As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.
ALBERT EINSTEIN (1879 - 1955)

3

Continuous Symmetries and Conservation Laws. Noether's Theorem

In many physical systems, the action is invariant under some continuous set of transformations. Then there exist local and global *conservation laws* analogous to current and charge conservation in electrodynamics. With the help of Poisson brackets, the analogs of the charges can be used to generate the symmetry transformation, from which they were derived. After field quantization, the Poisson brackets become commutators of operators associated with these charges.

3.1 Continuous Symmetries and Conservation Laws

Consider first a simple mechanical system with a generic action

$$\mathcal{A} = \int_{t_a}^{t_b} dt\, L(q(t), \dot{q}(t)), \tag{3.1}$$

and subject it to a continuous set of local transformations of the dynamical variables:

$$q(t) \to q'(t) = f(q(t), \dot{q}(t)), \tag{3.2}$$

where $f(q(t), \dot{q}(t))$ is some function of $q(t)$ and $\dot{q}(t)$. In general, $q(t)$ will carry various labels as in (2.1) which are suppressed, for brevity. If the transformed action

$$\mathcal{A}' \equiv \int_{t_a}^{t_b} dt\, L(q'(t), \dot{q}'(t)) \tag{3.3}$$

is the same as $\mathcal{A}$, up to boundary terms, then (3.2) is called a symmetry transformation.

3.1.1 Group Structure of Symmetry Transformations

For any two *symmetry transformations*, we may defined a product by performing the transformations successively. The result is certainly again a symmetry transformation. Since all transformations can be undone, they possess an inverse. Thus, symmetry transformations form a group called the *symmetry group* of the system. It is important that the equations of motion are *not* used when showing that the action $\mathcal{A}'$ is equal to $\mathcal{A}$, up to boundary terms.

3.1.2 Substantial Variations

For infinitesimal symmetry transformations (3.2), the difference

$$\delta_{\rm s} q(t) \equiv q'(t) - q(t) \tag{3.4}$$

will be called a *symmetry variation*. It has the general form

$$\delta_{\rm s} q(t) = \epsilon \Delta(q(t), \dot{q}(t)), \tag{3.5}$$

where ϵ is a small parameter. Symmetry variations must not be confused with the variations $\delta q(t)$ used in Section 2.1 to derive the Euler-Lagrange equations (2.8), which always vanish at the ends, $\delta q(t_b) = \delta q(t_a) = 0$ [recall (1.4)]. This is usually not true for symmetry variation $\delta_{\rm s} q(t)$.

Another name for the symmetry variation (3.5) is *substantial variation*. It is defined for any function of spacetime $f(x)$ as the difference between $f(x)$ and a transformed function $f'(x)$ when evaluated at the *same numerical values of the coordinates* x (which usually correspond to two *different points* in space):

$$\delta_s f(x) \equiv f(x) - f'(x). \tag{3.6}$$

3.1.3 Conservation Laws

Let us calculate the change of the action under a substantial variation (3.5). Using the chain rule of differentiation and a partial integration we obtain

$$\delta_{\rm s} \mathcal{A} = \int_{t_a}^{t_b} dt \left[\frac{\partial L}{\partial q(t)} - \partial_t \frac{\partial L}{\partial \dot{q}(t)} \right] \delta_{\rm s} q(t) + \left. \frac{\partial L}{\partial \dot{q}(t)} \right|_{t_a}^{t_b}. \tag{3.7}$$

Let us denote the solutions of the Euler-Lagrange equations (2.8) by $q_c(t)$ and call them *classical orbits*, or briefly *orbits*. For orbits, only the boundary terms in (3.7) survive, and we are left with

$$\delta_{\rm s} \mathcal{A} = \epsilon \left. \frac{\partial L}{\partial \dot{q}} \Delta(q, \dot{q}) \right|_{t_b}^{t_a}, \quad \text{for} \;\; q(t) = q_c(t). \tag{3.8}$$

By the symmetry assumptions, $\delta_{\rm s} \mathcal{A}$ vanishes or is equal to a surface term. In the first case, the quantity

$$Q(t) \equiv \frac{\partial L}{\partial \dot{q}} \Delta(q, \dot{q}), \quad \text{for} \;\; q(t) = q_c(t) \tag{3.9}$$

is the same at times $t = t_a$ and $t = t_b$. Since t_b is arbitrary, $Q(t)$ is *independent* of the time t, i.e., it satisfies

$$Q(t) \equiv Q. \tag{3.10}$$

It is a *conserved quantity*, a *constant of motion* along the orbit. The expression on the right-hand side of (3.9) is called a *Noether charge*.

In the second case, $\delta_s q(t)$ is equal to a boundary term

$$\delta_s \mathcal{A} = \epsilon\, \Lambda(q,\dot q)\Big|_{t_a}^{t_b}, \tag{3.11}$$

and the conserved Noether charge becomes

$$Q(t) = \frac{\partial L}{\partial \dot q}\Delta(q,\dot q) - \Lambda(q,\dot q), \quad \text{for} \quad q(t)=q_c(t). \tag{3.12}$$

It is possible to derive the constant of motion (3.12) also without invoking the action, starting from the Lagrangian $L(q,\dot q)$. We expand its substantial variation of $L(q,\dot q)$ as follows:

$$\delta_s L \equiv L\left(q+\delta_s q, \dot q+\delta_s \dot q\right) - L(q,\dot q) = \left[\frac{\partial L}{\partial q(t)} - \partial_t \frac{\partial L}{\partial \dot q(t)}\right]\delta_s q(t) + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot q(t)}\delta_s q(t)\right]. \tag{3.13}$$

On account of the Euler-Lagrange equations (2.8), the first term on the right-hand side vanishes as before, and only the last term survives. The assumption of invariance of the action up to a possible surface term in Eq. (3.11) is equivalent to assuming that the substantial variation of the Lagrangian is at most a *total time derivative* of some function $\Lambda(q,\dot q)$:

$$\delta_s L(q,\dot q,t) = \epsilon \frac{d}{dt}\Lambda(q,\dot q). \tag{3.14}$$

Inserting this into the left-hand side of (3.13), we find the equation

$$\epsilon \frac{d}{dt}\left[\frac{\partial L}{\partial \dot q}\,\Delta(q,\dot q) - \Lambda(q,\dot q)\right] = 0, \quad \text{for} \quad q(t)=q_c(t) \tag{3.15}$$

thus recovering again the conserved Noether charge (3.12).

3.1.4 Alternative Derivation of Conservation Laws

Let us subject the action (3.1) to an arbitrary variation $\delta q(t)$, which may be nonzero at the boundaries. Along a classical orbit $q_c(t)$, the first term in (3.7) vanishes, and the action changes at most by the boundary term:

$$\delta \mathcal{A} = \frac{\partial L}{\partial \dot q}\delta q\Big|_{t_a}^{t_b}, \quad \text{for} \quad q(t)=q_c(t). \tag{3.16}$$

This observation leads to another derivation of Noether's theorem. Suppose we perform on $q(t)$ a so-called *local symmetry transformations*, which generalizes the previous substantial variations (3.5) to a *time-dependent* parameter ϵ:

$$\delta_s^t q(t) = \epsilon(t)\Delta(q(t),\dot q(t)). \tag{3.17}$$

The superscript t on δ_s^t emphasized the extra time dependence in $\epsilon(t)$. If the variations (3.17) are applied to a classical orbit $q_c(t)$, the action changes by the boundary term (3.16).

This will now be expressed in a more convenient way. For this purpose we introduce the infinitesimally transformed orbit

$$q^{\epsilon(t)}(t) \equiv q(t) + \delta_s^t q(t) = q(t) + \epsilon(t)\Delta(q(t), \dot{q}(t)), \tag{3.18}$$

and the transformed Lagrangian

$$L^{\epsilon(t)} \equiv L(q^{\epsilon(t)}(t), \dot{q}^{\epsilon(t)}(t)). \tag{3.19}$$

Then the local substantial variation of the action with respect to the time-dependent parameter $\epsilon(t)$ is

$$\delta_s^t \mathcal{A} = \int_{t_a}^{t_b} dt \left[\frac{\partial L^{\epsilon(t)}}{\partial \epsilon(t)} - \frac{d}{dt} \frac{\partial L^{\epsilon(t)}}{\partial \dot{\epsilon}(t)} \right] \epsilon(t) + \frac{d}{dt} \left[\frac{\partial L^{\epsilon(t)}}{\partial \dot{\epsilon}} \right] \epsilon(t) \Bigg|_{t_a}^{t_b}. \tag{3.20}$$

Along a classical orbit, the action is extremal. Hence the infinitesimally transformed action

$$\mathcal{A}^\epsilon \equiv \int_{t_a}^{t_b} dt \, L(q^{\epsilon(t)}(t), \dot{q}^{\epsilon(t)}(t)) \tag{3.21}$$

must satisfy the equation

$$\frac{\delta \mathcal{A}^\epsilon}{\delta \epsilon(t)} = 0. \tag{3.22}$$

This holds for an arbitrary time dependence of $\epsilon(t)$, in particular for $\epsilon(t)$ which vanishes at the ends. In this case, (3.22) leads to an Euler-Lagrange type of equation

$$\frac{\partial L^{\epsilon(t)}}{\partial \epsilon(t)} - \frac{d}{dt} \frac{\partial L^{\epsilon(t)}}{\partial \dot{\epsilon}(t)} = 0, \quad \text{for} \quad q(t) = q_c(t). \tag{3.23}$$

This can also be checked explicitly differentiating (3.19) according to the chain rule of differentiation:

$$\frac{\partial L^{\epsilon(t)}}{\partial \epsilon(t)} = \frac{\partial L}{\partial q(t)} \Delta(q, \dot{q}) + \frac{\partial L}{\partial \dot{q}(t)} \dot{\Delta}(q, \dot{q}), \tag{3.24}$$

$$\frac{\partial L^{\epsilon(t)}}{\partial \dot{\epsilon}(t)} = \frac{\partial L}{\partial \dot{q}(t)} \Delta(q, \dot{q}), \tag{3.25}$$

and inserting on the right-hand side the ordinary Euler-Lagrange equations (1.5). Note that (3.25) can also be written as

$$\frac{\partial L^{\epsilon(t)}}{\partial \dot{\epsilon}(t)} = \frac{\partial L}{\partial \dot{q}(t)} \frac{\delta_s q(t)}{\epsilon(t)}. \tag{3.26}$$

We now invoke the symmetry assumption, that the action is a pure surface term under the time-independent transformations (3.17). This implies that

$$\frac{\partial L^{\epsilon}}{\partial \epsilon} = \frac{\partial L^{\epsilon(t)}}{\partial \epsilon(t)} = \frac{d}{dt}\Lambda. \tag{3.27}$$

Combining this with (3.23), we derive a conservation law for the charge:

$$Q = \frac{\partial L^{\epsilon(t)}}{\partial \dot{\epsilon}(t)} - \Lambda, \quad \text{for} \;\; q(t) = q_c(t). \tag{3.28}$$

Inserting here Eq. (3.25) we find that this is the same charge as the previous (3.12).

3.2 Time Translation Invariance and Energy Conservation

As a simple but physically important example consider the case that the Lagrangian does not depend explicitly on time, i.e., that $L(q,\dot{q}) \equiv L(q,\dot{q})$. Let us perform a time translation on the system, so that the same events happen at a new time

$$t' = t - \epsilon. \tag{3.29}$$

The time-translated orbit has the time dependence

$$q'(t') = q(t), \tag{3.30}$$

i.e., the translated orbit $q'(t)$ has at the time t' the same value as the orbit $q(t)$ had at the original time t. For the Lagrangian, this implies that

$$L'(t') \equiv L(q'(t'), \dot{q}'(t')) = L(q(t), \dot{q}(t)) \equiv L(t). \tag{3.31}$$

This makes the action (3.3) equal to (3.1), up to boundary terms. Thus time-independent Lagrangians possess time translation symmetry.

The associated substantial variations of the form (3.5) read

$$\begin{aligned} \delta_{\rm s} q(t) &= q'(t) - q(t) = q(t'+\epsilon) - q(t) \\ &= q(t') + \epsilon\dot{q}(t') - q(t) = \epsilon\dot{q}(t), \end{aligned} \tag{3.32}$$

Under these, the Lagrangian changes by

$$\delta_{\rm s} L = L(q'(t), \dot{q}'(t)) - L(q(t), \dot{q}(t)) = \frac{\partial L}{\partial q}\delta_{\rm s} q(t) + \frac{\partial L}{\partial \dot{q}}\delta_{\rm s}\dot{q}(t). \tag{3.33}$$

Inserting $\delta_{\rm s} q(t)$ from (3.32) we find, without using the Euler-Lagrange equation,

$$\delta_{\rm s} L = \epsilon\left(\frac{\partial L}{\partial \dot{q}}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q}\right) = \epsilon\frac{d}{dt}L. \tag{3.34}$$

This has precisely the derivative form (3.14) with $\Lambda = L$, thus confirming that time translations are symmetry transformations.

