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On the Dimension of Chiral and Conformal Symmetry Breaking.

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Summary. — We investigate the general consequences of Gell-Mann's decomposition of the energy density $\theta_{00}(x) = \theta^*_{00}(x) + \sum_n \delta_n(x) + u(x)$, where u is the $SU_3 \times SU_3$ breaking term transforming according to a $(3\overline{3}) + (\overline{3}3)$ representation and being a Lorentz scalar of dimension d. We show that there must be at least two terms among θ^*_{00}, δ_n with non-vanishing vacuum expectation values. If we further assume that there are only two such terms with dimensions d' and d'' we can obtain a new sum rule involving the spectral functions of the propagators of θ^μ_μ, $\partial^\mu A^\pi_\mu$ and $\partial^\mu A^K_\mu$, which reads

$$\int \varrho_\theta(\mu^2)\,\frac{\mathrm{d}\mu^2}{\mu^2} = X\left[\frac{r-2}{2r}\int \varrho_\pi(\mu^2)\,\frac{\mathrm{d}\mu^2}{\mu^2} + \frac{r+2}{r+1}\int \varrho_K(\mu^2)\,\frac{\mathrm{d}\mu^2}{\mu^2}\right],$$

where $r = -\frac{2}{3}(c+\sqrt{2})/c$ and $X = (d-d')(d''-d)$. Saturating with σ, π, K, one finds, using an earlier result on $g_{\sigma\pi\pi}$,

$$\frac{m_\sigma}{\Gamma_{\sigma\pi\pi}} = \frac{32\pi}{3m_\sigma^4}\,X\left[\frac{r-2}{2r}\,f_\pi^2 m_\pi^2 + \frac{r+2}{r+1}\,f_K^2 m_K^2\right]\left(1 + [d-2]\,\frac{m_\pi^2}{m_\sigma^2}\right)^{-2}.$$

We use the experimental values $m_\sigma \approx 700$ and $\Gamma_{\sigma\pi\pi} \approx 400$ to estimate X in both the Gell-Mann–Oakes–Renner and the Brandt-Preparata $SU_3 \times SU_3$ symmetry-breaking schemes. Unfortunately, for reasonable values of f_K/f_π our sum rule does not distinguish between these two schemes, *e.g.* for $f_K/f_\pi = 1.25$, both schemes give $X \approx 3$. Nevertheless, this typical value allows us to conclude that σ dominance is consistent with many models not involving operators of anomalously high dimensions and at the same time it allows us to exclude many other models.

1. – Introduction.

Recently, a great deal of work has been invested in linking up chiral and dimensional information and extracting its physical consequences [1,2]. Chiral Lagrangians provided the first models for these studies and have until now given the bulk of useful insights [3]. It is well known that phenomenological Lagrangians provide a simple mnemonic device to enforce low-energy theorems on physical coupling constants given arbitrary current field commutation rules [4]. Indeed, for the complicated commutation rules of $SU_3 \times SU_3$, the Lagrangian methods have made the involved consistency equations of Ward identities [5] an easy SU_3 coupling exercise.

For evaluating consequence of dimensional assumptions, however, Lagrangians probably provide somewhat too big an apparatus. The reason is that the commutator carrying all information on the dimension of a field is extremely simple:

$$i[D_0(x), \varphi(y)]_{x_0=y_0} = (x\delta + d_\varphi)\varphi(x)\,\delta^3(x-y) + \text{Schwinger term}\,, \tag{1}$$

where $D_\mu(x)$ is the current density of dilatations, defined in terms of the symmetric local energy-momentum tensor $\theta_{\mu\nu}(x)$ [6] by

$$D_\mu(x) = x^\nu \theta_{\mu\nu}(x)\,, \tag{2}$$

such that the conservation condition reads

$$\partial^\mu D_\mu(x) = \theta^\mu_\mu(x) = \theta(x)\,. \tag{3}$$

The trivial algebraic structure of (1) allows one to solve many Ward identities at once. Thus for the three-point function $T(D_\mu(x)\varphi(y)\varphi(0))$ one finds imme-

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(1) G. Mack: *Nucl. Phys.*, **5** B, 499 (1968); G. Mack and A. Salam: *Ann. of Phys.*, **53**, 174 (1969); K. Wilson: *Phys. Rev.*, **179**, 1499 (1969); M. Dal-Cin and H. A. Kastrup: *Nucl. Phys.*, **15** B, 189 (1970).

(2) M. Gell-Mann: *Proceedings of the Third Hawaii Topical Conference on Elementary-Particle Physics* (Los Angeles, Cal., 1969).

