_N(\hat{g}_0, \sigma) = K^p \sum_{n=0}^N \varepsilon_n(\sigma) (\hat{g}_0)^n, \quad (8)$$

with the coefficients

$$\varepsilon_n(\sigma) = \sum_{j=0}^n a_j \binom{(p - qj)/2}{n - j} (-\sigma)^{n-j}. \quad (9)$$

For any fixed g_0 , we form the first and second derivatives of $W_N(g_0, K)$ with respect to K , calculate the K -values of the extrema and the turning points. If there is a unique extremum, this supplies us with an optimal scaling parameter K_N . If no extremum exists, we use the turning point to determine K_N . If there are more than one extremum or turning point, we take the smallest of these as K_N . This procedure is called *optimization*. The function $W_N(g_0) \equiv W_N(g_0, K_N)$ constitutes the N th variational approximation $E_N(g_0)$ to the function $E(g_0)$.

It is easy to take this approximation to the strong-coupling limit $g_0 \rightarrow \infty$. For this we observe that (8) has the scaling form

$$W_N(g_0, K) = K^p w_N(\hat{g}_0, \hat{\kappa}^2). \quad (10)$$

For dimensional reasons, the optimal K_N increases with g_0 like $K_N \approx g_0^{1/q} c_N$, so that $\hat{g}_0 = c_N^{-q}$ and $\sigma = 1/\hat{g}_0 = c_N^q$ remain finite in the strong-coupling limit, whereas $\hat{\kappa}^2$ goes to zero like $1/[c_N(g_0/\kappa^q)^{1/q}]^2$. Hence

$$W_N(g_0, K_N) \approx g_0^{p/q} c_N^p w_N(c_N^{-q}, 0). \quad (11)$$

Here c_N plays the role of the variational parameter to be determined by applying the optimization process described above to the function $c_N^p w_N(c_N^{-q}, 0)$. The full strong-coupling expansion is obtained from the Taylor series of $w_N(\hat{g}_0, \hat{\kappa}^2)$ in powers of $\hat{\kappa}^2 = (g_0/\kappa^q \hat{g}_0)^{-2/q}$, which yields

$$W_N(g_0) = g_0^{p/q} \left[b_0(\hat{g}_0) + b_1(\hat{g}_0) \left(\frac{g_0}{\kappa^q} \right)^{-2/q} + b_2(\hat{g}_0) \left(\frac{g_0}{\kappa^q} \right)^{-4/q} + \dots \right] \quad (12)$$

with

$$b_n(\hat{g}_0) = \frac{1}{n!} w_N^{(n)}(\hat{g}_0, 0) \hat{g}_0^{(2n-p)/q}, \quad (13)$$

where $w_N^{(n)}(\hat{g}_0, \hat{\kappa}^2)$ is the n th derivatives of $w_N(\hat{g}_0, \hat{\kappa}^2)$ with respect to $\hat{\kappa}^2$. Explicitly:

$$\frac{1}{n!} w_N^{(n)}(\hat{g}_0, 0) = \sum_{l=0}^N (-1)^{l+n} \sum_{j=0}^{l-n} a_j \binom{(p - qj)/2}{l - j} \times \binom{l - j}{n} (-\hat{g}_0)^j. \quad (14)$$

The optimal expansion of the energy (12) is obtained by expanding

$$\hat{g}_0 = \gamma_0 + \gamma_1 \left(\frac{g_0}{\kappa^q} \right)^{-2/q} + \gamma_2 \left(\frac{g_0}{\kappa^q} \right)^{-4/q} + \dots, \quad (15)$$

where $\gamma_0 = c_N^{-q}$, and finding the optimal extremum (or turning point) in the resulting polynomials of $\gamma_1, \gamma_2, \dots$. In this way we obtain a systematic strong-coupling expansion in powers of $(g_0/\kappa^q)^{-2/q}$. This is done as follows: We first optimize the leading strong-coupling coefficient $b_0(\hat{g}_0)$ in \hat{g}_0 , and identify the optimal position by γ_0 . Optimizing $W_N(g_0)$ with the expansion (15), in $\gamma_1, \gamma_2, \dots$, yields for the parameters $p = -2, q = 2$ at the coefficients $\gamma_1, \gamma_2, \dots$ and optimal $b_n(\hat{g}_0)$'s by the equations listed in Table 1.