According to Eq. (3.12), we find the Noether charge

$$Q = H \equiv \frac{\partial L}{\partial \dot{q}}\dot{q} - L(q, \dot{q}), \quad \text{for} \;\; q(t) = q_c(t) \tag{3.35}$$

to be a constant of motion. This is recognized as the *Legendre transform* of the Lagrangian, which is the *Hamiltonian* (2.10) of the system.

Let us briefly check how this Noether charge is obtained from the alternative formula (3.12). The time-dependent substantial variation (3.17) is here

$$\delta_s^t q(t) \;\; = \;\; \epsilon(t)\dot{q}(t) \tag{3.36}$$

under which the Lagrangian is changed by

$$\delta_s^t L = \frac{\partial L}{\partial q}\epsilon\dot{q} + \frac{\partial L}{\partial \dot{q}}(\dot{\epsilon}\dot{q} + \epsilon\ddot{q}) = \frac{\partial L^\epsilon}{\partial \dot{\epsilon}}\epsilon + \frac{\partial L^\epsilon}{\partial \dot{\epsilon}}\dot{\epsilon}, \tag{3.37}$$

with

$$\frac{\partial L^\epsilon}{\partial \dot{\epsilon}} = \frac{\partial L}{\partial \dot{q}}\dot{q} \tag{3.38}$$

and

$$\frac{\partial L^\epsilon}{\partial \epsilon} = \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\epsilon\ddot{q} = \frac{d}{dt}L. \tag{3.39}$$

The last equation confirms that time translations fulfill the symmetry condition (3.27), and from (3.38) we see that the Noether charge (3.28) coincides with the Hamiltonian found in Eq. (3.12).

3.3 Momentum and Angular Momentum

While the conservation law of energy follow from the symmetry of the action under time translations, conservation laws of momentum and angular momentum are found if the action is invariant under translations and rotations, respectively.

Consider a Lagrangian of a point particle in a Euclidean space

$$L = L(q^i(t), \dot{q}^i(t)). \tag{3.40}$$

In contrast to the previous discussion of time translation invariance, which was applicable to systems with arbitrary Lagrange coordinates $q(t)$, we denote the coordinates here by q^i, with the superscripts i emphasizing that we are now dealing with Cartesian coordinates. If the Lagrangian depends only on the velocities $\dot{q}^i$ and not on the coordinates q^i themselves, the system is *translationally invariant*. If it depends, in addition, only on $\dot{\mathbf{q}}^2 = \dot{q}^i\dot{q}^i$, it is also rotationally invariant.

The simplest example is the Lagrangian of a point particle of mass m in Euclidean space:

$$L = \frac{m}{2}\dot{\mathbf{q}}^2. \tag{3.41}$$

It exhibits both invariances, leading to conserved Noether charges of momentum and angular momentum, as we shall now demonstrate.

3.3.1 Translational Invariance in Space

Under a spatial translation, the coordinates q^i of the particle change to

$$q'^i = q^i + \epsilon^i, \tag{3.42}$$

where ϵ^i are small numbers. The infinitesimal translations of a particle path are [compare (3.5)]

$$\delta_s q^i(t) = \epsilon^i. \tag{3.43}$$

Under these, the Lagrangian changes by

$$\begin{aligned} \delta_s L &= L(q'^i(t), \dot{q}'^i(t)) - L(q^i(t), \dot{q}^i(t)) \\ &= \frac{\partial L}{\partial q^i}\delta_s q^i = \frac{\partial L}{\partial q^i}\epsilon^i = 0. \end{aligned} \tag{3.44}$$

By assumption, the Lagrangian is independent of q^i, so that the right-hand side vanishes. This is to be compared with the substantial variation of the Lagrangian around a classical orbit calculated with the help of the Euler-Lagrange equation:

$$\delta_s L = \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}\right)\delta_s q^i + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^i}\delta_s q^i\right] = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^i}\right]\epsilon^i \tag{3.45}$$

This has the form (3.8), from which we extract a conserved Noether charge (3.9) for each coordinate q^i, to be called p^i:

$$p^i = \frac{\partial L}{\partial \dot{q}^i}. \tag{3.46}$$

Thus the Noether charges associated with translational invariance are simply the canonical momenta of the point particle.

3.3.2 Rotational Invariance

Under rotations, the coordinates q^i of the particle change to

$$q'^i = R^i{}_j q^j \tag{3.47}$$

where $R^i{}_j$ are the orthogonal 3×3 -matrices (1.8). Infinitesimally, these can be written as

$$R^i{}_j = \delta^i{}_j - \varphi_k \epsilon_{kij} \tag{3.48}$$

where $\boldsymbol{\varphi}$ is the infinitesimal rotation vector in Eq. (1.57). The corresponding rotation of a particle path is

$$\delta_s q^i(t) = q'^i(t) - q^i(t) = -\varphi^k \epsilon_{kij} q^j(\tau). \tag{3.49}$$

In the antisymmetric tensor notation (1.55) with $\omega_{ij} \equiv \varphi_k \epsilon_{kij}$, we write

$$\delta_s q^i = -\omega_{ij} q^j. \tag{3.50}$$

Under this, the substantial variation of the Lagrangian (3.41)

$$\begin{aligned} \delta_s L &= L(q'^i(t), \dot{q}'^i(t)) - L(q^i(t), \dot{q}^i(t)) \\ &= \frac{\partial L}{\partial q^i}\delta_s q^i + \frac{\partial L}{\partial \dot{q}^i}\delta_s \dot{q}^i \end{aligned} \tag{3.51}$$

becomes

$$\delta_s L = -\left(\frac{\partial L}{\partial q^i} q^j + \frac{\partial L}{\partial \dot{q}^i}\dot{q}^j\right)\omega_{ij} = 0. \tag{3.52}$$

For any Lagrangian depending only on the rotational invariants $\mathbf{q}^2, \dot{\mathbf{q}}^2, \mathbf{q}\cdot\dot{\mathbf{q}}$ and powers thereof, the right-hand side vanishes on account of the antisymmetry of ω_{ij}. This ensures the rotational symmetry for the Lagrangian (3.41).

We now calculate the substantial variation of the Lagrangian once more using the Euler-Lagrange equations:

$$\begin{aligned} \delta_s L &= \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}\right)\delta_s q^i + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^i}\delta_s q^i\right] \\ &= -\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^i} q^j\right]\omega_{ij} = \frac{1}{2}\frac{d}{dt}\left[q_i \frac{\partial L}{\partial \dot{q}^j} - (i \leftrightarrow j)\right]\omega_{ij}. \end{aligned} \tag{3.53}$$

The right-hand side yields the conserved Noether charges of the type (3.9), one for each antisymmetric pair i, j:

$$L^{ij} = q^i \frac{\partial L}{\partial \dot{q}^j} - q^j \frac{\partial L}{\partial \dot{q}^i} \equiv q^i p^j - q^j p^i. \tag{3.54}$$

These are the conserved components of angular momentum for a Cartesian system in any dimension.

In three dimensions, we may prefer working with the original rotation angles φ^k, in which case we would have found the angular momentum in the standard form

$$L_k = \frac{1}{2}\epsilon_{kij} L^{ij} = (\mathbf{q} \times \mathbf{p})^k. \tag{3.55}$$

3.3.3 Center-of-Mass Theorem

Let us now study symmetry transformations corresponding to a uniform motion of the coordinate system described by Galilei transformations (1.11), (1.12). Consider a set of free massive point particles in Euclidean space described by the Lagrangian

$$L(\dot{q}^i_n) = \sum_n \frac{m_n}{2}\dot{q}^{i\,2}_n. \tag{3.56}$$

The infinitesimal substantial variation associated with the Galilei transformations are

$$\delta_{\rm s} q_n^i(t) = \dot{q}_n^i(t) - q_n^i(t) = -v^i t, \tag{3.57}$$

where v^i is a small relative velocity along the ith axis. This changes the Lagrangian by

$$\delta_{\rm s} L = L(q_n^i - v^i t, \dot{q}_n^i - v^i) - L(q_n^i, \dot{q}_n^i). \tag{3.58}$$

Inserting here (3.56), we find

$$\delta_{\rm s} L = \sum_n \frac{m_n}{2} \left[(\dot{q}_n^i - v^i)^2 - (\dot{q}_n{}^i)^2 \right], \tag{3.59}$$

which can be written as a total time derivative

$$\delta_{\rm s} L = \frac{d}{dt}\Lambda = \frac{d}{dt} \sum_n m_n \left[-\dot{q}_n^i v^i + \frac{v^2}{2} t \right] \tag{3.60}$$

proving that Galilei transformations are symmetry transformations in the Noether sense. Note that terms quadratic in v^i are omitted in the last expression since the velocities v^i in (3.57) are infinitesimal, by assumption.

By calculating $\delta_{\rm s} L$ once more via the chain rule with the help of the Euler-Lagrange equations, and equating the result with (3.60), we find the conserved Noether charge

$$\begin{aligned} Q &= \sum_n \frac{\partial L}{\partial \dot{q}_n^i} \delta_{\rm s} q_n^i - \Lambda \\ &= \left(-\sum_n m_n \dot{q}_n^i \, t + \sum_n m_n q_n^i \right) v^i. \end{aligned} \tag{3.61}$$

Since the direction of the velocities v^i is arbitrary, each component is separately a constant of motion:

$$N^i = -\sum_n m_n \dot{q}^i \, t + \sum_n m_n q_n{}^i = \text{const.} \tag{3.62}$$

This is the well-known *center-of-mass theorem* [1]. Indeed, introducing the center-of-mass coordinates

$$q_{\rm CM}^i \equiv \frac{\sum_n m_n q_n{}^i}{\sum_n m_n}, \tag{3.63}$$

and velocities

$$v_{\rm CM}^i = \frac{\sum_n m_n \dot{q}_n{}^i}{\sum_n m_n}, \tag{3.64}$$

the conserved charge (3.62) can be written as

$$N^i = \sum_n m_n(-v^i_{\rm CM}\, t + q^i_{\rm CM}). \tag{3.65}$$

The time-independence of N^i implies that the center-of-mass moves with uniform velocity according to the law

$$q^i_{\rm CM}(t) = q^i_{\rm CM,0} + v^i_{\rm CM} t, \tag{3.66}$$

where

$$q^i_{\rm CM,0} = \frac{N^i}{\sum_n m_n} \tag{3.67}$$

is the position of the center of mass at $t = 0$.

Note that in non-relativistic physics, the center of mass theorem is a consequence of momentum conservation since momentum $\equiv$ mass $\times$ velocity. In relativistic physics, this is no longer true.

3.3.4 Conservation Laws from Lorentz Invariance

In relativistic physics, particle orbits are described by functions in Minkowski spacetime $q^a(\sigma)$, where σ is a Lorentz-invariant length parameter. The action is an integral over some Lagrangian:

$$\mathcal{A} = \int_{\sigma_a}^{\sigma_b} d\sigma L\left(q^a(\sigma), \dot q^a(\sigma)\right), \tag{3.68}$$

where the dot denotes the derivative with respect to the parameter σ. If the Lagrangian depends only on invariant scalar products $q^a q_a, q^a \dot q_a,\ \dot q^a \dot q_a,$ then it is invariant under Lorentz transformations

$$q^a \to q'^a = \Lambda^a{}_b\, q^b \tag{3.69}$$

where $\Lambda^a{}_b$ are the pseudo-orthogonal 4×4 -matrices (1.28).