(3) A. Salam and J. Strathdee: *Phys. Rev.*, **184**, 174 (1969); C. J. Isham, A. Salam and J. Strathdee: *Phys. Lett.*, **31** B, 300 (1970); S. P. De Alwis and P. J. O'Donnell: Toronto preprint (1970); J. Ellis: *Nucl. Phys.*, **22** B, 478 (1970) (there is a normalization error in this paper—see eq. (14)—ref. (4) agrees with us. *Note added in proof*: see erratum in *Nucl. Phys.*, **25** B, 639 (1971)); J. Wess and B. Zumino: to be published and ref. (4).

(4) For an elegant proof of the equivalence of Ward identities and low-energy theorems extracted from phenomenological Lagrangians in the tree approximation see, for example, B. Zumino: *Brandeis Lectures* (1970).

(5) See, for example, I. Gerstein and H. J. Schnitzer: *Phys. Rev.*, **175**, 1876 (1968).

(6) E. Huggins: Ph. D. Thesis, Caltech (1962); F. Gürsey: *Ann. of Phys.*, **24**, 211 (1963); C. Callan, S. Coleman and R. Jackiw: *Ann. of Phys.*, **59**, 42 (1970).

diately the low-energy theorem [2]

(4) $$\Gamma(0, p^2, p^2) = -2\Delta^{-1}(p^2)\left[(2-d_\varphi) + p^2\Delta^{-1}(p^2)\frac{\partial}{\partial p^2}\Delta(p^2)\right],$$

where $\Delta(p^2)$ is the propagator of the φ field

$$\Delta(p^2) = \int\frac{\varrho_\varphi(\mu^2)\,\mathrm{d}\mu^2}{p^2-\mu^2}$$

and Γ is the reduced vertex function defined by

(5) $$-\Delta(p^2)\,\Delta\big((p-q)^2\big)\,\Gamma(q^2, p^2, (p-q)^2) =$$
$$= \int \mathrm{d}y\,\mathrm{d}x \exp\left[-i(qx-py)\right]\langle 0|T\big(\theta(x)\,\varphi(y)\,\varphi(0)\big)|0\rangle\,.$$

On the mass shell of a particle of mass μ this gives the universal result

(6) $$\Gamma(0, \mu^2, \mu^2) = 2\mu^2 Z^{-1}\,.$$

Here Z is the wave function normalization of the single-particle state of mass μ. For convenience, we shall sometimes state the results in terms of the properly normalized vertex function $\hat{\Gamma} = Z\Gamma$. There is no dependence on the dimension of the interpolating field (as any on-mass shell result should be). The dimension enters, however, in the off-mass shell continuation, determining

(7) $$\frac{\partial}{\partial p^2}\hat{\Gamma}(0, p^2, \mu^2)|_{p^2=\mu^2} = d_\varphi - 1\,.$$

One might suspect that a similar result could also be obtained for the derivative of $\Gamma(q^2, \mu^2, \mu^2)$ with respect to q^2, in terms of the dimensional content of the *energy-momentum tensor*. Indeed, in a recent paper it was shown [7] that using the *additional* assumptions of

1) standard chiral-symmetry properties of the Hamiltonian,

2) σ dominance of the $\Gamma(q^2, p^2, p'^2)$ vertex (with $\varphi = \partial^\mu A_\mu$),

3) smoothness of the vertex function,

one can in fact derive such a result for $\partial\Gamma/\partial q^2$, involving only the dimension d of the chiral-symmetry-breaking part in θ_{00}:

(8) $$\frac{\partial\hat{\Gamma}}{\partial q^2}(0, m_\pi^2, m_\pi^2) = 1 + (d-2)\frac{m_\pi^2}{m_\sigma^2}\,.$$

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[7] H. Kleinert and P. H. Weisz: *Nucl. Phys.*, **27** B, 23 (1971).

Obviously this equation implies, even for an once-subtracted dispersion relation for $\Gamma(q^2, \mu^2, \mu^2)$, that the $\sigma\pi\pi$ coupling constant ([8]) (normalized by $\mathscr{L} = g_{\sigma\pi\pi}\cdot (m_\sigma/2)\,\sigma\pi^a\pi^a$) is

$$g_{\sigma\pi\pi} = \gamma\left(1 + [d-2]\frac{m_\pi^2}{m_\sigma^2}\right), \tag{9}$$

where $(m_\sigma^3\gamma^{-1})$ is the coupling $\langle 0|\theta|\sigma\rangle$ ([9]).

