21

New Resummation Algorithm

A rapidly convergent resummation algorithm has recently been developed by Jasch and Kleinert [1]. It is applicable to many types of divergent power series, and yields critical exponents with high accuracy. It possesses similar virtues as variational perturbation theory: the approach to scaling at large bare couplings $\bar{g}_B$ is of the form $c + c'/g_B^\omega$, and there are variational parameters to optimize the results. In addition, it exploits the knowledge on the large-order behavior of the expansion coefficients, or equivalently, on the discontinuity across the tip of the left-hand cut in the complex coupling constant plane. Moreover, it is formulated in a simple algebraic way similar to the Janke-Kleinert algorithm in Section 16.5.

The combination of these properties leads to a considerable increase in the speed of convergence and a high accuracy of the results.

In this chapter we explain this algorithm and illustrate its power by calculating the critical exponents of O(N)-symmetric ϕ^4-theories from the six- and seven-loop perturbation expansions.

21.1 Hyper-Borel Transformation

The basis for this algorithm is a generalization of the Borel transformation to a *hyper-Borel transformation*. Like the ordinary Borel or Borel-Leroy transforms in Section 16.3 [recall (16.25) and (16.43)] it is constructed by removing from the expansion coefficients f_n in Eq. (19.1) [recall (17.23)]

$$f_n = \gamma(-\alpha)^n n^\beta n! [1 + O(1/n)] \tag{21.1}$$

the factorial growth which produces a convergent series denoted by $\tilde{B}(t)$. From this the original function $f(g_B)$ can be recovered by some integral over $\tilde{B}(t)$ along the positive t-axis. Since $\tilde{B}(t)$ is initially obtained from $f(g_B)$ as a power series with a finite convergence radius, an analytic continuation is necessary to achieve convergence. This is done as before by a transformation like (16.45) from the complex t- into a complex w-plane such that the image of the positive t-axis lies inside the maximal circle of convergence. As before, a suitable modification of the expansion for $\tilde{B}(t)$ of the type (16.55) allows us to account for the leading strong-coupling behavior g_B^s.

The important additional property of the new transformation is that the nonleading powers of the strong-coupling expansion (19.2) are also matched. All previous approaches based on a Borel or Borel-Leroy transformation do not allow this. In order to understand the difficulty, let us insert the strong-coupling expansion (19.2) in the form

$$f(g_B) = g_B^s \sum_{m=0}^{\infty} b_m \, g_B^{-\omega m} \tag{21.2}$$

into Eq. (16.28) for the Borel transform. We have set the leading power p/q equal to s, as in Eq. (16.54), and identified the powers $(g_B^{-2/q})^m$ directly with $(g_B^{-\omega})^m$, where ω is the exponent of approach to scaling. Performing the contour integral, we find

$$B(t) = t^s \sum_{m=0}^{M} \frac{\sin \pi(\omega m - s)}{\pi} \Gamma(\omega m - s) b_m \, t^{-\omega m}, \tag{21.3}$$

For dimensional reasons, the large-t expansion of the Borel transform $B(t)$ contains the same powers of t as the original function $f(g_B)$ does in g_B. However, while the series (19.2) has a finite radius of convergence, the large-t expansion of $B(t)$ is a divergent asymptotic one, due to the factor $\Gamma(m\omega - s)$ in each expansion coefficient. Thus, if we want to account for the power structure of the strong-coupling expansion (19.2) within the ordinary Borel framework, we have to find a way of re-expanding the Borel series which will guarantee the large-t property (21.3). This is an interesting open problem.

Note that, in general, an expansion in the Borel-plane with a power sequence in t as in (21.3) is not sufficient to ensure an expansion in the same powers in the g_B-plane as in (19.2), because of the appearance of extra integer powers in g_B. This is illustrated by the simple function $B(t) = (1+t)^s$, which possesses a strong-coupling expansion in the powers t^{s-k}. If s is non-integer the expansion of the corresponding function $f(g_B)$ reads

$$f(g_B) = \int_0^\infty dt \, e^{-t}(1 + g_B t)^s = e^{1/g_B}\Gamma(s+1) g_B^s + e^{1/g_B} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+s+1)k!} g_B^{-k-1}. \tag{21.4}$$

Expanding the exponential we see that the sum contains integer powers which are not contained in the strong-coupling expansion of $B(t)$.

A match of the strong-coupling power structure can be achieved with the help of the following hyper-Borel transform defined by

$$\tilde{B}(y) = \sum_{k=0}^{\infty} \tilde{B}_k \, y^k, \tag{21.5}$$

with coefficients

$$\tilde{B}_k \equiv \omega \frac{\Gamma\left(k(1/\omega - 1) + \beta_0\right)}{\Gamma\left(k/\omega - s/\omega\right)\Gamma(\beta_0)} f_k. \tag{21.6}$$

The original function $f(g_B)$ is recovered from $\tilde{B}(y)$ by the integral

$$f(g_B) = \frac{\Gamma(\beta_0)}{2\pi i} \oint_C dt \, e^t \, t^{-\beta_0} \int_0^\infty \frac{dy}{y} \left[\frac{g_B}{y \, t^{(1-\omega)/\omega}}\right]^s \exp\left[\frac{y \, t^{(1-\omega)/\omega}}{g_B}\right]^\omega \tilde{B}(y). \tag{21.7}$$

The proof of this is straightforward with the help of the integral representation of the inverse Gamma function

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_C dt \, e^t \, t^{-z}. \tag{21.8}$$

The transformation possesses a free parameter β_0 which can be used to optimize the approximation $f_L(g_B)$ for each order L. The power ω of the strong-coupling expansion is assumed to lie in the interval $0 < \omega < 1$.

The hyper-Borel transformation has the desired property of allowing for a resummation of $f_L(g_B)$ with the correct powers of g_B of the strong-coupling expansion (19.2). In order to show this we first observe that, as in the ordinary Borel transform (16.25), the large-argument behavior of the Gamma function [recall Sterling's formula (16.11)] removes the factorial growth (21.1) from the expansion coefficients f_k, and leads to a simple power behavior of the coefficients $\tilde{B}_k$:

$$\tilde{B}_k \stackrel{k\to\infty}{=} \text{const.} \times \left[\alpha\omega(1-\omega)^{1/\omega-1}\right]^k k^{\beta+\beta_0+1/2+s/\omega}[1+\mathcal{O}(1/k)]. \tag{21.9}$$

Thus our new transform $\tilde{B}(y)$ shares with the ordinary Borel transform $B(t)$ the property of being analytic at the origin. Its radius of convergence is determined by the singularity on the negative real axis at

$$y_s = -\frac{1}{\sigma} \equiv -\frac{1}{\alpha}\frac{1}{\omega(1-\omega)^{1/\omega-1}}. \tag{21.10}$$

A resummation procedure can now be set up on the basis of the transform $\tilde{B}(y)$ as before. The inverse transformation (21.7) contains an integral over the entire positive axis, requiring again an analytic continuation of the Taylor expansion of $\tilde{B}(y)$ beyond the convergence radius. In the previous resummation scheme using the ordinary Borel of Borel-Leroy transforms, this is achieved in two ways: either by a conformal mapping such as (16.45), or by a re-expansion of $B(g)$ in terms of a complete set of basis functions (16.75), (16.76). The relation between the two methods was explained in Section 16.5.3.

Here we shall follow the second method and re-expand the hyper-Borel transform $\tilde{B}(y)$ in a complete set of basis functions $\tilde{B}_{I_n}(y)$, which all possess a singularity at $y = -1/\sigma$. As before, we choose $\tilde{B}_{I_n}(y)$ to have Taylor series starting with y^n, which will lead to a triangular matrix to be inverted when re-expanding $\tilde{B}(y)$ in terms of $\tilde{B}_{I_n}(y)$.