A free massive point particle in spacetime has the Lagrangian [see (2.19)]

$$L(\dot q(\sigma)) = -mc\sqrt{g_{ab}\dot q^a \dot q^b}, \tag{3.70}$$

so that the action (3.68) is invariant under arbitrary reparametrizations $\sigma \to f(\sigma)$. Since the Lagrangian depends only on $\dot q(\sigma)$, it is invariant under arbitrary translations of the coordinates:

$$\delta_{\rm s} q^a(\sigma) = q^a(\sigma) - \epsilon^a(\sigma), \tag{3.71}$$

for which $\delta_{\rm s} L = 0$. Calculating this variation once more with the help of the Euler-Lagrange equations, we find

$$\delta_{\rm s} L = \int_{\sigma_a}^{\sigma_b} d\sigma \left(\frac{\partial L}{\partial q^a}\delta_{\rm s} q^a + \frac{\partial L}{\partial \dot q^a}\delta_{\rm s}\dot q^a \right) = -\epsilon^a \int_{\sigma_a}^{\sigma_b} d\sigma \frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot q^a}\right). \tag{3.72}$$

From this we obtain the conserved Noether charges

$$p_a \equiv -\frac{\partial L}{\partial \dot{q}^a} = m\frac{\dot{q}_a}{\sqrt{g_{ab}\dot{q}^a\dot{q}^b/c^2}} = mu^a, \tag{3.73}$$

which satisfy the conservation law

$$\frac{d}{d\sigma}p_a(\sigma) = 0. \tag{3.74}$$

The Noether charges $p_a(\sigma)$ are the conserved four-momenta (1.144) of the free relativistic particle, derived in Eq. (2.20) from the canonical formalism. The four-vector

$$u^a \equiv \frac{\dot{q}^a}{\sqrt{g_{ab}\dot{q}^a\dot{q}^b/c^2}} \tag{3.75}$$

is the relativistic four-velocity of the particle. It is the reparametrization-invariant expression for the four-velocity $\dot{q}_a(\tau) = u_a(\tau)$ in Eqs. (2.22) and (1.144). A sign change is made in Eq. (3.73) to agree with the canonical definition of the covariant momentum components in (2.20). By choosing for σ the physical time $t = q^0/c$, we can express u^a in terms of the physical velocities $v^i = dq^i/dt$, as in (1.145):

$$u^a = \gamma(1, v^i/c), \qquad \text{with} \qquad \gamma \equiv \sqrt{1-v^2/c^2}. \tag{3.76}$$

For small Lorentz transformations near the identity we write

$$\Lambda^a{}_b = \delta^a{}_b + \omega^a{}_b \tag{3.77}$$

where

$$\omega^a{}_b = g^{ac}\omega_{cb} \tag{3.78}$$

is an arbitrary infinitesimal antisymmetric matrix. An infinitesimal Lorentz transformation of the particle path is

$$\begin{aligned} \delta_s q^a(\sigma) &= q'^a(\sigma) - q^a(\sigma) \\ &= \omega^a{}_b q^b(\sigma). \end{aligned} \tag{3.79}$$

Under it, the substantial variation of a Lorentz-invariant Lagrangian vanishes:

$$\delta_s L = \left(\frac{\partial L}{\partial q^a}q^b + \frac{\partial L}{\partial \dot{q}^a}\dot{q}^b\right)\omega^a{}_b = 0. \tag{3.80}$$

This is to be compared with the substantial variation of the Lagrangian calculated via the chain rule with the help of the Euler-Lagrange equation

$$\begin{aligned} \delta_s L &= \left(\frac{\partial L}{\partial q^a} - \frac{d}{d\sigma}\frac{\partial L}{\partial \dot{q}^a}\right)\delta_s q^a + \frac{d}{d\sigma}\left[\frac{\partial L}{\partial \dot{q}^a}\delta_s q^a\right] \\ &= \frac{d}{d\sigma}\left[\frac{\partial L}{\partial \dot{q}^a}\dot{q}^b\right]\omega^a{}_b \\ &= \frac{1}{2}\omega_a{}^b\frac{d}{d\sigma}\left(q^a\frac{\partial L}{\partial \dot{q}_b} - q^b\frac{\partial L}{\partial \dot{q}_a}\right). \end{aligned} \tag{3.81}$$

By equating this with (3.80) we obtain the conserved rotational Noether charges

$$L^{ab} = -q^a \frac{\partial L}{\partial \dot{q}_b} + q^b \frac{\partial L}{\partial \dot{q}_a} = q^a p^b - q^b p^a. \tag{3.82}$$

They are the four-dimensional generalizations of the angular momenta (3.54).

The Noether charges L^{ij} coincide with the components (3.54) of angular momentum. The conserved components

$$L^{0i} = q^0 p^i - q^i p^0 \equiv M_i \tag{3.83}$$

yield the relativistic generalization of the center-of-mass theorem (3.62):

$$M_i = \text{const.} \tag{3.84}$$

3.4 Generating the Symmetries

As mentioned in the introduction to this chapter, there is a second important relation between invariances and conservation laws. The charges associated with continuous symmetry transformations can be used to *generate* the symmetry transformation from which they it was derived. In the classical theory, this is done with the help of Poisson brackets:

$$\delta_s \hat{q} = \epsilon \{\hat{Q}, \hat{q}(t)\}. \tag{3.85}$$

After canonical quantization, the Poisson brackets turn into $-i$ times commutators, and the charges become operators, generating the symmetry transformation by the operation

$$\delta_s \hat{q} = -i\epsilon [\hat{Q}, \hat{q}(t)]. \tag{3.86}$$

The most important example for this quantum-mechanical generation of symmetry transformations is the effect of the Noether charge (3.35) derived in Section 3.2 from the invariance of the system under time displacement. That Noether charge Q was the *Hamiltonian* H, whose operator version generates the infinitesimal time displacements (3.32) by the *Heisenberg equation of motion*

$$\delta_s q(t) = \epsilon \dot{\hat{q}}(t) = -i\epsilon [\hat{H}, \hat{q}(t)], \tag{3.87}$$

as a special case of the general Noether relation (3.86).

The canonical quantization is straightforward if the Lagrangian has the standard form

$$L(q, \dot{q}) = \frac{m}{2} \dot{q}^2 - V(q). \tag{3.88}$$

Then the operator version of the canonical momentum $p \equiv \dot{q}$ satisfies the equal-time commutation rules

$$[\hat{p}(t), \hat{q}(t)] = -i, \quad [\hat{p}(t), \hat{p}(t)] = 0, \quad [\hat{q}(t), \hat{q}(t)] = -i. \tag{3.89}$$

The Hamiltonian

$$H=\frac{p^2}{2m}+V(\hat{q}) \tag{3.90}$$

turns directly into the *Hamiltonian operator*

$$\hat{H}=\frac{\hat{p}^2}{2m}+V(\hat{q}). \tag{3.91}$$

If the Lagrangian does not have the standard form (3.88), quantization is a nontrivial problem, solved in the textbook [2].

Another important example is provided by the charges (3.46) derived in Section 3.3.1 from translational symmetry. After quantization, the commutator (3.86) generating the transformation (3.43) becomes

$$\epsilon^j=i\epsilon^i[\hat{p}^i(t),\hat{q}^j(t)]. \tag{3.92}$$

This coincides with one of the canonical commutation relations (3.89) in three dimensions.

The relativistic charges (3.73) of spacetime generate translations via

$$\delta_{\rm s}\hat{q}^a=\epsilon^a=-i\epsilon^b[\hat{p}_b(t),\hat{q}^a(\tau)], \tag{3.93}$$

implying the relativistic commutation rules

$$[\hat{p}_b(t),\hat{q}^a(\tau)]=i\delta_b{}^a, \tag{3.94}$$

in agreement with the relativistic canonical commutation rules (1.157) (in natural units with $\hbar=1$).

Note that all commutation rules derived from the Noether charge according to the rule (3.86) hold for the operators in the Heisenberg picture, where they are time-dependent. The commutation rules in the purely algebraic discussion in Chapter 3, on the other hand, apply to the time-independent Schrödinger picture of the operators.

Similarly we find that the quantized versions of the conserved charges L_i in Eq. (3.55) generate infinitesimal rotations:

$$\delta_{\rm s}\hat{q}^j=-\omega^i\epsilon_{ijk}\hat{q}^k(t)=i\omega^i[\hat{L}_i,\hat{q}^j(t)], \tag{3.95}$$

whereas the quantized conserved charges N^i of Eq. (3.62) generate infinitesimal Galilei transformations, and that the charges M_i of Eq. (3.83) generate pure Lorentz transformations:

$$\begin{aligned}\delta_{\rm s}\hat{q}^j &= \epsilon_i\hat{q}^0=i\epsilon_i[M_i,\hat{q}^j],\\ \delta_{\rm s}\hat{q}^0 &= \epsilon_i\hat{q}^i=i\epsilon_i[M_i,\hat{q}^0].\end{aligned} \tag{3.96}$$

Since the quantized charges generate the symmetry transformations, they form a *representation* of the generators of the Lorentz group. As such they must have the same commutation rules between each other as the generators of the symmetry group in Eq. (1.71) or their short version (1.72). This is indeed true, since the operator versions of the Noether charges (3.82) correspond to the operators (1.158) (in natural units).

3.5 Field Theory

A similar relation between continuous symmetries and constants of motion holds in field theories, where the role of the Lagrange coordinates is played by fields $q_{\mathbf{x}}(t) = \varphi(\mathbf{x}, t)$.

3.5.1 Continuous Symmetry and Conserved Currents

Let $\mathcal{A}$ be the local action of an arbitrary field $\varphi(x) \to \varphi(\mathbf{x}, t)$,

$$\mathcal{A} = \int d^4x \, \mathcal{L}(\varphi, \partial\varphi, x), \tag{3.97}$$

and suppose that a transformation of the field

$$\delta_{\rm s}\varphi(x) = \epsilon\Delta(\varphi, \partial\varphi, x) \tag{3.98}$$

changes the Lagrangian density $\mathcal{L}$ merely by a total derivative

$$\delta_{\rm s}\mathcal{L} = \epsilon\partial_a\Lambda^a, \tag{3.99}$$

which makes the change of the action $\mathcal{A}$ a surface integral, by Gauss's divergence theorem:

$$\delta_{\rm s}\mathcal{A} = \epsilon \int d^4x \, \partial_a\Lambda^a = \epsilon \int_S ds_a \, \Lambda^a, \tag{3.100}$$

where S is the surface of the total spacetime volume. Then $\delta_{\rm s}\varphi$ is called a *symmetry transformation*.

Given such a transformation, we see that the four-dimensional current density

$$j^a = \frac{\partial\mathcal{L}}{\partial\partial_a\varphi}\Delta - \Lambda^a \tag{3.101}$$

that has no divergence

$$\partial_a j^a(x) = 0. \tag{3.102}$$

The expression (3.101) is called a *Noether current density* and (3.102) is a *local conservation law*, just as in the electromagnetic equation (1.196).

We have seen in Eq. (1.198) that a local conservation law (3.102) always implies a global conservation law of the type (3.9) for the charge, which is now the Noether charge $Q(t)$ defined as in (1.199) by the spatial integral over the zeroth component (here in natural units with $c = 1$)

$$Q(t) = \int d^3x \, j^0(\mathbf{x}, t). \tag{3.103}$$

The proof of the local conservation law (3.102) is just as easy as for the mechanical action (3.1). We calculate the variation of $\mathcal{L}$ under infinitesimal symmetry transformations (3.98) in a similar way as in Eq. (3.13), and find

$$\begin{aligned}\delta_{\rm s}\mathcal{L} &= \left(\frac{\partial\mathcal{L}}{\partial\varphi}-\partial_a\frac{\partial\mathcal{L}}{\partial\partial_a\varphi}\right)\delta_{\rm s}\varphi+\partial_a\left(\frac{\partial\mathcal{L}}{\partial\partial_a\varphi}\delta_{\rm s}\varphi\right)\\ &= \epsilon\left(\frac{\partial\mathcal{L}}{\partial\varphi}-\partial_a\frac{\partial\mathcal{L}}{\partial\partial_a\varphi}\right)\Delta+\epsilon\,\partial_a\left(\frac{\partial\mathcal{L}}{\partial\partial_a\varphi}\Delta\right).\end{aligned} \tag{3.104}$$

The Euler-Lagrange equation removes the first term and, equating the second term with (3.99), we obtain

$$\partial_a j^a \equiv \partial_a\left(\frac{\partial\mathcal{L}}{\partial\partial_a\varphi}\Delta-\Lambda^a\right)=0. \tag{3.105}$$

The relation between continuous symmetries and conservation is called *Noether's theorem* [3].

3.5.2 Alternative Derivation

There is again an alternative derivative of the conserved current analogous to Eqs. (3.17)–(3.28). It is based on a variation of the fields under symmetry transformations whose parameter ϵ is made artificially spacetime-dependent $\epsilon(x)$, thus extending (3.17) to

$$\delta_{\rm s}^x\varphi(x)=\epsilon(x)\Delta(\varphi(x),\partial_a\varphi(x)). \tag{3.106}$$

As before in Eq. (3.19), let us calculate the Lagrangian density for a slightly transformed field

$$\varphi^{\epsilon(x)}(x)\equiv\varphi(x)+\delta_{\rm s}^x\varphi(x), \tag{3.107}$$

calling it

$$\mathcal{L}^{\epsilon(x)}\equiv\mathcal{L}(\varphi^{\epsilon(x)},\partial\varphi^{\epsilon(x)}). \tag{3.108}$$

The associated action differs from the original one by

$$\delta_{\rm s}^x\mathcal{A} = \int dx\left\{\left[\frac{\partial\mathcal{L}^{\epsilon(x)}}{\partial\epsilon(x)}-\partial_a\frac{\partial\mathcal{L}^{\epsilon(x)}}{\partial\partial_a\epsilon(x)}\right]\delta\epsilon(x)+\partial_a\left[\frac{\partial\mathcal{L}^{\epsilon(x)}}{\partial\partial_a\epsilon(x)}\delta\epsilon(x)\right]\right\}. \tag{3.109}$$

For classical fields $\varphi(x)=\varphi_c(x)$ satisfying the Euler-Lagrange equation (2.40), the extremality of the action implies the vanishing of the first term, and thus the Euler-Lagrange-like equation

$$\frac{\partial\mathcal{L}^{\epsilon(x)}}{\partial\epsilon(x)}-\partial_a\frac{\partial\mathcal{L}^{\epsilon(x)}}{\partial\partial_a\epsilon(x)}=0. \tag{3.110}$$

By assumption, the action changes by a pure surface term under the x-independent transformation (3.106), implying that

$$\frac{\partial\mathcal{L}^{\epsilon}}{\partial\epsilon}=\partial_a\Lambda^a. \tag{3.111}$$

Inserting this into (3.110) we find that

$$j^a = \frac{\partial \mathcal{L}^{\epsilon(x)}}{\partial \partial_a \epsilon(x)} - \Lambda^a \tag{3.112}$$

has no four-divergence. This coincides with the previous Noether current density (3.101), as can be seen by differentiating (3.108) with respect to $\partial_a \epsilon(x)$:

$$\frac{\partial \mathcal{L}^{\epsilon(x)}}{\partial \partial_a \epsilon(x)} = \frac{\partial \mathcal{L}}{\partial \partial_a \varphi} \Delta(\varphi, \partial\varphi). \tag{3.113}$$

3.5.3 Local Symmetries

In Chapter 2 we observed that charged particles and fields coupled to electromagnetism possess a more general symmetry. They are invariant under local gauge transformations (2.103). The scalar Lagrangian (2.135), for example, is invariant under the gauge transformations (2.103) and (2.136), and the Dirac Lagrange density (2.150) under (2.103) and (2.151). These are all of the form (3.98), but with a parameter ϵ which depends on spacetime. Thus the action is invariant under local substantial variations of the type (3.106), which were introduced in the last section only as an auxiliary tool for an alternative derivation of the Noether current density (2.135). For a locally invariant Lagrangian density, the Noether expression (3.112) vanishes identically. This does not mean, however, that the system does not possess a conserved current, as we have seen in Eqs. (2.139) and (2.155). Only Noether's derivation breaks down. Let us discuss this phenomenon in more detail for the Lagrangian density (2.150).