A second very simple Ward identity can be derived for the two-point function $\langle 0|T(D_\mu(x)\theta(0))|0\rangle$. Taking its divergence leads directly to

$$\Delta_{\theta\theta}(0) = \langle 0|i\left[\int d^3x\, D_0(\boldsymbol{x}, 0), \theta(0)\right]|0\rangle, \tag{10}$$

where $\Delta_{\theta\theta}$ is the propagator of the $\theta(x)$ field. Under the above assumption 2), the left-hand side becomes $-\gamma^{-2}m_\sigma^4$ such that

$$m_\sigma^4\gamma^{-2} = -\langle 0|i\left[\int d^3x\, D_0(\boldsymbol{x}, 0), \theta(0)\right]|0\rangle. \tag{11}$$

Equations (9) and (11) can be used to eliminate the unknown σ graviton coupling γ. The commutator in (11), on the other hand, can be evaluated by making dimensional assumptions on θ_{00}.

It is the purpose of this paper to discuss the consequences of (9) and (11) assuming only dimensional properties of the decomposition of θ_{00}, PCAC for π and K, and σ dominance of θ. In spite of the roughness of the information on possibile $I = 0$ s-wave resonances ([10]) the presently assumed principal candidate at $m_\sigma = 700$, $\Gamma_{\sigma\pi\pi} \approx 400$ helps considerably to distinguish between the possible dimensions of the different parts contained in θ_{00}.

2. – Dimensional and chiral content of $\theta_{00}(x)$.

We adopt the assumptions on the $SU_3\times SU_3$ decomposition of the energy density conjectured a long time ago by GELL-MANN ([11])

$$\theta_{00} = \theta_{00}^* + \delta + u^0 + cu^8, \tag{12}$$

([8]) Notice that our result holds up to order $O(m_\pi^2/M^2)$ and $O(m_\sigma^2/M'^2)$, where M, M' are the next significant mass contribution in $\Delta_{\partial^\mu A_\mu \partial^\nu A_\nu}$ and $\Delta_{\theta\theta_4}$. They are, in principle, valid for any m_π^2/m_σ^2. We are aware that such a term is physically as small as the neglected terms. However, we shall carry it along for the sake of allowing for a comparison with Lagrangian results.

([9]) Notice that this is the same result as was obtained by SALAM *et al.* ([3]), from phenomenological Lagrangians for the special value $\gamma = m_\sigma/f_\pi$. But, as was pointed out by ELLIS ([3]), the conformal covariance properties assumed by SALAM *et al.* do not determine γ. However, the physical significance of the arbitrary constant γ appearing in the construction of the chiral Lagrangian is perhaps not as manifest as here.

([10]) PARTICLE DATA GROUP: *Phys. Rev. Lett.*, **34** B, 1 (1971).

([11]) M. GELL-MANN: *Phys. Rev.*, **125**, 1067 (1962).

where θ_{00}^* and δ are $SU_3\times SU_3$ singlets. The terms u^0, u^8 are Lorentz scalars transforming like members of the $(3, \overline{3}) + (\overline{3}, 3)$ representation, which they form together with a nonet of pseudoscalar operators v^a $(a = 0, \dots, 8)$

$$\begin{cases} [A_0^a(x),\, u^b(0)]_{x_0=0} = -id^{abc}v^c(0)\,\delta^3(x)\,, \\ [A_0^a(x),\, v^b(0)]_{x_0=0} = id^{abc}u^c(0)\,\delta^3(x)\,. \end{cases} \tag{13}$$

The term u can conveniently be split into a term S conserving $SU_2\times SU_2$ and another term Σ, transforming like the fourth component of an $(\frac{1}{2}, \frac{1}{2})$ representation:

$$u \equiv u^0 + cu^8 = S + \Sigma\,.$$

From this assumption one readily derives, using the equation of motion for A_0^i, $i = 1, \dots, 8$,

$$\partial^\mu A_\mu^i = -(d^{ii0} + cd^{ii8})v^i - \delta^{i8}\sqrt{\tfrac{2}{3}}\,cv^0\,, \tag{14}$$

which in turn implies the so-called Σ commutator to be

$$\Sigma^{ij} \equiv i\left[\int \mathrm{d}^3x\, A(\boldsymbol{x}, 0),\, \partial^\mu A_\mu^j(0)\right] = \tag{15}$$
$$= \left(\tfrac{2}{3}\delta^{ij} + \sqrt{\tfrac{2}{3}}\,cd^{ij8}\right)u^0(0) + \left(\sqrt{\tfrac{2}{3}}\,d^{ijk} + cd^{j8l}d^{lik} + \tfrac{2}{3}\,c\delta^{j8}\delta^{ik}\right)u^k(0)\,.$$