The reason for introducing the new transform $\tilde{B}(y)$ is to allow us to reproduce the complete power structure of the strong-coupling expansion (19.2), with a leading power g^s and a sub-leading string of powers $g^{s-k\omega}$, $k = 1, 2, 3, \ldots$. This is achieved by the following basis functions to span the space of transforms $\tilde{B}(y)$:

$$\tilde{B}_{I_n}(y) = e^{-\rho\sigma y}\frac{(\sigma y)^n}{(1+\sigma y)^{n+\delta}}. \tag{21.11}$$

By analogy with the re-expansion in the Borel-Leroy case (16.55), this may be written as

$$\tilde{B}(y) \equiv \sum_{k=0}^{\infty} \tilde{B}_k\, y^k = e^{-\rho\sigma y}[1-w(y)]^\delta \sum_{k=0}^{\infty} W_k[w(y)]^k, \tag{21.12}$$

with

$$w(y) = \frac{\sigma y}{1+\sigma y}. \tag{21.13}$$

The inverse transform of $\tilde{B}_{I_n}(y)$ yields the basis functions

$$I_n(g_B) = \frac{\Gamma(\beta_0)}{2\pi i}\oint_C dt\, e^t\, t^{-\beta_0}\int_0^\infty \frac{dy}{y}\left[\frac{g_B}{y\,t^{1/\omega-1}}\right]^s \exp\left[-\frac{y\,t^{1/\omega-1}}{g_B}\right]^\omega \tilde{B}_{I_n}(y), \tag{21.14}$$

which span the space of functions $f(g_B)$. The functions $I_n(g_B)$ are used as basis functions to re-expand the truncated expansions $f_L(g_B)$ in the form

$$f_L(g_B) = \sum_{n=0}^{L} h_n I_n(g_B). \tag{21.15}$$

The complete list of parameters of the functions $I_n(g_B)$ reads as follows:

$$I_n(g_B) = I_n(g_B, \omega, s, \rho, \sigma, \delta, \beta_0) = I_n(\sigma g_B, \omega, s, \rho, 1, \delta, \beta_0), \tag{21.16}$$

but in the following we shall mostly use the shorter notation $I_n(g_B)$.

The integral representation of $I_n(g_B)$ breaks down at $s = n$, requiring an analytical continuation. For our applications it will be sufficient to perform this continuation in the convergent strong-coupling expansion of $I_n(g_B)$. This is obtained by performing a Taylor expansion of the exponential function in (21.14), which is an expansion in powers of $1/g_B^\omega$. After integrating over t and y using (21.8), we obtain the expansion

$$I_n(g_B) = g_B^p \sum_{k=0}^{\infty} b_k^{(n)} g_B^{-k\omega}, \tag{21.17}$$

which agrees with the strong-coupling behavior (19.2). The expansion coefficients are

$$b_k^{(n)} = \frac{(-1)^k}{k!} \frac{\sigma^{s-k\omega}\Gamma(\beta_0)}{\Gamma[(\omega-1)k + \beta_0 + (1/\omega - 1)s]} i_k^{(n)}, \tag{21.18}$$

where $i_k^{(n)}$ denotes the integral

$$i_k^{(n)} = \int_0^\infty dy e^{-\rho y}(1+y)^{-\delta-n} y^{k\omega+n-s-1}. \tag{21.19}$$

This integral is seen to coincide with a Kummer function

$$U(\alpha,\gamma,z) \equiv \frac{1}{\Gamma(\alpha)} \int_0^\infty dy e^{-zy} y^{\alpha-1}(1+y)^{\gamma-\alpha-1}, \tag{21.20}$$

so that we can write

$$i_k^{(n)} = \Gamma(k\omega+n-s)\, U(k\omega+n-s, k\omega-s-\delta+1, \rho). \tag{21.21}$$

The latter expression is useful since in some applications the integral (21.19) may diverge requiring an analytic continuation of the contour of integration. Such deformations are automatically supplied by other representations of the Kummer function, for instance

$$U(\alpha,\gamma,z) = \frac{\pi}{\sin\pi\gamma}\left[\frac{M(\alpha,\gamma,z)}{\Gamma(1+\alpha-\gamma)\Gamma(\gamma)} - z^{1-\gamma}\frac{M(1+\alpha-\gamma,2-\gamma,z)}{\Gamma(\alpha)\Gamma(2-\gamma)}\right], \tag{21.22}$$

where $M(\alpha,\gamma,z)$ is the confluent hypergeometric function which has the Taylor expansion

$$M(\alpha,\gamma,z) = 1 + \frac{\alpha}{\gamma}\frac{z}{1!} + \frac{\alpha(\alpha-1)}{\gamma(\gamma-1)}\frac{z^2}{2!} + \ldots\,. \tag{21.23}$$

The alternative expression (21.21) for $i_k^{(n)}$ with (21.22) and (21.23) is useful for resumming various asymptotic expansions, for example, that of the ground state energy of the anharmonic oscillator, in which case the leading strong-coupling power s has the value 1/3. There the integral representation (21.19) will have to be evaluated for values $n = 0$, $k = 0$, where the integral does not exist, whereas formula (21.21) with (21.22), (21.23) is well-defined.

For large k, the integral on the right-hand side of (21.19) can be estimated with the help of the saddle point approximation. The saddle point lies at

$$y_s \approx \frac{k\omega}{\rho}, \tag{21.24}$$

leading to the asymptotic estimate

$$
\begin{aligned}
i_k^{(n)} &\overset{k\to\infty}{=} \left(\frac{k\omega}{\rho}\right)^{-\delta-n} \int_0^\infty dy e^{-\rho y} y^{\omega k+n-s-1}\,[1+\mathcal{O}(1/k)] \\
&= \left(\frac{\omega k}{\rho}\right)^{-\delta-n} \rho^{-k\omega-n+s}\Gamma(k\omega+n-s)\,[1+\mathcal{O}(1/k)]\,. \qquad (21.25)
\end{aligned}
$$

The behavior of the strong-coupling coefficients $b_k^{(n)}$ for large k is obtained with the help of the identity

$$
\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z} \qquad (21.26)
$$

and Stirling's formula (16.11), yielding

$$
b_k^{(n)} \overset{k\to\infty}{=} \gamma\sin\pi[k(\omega-1)+\beta_0+(1/\omega-1)s]\left[-\frac{(1-\omega)^{(1-\omega)}}{(\sigma\rho)^\omega}\right]^k k^{\gamma_1}[1+\mathcal{O}(1/k)]. \qquad (21.27)
$$

The real constants γ, γ_1 will not be needed for the upcoming discussions and are therefore not calculated explicitly. Equation (21.27) shows that the strong-coupling expansion (19.2) has a convergence radius

$$
|g_B| > \frac{(\rho\sigma)^\omega}{(1-\omega)^{1-\omega}}, \qquad (21.28)
$$

which means that the basis functions $I_n(g_B)$, and certainly also $f(g_B)$ itself, possess additional singularities beside $g_B=0$. The parameter ρ will be optimally adjusted to match the positions of these singularities.