If we restrict the gauge transformations (2.151) to x-spacetime-independent gauge transformations

$$\psi(x) \to e^{ie\Lambda/c}\psi(x), \tag{3.114}$$

we can easily derive a conserved Noether current density of the type (3.101) for the Dirac field. The result is the known Dirac current density (2.153). It is the source of the electromagnetic field, with a minimal coupling between them. A similar structure exists for many internal symmetries giving rise to nonabelian versions of electromagnetism, which govern strong and weak interactions. What happens to Noether's derivation of conservation laws in such theories.

As observed above, the formula (3.112) for the current density which was so useful in the globally invariant theory would yield a Noether current density

$$j_a = \frac{\delta \mathcal{L}}{\partial \partial_a \Lambda} \tag{3.115}$$

which vanishes identically, due to local gauge invariance, Thus it woul not provide us with a current density. We may, however, subject *only* the Dirac field to a local gauge transformation at *fixed* gauge fields. Then we obtain the conserved current

$$j_a \equiv \left.\frac{\partial \mathcal{L}}{\partial \partial_a \Lambda}\right|_{A^a}. \tag{3.116}$$

Since the complete change under local gauge transformations $\delta_s^x \mathcal{L}$ vanishes identically, we can alternatively vary *only* the gauge fields and keep the particle orbit fixed

$$j_a = -\left.\frac{\partial \mathcal{L}}{\partial \partial_a \Lambda}\right|_\psi . \tag{3.117}$$

This is done most simply by forming the functional derivative with respect to the gauge field and omitting the contribution of $\stackrel{\rm em}{\mathcal{L}}$, i.e., by applying it only to the Lagrangian of the charge particles $\stackrel{\rm e}{\mathcal{L}} \equiv \mathcal{L} - \stackrel{\rm em}{\mathcal{L}}$:

$$j^a = -\frac{\partial \stackrel{\rm e}{\mathcal{L}}}{\partial \partial_a \Lambda} = -\frac{\partial \stackrel{\rm e}{\mathcal{L}}}{\partial A_a}. \tag{3.118}$$

As a check we apply the rule (3.118) to Dirac complex Klein-Gordon fields with the actions (2.140) and (2.27), and re-obtain the conserved current densities (2.153) and (2.138) (the extra factor c is a convention). From the Schrödinger action (2.50) we derive the conserved charge current density

$$\mathbf{j}(\mathbf{x}, t) \equiv e\frac{i}{2m}\psi^*(\mathbf{x}, t)\stackrel{\leftrightarrow}{\boldsymbol{\nabla}}\psi(\mathbf{x}, t) - \frac{e^2}{c}\mathbf{A}\,\psi^*(\mathbf{x}, t)\psi(\mathbf{x}, t), \tag{3.119}$$

to be compared with the particle current density (2.70) which satisfied the conservation law (2.71) together with the charge density $\rho(\mathbf{x}, t) \equiv e\psi^*(\mathbf{x}, t)\psi(\mathbf{x}, t)$.

An important consequence of local gauge invariance can be found for the gauge field itself. If we form the variation of the pure gauge field action

$$\delta_s \stackrel{\rm em}{\mathcal{A}} = \int d^4x \,\mathrm{tr}\left(\delta_s^x A_a \frac{\delta \stackrel{\rm em}{\mathcal{A}}}{\delta A_a}\right), \tag{3.120}$$

and insert for $\delta_s^x A$ an infinitesimal pure gauge field configuration

$$\delta_s^x A_a = -\partial_a \Lambda(x) \tag{3.121}$$

the right-hand side must vanish for all $\Lambda(x)$. After a partial integration this implies the local conservation law $\partial_a j^a(x) = 0$ for the Noether current

$$\stackrel{\rm em}{j}{}^a(x) = -\frac{\delta \stackrel{\rm em}{\mathcal{A}}}{\delta A_a}. \tag{3.122}$$

Recalling the explicit form of the action in Eqs. (16.20) and (2.83), we find

$$\stackrel{\rm em}{j}{}^a(x) = -\partial_b F^{ab}. \tag{3.123}$$

The Maxwell equation (2.86) can therefore be written as

$$\stackrel{\rm em}{j}{}^a(x) = -\stackrel{\rm e}{j}{}^a(x), \tag{3.124}$$

where we have emphasized the fact that the current j^a contains only the fields of the charge particles by a superscript e. In the form (3.124), the Maxwell equation implies the vanishing of the total current density consisting of the sum of the conserved current (3.117) of the charges and the Noether current (3.122) of the electromagnetic field:

$$\overset{\text{tot}}{j}{}^a(x) = \overset{\text{e}}{j}{}^a(x) + \overset{\text{em}}{j}{}^a(x) = 0. \tag{3.125}$$

This unconventional way of phrasing the Maxwell equation (2.86) will be useful for understanding later the Einstein field equation (17.149) by analogy.

At this place we make an important observation. In contrast to the conservation laws derived for matter fields, which are valid only if the matter fields obey the Euler-Lagrange equations, the current conservation law for the Noether current (3.123) of the gauge fields

$$\partial_a \overset{\text{em}}{j}{}^a(x) = -\partial_a \partial_b F^{ab} = 0 \tag{3.126}$$

is valid for *all* field configurations. The right-hand side vanishes *identically* since the vector potential A^a as an observable field in any fixed gauge satisfies the *Schwarz integrability condition* (2.88).

3.6 Canonical Energy-Momentum Tensor

As an important example for the field-theoretic version of the Noether theorem consider a Lagrangian density that does not depend explicitly on the spacetime coordinates x:

$$\mathcal{L}(x) = \mathcal{L}(\varphi(x), \partial\varphi(x)). \tag{3.127}$$

We then perform a translation of the coordinates along an arbitrary direction $b = 0, 1, 2, 3$ of spacetime

$$x'^a = x^a - \epsilon^a, \tag{3.128}$$

under which field $\varphi(x)$ transforms as

$$\varphi'(x') = \varphi(x), \tag{3.129}$$

so that

$$\mathcal{L}'(x') = \mathcal{L}(x). \tag{3.130}$$

If ϵ^a is infinitesimally small, the field changes by

$$\delta_s\varphi(x) = \varphi'(x) - \varphi(x) = \epsilon^b \partial_b \varphi(x), \tag{3.131}$$

and the Lagrangian density by

$$\begin{aligned} \delta_s\mathcal{L} &\equiv \mathcal{L}(\varphi'(x), \partial\varphi'(x)) - \mathcal{L}(\varphi(x), \partial\varphi(x)) \\ &= \frac{\partial\mathcal{L}}{\partial\varphi(x)}\delta_s\varphi(x) + \frac{\partial\mathcal{L}}{\partial\partial_a\varphi}\partial_a\delta_s\varphi(x), \end{aligned} \tag{3.132}$$

which is a pure divergence term

$$\delta_{\rm s}\mathcal{L}(x) = \epsilon^b \partial_b \mathcal{L}(x). \tag{3.133}$$

Hence the requirement (3.99) is satisfied and $\delta_{\rm s}\varphi(x)$ is a symmetry transformation, with a function Λ which happens to coincide with the Lagrangian density

$$\Lambda = \mathcal{L}. \tag{3.134}$$

We can now define four four-vectors of current densities ${j_b}^a$, one for each component of ϵ^b. For the spacetime translation symmetry, they are denoted by ${\Theta_b}^a$:

$${\Theta_b}^a = \frac{\partial \mathcal{L}}{\partial \partial_a \varphi} \partial_b \varphi - {\delta_b}^a \mathcal{L}. \tag{3.135}$$

Since ϵ^b is a vector, this 4×4 object is a tensor field, the so-called *energy-momentum tensor* of the scalar field $\varphi(x)$. According to Noether's theorem, this has no divergence in the index a [compare (3.102)]:

$$\partial_a {\Theta_b}^a(x) = 0. \tag{3.136}$$

The four conserved charges Q_b associated with these current densities [see the definition (3.103)]

$$P_b = \int d^3x \, {\Theta_b}^0(x), \tag{3.137}$$

are the components of the *total four-momentum* of the system.

The alternative derivation of this conservation law follows Subsection 3.1.4 by introducing the local variations

$$\delta_{\rm s}^x \varphi(x) = \epsilon^b(x) \partial_b \varphi(x) \tag{3.138}$$

under which the Lagrangian density changes by

$$\delta_{\rm s}^x \mathcal{L}(x) = \epsilon^b(x) \partial_b \mathcal{L}(x). \tag{3.139}$$

Applying the chain rule of differentiation we obtain, on the other hand,

$$\delta_{\rm s}^x \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \varphi(x)} \epsilon^b(x) \partial_b \varphi(x) + \frac{\partial \mathcal{L}}{\partial \partial_a \varphi(x)} \left\{ [\partial_a \epsilon^b(x)] \partial_b \varphi + \epsilon^b \partial_a \partial_b \varphi(x) \right\}, \tag{3.140}$$

which shows that

$$\frac{\partial \mathcal{L}^\epsilon}{\partial \partial_a \epsilon^b(x)} = \frac{\partial \mathcal{L}}{\partial \partial_a \varphi} \partial_b \varphi. \tag{3.141}$$

Forming for each b the combination (3.101), we obtain again the conserved energy-momentum tensor (3.135).

Note that by analogy with (3.26), we can write (3.141) as

$$\frac{\partial \mathcal{L}^{\epsilon}}{\partial \partial_a \epsilon^b(x)} = \frac{\partial \mathcal{L}}{\partial \partial_a \varphi} \frac{\partial \delta_s^x \varphi}{\partial \epsilon^b(x)}. \tag{3.142}$$

Note further that the component ${\Theta_0}^0$ of the *canonical energy momentum tensor*

$${\Theta_0}^0 = \frac{\partial \mathcal{L}}{\partial \partial_0 \varphi} \partial_0 \varphi - \mathcal{L} \tag{3.143}$$

coincides with Hamiltonian density (2.61) derived in the canonical formalism by a Legendre transformation of the Lagrangian density.