It was demonstrated in [11] how one can now find a variational perturbation series for functions for which one knows N weak-coupling and M strong-coupling expansion coefficients. We must merely extend the set of coefficients a_1, \dots, a_N by M unknown ones a_{N+1}, \dots, a_{N+M} , and determine the latter via a fit of the resulting strong-coupling coefficients b_0, \dots, b_{M-1} to the known ones.

5. This interpolation procedure will now be applied to the perturbation expansion (1) in $2 + \epsilon$ dimensions, considering it as the *strong-coupling*

Table 1

Equations determining the coefficients $b_n(\hat{g}_0)$ in the strong-coupling expansion (12) and the associated $\gamma_i \equiv c_n^{-q} \delta_i$ in (15) from the functions $\bar{b}_n \equiv b_n(\gamma_0)$ and their derivatives. For brevity, we have suppressed the argument γ_0 in these functions.

n	b_n	$-\gamma_{n-1}$
2	$\bar{b}_2 + \gamma_1 \bar{b}'_1 + \frac{1}{2} \gamma_1^2 \bar{b}''_0$	\bar{b}'_1 / \bar{b}''_0
3	$\bar{b}_3 + \gamma_2 \bar{b}'_1 + \gamma_1 \bar{b}'_2 + \gamma_1 \gamma_2 \bar{b}''_0 + \frac{1}{2} \gamma_1^2 \bar{b}''_1 + \frac{1}{6} \gamma_1^3 \bar{b}'''_0$	$(\bar{b}'_2 + \gamma_1 \bar{b}'_1 + \frac{1}{2} \gamma_1^2 \bar{b}'''_0) / \bar{b}''_0$
4	$\bar{b}_4 + \gamma_3 \bar{b}'_1 + \gamma_2 \bar{b}'_2 + \gamma_1 \bar{b}'_3 + (\frac{1}{2} \gamma_2^2 + \gamma_1 \gamma_3) \bar{b}''_0 + \gamma_1 \gamma_2 \bar{b}''_1 + \frac{1}{2} \gamma_1^2 \bar{b}''_2 + \frac{1}{2} \gamma_1^2 \gamma_2 \bar{b}'''_0 + \frac{1}{6} \gamma_1^3 \bar{b}'''_1 + \frac{1}{24} \gamma_1^4 \bar{b}''''_0$	$(\bar{b}'_3 + \gamma_2 \bar{b}'_1 + \gamma_1 \bar{b}'_2 + \gamma_1 \gamma_2 \bar{b}'''_0) / \bar{b}''_0 + \frac{1}{2} \gamma_1^2 \bar{b}'''_1 + \frac{1}{6} \gamma_1^3 \bar{b}''''_0) / \bar{b}''_0$

expansion of a series in the variable $\tilde{\epsilon} = 2(4 - D) / (D - 2) = 4(1 - \epsilon/2) / \epsilon = \epsilon / (1 - \epsilon/2)$:

$$\nu^{-1}(\tilde{\epsilon}) = 4 \tilde{\epsilon}^{-1} - 8 \frac{n-4}{n-2} \tilde{\epsilon}^{-2} + 16 \frac{n-4}{n-2} \tilde{\epsilon}^{-3} - 32 \{ [52 + 108\zeta(3)] - [16 + 36\zeta(3)] n + n^2 \} \frac{\tilde{\epsilon}^{-4}}{(n-2)^3} + \dots \tag{16}$$