For re-expanding $f_L(g_B)$ in terms of the basis functions $I_n(g_B)$, we must know the Taylor series of the basis functions $I_n(g_B)$. This is obtained by substituting into (21.14) the variable y with $g_B y'$, and expanding the integrand of (21.14) in powers of g_B. After performing the integrals over y' and t, we obtain the expansion

$$
I_n(g_B)=\sum_{k=n}^\infty f_k^{(n)} g_B^k, \qquad (21.29)
$$

with the coefficients

$$
f_k^{(n)}=\sum_{k=n}^\infty \frac{1}{\omega}\frac{\Gamma(\beta_0)\Gamma(k/\omega-s/\omega)}{\Gamma(k(1/\omega-1)+\beta_0)}\sum_{j=0}^{k-n}\binom{-\delta-n}{j}\frac{(-\rho)^{k-n-j}}{(k-n-j)!}\sigma^k. \qquad (21.30)
$$

The coefficients in the second sum come from the t integral:

$$
\sum_{j=0}^{k-n}\binom{-n-\delta}{j}\frac{(-\rho)^{k-n-j}}{(k-n-j)!} = \frac{(-1)^{k-n}}{\Gamma(k-n+1)\Gamma(n+\delta)}\int_0^\infty dt\, e^{-t}\, t^{\delta+n-1}(\rho+t)^{k-n}. \qquad (21.31)
$$

For large k, we may evaluate the integral with the help of the saddle-point approximation. Using also Stirling's formula (16.11), we find

$$
\sum_{j=0}^{k-n}\binom{-n-\delta}{j}\frac{(-\rho)^{k-n-j}}{(k-n-j)!} \overset{k\to\infty}{=} \frac{(-1)^{k-n}e^\rho}{\Gamma(\delta+n)}k^{\delta+n-1}\,[1+\mathcal{O}(1/k)]\,. \qquad (21.32)
$$

Inserted into (21.30) and using once more Stirling's formula we obtain for the expansion coefficients $f_k^{(n)}$ the following factorial growth

$$f_k^{(n)} \overset{k\to\infty}{=} \frac{(-1)^n e^\rho \Gamma(\beta_0)}{\sqrt{2\pi}\Gamma(\delta+n)}(1-\omega)^{1/2-\beta_0}\omega^{\beta_0-1+s/\omega}k^{\delta-\beta_0+n-3/2-s/\omega}\left[-\frac{\sigma}{\omega(1-\omega)^{1/\omega-1}}\right]^k k! \times[1+\mathcal{O}(1/k)]. \tag{21.33}$$

For an optimal re-expansion (21.15), we shall choose the free parameters of the basis functions $I_n(g_B, \omega, s, \rho, \sigma, \delta, \beta_0)$ to match the large-order behavior of the coefficients f_k in (21.1).

For an understanding of the convergence properties of the resummed $f_L(g_B)$ to the correct strong-coupling expansions (19.2) we shall also need the large-n behavior of the expansion coefficients $b_k^{(n)}$ in the strong-coupling expansion (21.17) of $I_n(g_B)$. This is determined by the saddle point approximation to the integral $i_k^{(n)}$ in Eq. (21.19), which we rewrite as

$$i_k^{(n)} = \int_0^\infty dy e^{-\rho y - n\ln(1+1/y)}(1+y)^{-\delta}y^{k\omega-s-1}. \tag{21.34}$$

The saddle point lies at

$$y_s = \sqrt{\frac{n}{\rho}}\left[1+\mathcal{O}(1/\sqrt{n})\right]. \tag{21.35}$$

At this point, the total exponent in the integral is

$$-\rho y_s - n\ln\left(1+\frac{1}{y_s}\right) = -2\sqrt{\rho n}\left[1+\mathcal{O}(1/\sqrt{n})\right], \tag{21.36}$$

implying the large-n behavior

$$b_k^{(n)} \overset{k\to\infty}{=} \text{const.} \times n^{k\omega-s-1-\delta}e^{-2\sqrt{\rho n}}\left[1+\mathcal{O}(1/\sqrt{n})\right]. \tag{21.37}$$

21.2 Convergence Properties

Let us denote the strong-coupling coefficients b_k of the truncated function $f_L(g)$ by b_k^L. They are linear combinations of the coefficients $b_k^{(n)}$ of the basis functions $I_n(g_B)$:

$$b_k^L = \sum_{n=0}^{L} b_k^{(n)}h_n. \tag{21.38}$$

Let us estimate the expected speed of convergence with which the coefficients b_k^L converge against the strong-coupling coefficients b_k of $f(g_B)$ as the number L goes to infinity. It is governed by the growth with n of the re-expansion coefficients h_n, and of the coefficients $b_k^{(n)}$ in Eq. (21.37). For the series to be resummed, the re-expansion coefficients h_n will have to grow at most like some power n^r, implying that the approximations b_k^L approach their $L\to\infty$ -limit b_k with an error proportional to

$$b_k^L - b_k \sim L^{r+k\omega-s-\delta-1/2} \times e^{-2\sqrt{\rho L}}. \tag{21.39}$$

The leading exponential falloff of the error $e^{-2\sqrt{\rho L}}$ is independent of the other parameters in the basis functions $I_n(g_B, \omega, p, \rho, \sigma, \delta, \beta_0)$ to be adjusted below. This is the important advantage of the present resummation method with respect to variational perturbation theory where the error decreases like $e^{-L^{1-\omega}}$ with $1-\omega$ close to $1/4$ [recall (19.51) as well as Figs. 19.10–19.16 in $4-\varepsilon$ and Figs. 20.14–20.17 in $D=3$ dimensions].

21.2.1 Parameters s and ω

The perturbation expansions for the critical exponents are power series in the bare coupling constant g_B whose strong-coupling limit is a constant. The same is true for the series expressing the renormalized coupling constant g in powers of the bare coupling constant. This implies that the growth parameter s for the basis functions $I_n(g_B)$ should be set equal to zero. The constant asymptotic values are approached with a subleading power behavior $\propto 1/g_B^{k\omega-s}$, where ω is the universal experimentally measurable critical exponent governing the approach to scaling.

21.2.2 Parameter σ

In the ordinary Borel-transformation, the parameter α in the large-order behavior of the expansion coefficients f_k in Eq. (21.1) specifies also the position of the singularity on the negative y-axis in $B(y)$. It is determined directly by the inverse reduced action of the classical solution to the field equations. In our new transform $\tilde{B}(y)$, the growth parameter and the inverse reduced action α is no longer directly given by the nearest singularity in $\tilde{B}(y)$, which now lies at [see Eq. (21.10)]

$$\sigma = \alpha\omega(1-\omega)^{1/\omega-1}. \tag{21.40}$$

This value of σ ensures that the expansion coefficients f_k^n of the basis functions $I_n(g_B)$ in Eq. (21.33) grow for large k with the same factor $(-\alpha)^k$ as the expansion coefficients for $f(g_B)$ in Eq. (21.1).

The conformal mapping (21.13) maps the singularity at $y=-1/\sigma$ to $w=\infty$, and converts the cut along the negative y-axis into a cut in the w-plane from 1 to ∞. The growth of the re-expansion coefficients h_n in (21.15) with n is therefore determined by the nature of the singularity of $\tilde{B}(y)$ at ∞.

In the upcoming applications to critical exponents both, the value (21.40) following from the inverse action of the solution to the classical field equations and ω, will not yield the fastest convergence of the approximations $f^L(g_B)$ towards $f(g_B)$. A slightly smaller value will turn out to give better results. This seems to be due to the fact that the classical solution gives only the nearest singularity in the hyper-Borel transform $\tilde{B}(y)$ of $f(g_B)$. In reality, there are many additional cuts from other fluctuating field configurations which determine the size of the expansion coefficients f_k at asymptotic orders k. Since the few known f_ks are in praxis quite far from the asymptotic estimates, they are best accounted for by an effective shift of the position of the singularity in the direction of the additional cuts at larger negative y, corresponding to a smaller σ.