3.6.1 Electromagnetism

As a an important physical application of the field-theoretic Noether theorem, consider the free electromagnetic field with the action

$$\mathcal{L} = -\frac{1}{4} F_{cd} F^{cd}, \tag{3.144}$$

where F_d are the field strength $F_{cd} \equiv \partial_c A_d - \partial_d A_c$. Under a translation of the spacetime coordinates from x^a to $x^a - \epsilon^a$, the vector potential undergoes a similar change as the scalar field in (3.129):

$$A'^a(x') = A^a(x). \tag{3.145}$$

For infinitesimal translations, this can be written as

$$\begin{aligned} \delta_s A^c(x) &\equiv A'^c(x) - A^c(x) \\ &= A'^c(x' + \epsilon) - A^c(x) \\ &= \epsilon^b \partial_b A^c(x). \end{aligned} \tag{3.146}$$

Under this, the field tensor changes as follows

$$\delta_s F^{cd} = \epsilon^b \partial_b F^{cd}, \tag{3.147}$$

and we verify that the Lagrangian density (3.144) is a total four divergence:

$$\delta_s \mathcal{L} = -\epsilon^b \frac{1}{2} \left(\partial_b F_{cd} F^{cd} + F_{cd} \partial_b F^{cd} \right) = \epsilon^b \partial_b \mathcal{L}. \tag{3.148}$$

Thus, the spacetime translations (3.146) are symmetry transformations, and Eq. (3.100) yield the four Noether current densities, one for each ϵ^b:

$${\Theta_b}^a = \frac{1}{c} \left[\frac{\partial \mathcal{L}}{\partial \partial_a A^c} \partial_b A^c - {\delta_b}^a \mathcal{L} \right]. \tag{3.149}$$

The factor $1/c$ is introduced to give the Noether current the dimension of the energy-momentum tensors introduced in Section 1.13, which are momentum densities. Here we have found the *canonical energy-momentum tensor* of the electromagnetic field, which satisfy the local conservation laws

$$\partial_a \Theta_b{}^a(x) = 0. \tag{3.150}$$

Inserting the derivatives $\partial\mathcal{L}/\partial\partial_a A^c = -F^a{}_c$, we obtain

$$\Theta_b{}^a = \frac{1}{c}\left[-F^a{}_c\partial_b A^c + \frac{1}{4}\delta_b{}^a F^{cd}F_{cd}\right]. \tag{3.151}$$

3.6.2 Dirac Field

We now turn to the Dirac field whose transformation law under spacetime translations

$$x'^a = x^a - \epsilon^a \tag{3.152}$$

is

$$\psi'(x') = \psi(x). \tag{3.153}$$

Since the Lagrangian density in (2.140) does not depend explicitly on x we calculate, as in (3.130):

$$\overset{\rm D}{\mathcal{L}}{}'(x') = \overset{\rm D}{\mathcal{L}}(x). \tag{3.154}$$

The infinitesimal variations

$$\delta_{\rm s}\psi(x) = \epsilon^a\partial_a\psi(x). \tag{3.155}$$

produce the pure derivative term

$$\delta_{\rm s}\overset{\rm D}{\mathcal{L}}(x) = \epsilon^a\partial_a\overset{\rm D}{\mathcal{L}}(x), \tag{3.156}$$

and the combination (3.101) yields the Noether current densities

$$\Theta_b{}^a = \frac{\partial\overset{\rm D}{\mathcal{L}}}{\partial\partial_a\psi^c}\partial_b\psi^c + {\rm cc} - \delta_b{}^a\overset{\rm D}{\mathcal{L}}, \tag{3.157}$$

which satisfies local conservation laws

$$\partial_a\Theta_b{}^a(x) = 0. \tag{3.158}$$

From (2.140) we see that

$$\frac{\partial\overset{\rm D}{\mathcal{L}}}{\partial\partial_a\psi^c} = \frac{1}{2}\bar\psi\gamma^a \tag{3.159}$$

so that we obtain the *canonical energy-momentum tensor* of the Dirac field:

$$\Theta_b{}^a = \frac{1}{2}\bar\psi\gamma^a\partial_b\psi^c + {\rm cc} - \delta_b{}^a\overset{\rm D}{\mathcal{L}}. \tag{3.160}$$

3.7 Angular Momentum

Let us now turn to angular momentum in field theory. Consider first the case of a scalar field $\varphi(x)$. Under a rotation of the coordinates,

$$x'^i = R^i{}_j x^j \tag{3.161}$$

the field does not change, if considered at the same space point, i.e.,

$$\varphi'(x'^i) = \varphi(x^i). \tag{3.162}$$

The infinitesimal substantial variation is:

$$\delta_s\varphi(x) = \varphi'(x) - \varphi(x). \tag{3.163}$$

For infinitesimal rotations (3.48),

$$\delta_s x^i = -\varphi_k \epsilon_{kij} x^j = -\omega_{ij} x^j, \tag{3.164}$$

we see that

$$\begin{aligned} \delta_s\varphi(x) &= \varphi'(x^0, x'^i - \delta x^i) - \varphi(x) \\ &= \partial_i\varphi(x) x^j \omega_{ij}. \end{aligned} \tag{3.165}$$

For a rotationally Lorentz-invariant Lagrangian density which has no explicit x-dependence:

$$\mathcal{L}(x) = \mathcal{L}(\varphi(x), \partial\varphi(x)), \tag{3.166}$$

the substantial variation is

$$\begin{aligned} \delta_s\mathcal{L}(x) &= \mathcal{L}(\varphi'(x), \partial\varphi'(x)) - \varphi(\varphi(x), \partial\varphi(x)) \\ &= \frac{\partial\mathcal{L}}{\partial\varphi(x)}\delta_s\varphi(x) + \frac{\partial\mathcal{L}}{\partial\partial_a\varphi(x)}\partial_a\delta_s\varphi(x). \end{aligned} \tag{3.167}$$

Inserting (3.165), this becomes

$$\begin{aligned} \delta_s\mathcal{L} &= \left[\frac{\partial\mathcal{L}}{\partial\varphi}\partial_i\varphi x^j + \frac{\partial\mathcal{L}}{\partial_a\varphi}\partial_a(\partial_i\varphi x^j)\right]\omega_{ij} \\ &= \left[(\partial_i\mathcal{L})x^j + \frac{\partial\mathcal{L}}{\partial\partial_j\varphi}\partial_i\varphi\right]\omega_{ij}. \end{aligned} \tag{3.168}$$

Since we are dealing with a rotation-invariant local Lagrangian density $\mathcal{L}(x)$ by assumption, the derivative $\partial\mathcal{L}/\partial\partial_a\varphi$ is a vector proportional to $\partial_a\varphi$. Hence the second term in the brackets is symmetric and vanishes upon contraction with the antisymmetric ω_{ij}. This allows us to express $\delta_s\mathcal{L}$ as a pure derivative term

$$\delta_s\mathcal{L} = \partial_i\left(\mathcal{L}\, x^j\omega_{ij}\right). \tag{3.169}$$

Calculating $\delta_s\mathcal{L}$ once more using the Euler-Lagrange equations gives

$$\begin{aligned}\delta_s\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\mathcal{L}}\delta_s\varphi + \frac{\partial\mathcal{L}}{\partial\partial_a\varphi}\partial_a\delta_s\varphi \\ &= \left(\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_a\frac{\partial\mathcal{L}}{\partial\partial_a\varphi}\right)\delta_s\varphi + \partial_a\left(\frac{\partial\mathcal{L}}{\partial\partial_a\varphi}\delta_s\varphi\right) \\ &= \partial_a\left(\frac{\partial\mathcal{L}}{\partial\partial_a\varphi}\partial_i\varphi\, x^j\right)\omega_{ij}.\end{aligned} \tag{3.170}$$

Thus we find the Noether current densities (3.101):

$$L^{ij,a} = \left(\frac{\partial\mathcal{L}}{\partial\partial_a\varphi}\partial_i\varphi\, x^j - \delta_i{}^a\mathcal{L}\, x^j\right) - (i \leftrightarrow j), \tag{3.171}$$

which have no four-divergence

$$\partial_a L^{ij,a} = 0. \tag{3.172}$$

The current densities can be expressed in terms of the canonical energy-momentum tensor (3.135) as

$$L^{ij,a} = x^i\Theta^{ja} - x^j\Theta^{ia}. \tag{3.173}$$

The associated Noether charges

$$L^{ij} = \int d^3x L^{ij,a} \tag{3.174}$$

are the time-independent components of the *total angular momentum* of the field system.

3.8 Four-Dimensional Angular Momentum

Consider now pure Lorentz transformations (1.27). An infinitesimal boost to a rapidity ζ^i is described by a coordinate change [recall (1.34)]

$$x'^a = \Lambda^a{}_b x^b = x^a - \delta^a{}_0\zeta^i x^i - \delta^a{}_i\zeta^i x^0. \tag{3.175}$$

This can be written as

$$\delta x^a = \omega^a{}_b x^b, \tag{3.176}$$

where for passive boosts

$$\omega_{ij} = 0, \quad \omega_{0i} = -\omega_{i0} = \zeta^i. \tag{3.177}$$

With the help of the tensor $\omega^a{}_b$, the boosts can be treated on the same footing with the passive rotations (1.36), for which (3.176) holds with

$$\omega_{ij} = \omega_{ij} = \epsilon_{ijk}\varphi^k, \quad \omega_{0i} = \omega_{i0} = 0. \tag{3.178}$$

For both types of transformations, the substantial variations of the field are

$$\begin{aligned}\delta_s\varphi(x) &= \varphi'(x'^a - \delta x^a) - \varphi(x)\\ &= -\partial_a\varphi(x)x^b\omega^a{}_b.\end{aligned} \tag{3.179}$$

For a Lorentz-invariant Lagrangian density, the substantial variation can be shown, as in (3.169), to be a total derivative:

$$\delta_s\varphi = -\partial_a(\mathcal{L}x^b)\omega^a{}_b, \tag{3.180}$$

and we obtain the Noether current densities

$$L^{ab,c} = -\left(\frac{\partial\mathcal{L}}{\partial\partial_c\varphi}\partial^c\varphi\, x^b - \delta^{ac}\mathcal{L}\, x^b\right) - (a \leftrightarrow b). \tag{3.181}$$

The right-hand side can be expressed in terms of the canonical energy-momentum tensor (3.135), yielding

$$\begin{aligned}L^{ab,c} &= -\left(\frac{\partial\mathcal{L}}{\partial\partial_c\varphi}\partial^c\varphi\, x^b - \delta^{ac}\mathcal{L}x^b\right) - (a \leftrightarrow b)\\ &= x^a\Theta^{bc} - x^b\Theta^{ac}.\end{aligned} \tag{3.182}$$

According to Noether's theorem (3.102), these current densities have no four-divergence:

$$\partial_c L^{ab,c} = 0. \tag{3.183}$$

The charges associated with these current densities:

$$L^{ab} \equiv \int d^3x\, L^{ab,0} \tag{3.184}$$

are independent of time. For the particular form (3.177) of ω_{ab}, we recover the time independent components L^{ij} of angular momentum.

The time-independence of L^{i0} is the relativistic version of the *center-of-mass theorem* (3.66). Indeed, since

$$L^{i0} = \int d^3x\,(x^i\Theta^{00} - x^0\Theta^{i0}), \tag{3.185}$$

we can then define the relativistic center of mass

$$x^i_{\rm CM} = \frac{\int d^3x\,\Theta^{00}x^i}{\int d^3x\,\Theta^{00}} \tag{3.186}$$

and the average velocity

$$v^i_{\rm CM} = c\frac{d^3x\Theta^{i0}}{\int d^3x\,\Theta^{00}} = c\frac{P^i}{P^0}. \tag{3.187}$$

Since $\int d^3x\, \Theta^{i0} = P^i$ is the constant momentum of the system, also $v^i_{\rm CM}$ is a constant. Thus, the constancy of L^{0i} implies the center of mass to move with the constant velocity

$$x^i_{\rm CM}(t) = x^i_{\rm CM,0} + v^i_{\rm CM,0}t \tag{3.188}$$

with $x^i_{\rm CM,0} = L^{0i}/P^0$.

The Noether charges L^{ab} are the four-dimensional angular momenta of the system.

It is important to point out that the vanishing divergence of $L^{ab,c}$ makes Θ^{ba} symmetric:

$$\begin{aligned}\partial_c L^{ab,c} &= \partial_c(x^a\Theta^{bc} - x^b\Theta^{ac}) \\ &= \Theta^{ba} - \Theta^{ba} = 0.\end{aligned} \tag{3.189}$$

Thus, field theories which are invariant under spacetime translations and Lorentz transformations must have a symmetric canonical energy-momentum tensor.

$$\Theta^{ab} = \Theta^{ba} \tag{3.190}$$

3.9 Spin Current

If the field $\varphi(x)$ is no longer a scalar but has several spatial components, then the derivation of the four-dimensional angular momentum becomes slightly more involved.