The variable $\tilde{\epsilon}$ plays the role of the variable g_0 in the general formulas of the last section. The weak-coupling expansion of $\nu^{-1}(\tilde{\epsilon})$ in powers of $\tilde{\epsilon}$ can be obtained directly from the ϵ -expansions of Refs. [3,4], and has for $n = 3, 4, 5, 1$, the numerical form

$$n = 3: \quad \nu^{-1} = 2 - 0.45455 \tilde{\epsilon} + 0.071375 \tilde{\epsilon}^2 + 0.15733 \tilde{\epsilon}^3 - 0.52631 \tilde{\epsilon}^4 + \dots, \tag{17}$$

$$n = 4: \quad \nu^{-1} = 2 - 0.5 \tilde{\epsilon} + 0.0833333 \tilde{\epsilon}^2 + 0.147522 \tilde{\epsilon}^3 - 0.499944 \tilde{\epsilon}^4 + 1.47036 \tilde{\epsilon}^5 + \dots, \tag{18}$$

Table 2

Coefficients of the successive extension of the expansion coefficients in Eq. (20) for $n = 3$ determined from $M = 1, 2, 3, 4$ strong-coupling coefficients (4, 8, -16, 160) of Eq. (16).

n	a_6	a_7	a_8	a_9
1	-203.827			
2	-5.67653	17.6165		
3	-4.25622	9.04109	-15.7331	
4	-3.80331	6.87304	-10.0012	12.3552

$$n = 5: \quad \nu^{-1} = 2 - 0.538462 \tilde{\epsilon} + 0.0955849 \tilde{\epsilon}^2 + 0.135442 \tilde{\epsilon}^3 - 0.469842 \tilde{\epsilon}^4 + 1.34491 \tilde{\epsilon}^5 + \dots, \tag{19}$$

$$n = 1: \quad \nu^{-1} = 2 - 0.333333 \tilde{\epsilon} + 0.0493827 \tilde{\epsilon}^2 + 0.158478 \tilde{\epsilon}^3 - 0.539937 \tilde{\epsilon}^4 + 1.78954 \tilde{\epsilon}^5 + \dots \tag{20}$$

Extending these series by four more terms $a_6 \tilde{\epsilon}^6 + a_7 \tilde{\epsilon}^7 + a_8 \tilde{\epsilon}^8 + a_9 \tilde{\epsilon}^9$, we calculate the strong-coupling coefficients (13) by extremizing (12) with (15), after identifying g_0 with $\tilde{\epsilon}$. The parameters (p, q) are equal to $(-2, 2)$, as follows directly from a comparison of the strong-coupling powers

Table 3

Coefficients of the successive extension of the expansion coefficients in Eq. (20) for $n = 4$ determined from $M = 1, 2, 3, 4$ strong-coupling coefficients (4, 0, 0, 221.096) of Eq. (16).

n	a_6	a_7	a_8	a_9
1	-147.508			
2	-7.91064	37.1745		
3	-4.59388	12.3044	-27.0837	
4	-3.72613	7.47851	-12.2129	16.9547

Table 4

Coefficients of the successive extension of the expansion coefficients in Eq. (20) for $n = 5$ determined from $M = 1, 2, 3, 4$ strong-coupling coefficients (8, -8/3, 16/3, 106.131) of Eq. (16).

n	a_6	a_7	a_8	a_9
1	-108.648			
2	-10.1408	60.7217		
3	-4.75598	15.1045	-38.9689	
4	-3.57909	7.84272	-14.1142	21.6045

Table 5

Coefficients of the successive extension of the expansion coefficients in Eq. (20) for $n = 1$ determined from $M = 1, 2, 3, 4$ strong-coupling coefficients (4, -24, 48, 3825.54) of Eq. (16).

n	a_6	a_7	a_8	a_9
1	-413.921			
2	-5.25285	12.1104		
3	-442759	12450066	-196950675	
4	-5.7343	13.7134	-25.226	38.0976