21.2.3 Parameter ρ

According to Eq. (21.28), the parameter ρ determines the radius of convergence of the strong-coupling expansion of the basis functions $I_n(g_B)$. It should therefore be adjusted to fit optimally the corresponding radius of the original function $f(g_B)$. Since we do not know this radius, this adjustment will be done phenomenologically by varying ρ to optimize the speed of convergence. Specifically, we shall search at each order L for a vanishing highest re-expansion coefficient h_L or, if it does not vanish anywhere, for a vanishing derivative with respect to ρ:

$$h_L(\rho) = 0, \quad \text{or} \quad \frac{dh_L(\rho)}{d\rho} = 0. \tag{21.41}$$

21.2.4 Parameter δ

From Eq. (21.33) we see that the parameter δ influences the power k^β in the large-order behavior (21.1). By comparing the two equations, we identify the growth parameter β of $I_n(g_B)$ as being

$$\beta = \delta - \beta_0 - 3/2 - s/\omega + n. \tag{21.42}$$

At first it would appear impossible to give *all* basis functions $I_n(g_B)$ the same growth power β in (21.33) by simply letting δ depend on the order n as required by (21.42). If we were to do this, we will have to assign to δ the value

$$\delta = \delta_n \equiv \beta + \beta_0 + 3/2 + s/\omega - n, \tag{21.43}$$

which depends on the index n of the function $I_n(g_B)$. This means that we perform an analytical continuation of the power series expansion of $\tilde{B}(y)$ by re-expanding it as follows:

$$\tilde{B}(y) = \sum_{k=0}^{\infty} \tilde{B}_k\, y^k = e^{-\rho\sigma y}(1+\sigma y)^{-\delta} \sum_{k=0}^{\infty} h_k(\sigma y)^k. \tag{21.44}$$

But the series in this formula which is obtained from the series of $\tilde{B}(y)$ by removing a simple factor still has the same finite radius of convergence and cannot be used to estimate $\tilde{B}(y)$ for large values of y needed to perform the back-transform (21.7). It is, however, possible to sidetrack this problem by letting ρ grow linearly with the order L. Then the exponential factor of (21.44) suppresses the integrals over y for large y sufficiently to make the divergence of the re-expanded series (21.44) at large y irrelevant. If we determine ρ from the condition (21.41), the growth of ρ with L will come about by itself.

21.2.5 Parameter β_0

The parameter β_0 has two effects. From Eq. (21.18) we see that for

$$k > k_c \equiv \frac{\beta_0 + (1/\omega - 1)s}{1-\omega} \tag{21.45}$$

the signs of the strong-coupling expansion coefficients start to alternate irregularly. This irregularity weakens the convergence of the higher strong-coupling coefficients b_k^L with $k > k_c$ against b_k. The convergence can therefore be improved by choosing a β_0 which grows proportionally to the order L of the approximation.

In addition, β_0 appears in the power of k in (21.33) because it determines the nature of the cut in $\tilde{B}(y)$ in the complex y-plane starting at $y = -1/\sigma$ [see Eq. (21.12)].

If we expand both sides of (21.12) in powers of $w = \sigma y/(1+\sigma y)$ and compare the coefficients of powers of w, it is easy to write down an explicit formula for the re-expansion coefficients h_n in terms of the coefficients $\tilde{B}_j$ of $\tilde{B}(y)$:

$$h_n = \sum_{k=0}^{n}\sum_{j=0}^{k} \frac{\tilde{B}_j \sigma^{-j}\rho^{k-j}}{(k-j)!}\binom{\delta+n-1}{n-k}, \tag{21.46}$$

where $\tilde{B}_j$ are obtained from the original expansion coefficients f_k of $f(g_B)$ by the relation (21.6).

Before beginning with the resummation of the perturbation expansions for the critical exponents of ϕ^4-field theories, it will be useful to have a feel for the quality of the new resummation procedure and for the significance of the parameters to the speed of convergence by resumming the asymptotic expansion for the ground state energy of the anharmonic oscillator.

21.3 Resummation of Ground State Energy of Anharmonic Oscillator

Consider the one-dimensional anharmonic oscillator with the Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{m^2}{2}x^2 + g_B x^4. \tag{21.47}$$

The ground state energy has a perturbation expansion

$$E^{(0)}(g_B) = \sum_k^{\infty} f_k g_B^k, \tag{21.48}$$

whose coefficients can be calculated by a recursion relation of Bender and Wu [2] to arbitrarily high orders, with a large-order behavior

$$f_k = -\sqrt{\frac{6}{\pi^3}}k!(-3)^k k^{-1/2}[1 + \mathcal{O}(1/k)]. \tag{21.49}$$

By comparison with (21.1) we identify the growth parameters

$$\alpha = 3, \qquad \beta = -1/2. \tag{21.50}$$

By performing a scale transformation $x \to g^{1/6}x$ on the Hamiltonian (21.47), one finds the scaling property

$$E(m^2, g_B) = g_B^{1/3} E(g_B^{-2/3} m^2, 1) \tag{21.51}$$

for the energy considered as a function of g_B and m^2. Combining this with the knowledge [3] that $E(m^2, 1)$ is an analytic function at $m^2 = 0$, we see that $E(1, g)$ possesses an expansion of the form (19.2) with the powers $s = 1/3$, $\omega = 2/3$. Inserting the latter number together with α from Eq. (21.50) into (21.40), we identify

$$\sigma = \frac{2}{\sqrt{3}}. \tag{21.52}$$

The ground state energy $E^{(0)}(g)$ obeys a once-subtracted dispersion relation

$$E^{(0)}(g) = \frac{1}{2} + \frac{g}{\pi}\int_0^\infty \frac{dg'}{g'} \frac{\mathrm{Im} E^{(0)}(-g')}{g' + g}. \tag{21.53}$$

The perturbative expansion (21.48) is obtained from this by expanding the integrand in powers of g, and performing the integral term by term. This shows explicitly that the large-order behavior (21.49) corresponds to an imaginary part

$$\mathrm{Im} E^{(0)}(-|g_B|) = \sqrt{\frac{6}{\pi}}\sqrt{\frac{1}{3|g_B|}}e^{-1/3|g_B|}[1 + \mathcal{O}(|g_B|)] \tag{21.54}$$

near the tip of the left-hand cut in the complex g_B-plane, in agreement with the general form (16.7) associated with the large-order behavior (21.1) [compare (16.12)]. Equation (21.33) shows us that if we want to match the imaginary part of $E^{(0)}(g_B)$ at small negative g_B, we have to set $\delta = \beta_0 + 3/2 - n$, in accordance with the general relation (21.42). Thus we resum the perturbative expansion by a re-expansion in terms of the basis functions

$$I_n(g, 2/3, 1/3, \rho, 2/\sqrt{3}, \beta_0 + 3/2 - n, \beta_0), \tag{21.55}$$

where the parameters ρ and β_0 are still undetermined.