3.9.1 Electromagnetic Fields

Consider first the case of electromagnetism where the relevant field is the four-vector potential $A^a(x)$. When going to a new coordinate frame

$$x'^a = \Lambda^a{}_b x^b \tag{3.191}$$

the vector field at the same point remains unchanged in absolute spacetime. But since the components A^a refer to two different basic vectors in the different frames, they must be transformed simultaneously with x^a. Since $A^a(x)$ is a vector, it transforms as follows:

$$A'^a(x') = \Lambda^a{}_b A^b(x). \tag{3.192}$$

For an infinitesimal transformation

$$\delta_{\rm s} x^a = \omega^a{}_b x^b \tag{3.193}$$

this implies the substantial variation

$$\begin{aligned}\delta_{\rm s} A^a(x) &= A'^a(x) - A^a(x) = A'^a(x-\delta x) - A^a(x) \\ &= \omega^a{}_b A^b(x) - \omega^c{}_b x^b \partial_c A^a.\end{aligned} \tag{3.194}$$

The first term is a *spin transformation*, the other an *orbital transformation*. The orbital transformation can also be written in terms of the generators $\hat{L}_{ab}$ of the Lorentz group defined in (3.82) as

$$\delta_{\rm s}^{\rm orb} A^a(x) \quad = \quad -i\omega^{bc}\hat{L}_{bc}A^a(x). \tag{3.195}$$

The spin transformation of the vector field is conveniently rewritten with the help of the 4×4 generators L_{ab} in Eq. (1.51). Adding the two together, we form the operator of total four-dimensional angular momentum

$$\hat{J}_{ab} \equiv 1 \times \hat{L}_{ab} + L_{ab} \times 1, \tag{3.196}$$

and can write the transformation (3.194) as

$$\delta_{\rm s}^{\rm orb} A^a(x) \quad = \quad -i\omega^{ab}\hat{J}_{ab}A(x). \tag{3.197}$$

If the Lagrangian density involves only scalar combinations of four-vectors A^a and if it has no explicit x-dependence, it changes under Lorentz transformations like a scalar field:

$$\mathcal{L}'(x') \equiv \mathcal{L}(A'(x'), \partial' A'(x')) = \mathcal{L}(A(x), \partial A(x)) \equiv \mathcal{L}(x). \tag{3.198}$$

Infinitesimally, this amounts to

$$\delta_{\rm s}\mathcal{L} = -(\partial_a \mathcal{L}\, x^b)\omega^a{}_b. \tag{3.199}$$

With the Lorentz transformations being symmetry transformations in the Noether sense, we calculate as in (3.171) the *current of total four-dimensional angular momentum*:

$$J^{ab,c} = \frac{1}{c}\left[\frac{\partial \mathcal{L}}{\partial \partial_c A_a} A^b - \left(\frac{\partial \mathcal{L}}{\partial \partial_c A^d}\partial^a A^d x^b - \delta^{ac}\mathcal{L}\, x^b\right) - (a \leftrightarrow b)\right]. \tag{3.200}$$

The prefactor $1/c$ is chosen to give these Noether currents of the electromagnetic field the conventional physical dimension. In fact, the last two terms have the same form as the current $L^{ab,c}$ of the four-dimensional angular momentum of the scalar field. Here they are the corresponding quantities for the vector potential $A^a(x)$:

$$L^{ab,c} = -\frac{1}{c}\left(\frac{\partial \mathcal{L}}{\partial \partial_c A^d}\partial^a A^d x^b - \delta^{ac}\mathcal{L}\, x^b\right) + (a \leftrightarrow b). \tag{3.201}$$

Note that this current has the form

$$L^{ab,c} = \frac{1}{c}\left\{-i\frac{\partial \mathcal{L}}{\partial \partial_c A^d}\hat{L}^{ab}A^d + \left[\delta^{ac}\mathcal{L}x^b - (a \leftrightarrow b)\right]\right\}, \tag{3.202}$$

where $\hat{L}^{ab}$ are the differential operators of four-dimensional angular momentum (1.103) satisfying the commutation rules (1.71) and (1.72).

Just as the scalar case (3.182), the currents (3.201) can be expressed in terms of the canonical energy-momentum tensor as

$$L^{ab,c} = x^a \Theta^{bc} - x^b \Theta^{ac}. \tag{3.203}$$

The first term in (3.200),

$$\Sigma^{ab,c} = \frac{1}{c}\left[\frac{\partial \mathcal{L}}{\partial \partial_c A_b} A^b - (a \leftrightarrow b)\right], \tag{3.204}$$

is referred to as the *spin current*. It can be written in terms of the 4×4-generators (1.51) of the Lorentz group as

$$\Sigma^{ab,c} = -\frac{i}{c}\frac{\partial \mathcal{L}}{\partial \partial_c A^d}(L^{ab})_{d\sigma} A^\sigma. \tag{3.205}$$

The two currents together

$$J^{ab,c}(x) \equiv L^{ab,c}(x) + \Sigma^{ab,c}(x) \tag{3.206}$$

are conserved, $\partial_c J^{ab,c}(x) = 0$. Individually, the terms are not conserved.

The total angular momentum is given by the charge

$$J^{ab} = \int d^3x \, J^{ab,0}(x). \tag{3.207}$$

It is a constant of motion. Using the conservation law of the energy-momentum tensor we find, just as in (3.189), that the orbital angular momentum satisfies

$$\partial_c L^{ab,c}(x) = -\left[\Theta^{ab}(x) - \Theta^{ba}(x)\right]. \tag{3.208}$$

From this we find the divergence of the spin current

$$\partial_c \Sigma^{ab,c}(x) = -\left[\Theta^{ab}(x) - \Theta^{ba}(x)\right]. \tag{3.209}$$

For the charges associated with orbital and spin currents

$$L^{ab}(t) \equiv \int d^3x L^{ab,0}(x), \qquad \Sigma^{ab}(t) \equiv \int d^3x \Sigma^{ab,0}(x), \tag{3.210}$$

this implies the following time dependence:

$$\begin{aligned} \dot{L}^{ab}(t) &= -\int d^3x \left[\Theta^{ab}(x) - \Theta^{ba}(x)\right], \\ \dot{\Sigma}^{ab}(t) &= \int d^3x \left[\Theta^{ab}(x) - \Theta^{ba}(x)\right]. \end{aligned} \tag{3.211}$$

Fields with spin have always have a non-symmetric energy momentum tensor.

Then the current $J^{ab,c}$ becomes, now back in natural units,

$$J^{ab,c} = \left(\frac{\partial \delta_s^x \mathcal{L}}{\partial \partial_c \omega_{ab}(x)} - \delta^{ac} \mathcal{L} x^b\right) - (a \leftrightarrow b) \tag{3.212}$$

By the chain rule of differentiation, the derivative with respect to $\partial, \omega_{ab}(x)$ can come only from field derivatives. For a scalar field

$$\frac{\partial \delta_{\rm s}^x \mathcal{L}}{\partial \partial_c \omega_{ab}(x)} = \frac{\partial \mathcal{L}}{\partial \partial_c \varphi} \frac{\partial \delta_{\rm s}^x \varphi}{\partial \omega_{ab}(x)}, \tag{3.213}$$

and for a vector field

$$\frac{\partial \delta_{\rm s}^x \mathcal{L}}{\partial \partial_c \omega_{ab}(x)} = \frac{\partial \mathcal{L}}{\partial \partial_c A^d} \frac{\partial \delta_{\rm s}^x A^d}{\partial \omega_{ab}}. \tag{3.214}$$

The alternative rule of calculating angular momenta is to introduce spacetime-dependent transformations

$$\delta^x x = \omega^a{}_b(x) x^b \tag{3.215}$$

under which the scalar fields transform as

$$\delta_{\rm s} \varphi = -\partial_c \varphi \omega^c{}_b(x) x^b \tag{3.216}$$

and the Lagrangian density as

$$\delta_{\rm s}^x \varphi = -\partial_c \mathcal{L}\, \omega^c{}_b(x) x^b = -\partial_c (x^b \mathcal{L}) \omega^c{}_b(x). \tag{3.217}$$

By separating spin and orbital transformations of $\delta_{\rm s}^x A^d$ we find the two contributions $\sigma^{ab,c}$ and $L^{ab,c}$ to the current $J^{ab,c}$ of the total angular momentum, the latter receiving a contribution from the second term in (3.212).

3.9.2 Dirac Field

We now turn to the Dirac field. Under a Lorentz transformation (3.191), this transforms according to the law

$$\psi(x') \xrightarrow{\Lambda} \psi'_\Lambda(x) = D(\Lambda)\psi(x), \tag{3.218}$$

where $D(\Lambda)$ are the 4×4 spinor representation matrices of the Lorentz group. Their matrix elements can most easily be specified for infinitesimal transformations. For an infinitesimal Lorentz transformation

$$\Lambda_a{}^b = \delta_a{}^b + \omega_a{}^b, \tag{3.219}$$

under which the coordinates are changed by

$$\delta_{\rm s} x^a = \omega^a{}_b x^b \tag{3.220}$$

the spin transforms under the representation matrix

$$D(\delta_a{}^b + \omega_a{}^b) = \left(1 - i\frac{1}{2}\omega_{ab}\sigma^{ab}\right)\psi(x), \tag{3.221}$$

where σ_{ab} are the 4×4 matrices acting on the spinor space defined in Eq. (1.222). We have shown in (1.220) that the spin matrices $\Sigma_{ab} \equiv \sigma_{ab}/2$ satisfy the same commutation rules (1.71) and (1.72) as the previous orbital and spin-1 generators $\hat{L}_{aba}$ and L_{ab} of Lorentz transformations.

The field has the substantial variation [compare (3.194)]:

$$\begin{aligned}\delta_s\psi(x) &= \psi'(x)-\psi(x) = D(\delta_a{}^b+\omega_a{}^b)\psi(x-\delta x)-\psi(x)\\ &= -i\frac{1}{2}\omega_{ab}\sigma^{ab}\psi(x)-\omega^c{}_b x^b\partial_c\psi(x)\\ &= -i\frac{1}{2}\omega_{ab}\left[S^{ab}+\hat{L}^{ab}\right]\psi(x)\equiv -i\frac{1}{2}\omega_{ab}\hat{J}^{ab}\psi(x),\end{aligned} \tag{3.222}$$

the last line showing the separation into spin and orbital transformation for a Dirac particle.

Since the Dirac Lagrangian is Lorentz-invariant, it changes under Lorentz transformations like a scalar field:

$$\mathcal{L}'(x') = \mathcal{L}(x). \tag{3.223}$$

Infinitesimally, this amounts to

$$\delta_s\mathcal{L} = -(\partial_a\mathcal{L}x^b)\omega^a{}_b. \tag{3.224}$$

With the Lorentz transformations being symmetry transformations in the Noether sense, we calculate the *current of total four-dimensional angular momentum* extending the formulas (3.182) and (3.200) for scalar field and vector potential. The result is

$$J^{ab,c} = \left(-i\frac{\partial\mathcal{L}}{\partial\partial_c\psi}\sigma^{ab}\psi - i\frac{\partial\mathcal{L}}{\partial\partial_c\psi}\hat{L}^{ab}\psi+\mathrm{cc}\right)+\left[\delta^{ac}\mathcal{L}x^b-(a\leftrightarrow b)\right]. \tag{3.225}$$

As before in (3.201) and (3.182), the orbital part of (3.225) can be expressed in terms of the canonical energy-momentum tensor as

$$L^{ab,c} = x^a\Theta^{bc}-x^b\Theta^{ac}. \tag{3.226}$$

The first term in (3.225) is the *spin current*

$$\Sigma^{ab,c} = \frac{1}{2}\left(-i\frac{\partial\mathcal{L}}{\partial\partial_c\psi}\sigma^{ab}\psi+\mathrm{cc}\right). \tag{3.227}$$

Inserting (3.159), this becomes explicitly

$$\Sigma^{ab,c} = -\frac{i}{2}\bar{\psi}\gamma^c\sigma^{ab}\psi = \frac{1}{2}\bar{\psi}\gamma^{[a}\gamma^b\gamma^{c\}]}\psi = \frac{1}{2}\epsilon^{abcd}\bar{\psi}\gamma^d\psi. \tag{3.228}$$

The spin density is completely antisymmetric in the three indices[1]

The conservation properties of the three currents are the same as in Eqs. (3.207)–(3.211).

Due to the presence of spin, the energy-momentum tensor is not symmetric.

[1] This property is important for being able to construct a consistent quantum mechanics in a space with torsion. See Ref. [2].

3.10 Symmetric Energy-Momentum Tensor

Since the presence of spin is the cause for the asymmetry of the canonical energy-momentum tensor, it is suggestive that by an appropriate use of the spin current it should be possible to construct a new modified momentum tensor

$$T^{ab} = \Theta^{ab} + \Delta\Theta^{ba} \tag{3.229}$$

which is symmetric, while still having the fundamental property of Θ^{ab}, that the integral $P^a = \int d^3x\, T^{a0}$ is the total energy-momentum vector of the system. This is the case if $\Delta\Theta^{a0}$ is a three-divergence of a spatial vector. Such a construction was found by Belinfante in 1939. He introduced the tensor [4]

$$T^{ab} = \Theta^{ab} - \frac{1}{2}\partial_c(\Sigma^{ab,c} - \Sigma^{bc,a} + \Sigma^{ca,b}), \tag{3.230}$$

whose symmetry is manifest, due to (3.209) and the symmetry of the last two terms under $a \leftrightarrow b$. Moreover, by the components

$$T^{a0} = \Theta^{a0} - \frac{1}{2}\partial_c(\Sigma^{a0,c} - \Sigma^{0c,a} + \Sigma^{ca,0}) = x^a T^{bc} - x^b T^{ac} \tag{3.231}$$

which gives the same total angular momentum as the canonical expression (3.206):

$$J^{ab} = \int d^3x\, J^{ab,0}. \tag{3.232}$$

Indeed, the zeroth component of (3.231) is

$$x^a\Theta^{b0} - x^b\Theta^{a0} - \frac{1}{2}\left[\partial_k(\Sigma^{a0,k} - \Sigma^{0k,a} + \Sigma^{ka,0})x^b - (a \leftrightarrow b)\right]. \tag{3.233}$$

Integrating the second term over d^3x and performing a partial integration gives, for $a = 0, b = i$:

$$-\frac{1}{2}\int d^3x\, \left[x^0\partial_k(\Sigma^{i0,k} - \Sigma^{0k,i} + \Sigma^{ki,0}) - x^i\partial_k(\Sigma^{00,k} - \Sigma^{0k,0} + \Sigma^{k0,0})\right] = \int d^3x\, \Sigma^{0i,0}, \tag{3.234}$$

and for $a = i, b = j$

$$-\frac{1}{2}\int d^3x\, \left[x^i\partial_k(\Sigma^{j0,k} - \Sigma^{0k,j} + \Sigma^{kj,0}) - (i \leftrightarrow j)\right] = \int d^3x\, \Sigma^{ij,0}. \tag{3.235}$$

The right-hand sides are the contributions of the spin to the total angular momentum.