This equation can be used to derive the well-known low-energy theorem for the propagator

$$\Delta^i(q^2) \equiv -i\int_x \exp[-iqx]\langle 0|T(\partial^\mu A_\mu^i(x)\,\partial^\nu A_\nu^i(0))|0\rangle\,,$$

$$\Delta^i(0) = \langle \Sigma^{ii}\rangle = a^i\langle S\rangle + b^i\langle\Sigma\rangle\,, \tag{16}$$

where

$$a^i = \begin{cases} 0 \\[2ex] \dfrac{r+1}{r+2}\,, \\[2ex] \dfrac{4}{3} \end{cases} \qquad b^i = \begin{cases} 1 \\[2ex] \dfrac{r+1}{2r} \\[2ex] \dfrac{1}{3} \end{cases} \qquad \text{for } i = \begin{cases} 1, 2, 3\,, \\[2ex] 4, 5, 6, 7\,, \\[2ex] 8\,, \end{cases}$$

and

$$r = -\frac{2}{3}\,\frac{c + \sqrt{2}}{c}\,. \tag{17}$$

From eq. (16), we find for $i = \pi$, K that the vacuum expectation values $\langle S\rangle$ and $\langle\Sigma\rangle$ can be expressed as

$$\langle\Sigma\rangle = \Delta^{\pi}(0)\,, \tag{18}$$

$$\langle s\rangle = (r+2)\left(\frac{1}{r+1}\,\Delta^{K}(0) - \frac{1}{2r}\,\Delta^{\pi}(0)\right). \tag{19}$$

These two equations allow us to obtain for $u = S + \Sigma$

$$\langle u\rangle = \frac{r-2}{2r}\,\Delta^{\pi}(0) + \frac{r+2}{r+1}\,\Delta^{K}(0)\,. \tag{20}$$

After these preliminaries, let us proceed to the evaluation of the commutator in eq. (10)

$$i\left[\int D_0(\boldsymbol{x}, 0)\,\mathrm{d}^3x, \theta(0)\right].$$

To do so, we obviously need information on the dimensional content of $\theta(0)$. Let us assume that δ is a sum of scalars δ_n of definite dimension d_n, while u has the dimension d ([12]). Then it can be shown that θ has the form ([2])

$$\theta = \sum_n (4 - d_n)\,\delta_n + (4 - d)\,u\,. \tag{21}$$

As a consequence, eq. (10) is immediately evaluated to give

$$\Delta_{\theta\theta}(0) = \sum_n (4 - d_n)\,d_n\langle\delta_n\rangle + (4 - d)\,d\langle u\rangle\,. \tag{22}$$

Now we also have the following trivial identities:

$$\left\{\begin{aligned} \langle\theta\rangle &= \sum_n (4 - d_n)\langle\delta_n\rangle + (4 - d)\langle u\rangle = 0\,,\\ \langle\theta_{00}\rangle &= \langle\theta^*_{00}\rangle + \sum_n \langle\delta_n\rangle + \langle u\rangle = 0\,,\end{aligned}\right. \tag{23}$$

([12]) Notice, however, that many popular models have a more complicated vacuum expectation value structure. For example, in the SU_2 π-σ model, the term δ consists of the terms $(1/4)(3m_\pi^2 - m_\sigma^2)\times(\sigma^2 + \pi^2)$ and a constant $= f_\pi^2(3m_\pi^2 + m_\sigma^2)/8 > 0$ (necessary to make the vacuum expectation value of the Lagrangian vanish). The first term has dimension two, the second dimension zero. The vacuum expectation value of θ^*_{00} is $f_\pi^2(m_\sigma^2 - m_\pi^2)/8$. Thus in the σ-model there are just two terms among θ^*_{00} and δ_n with nonvanishing vacuum expectation value only when $m_\sigma^2 = m_\pi^2$ or $m_\sigma^2 = 3m_\pi^2$.

from which we immediately see that if among $\langle\theta^*_{00}\rangle$ and $\langle\delta_n\rangle$ there are less than two nonvanishing terms, then $\Delta_{\theta\theta}(0)=0$. We reject this case for it would require the corresponding spectral function $\varrho_\theta(\mu^2)$ to vanish everywhere. A corollary of this simple result is that there must be at least one δ_n with a nonvanishing vacuum expectation value (*i.e.* there must be a δ), although this can be a constant.