Note that by (21.12), the resulting re-expansion corresponds to an analytic continuation of our transform $\tilde{B}(y)$ written as follows:

$$\tilde{B}(y) = \sum_{k=0}^{\infty} \tilde{B}_k \, y^k = e^{-2\rho y/\sqrt{3}}(1 + 2y/\sqrt{3})^{-\beta_0+3/2} \sum_{k=0}^{\infty} W_k \, (2y/\sqrt{3})^k. \tag{21.56}$$

The parameter ρ will be determined by an order-dependent optimization of the approximations. Specifically, we shall search in each order L for a vanishing highest re-expansion coefficient h_L with a vanishing derivative with respect to ρ:

$$h_L(\rho) = 0, \qquad \frac{dh_L(\rho)}{d\rho} = 0. \tag{21.57}$$

From Eq. (21.30) for the expansion coefficients $f_k^{(n)}$ of the basis functions $I_n(g_B)$, we see that a large parameter β_0 shifts the large-k-regime, where the large-order formula Eq. (21.33) begins dominating the growth, to increasing values of k. To match these growth properties with those of the expansion coefficients f_k of the ground state energy, we assign to β_0 the value $\beta_0 = 5$.

Let us test the convergence of our algorithm at small negative coupling constants g_B, i.e., near the tip of the left-hand cut in the complex g_B-plane by calculating the prefactor γ in the large-order behavior (21.1). With the large-order behavior (21.33) of the basis functions $I_n(g_B)$, we find the resummed functions $f_L(g_B)$ of Lth order $\sum_{n=0}^{L} h_n I_n(g_B)$ having a large-order behavior (21.1) with a prefactor

$$\gamma_L = \frac{e^{\rho}\Gamma(\beta_0)}{\sqrt{2\pi}\Gamma(\delta)} \sum_{k=0}^{L} (-1)^k w_k. \tag{21.58}$$

The values of these sums for increasing L are shown in Fig. 21.1. They converge exponentially

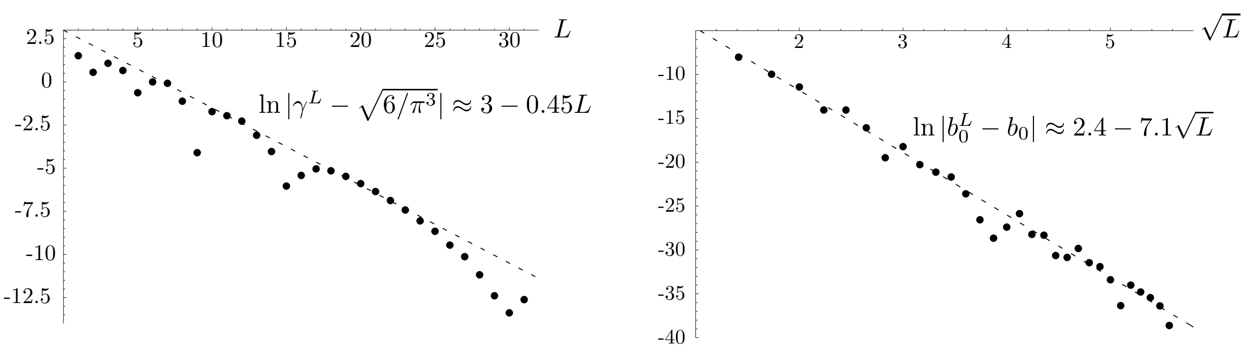

FIGURE 21.1 Logarithmic plot of the convergence behavior of the constant γ^L which normalizes the asymptotic behavior of the perturbative expansion, and of the leading strong coupling coefficient b_0^L.

fast against the exact limiting value

$$\gamma = -\sqrt{\frac{6}{\pi^3}}, \tag{21.59}$$

with superimposed oscillations. The oscillations are of the same kind as those observed in variational perturbation theory for the convergence of the approximations to the strong-coupling coefficients b_k (see Figs. 5.19 and 5.20 in Ref. [5]).

Here also, the strong-coupling coefficients b_k^L converge exponentially fast towards b_k, but with a larger power of L in the exponent of the last term $\approx e^{-\text{const}\sqrt{L}}$ [see Eq. (21.39)], rather

n	b_n
0	0.667 986 259 155 777 108 270 962 02
1	0.143 668 783 380 864 910 020 319
2	−0.008 627 565 680 802 279 127 963
3	0.000 818 208 905 756 349 542 41
4	−0.000 082 429 217 130 077 219 91
5	0.000 008 069 494 235 040 964 75
6	−0.000 000 727 977 005 945 772 63
7	0.000 000 056 145 997 222 351 17
8	−0.000 000 002 949 562 732 709 36
9	−0.000 000 000 064 215 331 956 97
10	0.000 000 000 048 214 263 789 07

TABLE 21.1 Strong-coupling coefficients b_n of the 70-th order approximants $E^0_{70}(g) = \sum_{n=0}^{70} h_n I_n(g)$ to ground state energy $E^0(g)$ of the anharmonic oscillator. They have the same accuracy as the variational perturbation theoretic calculations up to order 251 in Ref. [4].

than $\approx e^{-\text{const}L^{1/3}}$ for variational perturbation theory as in Eq. (19.51). This is seen on the right of Fig. 21.1.

We apply the resummation method to the first ten strong-coupling coefficients using the expansion coefficients f_k up to order 70. The results are shown in Table 21.3. Comparison with a similar table in Refs. [4, 5] shows that the new resummation method yields in 70th order the same accuracy as variational perturbation theory did in 251st order. In all cases the optimal parameter ρ turns out to be a slowly growing function with L.

In the strong-coupling regime, the convergence is fastest by choosing for β_0 an L-dependent value

$$\beta_0 = L. \tag{21.60}$$

Note that this choice of β_0 ruins the convergence to the imaginary part for small negative g_B which was resummed best with $\beta_0 = 5$.

21.4 Resummation for Critical Exponents

Having convinced ourselves of the fast convergence of our new resummation method, let us now apply our technique to the perturbation expansions for $\bar{g}(\bar{g}_B)$, and the critical exponents $\eta(\bar{g}_B)$, $\eta_m(\bar{g}_B)$, and $\gamma(\bar{g}_B)$ in Eqs. (20.15)–(20.19) of the O(N)-symmetric ϕ^4-theory in $D = 3$ dimensions, with the latter extended to seven loops in Eq. (20.60).

Now the power s in Eq. (19.2) is equal to zero for the critical exponents to approach a constant strong-coupling value. Similar to variational perturbation theory, we now use Eqs. (19.64) and (20.20), to determine the critical exponent of approach to scaling ω. In slight contrast to the earlier approach, we proceed here by resumming the power series of

$$-\bar{g}(\bar{g}_B)s(\bar{g}_B) = -\bar{g}_B \frac{\bar{g}'(\bar{g}_B)}{\bar{g}(\bar{g}_B)}\bigg|_{\bar{g}_B=\infty}, \tag{21.61}$$

which is simply the β-function:

$$\beta(\bar{g}_B) = -\bar{g}_B \frac{d\bar{g}(\bar{g}_B)}{d\bar{g}_B} \tag{21.62}$$

[recall (19.63)]. We do this for various values of ω and find that ω-value for which the approximations β_L^* extrapolate best to zero for $L \to \infty$. Thus we re-expand $\beta(\bar{g}_B)$ in terms of the basis functions $I_n(\bar{g}_B, \omega)$ up to the order L:

$$\beta_L(\bar{g}_B) = \sum_{n=0}^{L} h_n I_n(\bar{g}_B, \omega), \tag{21.63}$$

and plot β_L^* for various values of ω as shown in Fig. 21.2.