For the electromagnetic field, the spin current (3.204) reads, explicitly

$$\Sigma^{ab,c} = -\frac{1}{c}\left[F^{ca}A^b - (a \leftrightarrow b)\right]. \tag{3.236}$$

From this we calculate the Belinfante correction

$$\begin{aligned}\Delta\Theta^{ab} &= \frac{1}{2c}[\partial_c(F^{ca}A^b - F^{cb}A^a) - \partial_c(F^{ab}A^c - F^{ac}A^b) + \partial_c(F^{bc}A^a - F^{ba}A^c)] \\ &= \frac{1}{c}\partial_c(F^{bc}A^a). \end{aligned} \tag{3.237}$$

Adding this to the canonical energy-momentum tensor (3.151)

$$\Theta^{ab} = \frac{1}{c}\left[-F^b{}_c\partial^a A^c + \frac{1}{4}g^{ab}F^{cd}F_{cd}\right], \tag{3.238}$$

we find the symmetric energy-momentum tensor

$$T^{ab} = \frac{1}{c}\left[-F^b{}_cF^{ac} + \frac{1}{4}g^{ab}F^{cd}F_{cd} + (\partial_c F^{bc})A^a\right]. \tag{3.239}$$

The last term vanishes for a free Maxwell field which satisfies $\partial_c F^{ab} = 0$ [recall (2.86)], and can be dropped. In this case T^{ab} agrees with the previously constructed symmetric energy-momentum tensor (1.272) of the electromagnetic field. The symmetry of T^{ab} can easily be verified using once more the Maxwell equation $\partial_c F^{ab} = 0$.

We have seen in (1.269) that the component $cT^{00}(x)$ agrees with the well-known energy density $\mathcal{E}(x) = (\mathbf{E}^2 + \mathbf{B}^2)/2$ of the electromagnetic field, and that the components $c^2T^{0i}(x)$ are equal to the *Poynting vector* of energy current density $\mathbf{S}(x) = c\mathbf{E}\times\mathbf{B}$, so that the energy conservation law $c^2\partial_a T^{0a}(0)$ can be written as $\partial_t\mathcal{E}(x) + \boldsymbol{\nabla}\cdot\mathbf{S}(x) = 0$.

In the presence of an external current, where the Lagrangian density is (2.83), the canonical energy-momentum tensor becomes

$$\Theta^{ab} = \frac{1}{c}\left[-F^b{}_c\partial^a A^c + \frac{1}{4}g^{ab}F^{cd}F_{cd} + \frac{1}{c}g^{ab}j^cA_c\right], \tag{3.240}$$

generalizing (3.238).

The spin current is again given by Eq. (3.236), leading to the Belinfante energy-momentum tensor

$$\begin{aligned}T^{ab} &= \Theta^{ab} + \frac{1}{c}\partial_c(F^{bc}A^a) \\ &= \frac{1}{c}\left[-F^b{}_cF^{ac} + \frac{1}{4}g^{ab}F^{cd}F_{cd} + \frac{1}{c}g^{ab}j^cA_c - \frac{1}{c}j^bA^a\right]. \end{aligned} \tag{3.241}$$

The last term prevents T^{ab} from being symmetric, unless the current vanishes. Due to the external current, the conservation law $\partial_b T^{ab} = 0$ is modified to

$$\partial_b T^{ab} = \frac{1}{c^2}A_c(x)\partial^a j^c(x). \tag{3.242}$$

3.11 Internal Symmetries

In quantum field theory, an important role in classifying various actions is played by *internal symmetries*. They do not involve any change in the spacetime coordinate of the fields, i.e., they have the form

$$\phi'(x) = e^{-i\alpha_r G_r}\phi(x) \tag{3.243}$$

where G_r are the generators of some Lie group, and α_r the associated transformation parameters. If the field has N components, the generators G_r are $N \times N$ -matrices. They satisfy commutation rules of the general form [recall (1.65)]

$$[G_r, G_s] = if_{rst}\, G_t, \qquad (r, s, t = 1, \ldots, \text{rank}), \tag{3.244}$$

where f_{rst} are the structure constants of the Lie algebra.

The infinitesimal symmetry transformations are substantial variations of the form

$$\delta_s\varphi = -i\alpha_r G_r\varphi, \tag{3.245}$$

The associated conserved current densities read

$$j_r^a = -i\frac{\partial\mathcal{L}}{\partial\partial_a\varphi}G_r\varphi. \tag{3.246}$$

These can also be written as

$$j_r^a = -i\pi G_r\varphi, \tag{3.247}$$

where $\pi(x) \equiv \partial\mathcal{L}(x)/\partial\partial_a\varphi(x)$ is the canonical momentum of the field $\varphi(x)$ [compare (2.60)].

The most important example is that of a complex field ϕ and a generator $G = 1$, where the symmetry transformation (3.243) is simply a multiplication by a constant phase factor. One also speaks of U(1)-symmetry. Other important examples are those of a triplet or an octet of fields ϕ with G_r being the generators of an SU(2) or SU(3) representation. The U(1)-symmetry leads to charge conservation in electromagnetic interactions, the other two are responsible for isospin SU(2) and SU(3) invariance in strong interactions. The latter symmetries are, however, not exact.

3.11.1 U(1)-Symmetry and Charge Conservation

Consider a Lagrangian density $\mathcal{L}(x) = \mathcal{L}(\phi(x), \partial\phi(x), x)$ that is is invariant under U(1)-transformations

$$\delta_s\phi(x) = -i\alpha\phi(x) \tag{3.248}$$

i.e., $\delta_s\mathcal{L} = 0$. By the chain rule of differentiation we find, using the Euler-Lagrange equation (2.40)

$$\delta_s\mathcal{L} = \left(\frac{\partial\mathcal{L}}{\partial\phi} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial_a\phi}\right)\delta_s\phi + \left[\frac{\partial\mathcal{L}}{\partial\partial_a\phi}\delta_s\phi\right] = 0. \tag{3.249}$$

Inserting (3.248), we find that

$$j_\mu = -\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\phi \tag{3.250}$$

is a conserved current.

For a free relativistic complex scalar field with a Lagrangian density

$$\mathcal{L}(x) = \partial_\mu \varphi^* \partial_\mu \varphi - m^2 \varphi^* \varphi \tag{3.251}$$

we have to add the contributions of real and imaginary parts of the field ϕ in formula (3.250), and obtain the conserved current density

$$j_\mu = -i\varphi^* \overleftrightarrow{\partial_\mu} \varphi \tag{3.252}$$

where the symbol $\varphi^* \overleftrightarrow{\partial_\mu} \varphi$ denotes the left-minus-right derivative:

$$\varphi^* \overleftrightarrow{\partial} \varphi \equiv \varphi^* \partial_\mu \varphi - (\partial_\mu \varphi^*)\varphi. \tag{3.253}$$

For a free Dirac field, the current density (3.250) takes the form

$$j^\mu(x) = \bar{\psi}(x)\gamma^\mu \psi(x). \tag{3.254}$$

3.11.2 Broken Internal Symmetries

The physically important symmetries SU(2) of isospin and SU(3) are not exact. The symmetry variation of the Lagrange density is not strictly zero. In this case we remember the alternative derivation of the conservation law based on Eq. (3.110). We introduce the spacetime-dependent parameters $\alpha(x)$ and conclude from the extremality property of the action that

$$\partial_a \frac{\partial \mathcal{L}^\epsilon}{\partial \partial_a \alpha_r(x)} = \frac{\partial \mathcal{L}^\epsilon}{\partial \alpha_r(x)}. \tag{3.255}$$

This implies the divergence law for the above derived current

$$\partial_a j_r^a(x) = \frac{\partial \mathcal{L}^\epsilon}{\partial \alpha_r}. \tag{3.256}$$

3.12 Generating the Symmetry Transformations on Quantum Fields

As in quantum mechanical systems, the charges associated with the conserved currents obtained in the previous section can be used to generate the transformations of the fields from which they were derived. One merely has to invoke the canonical field commutation rules.

For the currents (3.246), the charges are

$$Q_r = -i \int d^3x \frac{\partial \mathcal{L}}{\partial \partial_a \varphi} G_r \varphi \tag{3.257}$$

and can be written as

$$Q_r = -i \int d^3x \, \pi G_r \varphi, \tag{3.258}$$

where $\pi(x) \equiv \partial \mathcal{L}(x)/\partial \partial_a \varphi(x)$ is the canonical momentum of the field $\varphi(x)$. After quantization, these fields satisfy the canonical commutation rules:

$$\begin{aligned} [\hat{\pi}(\mathbf{x}, t), \hat{\varphi}(\mathbf{x}', t)] &= -i\delta^{(3)}(\mathbf{x} - \mathbf{x}'), \\ [\hat{\varphi}(\mathbf{x}, t), \hat{\varphi}(\mathbf{x}', t)] &= 0, \\ [\hat{\pi}(\mathbf{x}, t), \hat{\pi}(\mathbf{x}', t)] &= 0. \end{aligned} \tag{3.259}$$

From this we derive directly the commutation rule between the quantized version of the charges (3.258) and the field operator $\hat{\varphi}(x)$:

$$[\hat{Q}_r, \hat{\varphi}(x)] = -\alpha_r G_r \varphi(x). \tag{3.260}$$

We also find that the commutation rules among the quantized charges $\hat{Q}_r$ are the same as those of the generators G_r in (3.244):

$$[\hat{Q}_r, \hat{Q}_s] = f_{rst} \hat{Q}_t, \qquad (r, s, t = 1, \ldots, \text{rank}). \tag{3.261}$$

Hence the operators $\hat{Q}_r$ form a representation of the generators of symmetry group in the many-particle Hilbert space generated by the quantized fields $\hat{\varphi}(x)$ (*Fock space*).

As an example, we may derive in this way the commutation rules of the conserved charges associated with the Lorentz generators (3.226):

$$J^{ab} \equiv \int d^3x J^{ab,0}(x). \tag{3.262}$$

They are obviously the same as those of the 4×4-matrices (1.51), and those of the quantum mechanical generators (1.103):

$$[\hat{J}^{ab}, \hat{J}^{ac}] = -ig^{aa} \hat{J}^{bc}. \tag{3.263}$$

The generators $J^{ab} \equiv \int d^3x J^{ab,0}(x)$, are sums $J^{ab} = L^{ab}(t) + \Sigma^{ab}(t)$ of charges (3.210) associated with orbital and spin rotations. According to (3.211), these individual charges are time dependent, only their sum being conserved. Nevertheless, they both generate Lorentz transformations: $L^{ab}(t)$ on the spacetime argument of the fields, and $\Sigma^{ab}(t)$ on the spin indices. As a consequence, they both satisfy the commutation relations (3.263):

$$[\hat{L}^{ab}, \hat{L}^{ac}] = -ig^{aa} \hat{L}^{bc}, \qquad [\hat{\Sigma}^{ab}, \hat{\Sigma}^{ac}] = -ig^{aa} \hat{\Sigma}^{bc}. \tag{3.264}$$

It is important to realize that the commutation relations (3.260) and (3.261) remain valid also in the presence of symmetry-breaking terms as long as these do not contribute to the canonical momentum of the theory. Such terms are called *soft symmetry-breaking terms*. The charges are no longer conserved, so that we must attach a time argument to the commutation relations (3.260) and (3.261). All times in these relations must be the same, in order to invoke the equal-time canonical commutation rules.

The commutators (3.261) have played an important role in developing a theory of strong interactions, where they first appeared in the form of a *charge algebra* of the broken symmetry SU(3) × SU(3) of weak and electromagnetic charges. This symmetry will be discussed in more detail in Chapter 10.