The important observation now is that if among θ^*_{00} and δ_n there are *two and only two* operators O', O'' with nonvanishing vacuum expectation values [12] then $\Delta_{\theta\theta}$ can be expressed completely in terms of $\langle u\rangle$ and the dimensions d' and d'' of O' and O'' respectively:

$$\Delta_{\theta\theta}(0)=(d-d')(d''-d)\langle u\rangle\,. \tag{24}$$

Equation (24), together with the expression (20) for $\langle u\rangle$ becomes our *central exact result*

$$\Delta_{\theta\theta}(0)=(d-d')(d''-d)\left[\frac{r-2}{2r}\,\Delta^\pi(0)+\frac{r+2}{r+1}\,\Delta^{\mathrm{K}}(0)\right]. \tag{25}$$

The result can be restated in terms of a sum rule relating the spectral function of the scalar operator θ to those of the pseudoscalar operators $\partial^\mu A^\pi_\mu$ and $\partial^\mu A^{\mathrm{K}}_\mu$:

$$\int\varrho_\theta(\mu^2)\,\frac{\mathrm{d}\mu^2}{\mu^2}=(d-d')(d''-d)\left[\frac{r-2}{2r}\int\varrho_\pi(\mu^2)\,\frac{\mathrm{d}\mu^2}{\mu^2}+\frac{r+2}{r+1}\int\varrho_{\mathrm{K}}(\mu^2)\,\frac{\mathrm{d}\mu^2}{\mu^2}\right]. \tag{26}$$

Let us discuss the consequences of this sum rule. We assume as usual that Δ^π, Δ^{K} are dominated by a π and K pole, respectively, *i.e.* the spectral functions are approximated by

$$\varrho_\pi(\mu^2)=f_\pi^2\,m_\pi^4\,\delta(\mu^2-m_\pi^2)\,,\qquad \varrho_{\mathrm{K}}(\mu^2)=f_{\mathrm{K}}^2\,m_{\mathrm{K}}^4\,\delta(\mu^2-m_{\mathrm{K}}^2)\,.$$

In addition, σ dominance of $\Delta_{\theta\theta}$ gives

$$\varrho_\theta(\mu^2)\equiv\frac{m_\sigma^6}{\gamma^2}\,\delta(\mu^2-m_\sigma^2)\,. \tag{27}$$

Therefore we find in this approximation

$$\frac{m_\sigma^4}{\gamma^2}=(d-d')(d''-d)\left[\frac{r-2}{2r}\,f_\pi^2 m_\pi^2+\frac{r+2}{r+1}\,f_{\mathrm{K}}^2 m_{\mathrm{K}}^2\right]. \tag{28}$$

3. – Discussion of the result.

Two types of ideas have been forwarded as to what the mechanism of $SU_3\times SU_3$ breaking amounts to. In the old Gell-Mann scheme, recently revived by BRANDT and PREPARATA ([13]), θ^*_{00} of dimension four creates a massless world which is made massive by a term δ breaking dilatational symmetry. Then $SU_3 \times SU_3$ is broken, conserving SU_3, by u^0, and finally a very small SU_3-breaking cu^8 is introduced to cause the SU_3 splitting of baryon and meson masses. In this scheme the smallness of the pion mass is an unexplained dynamical accident. The value r one finds from several considerations is $r \approx 3.3$ ([13]). Thus, neglecting the small pion contribution in (28), we obtain

$$\frac{m_\sigma^4}{\gamma^2} \approx (d-d')(d''-d)\,1.2 f_{\mathrm{K}}^2 m_{\mathrm{K}}^2 \,. \tag{29}$$

According to more recent ideas of GELL-MANN *et al.* ([14]), however, the small masses of the pseudoscalar mesons and the quality of PCAC can most naturally be explained by assuming the vacuum to break the $U_3 \times U_3$ symmetry of θ^*_{00} and creating a nonet of massless pseudoscalar Goldstone bosons. The term δ should commute with all $U_3 \times U_3$ generators, except for Q_5^0 to lift the singlet particle η' to its high mass of 985 MeV. Finally, an approximately $SU_2 \times SU_2$-invariant u term lifts K and η masses, leaving only the pion almost massless.

In this scheme r is estimated by the pseudoscalar masses as

$$r \approx \frac{m_\pi^2}{m_{\mathrm{K}}^2 - m_\pi^2}\,, \tag{30}$$

such that (29) becomes

$$\frac{m_\sigma^4}{\gamma^2} \approx (d-d')(d''-d)\left[\frac{m_\pi^2}{2}(3f_\pi^2 - 2f_{\mathrm{K}}^2) + m_{\mathrm{K}}^2(2f_{\mathrm{K}}^2 - f_\pi^2)\right]. \tag{31}$$

Notice that this result reduces to the one obtained by Lagrangian methods ([3,4]) if one sets $f_{\mathrm{K}} = f_\pi$, and $d' = 0$, $d'' = 4$ (ref. ([3,4]) assume δ is a constant).