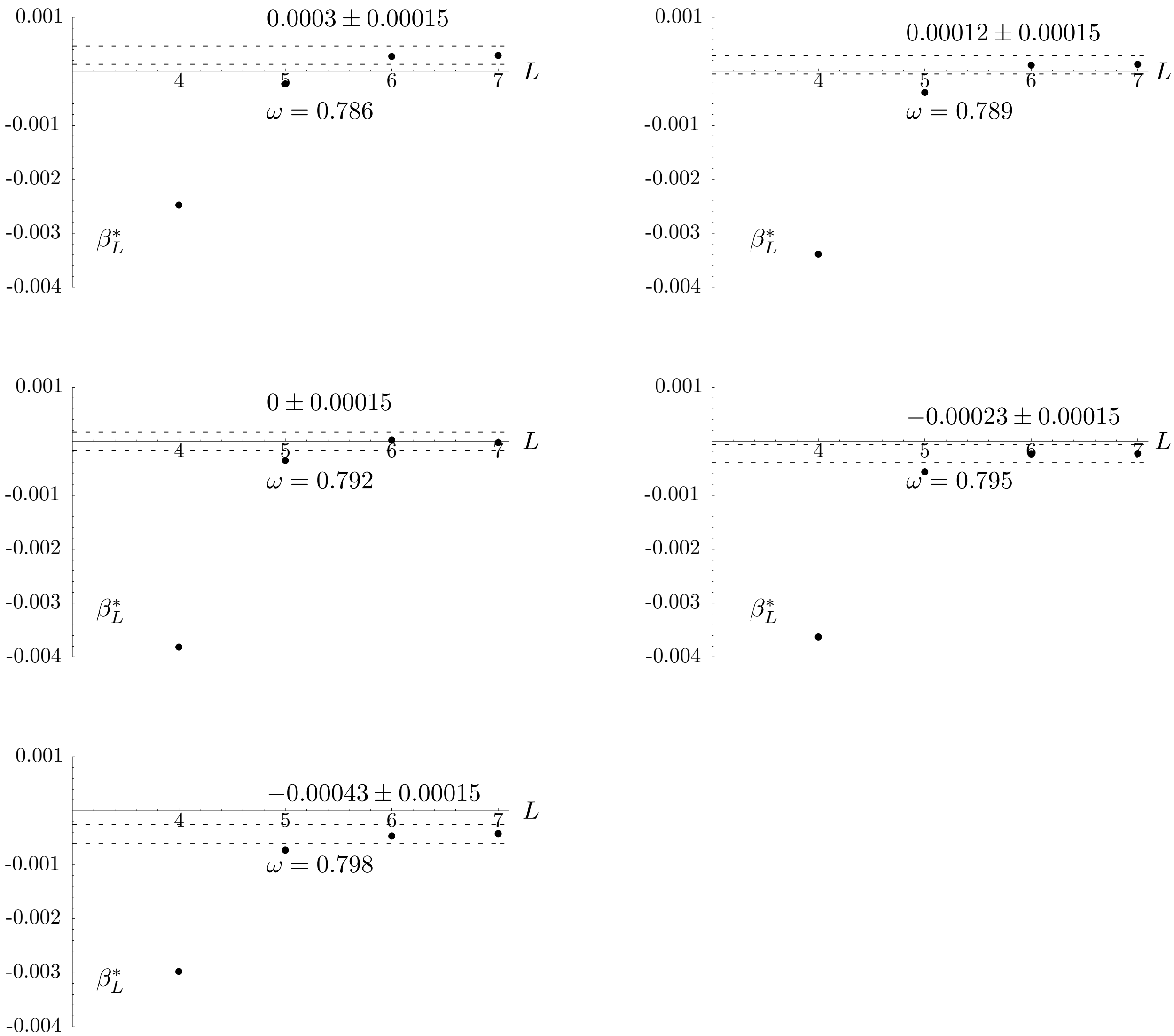

FIGURE 21.2 Convergence of strong-coupling limits of the β-function (21.62) for $N = 1$ and different values of ω. The upper and lower dashed lines denote the range of the $L \to \infty$ limit of β_L^* from which the value of ω is determined in Fig. 21.3 to be equal to $\omega = 0.792 \pm 0.003$.

We must explain how we have chosen the parameters in the basis functions $I_n(g_B) = I_n(g_B, \omega, s, \rho, \sigma, \delta, \beta_0)$. After trying out a few choices, we have given the parameters β and ρ the fixed values 1 and 10, respectively, to accelerate the convergence. To determine the remaining parameters, we recall the growth parameters of the large-order behavior of the perturbative series for the critical exponents listed in Table 20.4 . Having omitted a factor $1/(N+8)$ in the

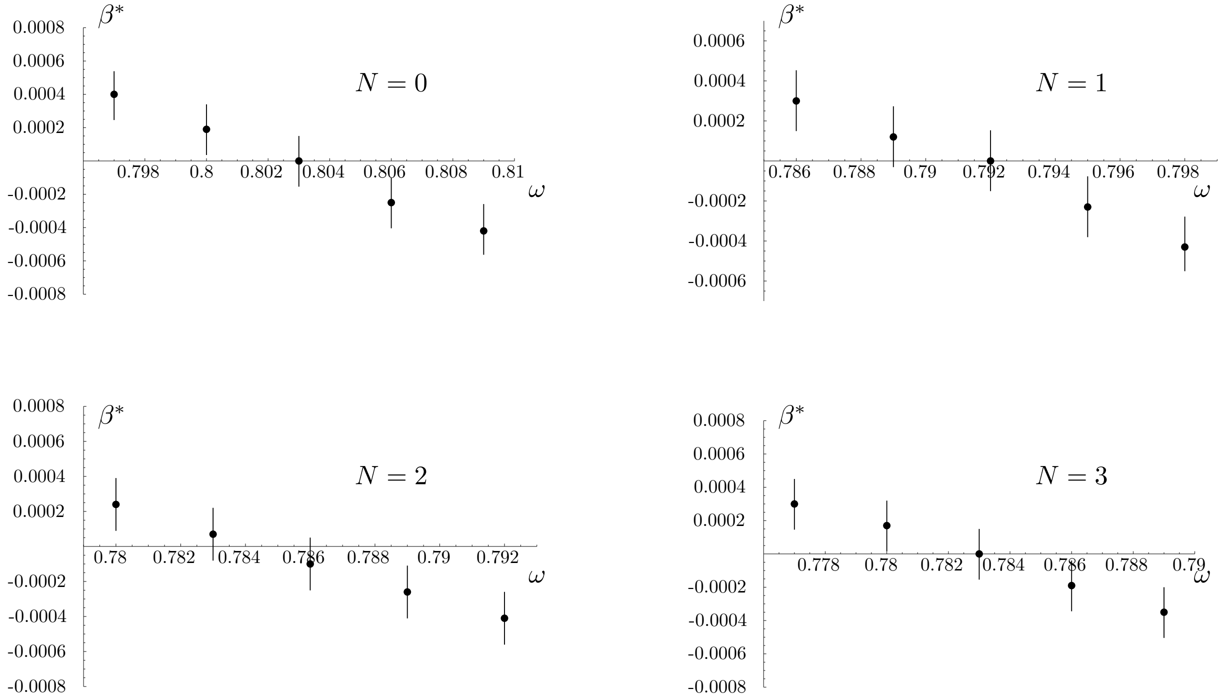

FIGURE 21.3 Plot of resummed values of β^* against ω. The true value of ω is deduced from the condition $\beta^* = 0$ and the errors are determined from the range of ω where the error bars from the resummation of β^* intersect with the x-axis.

bare coupling constants $\bar{g}_B$ on the right-hand sides in expansions (20.15)–(20.19), the α-value of these expansions is from Eq. (20.71):

$$\alpha = (D-1)\frac{16\pi}{I_4} \overset{D=3}{\equiv} 0.14777423 \cdot \frac{9}{N+8}, \tag{21.64}$$

while the parameter β_β is from Eq. (20.73)

$$\beta_\beta = 4 + N/2. \tag{21.65}$$

The prefactor γ_β in the asymptotic growth equation is listed for $N = 0, 1, 2, 3$ in the fourth row of Table 20.4 .