3.13 Energy-Momentum Tensor of Relativistic Massive Point Particle

If we want to study energy and momentum of charged relativistic point particles in an electromagnetic field it is useful to consider the action (3.68) with (3.70) as an integral over a Lagrangian density:

$$\mathcal{A} = \int d^4x\, \mathcal{L}(x), \qquad \text{with} \quad \mathcal{L}(x) = \int_{\tau_a}^{\tau_b} d\tau\, L(\dot{x}^a(\tau))\delta^{(4)}(x - x(\tau)). \tag{3.265}$$

This allows us to derive for point particles local conservation laws in the same way as for fields. Instead of doing this, however, we shall take advantage of the previously derived global conservation laws and convert them into local ones by inserting appropriate δ-functions with the help of the trivial identity

$$\int d^4x\, \delta^{(4)}(x - x(\tau)) = 1. \tag{3.266}$$

Consider for example the conservation law (3.72) for the momentum (3.73). With the help of (3.266) this becomes

$$0 = -\int d^4x \int_{-\infty}^{\infty} d\tau \left[\frac{d}{d\tau} p_c(\tau)\right] \delta^{(4)}(x - x(\tau)). \tag{3.267}$$

Note that in this expression the boundaries of the four-volume contain the information on initial and final times. We then perform a partial integration in τ, and rewrite (3.267) as

$$0 = -\int d^4x \int_{-\infty}^{\infty} d\tau \frac{d}{d\tau} \left[p_c(\tau)\delta^{(4)}(x - x(\tau))\right] + \int d^4x \int_{-\infty}^{\infty} d\tau p_c(\tau)\partial_\tau \delta^{(4)}(x - x(\tau)). \tag{3.268}$$

The first term vanishes if the orbits come from and disappear into infinity. The second term can be rewritten as

$$0 = -\int d^4x\, \partial_b \left[\int_{-\infty}^{\infty} d\tau p_c(\tau)\dot{x}^b(\tau)\delta^{(4)}(x - x(\tau))\right]. \tag{3.269}$$

This shows that

$$\Theta^{cb}(x) \equiv m \int_{-\infty}^{\infty} d\tau \, \dot{x}^c(\tau)\dot{x}^b(\tau)\delta^{(4)}(x - x(\tau)) \tag{3.270}$$

satisfies the local conservation law

$$\partial_b \Theta^{cb}(x) = 0. \tag{3.271}$$

This is the conservation law for the energy-momentum tensor of a massive point particle.

The total momenta are obtained from the spatial integrals over Θ^{c0}:

$$P^a(t) \equiv \int d^3x \, \Theta^{c0}(x). \tag{3.272}$$

For point particles, they coincide with the canonical momenta $p^a(t)$. If the Lagrangian depends only on the velocity $\dot{x}^a$ and not on the position $x^a(t)$, the momenta $p^a(t)$ are constants of motion: $p^a(t) \equiv p^a$.

The Lorentz invariant quantity

$$M^2 = P^2 = g_{ab}P^aP^b \tag{3.273}$$

is called the *total mass* of the system. For a single particle it coincides with the mass of the particle.

Subjecting the orbits $x^a(\tau)$ to Lorentz transformations according to the rules of the last section we find the currents of total angular momentum

$$L^{ab,c} \equiv x^a\Theta^{bc} - x^b\Theta^{ac}, \tag{3.274}$$

to satisfy the conservation law:

$$\partial_c L^{ab,c} = 0. \tag{3.275}$$

A spatial integral over the zeroth component of the current $L^{ab,c}$ yields the conserved charges:

$$L^{ab}(t) \equiv \int d^3x \, L^{ab,0}(x) = x^a p^b(t) - x^b p^a(t). \tag{3.276}$$

3.14 Energy-Momentum Tensor of Massive Charged Particle in Electromagnetic Field

Let us also consider an important combination of a charged point particle and an electromagnetic field Lagrangian

$$\mathcal{A} = -mc \int_{\tau_a}^{\tau_b} d\tau \sqrt{g_{ab}\dot{x}^a(\tau)\dot{x}^b(\tau)} - \frac{1}{4}\int d^4x F_{ab}F^{ab} - \frac{e}{c}\int_{\tau_a}^{\tau_b} d\tau \dot{x}^a(\tau)A_a(x(\tau)). \tag{3.277}$$

By varying the action in the particle orbits, we obtain the Lorentz equation of motion

$$\frac{dp^a}{d\tau} = \frac{e}{c} F^a{}_b \dot{x}^b(\tau). \tag{3.278}$$

By varying the action in the vector potential, we find the Maxwell-Lorentz equation

$$-\partial_b F^{ab} = \frac{e}{c} \dot{x}^b(\tau). \tag{3.279}$$

The action (3.277) is invariant under translations of the particle orbits and the electromagnetic fields. The first term is obviously invariant, since it depends only on the derivatives of the orbital variables $x^a(\tau)$. The second term changes under translations by a pure divergence [recall (3.133)]. Also the interaction term changes by a pure divergence, which is seen as follows: Since the substantial variation changes $x^b(\tau) \to x^b(\tau) - \epsilon^b$, under which $\dot{x}^a(\tau)$ is invariant,

$$\dot{x}^a(\tau) \quad \to \quad \dot{x}^a(\tau), \tag{3.280}$$

and $A_a(x^b)$ changes as follows:

$$A_a(x^b) \to A'_a(x^b) = A_a(x^b + \epsilon^b) = A_a(x^b) + \epsilon^b \partial_a A_a(x^b). \tag{3.281}$$

Altogether we obtain

$$\delta_s \mathcal{L} = \epsilon^b \partial_b \overset{m}{\mathcal{L}} . \tag{3.282}$$

We now we calculate the same variation once more invoking the equations of motion. This gives

$$\delta_s \mathcal{A} = \int d\tau \frac{d}{d\tau} \frac{\partial L^{\rm m}}{\partial x'^a} \delta_s x^a + \int d^4x \frac{\partial \overset{\rm em}{\mathcal{L}}}{\partial \partial_c A^a} \delta_s A^a. \tag{3.283}$$

The first term can be treated as in (3.268)–(3.269) after which it acquires the form

$$\begin{aligned} -\int_{\tau_a}^{\tau_b} d\tau \frac{d}{d\tau}\left(p_a + \frac{e}{c} A_a\right) &= -\int d^4x \int_{-\infty}^{\infty} d\tau \frac{d}{d\tau}\left[\left(p_a + \frac{e}{c} A_a\right) \delta^{(4)}(x - x(\tau)\right] \\ &+ \int d^4x \int_{-\infty}^{\infty} d\tau \left(p_a + \frac{e}{c} A_a\right) \frac{d}{d\tau} \delta^{(4)}(x - x(\tau)) \end{aligned} \tag{3.284}$$

and thus, after dropping boundary terms,

$$-\int_{\tau_a}^{\tau_b} d\tau \frac{d}{d\tau}\left(p_a + \frac{e}{c} A_a\right) = \partial_c \int d^4x \int_{-\infty}^{\infty} d\tau \left(p_a + \frac{e}{c} A_a\right) \frac{dx^c}{d\tau} \delta^{(4)}(x - x(\tau)). \tag{3.285}$$

The electromagnetic part is the same as before, since the interaction contains no derivative of the gauge field. In this way we find the canonical energy-momentum tensor

$$\Theta^{ab}(x) = \int d\tau \left(p^a + \frac{e}{c} A^a\right) \dot{x}^b(\tau) \delta^{(4)}(x - x(\tau)) - F^b{}_c \partial^a A^c + \frac{1}{4} g^{ab} F^{cd} F_{cd}. \tag{3.286}$$

Let us check its conservation by calculating the divergence:

$$\begin{aligned}\partial_b\Theta^{ab}(x) &= \int d\tau\left(p+\frac{e}{c}A_a\right)\dot{x}^b(\tau)\partial_b\delta^{(4)}(x-x(\tau))\\ &\quad -\partial_b F^b{}_c\partial^a A^c - F^b{}_c\partial_b\partial^a A^c + \frac{1}{4}\partial^a(F^{cd}F_{cd}).\end{aligned} \tag{3.287}$$

The first term is, up to a boundary term, equal to

$$-\int d\tau\left(p^a+\frac{e}{\tau}A^a\right)\frac{d}{d\tau}\delta^{(4)}(x-x(\tau))=\int d\tau\left[\frac{d}{d\tau}\left(p^a+\frac{e}{c}A^a\right)\right]\delta^{(4)}(x-x(\tau)). \tag{3.288}$$

Using the Lorentz equation of motion (3.278), this becomes

$$\frac{e}{c}\int_{-\infty}^{\infty} d\tau\left(F^a{}_b\dot{x}^b(\tau)+\frac{d}{d\tau}A^a\right)\delta^{(4)}(x-x(\tau)). \tag{3.289}$$

Inserting the Maxwell equation

$$\partial_b F^{ab} = -e\int d\tau(dx^a/d\tau)\delta^{(4)}(x-x(\tau)), \tag{3.290}$$

the second term in Eq. (3.287) can be rewritten as

$$-\frac{e}{c}\int_{-\infty}^{\infty} d\tau\frac{dx_c}{d\tau}\partial^a A^c\delta^{(4)}(x-x(\tau)), \tag{3.291}$$

which is the same as

$$-\frac{e}{c}\int d\tau\left(\frac{dx_a}{d\tau}F^{ac}+\frac{dx_c}{d\tau}\partial^c A^a\right)\delta^{(4)}(x-x(\tau)), \tag{3.292}$$

thus canceling (3.289). The third term in (3.287) is, finally, equal to

$$-F^b{}_c\partial^a F_b{}^c + \frac{1}{4}\partial^a(F^{cd}F_{cd}), \tag{3.293}$$

due to the antisymmetry of F^{bc}. By rewriting the homogeneous Maxwell equation, the Bianchi identity (2.88), in the form

$$\partial_c F_{ab} + \partial_a F_{bc} + \partial_b F_{ca} = 0, \tag{3.294}$$

and contracting it with F^{ab}, we see that the term (3.293) vanishes identically.

It is easy to construct from (3.286) Belinfante's symmetric energy momentum tensor. We merely observe that the spin density is entirely due to the vector potential, and hence the same as before [see (3.236)]

$$\Sigma^{ab,c} = -\left[F^{ca}A^b - (a\leftrightarrow b)\right]. \tag{3.295}$$

Hence the additional piece to be added to the canonical energy momentum tensor is again [see (3.237)]

$$\Delta\Theta^{ab} = \partial_c(F^{ab}A^a) = \frac{1}{2}(\partial_c F^{bc}A^a + F^{bc}\partial_c A^a). \tag{3.296}$$

The last term in this expression serves to symmetrize the electromagnetic part of the canonical energy-momentum tensor and brings it to the Belinfante form:

$$\overset{\text{em}}{T}{}^{ab} = -F^b{}_c F^{ac} + \frac{1}{4}g^{ab}F^{cd}F_{cd}. \tag{3.297}$$

The second-last term in (3.296), which in the absence of charges vanished, is needed to symmetrize the matter part of Θ^{ab}. Indeed, using once more Maxwell's equation, it becomes

$$-\frac{e}{c}\int d\tau\,\dot{x}^b(\tau)A^a\delta^{(4)}(x - x(\tau)), \tag{3.298}$$

thus canceling the corresponding term in (3.286). In this way we find that the total energy-momentum tensor of charged particles plus electromagnetic fields is simply the term of the two symmetric energy-momentum tensor.

$$\begin{aligned} T^{ab} &= \overset{\text{m}}{T}{}^{ab} + \overset{\text{em}}{T}{}^{ab} \\ &= m\int_{-\infty}^{\infty} d\tau\,\dot{x}^a\dot{x}^b\delta^{(4)}(x - x(\tau)) - F^b{}_c F^{ac} + \frac{1}{4}g^{ab}F^{cd}F_{cd}. \end{aligned} \tag{3.299}$$

For completeness, let us cross check also its conservation. Forming the divergence $\partial_b T^{ab}$, the first term gives now only

$$\frac{e}{c}\int d\tau\,\dot{x}^b(\tau)F^a{}_b(x(\tau)), \tag{3.300}$$

in contrast to (3.289), which is canceled by the divergence in the second term

$$-\partial_b F^b{}_c F^{ac} = -\frac{e}{c}\int d\tau\,\dot{x}_c(\tau)F^{ac}(x(\tau)), \tag{3.301}$$

in contrast to (3.292).

Notes and References

For more details on classical electromagnetic fields see

L.D. Landau, E.M. Lifshitz, *The Classical Theory of Fields*, Addison-Wesley, Reading, MA, 1951;
A.O. Barut, *Electrodynamics and Classical Theory of Fields and Particles*, MacMillan, New York, N.Y. 1964;
J.D. Jackson, *Classical Electrodynamics*, John Wiley & Sons, New York, N.Y., 1975.

The individual citations refer to:

[1] S. Coleman and J.H. Van Vleck, Phys. Rev. **171**, 1370 (1968).

[2] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets* World Scientific, Singapore 2006, 4th extended edition, pp. 1–1546 (kl/b5), where kl is short for the www address http://www.physik.fu-berlin.de/~kleinert.

[3] E. Noether, Nachr. d. vgl. Ges. d. Wiss. Göttingen, Math-Phys. Klasse, **2**, 235 (1918);
See also
E. Bessel-Hagen, Math. Ann. **84**, 258 (1926);
L. Rosenfeld, Me. Acad. Roy. Belg. **18**, 2 (1938);
F.J. Belinfante, Physica **6**, 887 (1939).

[4] The Belinfante energy-momentum tensor is discussed in detail in
H. Kleinert, *Gauge Fields in Condensed Matter*, Vol. II *Stresses and Defects*, World Scientific Publishing, Singapore 1989, pp. 744-1443 (kl/b2).