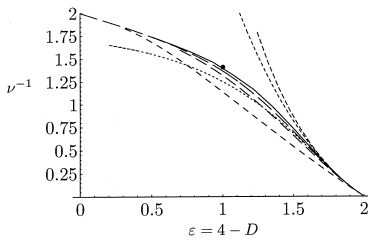


Fig. 2. Inverse of the critical exponent ν for the classical Heisenberg model in the $O(3)$ -universality class. Solid curve represents the interpolation result of fourth order. Lower dashed curves show interpolations of first, second, and third order. Upper short-dashed curves display, with decreasing dash length, the first three and four terms of the ϵ -expansion (2), respectively. Dotted curve is Padé [1,2]-Borel approximations. The fat dot corresponds to the seven-loop result in $D = 3$ dimensions, $\nu = 0.7073$ of Refs. [1,9,10]. The four interpolations give $(\nu_1, \nu_2, \nu_3, \nu_4) = (0.87917, 0.75899, 0.731431, 0.712152)$. These are extrapolated in Fig. 3 to infinite order, yielding $\nu = 0.695$.

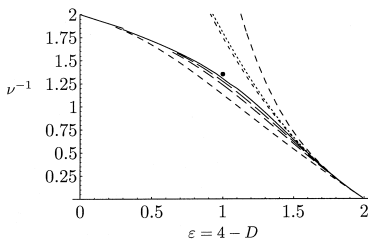


Fig. 3. Same plot as in Fig. 2, but for the $O(4)$ -universality class. Fat dot represents six-loop result in $D = 3$ dimensions $\nu = 0.737$ of Refs. [7,8]. The four interpolations give $(\nu_1, \nu_2, \nu_3, \nu_4) = (0.88635, 0.810441, 0.786099, 0.768565)$. The extrapolation to infinite order shown in Fig. 7 yields $\nu = 0.735$.

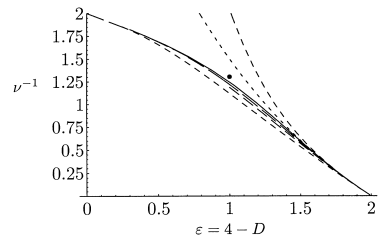


Fig. 4. Same plot as in Fig. 2, but for the $O(5)$ -universality class. There exists no Padé-Borel approximation. Fat dot represents ix-loop result in $D = 3$ dimensions $\nu = 0.767$ of Refs. [7,8]. The four interpolations give $(\nu_1, \nu_2, \nu_3, \nu_4) = (0.89278, 0.842391, 0.820491, 0.802416)$. The extrapolation to infinite order shown in Fig. 8 yields $\nu = 0.766$.

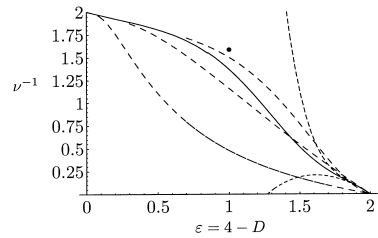


Fig. 5. Same plot as in Fig. 2, but for the $O(1)$ -universality class (of the Ising model). Again there is no Padé-Borel approximation. Fat dot represents seven-loop result in $D = 3$ dimensions $\nu = 0.6305$ of Refs. [1,9,10]. The four interpolations give $(\nu_1, \nu_2, \nu_3, \nu_4) = (0.862357, 0.665451, 2.08686, 0.729231)$. Their failure to converge is illustrated graphically in Fig. 9.

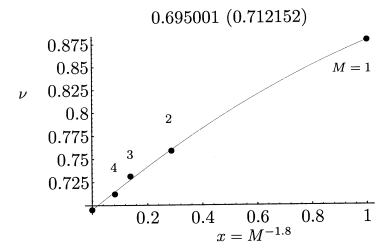


Fig. 6. The four successive approximations $(\nu_1, \nu_2, \nu_3, \nu_4) = (0.87917, 0.75899, 0.731431, 0.712152)$ for $n = 3$ (Heisenberg model) plotted as a function of $x = M^{-1.8}$ which makes them lie a smooth parabola line with the intercept $\nu_\infty = 0.695 \pm 0.010$. Numbers on top show extrapolated value and highest approximation (in parentheses).