Before starting with the resummation it is important to make the following observation: For the expansion coefficients of the critical exponents in powers of the bare coupling constant $\bar{g}_B$, large-order estimates of the type (20.68)–(20.70) become reliable only at much larger orders L than those in powers of the renormalized coupling constant, where the convergence was quite fast. The lack of precocity is illustrated for the β-function in Table 21.2 , which gives the β_k divided by their leading asymptotic estimates $f_k^{\beta\,\mathrm{as}}$:

$$f_k^\beta / f_k^{\beta\,\mathrm{as}} \equiv f_k^\beta / k!(-\alpha)^k k^\beta. \tag{21.66}$$

The first six approximations approach their large-order limits quite slowly. For this reason we prefer to adapt σ not from α by using Eq. (21.40), but by an optimization of the convergence. Since the re-expanded series converges for fixed values of δ and σ, it is reasonable to determine

N	0	1	2	3
k	$f_k^\beta/f_k^{\beta\,\mathrm{as}}$	$f_k^\beta/f_k^{\beta\,\mathrm{as}}$	$f_k^\beta/f_k^{\beta\,\mathrm{as}}$	$f_k^\beta/f_k^{\beta\,\mathrm{as}}$
2	0.57	0.45	0.35	0.27
3	0.61	0.45	0.32	0.22
4	0.73	0.51	0.34	0.22
5	0.89	0.61	0.40	0.25
6	1.07	0.73	0.47	0.29
7	1.26	0.88	0.56	0.34
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
γ_β	110.0	97.0	75.5	53.2

TABLE 21.2 First six perturbative coefficients in expansions of β-function in powers of bare coupling constant g_B, divided by their asymptotic large-order estimates $(-\alpha)^k k! k^{\beta_\beta}$. The ratios $f_k^\beta/f_k^{\beta\,\mathrm{as}}$ increase quite slowly towards the theoretically predicted normalization constant γ_β in the lowest row.

these parameters by searching for a point of least dependence in the largest available order L. This is done by imposing the conditions

$$\frac{d\kappa_L}{d\sigma} = 0 \quad \text{and} \quad \frac{d^2\kappa_L}{d\sigma^2} = 0 \tag{21.67}$$

to determine *both* parameters δ, σ, where κ_L denotes the Lth approximation to any of the exponents γ, ν or η. In accordance with the discussions in Subsection 21.2.2, this procedure provides a value of σ which is smaller than that given by (21.40).

The results for the critical exponents of all O(N)-symmetries are shown in Figs. 21.2–21.6 and Table 21.3 .

N	γ	η	ν	ω
0	1.1604[8] (4) {0.075}	0.0285[6] (4) {0.037}	0.5881[8] (4) {0.075}	0.803[3] {1}
1	1.2403[8] (4) {0.110}	0.0335[6] (3) {0.043}	0.6303[8] (4) {0.065}	0.792[3] {1}
2	1.3164[8] (5) {0.033}	0.0349[8] (5) {0.042}	0.6704[7] (4) {0.098}	0.784[3] {1}
3	1.3882[10] (7) {0.210}	0.0350[8] (5) {0.043}	0.7062[7] (4) {0.110}	0.783[3] {1}

TABLE 21.3 Critical exponents of O(N)-symmetric ϕ^4-theory from new resummation method. The numbers in square brackets indicate the total errors. They arise form the error of the resummation at fixed values of ω indicated in parentheses, and the errors coming from the inaccurate knowledge of ω. The former are estimated from the scattering of the approximants around the graphically determined large-L limit, the latter follow from the errors in ω and the derivatives of the critical exponents with respect to changes of ω indicated in the curly brackets.

The total errors are indicated in the square brackets. They are deduced from the errors of resummation of the critical exponents at a fixed value of ω indicated in the parentheses, and from the error $\Delta\omega$ of ω, using the derivative of the exponent with respect to ω given in curly brackets. Symbolically, the relation between these errors is

$$[\ldots] = (\ldots) + \Delta\omega\{\ldots\}. \tag{21.68}$$

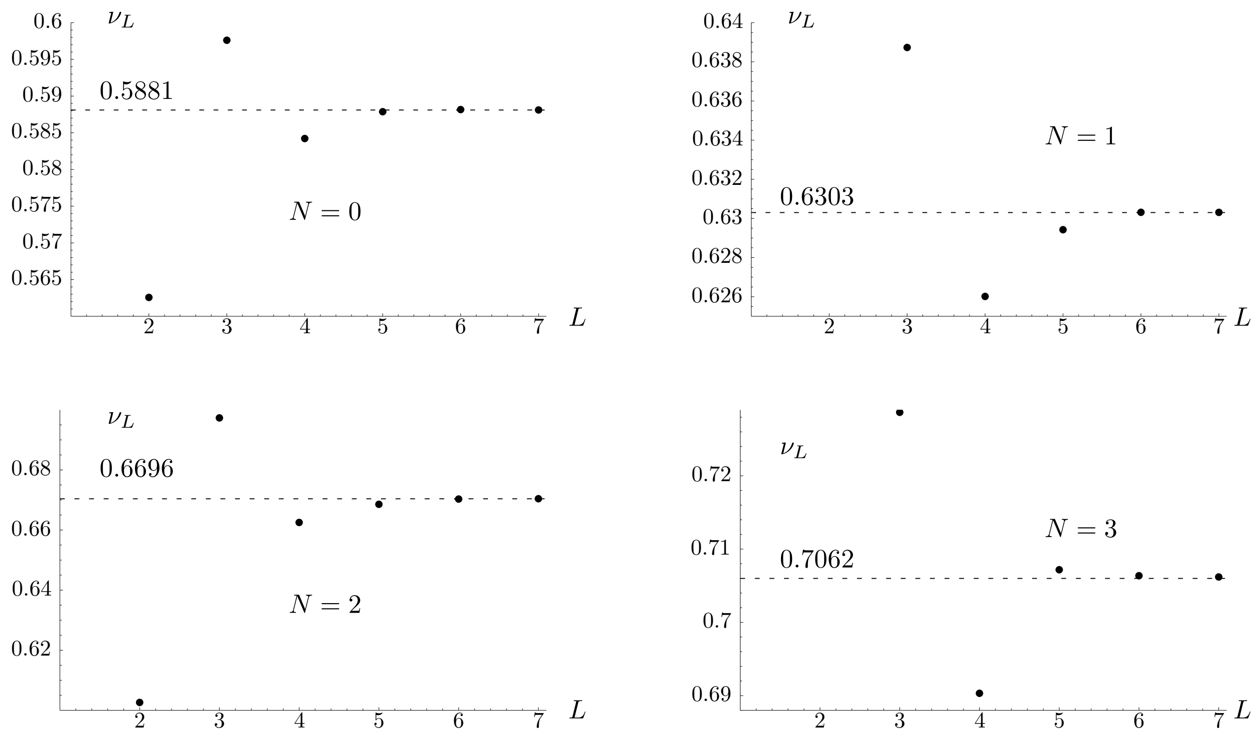

FIGURE 21.4 Convergence of critical exponent ν for different values of N.