In conjunction with eq. (9) for $g_{\sigma\pi\pi}$, we can eliminate γ from (29) and (31) and we find, in terms of mass and width of the σ-meson, our final result in the Brandt-Preparata scheme:

$$\frac{m_\sigma}{\Gamma_\sigma} \approx X\,\frac{32\pi}{3m_\sigma^4}\,1.2 f_{\mathrm{K}}^2 m_{\mathrm{K}}^2\left(1 + [d-2]\,\frac{m_\pi^2}{m_\sigma^2}\right)^{-2}, \tag{BP}$$

while the Gell-Mann–Oakes–Renner scheme gives

$$\frac{m_\sigma}{\Gamma_\sigma} \approx X\,\frac{32\pi}{3m_\sigma^4}\,(2f_{\mathrm{K}}^2 - f_\pi^2)\; m_{\mathrm{K}}^2\left(1 + [d-2]\,\frac{m_\pi^2}{m_\sigma^2}\right)^{-2}. \tag{GOR}$$

([13]) R. A. BRANDT and G. PREPARATA: *Ann. of Phys.*, **61**, 119 (1970).
([14]) M. GELL-MANN, R. OAKES and B. RENNER: *Phys. Rev.*, **175**, 2195 (1968).

Here we have introduced for brevity the parameter $X=(d-d')(d''-d)$. Notice that in both schemes $X>0$ and hence d necessarily lies between d' and d''.

These equations can be used to get some limits as to the possible dimensions of the symmetry breakers δ and u. We shall assume that only integer dimensions occur in these local fields. As we have discussed in our last paper (7), we believe that the naive dimensions of a field remain approximately a good concept in the presence of interactions (15). This has been proved to any finite order in renormalizable field theory (16). It has been pointed out by WILSON that in the Thirring model logarithmic factors arising in finite-order perturbation theory pile up to change the dimension completely (17). Also there is indication that the scale-invariant $\lambda\varphi^4$ theory has the same behaviour (18). From investigations of the non–scale-invariant φ^3 theory (16) we know, however, that such a pile-up does not occur. It is our belief that any theory containing massive constants from the beginning will have approximately integer dimensions. These considerations severely limit the possible values d', d'' and d can assume. From the Lehmann spectral representation we know that the dimension of a proper local operator has to be $\geqslant 1$. Allowing also constants, we have $0\leqslant d', d'', d\leqslant 4$. It has been argued (19) that no dimension higher than four should play an important role in the Hamiltonian. If we accept this point of view, we obtain for every ordered triplet (d', d, d'') the following possible values of X:

$(d'\ d\ d'')$	X
4 3 2	1
4 3 1	2
4 3 0	3
4 2 1	2
4 2 0	4
4 1 0	3
3 2 1	1
3 2 0	2
3 1 0	2
2 1 0	1

(15) If a local field $\varphi(x)$ has a dimension d, then the operator product $\varphi(x)\varphi(0)$ diverges for $x\to 0$, like $1/(x^2)^d$ (see K. WILSON, ref. (1)). It has been verified to finite order in perturbation theory that this singularity is modified only by logarithmic factors in the presence of interactions (see ref. (16)).

(16) K. SYMANZIK: *Comm. Math. Phys.*, **18**, 227 (1970).

(17) K. WILSON: *Phys. Rev. D*, **2**, 1473 (1970).

(18) R. GATTO: private communication.

(19) K. WILSON: *Phys. Rev.*, **179**, 1499 (1969).

Let us now assume that the σ-particle under consideration is the experimentally inferred $\varepsilon(700)$ of width $\Gamma_{\sigma\pi\pi} = 400$ [10,20]. Then we obtain the following values of X:

1) Brandt-Preparata scheme

a) $f_K/f_\pi = 1$: $X = 4.7$ ($\geqslant 3.3$),

b) $f_K/f_\pi = \frac{5}{4}$: $X = 3.0$ ($\geqslant 2.2$).

2) Gell-Mann–Oakes–Renner scheme

a) $f_K/f_\pi = 1$: $X = 5.8$ ($\geqslant 4.0$),

b) $f_K/f_\pi = \frac{5}{4}$: $X = 2.7$ ($\geqslant 1.9$).

The numbers in parentheses indicate the lower limits on X provided by the Adler-Weisberger relation ($g_{\sigma\pi\pi} \leqslant m_\sigma/\sqrt{2}f_\pi$) [21].

We note that in both schemes the values of X are extremely sensitive to the value of f_K/f_π taken. We consider two cases:

a) For $f_K/f_\pi = 1$ we see that the experimental values for X do not appear in our Table of possible X values, however, if $g_{\sigma\pi\pi}$ attains its Adler-Weisberger bound then $X \approx 3 \div 4$. This high value for X would exclude all but three (d', d, d'') combinations, each of which have $d' = 4$, $d'' = 0$. If $d'' = 0$, however, O'' is a constant and hence in the Gell-Mann scheme (where θ^*_{00} commutes with the whole $U_3 \times U_3$) it is not capable of raising the η' mass (since it commutes with Q_5^0); this would have to be done by other terms δ_n with vanishing vacuum expectation value.

b) For $f_K/f_\pi = \frac{5}{4}$ (which corresponds to $F_+(0)$ very close to one) we unfortunately find that our sum rule does not distinguish between the two proposed values for r [22], both schemes give $X = 2 \div 3$. This value for X allows

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(20) J. Engels and G. Höhler: *Nucl. Phys.*, **15** B, 365 (1970).