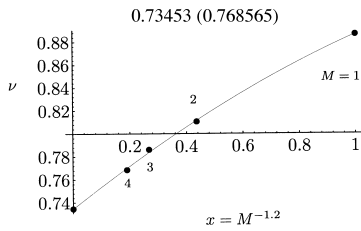


Fig. 7. The four successive approximations $(\nu_1, \nu_2, \nu_3, \nu_4) = (0.88635, 0.810441, 0.786099, 0.768565)$ for $n=4$ plotted as a function of $x = M^{-1.2}$ which puts them on a smooth parabola with the intercept $\nu_\infty = 0.735 \pm 0.010$. Numbers on top show extrapolated value and highest approximation (in parentheses).

$\tilde{\epsilon}^{p/q}(1 + \tilde{\epsilon}^{-2/q} + \dots)$ with (16). The coefficients a_6, a_7, a_8, a_9 are now determined to make $b_0(\hat{g}_0), b_1(\hat{g}_0), b_2(\hat{g}_0), b_3(\hat{g}_0)$ agree with (16). The technique of doing this is described in detail in Ref. [11].

In order to see how the result improves with the number M of additional terms in (15), we go through this procedure successively for $M = 1, 2, 3, 4$. The successive additional expansion coefficients for the $O(n)$ universality classes with $n = 3, 4, 5, 1$ are listed in Tables 2–5, respectively. The four resulting curves for $\nu^{-1}(\epsilon)$ are shown in Figs. 2–4. For $n=3$, the successive critical exponents ν at $\epsilon = 1$ taken from Fig. 2 are $(\nu_1, \nu_2, \nu_3, \nu_4) = (0.87917, 0.75899, 0.731431, 0.712152)$. Their M -dependence is plotted in Fig. 6 as a function of the variable $x = M^{-1.8}$ which makes them lie approximately on a smooth parabola intercepting the ν -axis at $\nu_\infty = 0.695 \pm 0.010$. This extrapolated value is in good agreement with the above-quoted value ≈ 0.705 from seven-loop calculations in $4 - \epsilon$ dimensions [1,9,10]. The

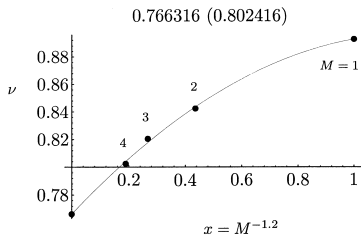


Fig. 8. The four successive approximations $(\nu_1, \nu_2, \nu_3, \nu_4) = (0.89278, 0.842391, 0.820491, 0.802416)$ for $n=5$ plotted as a function of $x = M^{-1.2}$ which puts them on a smooth parabola with intercept $\nu_\infty = 0.766 \pm 0.010$. Numbers on top show extrapolated value and highest approximation (in parentheses).

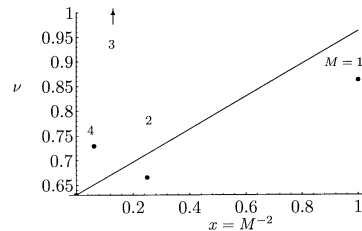


Fig. 9. The four successive approximations $(\nu_1, \nu_2, \nu_3, \nu_4) = (0.862357, 0.665451, 2.08686, 0.729231)$ for $n=1$ plotted as a function of $x = M^{-2}$. They show no tendency of convergence towards the known seven-loop exponent $\nu_\infty = 0.630$.

results for the other $O(4)$ and $O(5)$ universality classes are displayed analogously. The respective ν -values 0.735 ± 0.010 and 0.766 ± 0.010 agree well with the highest available six-loops results of Refs. [7,8], which are 0.737 and 0.767.

As discussed above, the relation between the ϵ - and ϵ -expansions is expected to be restricted to $n > 2$, for physical reasons. It is instructive to see that the variational interpolation method reflects this problem at two places. First, the expansion coefficients in Table 5 shows a large irregularity for $n = 1$. Second, the successive approximations for ν^{-1} in Fig. 5 display no tendency of convergence with increasing order M of approximation.

Finally, we plot our highest ($M = 4$) approximations for $n = 3, 4, 5$ together with the large- n approximations for $n = \infty, 20, 10, 6$ in Fig. 10 to see the change of the $\tilde{\epsilon}$ -behavior for increasing n , which shows that the latter for $n = 6$ is still far from the exact curve. This can also be seen in Fig. 3 of Ref. [7] (see also [8]).

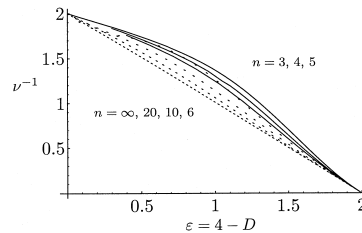


Fig. 10. Comparison of ν^{-1} from the highest approximations of our interpolating resummation for the $O(n)$ universality classes with $n = 3, 4, 5$ (counting from the top), with the values obtained from the $1/n$ -expansion to order $1/n^2$ for $n = \infty, 20, 10, 6$ (counting from the bottom). The $n = 6$ curve is still far from the exact one.

For the critical exponent η , the series (2) reads in the variable $\tilde{\epsilon}$:

$$\begin{aligned} \eta(\tilde{\epsilon}) = & \frac{\tilde{\epsilon}^{-1}}{n-2} - 2n \frac{\tilde{\epsilon}^{-2}}{(n-2)^2} \\ & + 8n(n-1) \frac{\tilde{\epsilon}^{-3}}{(n-2)^3} + 16(n-1) \\ & \times [(6-2n^2)[12-n-n^2]\zeta(3)] \\ & \times \frac{\tilde{\epsilon}^{-4}}{(n-2)^4} + \dots, \end{aligned} \quad (21)$$

whereas the weak-coupling expansion in powers of $\tilde{\epsilon}$ obtained from the ϵ -expansions of Refs. [3,4] has, for $N=3,4,5,1$, the numerical form

$$\begin{aligned} N=3: \quad \eta/\epsilon^2 = & 5/242 + 0.0183987\tilde{\epsilon} \\ & - 0.0166488\tilde{\epsilon}^2 + 0.032432\tilde{\epsilon}^3 \\ & + \dots, \end{aligned} \quad (22)$$

$$\begin{aligned} N=4: \quad \eta/\epsilon^2 = & 1/48 + 0.0173611\tilde{\epsilon} \\ & - 0.0157657\tilde{\epsilon}^2 + 0.029057\tilde{\epsilon}^3 \\ & + \dots, \end{aligned} \quad (23)$$

$$\begin{aligned} N=5: \quad \eta/\epsilon^2 = & 7/338 + 0.0161453\tilde{\epsilon} \\ & - 0.0148734\tilde{\epsilon}^2 + 0.0259628\tilde{\epsilon}^3 \\ & + \dots, \end{aligned} \quad (24)$$

$$\begin{aligned} N=1: \quad \eta/\epsilon^2 = & 1/54 + 0.01869\tilde{\epsilon} \\ & - 0.0176738\tilde{\epsilon}^2 + 0.0386577\tilde{\epsilon}^3 \\ & + \dots. \end{aligned} \quad (25)$$

These series can again be extended by four more terms $a_4\tilde{\epsilon}^4 + a_5\tilde{\epsilon}^5 + a_6\tilde{\epsilon}^6 + a_7\tilde{\epsilon}^7$, making the strong-coupling coefficients $\bar{b}_0, \bar{b}_1, \bar{b}_2, \bar{b}_3$, in Eq. (13) calculated for Eqs. (22)–(25) agree with those of the expansion of $\eta/\epsilon^2 = \eta/(2-\tilde{\epsilon})^2$ obtained from Eq. (21). Here, however, we encounter problems: The η -values from the interpolation come out too large by about a factor 2. Also γ does not interpolate well. A more convenient combination of critical exponents will have to be found to apply this method.

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