(21) In the σ-model, the $g_{\sigma\pi\pi}$ coupling constant can be obtained as $g_{\sigma\pi\pi} = (m_\sigma/M_{\mathcal{N}})\, g_{\pi\mathcal{N}\mathcal{N}} = (m_\sigma/m_{\mathcal{N}})\, g_{\sigma\mathcal{N}\mathcal{N}}$. The right part of this relation is very well satisfied (see ref. (20)) where $(g^2_{\sigma\mathcal{N}\mathcal{N}})/4\pi \approx 18 \pm 2$. The left-hand side yields, however, upon using the Goldberger-Treiman relation ($g_{\pi\mathcal{N}\mathcal{N}} = (m_{\mathcal{N}}/f_\pi) g_A$), the value $g_{\sigma\pi\pi} = (m_\sigma/f_\pi) g_A$. This result obviously violates the Adler-Weisberger limit $g_{\sigma\pi\pi} = m_\sigma/\sqrt{2} f_\pi$ giving a width of ≈ 1200 MeV. The reason is, clearly, that in the σ-model σ exchange alone makes up all the correctly normalized $\pi\pi$ scattering. There is no ρ-meson to fill the missing half of the sum rule. Notice that $g_{\sigma\pi\pi} = (m_\sigma/m_{\mathcal{N}})\, g_{\sigma\mathcal{N}\mathcal{N}}$ can also be derived assuming an unsubtracted dispersion relation for the $\langle \mathcal{N}|\theta|\mathcal{N}\rangle$ vertex. Thus its failure indicates the necessity for a subtraction just like in the $\langle\pi|\theta|\pi\rangle$ vertex.

(22) This is, however, a familiar situation. Thus in the pole approximation for the sum rule $\Delta^\kappa(0) = (1/(1+r))\,\Delta^\pi(0) - (1/r)\,\Delta^K(0)$, with $f_K/f_\pi = \frac{5}{4}$, one also obtains very sim-

us to reject many models, however, at the same time it is consistent with many (d', d, d'') combinations and hence the assumptions are consistent with no anomalously high dimensions entering the theory.

4. – Conclusion.

There has to be at least one operator δ in the Hamiltonian density with nonvanishing vacuum expectation value, although this operator can be a constant. When $F_+(0) \approx 1$ the σ dominance of $\Delta_{\theta\theta}$ is consistent with many dimensional structures of the Hamiltonian density in both the Gell-Mann–Oakes–Renner and the Brandt-Preparata symmetry-breaking schemes (in particular it is consistent with having just one δ term of definite dimension). If f_K/f_π is very close to one, however, the high η' mass requires at least two terms of different dimensions in δ in the Gell-Mann–Oakes–Renner scheme if θ^*_{00} commutes with $U_3 \times U_3$.

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ilar estimates for $m_\varkappa f_\varkappa$ in the two schemes:

$$m_\varkappa \approx \begin{cases} 0.75 \left|\dfrac{f_\pi}{f_\varkappa}\right| m_K\,, & \text{GOR}\,, \\[2ex] 0.6 \left|\dfrac{f_\pi}{f_\varkappa}\right| m_K\,, & \text{BP}\,, \end{cases}$$

i.e. both schemes require a small $|f_\varkappa/f_\pi|$ for a large $m_\varkappa$. (Recall that with $f_K = f_\pi$ the two values for r, $r = r_{(GOR)}$ and $r = \infty$, give exactly the same estimates for $\langle u \rangle$ and $f^2_\varkappa m^2_\varkappa = 0$.)

● RIASSUNTO (*)

Si esaminano le conseguenze della decomposizione di Gell-Mann della densità di energia $\theta_{00}(x) = \theta^*_{00}(x) + \sum_n \delta_n(x) = u(x)$, dove u è il termine di rottura di $SU_3 \times SU_3$ che si trasforma secondo una rappresentazione $(3\bar{3}) + (\bar{3}3)$ ed è uno scalare di Lorentz di dimensione d. Si dimostra che vi devono essere almeno due termini fra θ^*_{00}, δ_n con valori previsti del vuoto non nulli. Se si suppone inoltre che vi siano solo due di tali termini di dimensioni d' e d'', si può ottenere una nuova regola di somma che interessa le

(*) *Traduzione a cura della Redazione.*

funzioni spettrali dei propagatori di θ^{μ}_{μ}, $\partial^{\mu}A^{\pi}_{\mu}$ e $\partial^{\mu}A^{K}_{\mu}$, del tipo:

$$\int \varrho_{\theta}(\mu^2)\frac{d\mu^2}{\mu^2} = X\left[\frac{r-2}{2r}\int \varrho_{\pi}(\mu^2)\frac{d\mu^2}{\mu^2} + \frac{r+2}{r+1}\int \varrho_{K}(\mu^2)\frac{d\mu^2}{\mu^2}\right],$$

con $r = -\frac{2}{3}(c+\sqrt{2})/c$ e $X = (d-d')(d''-d)$. Saturando con σ, π, K, si trova, usando un risultato precedente per $g_{\sigma\pi\pi}$,

$$\frac{m_{\sigma}}{\Gamma_{\sigma\pi\pi}} = \frac{32\pi}{3m_{\sigma}^2}X\left[\frac{r-2}{2r}f_{\pi}^2m_{\pi}^2 + \frac{r+2}{r+1}f_{K}^2m_{K}^2\right]\left(1+[d-2]\frac{m_{\pi}^2}{m_{\sigma}^2}\right)^{-2}.$$

Si usano i valori sperimentali $m_{\sigma}\approx 700$ e $\Gamma_{\sigma\pi\pi}\approx 400$ per valutare X in entrambi gli schemi della rottura di simmetria $SU_3\times SU_3$ di Gell-Mann, Oakes e Renner e di Brandt e Preparata. Sfortunatamente per valori ragionevoli di f_K/f_{π} la regola di somma ottenuta non distingue fra i due schemi; per esempio, per $f_K/f_{\pi} = 1.25$, entrambi gli schemi danno $X\approx 3$. Nondimeno, questo valore tipico permette di concludere che il dominio di σ è consistente con molti modelli non interessanti operatori di dimensioni anormalmente alte ed allo stesso tempo permette di escludere molti altri modelli.

О размерности нарущения чиральной и конформной симметрий.

Резюме (*). — Мы исследуем общие следствия разложения Гелл-Манна плотности энергии $\theta_{00}(x) = \theta^{*}_{00}(x) + \sum_n \delta_n(x) + u(x)$, где u представляет член, нарушающий $SU_3\times SU_3$, который преобразуется согласно представлению $(3\bar{3})+(\bar{3}3)$, и является лорентцевским скаляром размерности d. Мы показываем, что должно быть, по крайней мере, два члена среди θ^{*}_{00}, δ_n с неисчезающими вакуумными ожидаемыми величинами. Если мы затем предположим, что существуют только два таких члена с размерностями d' и d'', то мы можем получить новое правило сумм, включающее спектральные функции пропагаторов θ^{μ}_{μ}, $\partial^{\mu}A^{\pi}_{\mu}$ и $\partial^{\mu}A^{K}_{\mu}$, которое имеет вид

$$\int \varrho_{\theta}(\mu^2)\frac{d\mu^2}{\mu^2} = X\left[\frac{r-2}{2r}\int \varrho_{\pi}(\mu^2)\frac{d\mu^2}{\mu^2} + \frac{r+2}{r+1}\int \varrho_{K}(\mu^2)\frac{d\mu^2}{\mu^2}\right],$$

где $r = -\frac{2}{3}(c+\sqrt{2})/c$ и $X = (d-d')(d''-d)$. Используя предыдущий результат для $g_{\sigma\pi\pi}$, при насыщении с σ, π, K получается

$$\frac{m_{\sigma}}{\Gamma_{\sigma\pi\pi}} = \frac{32\pi}{3m_{\sigma}^4}X\left[\frac{r-2}{2r}f_{\pi}^2m_{\pi}^2 + \frac{r+2}{r+1}f_{K}^{2}m_{K}^2\right]\left(1+[d-2]\frac{m_{\pi}^2}{m_{\sigma}^2}\right)^{-2}.$$

Мы используем экспериментальные значения $m_{\sigma}\approx 700$ и $\Gamma_{\sigma\pi\pi}\approx 400$, чтобы оценить X в схеме нарушения $SU_3\times SU_3$ симметрии Гелл-Манна, Оакса, Реннера и в схеме Брандта и Препараты. К сожалению, для соответствующих значений f_K/f_{π} наше правило сумм не дает различия между этими двумя схемами, т.е. для $f_K/f_{\pi}=1.25$ обе схемы приводят к $X\approx 3$. Тем не менее, это типичное значение позволяет нам утверждать, что σ доминантность согласуется с множеством моделей, не включающих операторы аномально высокой размерности, и в то же время позволяет нам исключить много других моделей.

(*) *Переведено редакцией.*