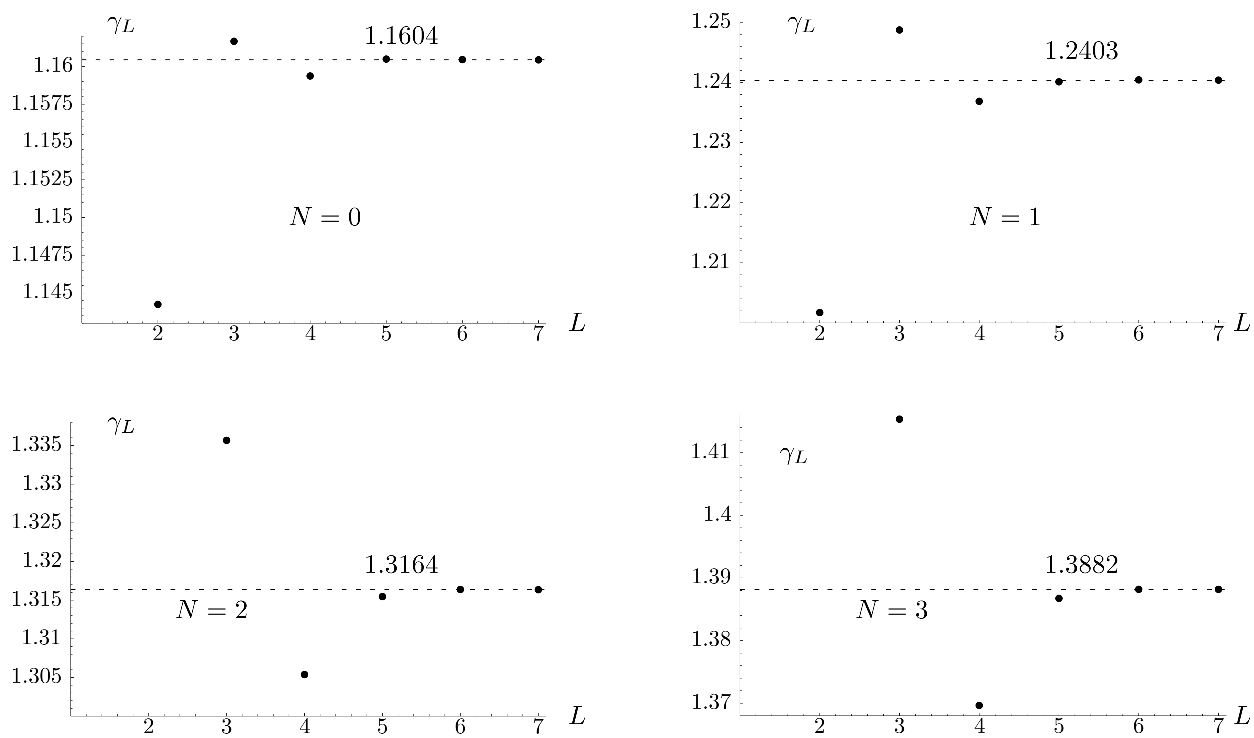

FIGURE 21.5 Convergence of critical exponent γ for different values of N.

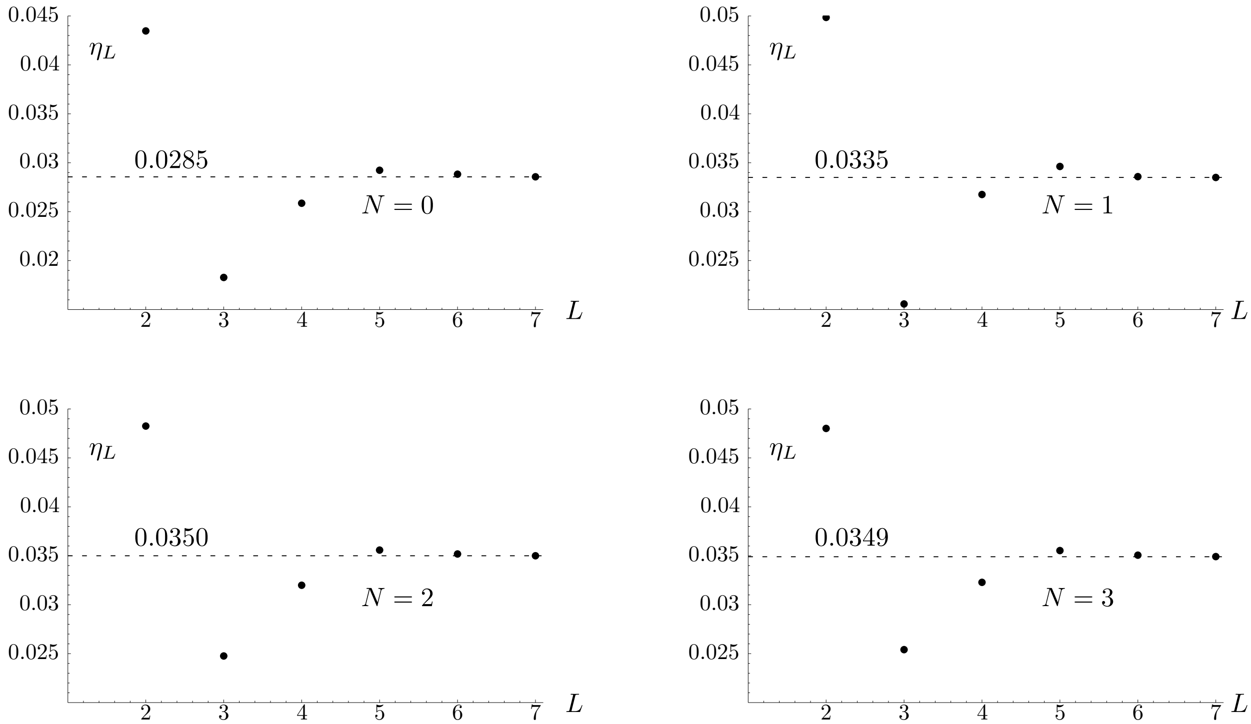

FIGURE 21.6 Convergence of critical exponent η for different values of N.

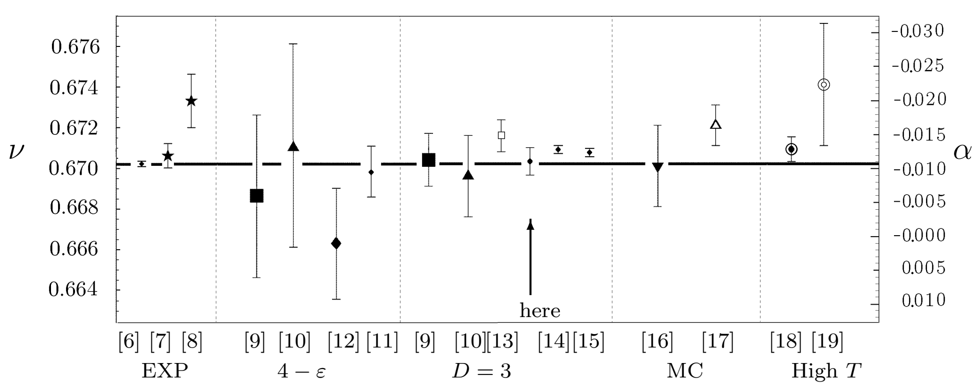

FIGURE 21.7 Comparison of our results for critical exponents α of superfluid helium with experiments and results from other theories. The value marked by an arrow points to the result (21.71) of the resummation in this chapter.

The accuracy of our results can be judged by comparison with the most accurately measured critical exponent α found in the space shuttle experiment in Ref. [6] [see Eq. (1.23)]:

$$\alpha = -0.01056 \pm 0.00038. \tag{21.69}$$

Our value for ν in Table 21.3 is

$$\nu = 0.6704 \pm 0.007 \tag{21.70}$$

and yields via the scaling relation $\alpha = 2 - 3\nu$:

$$\alpha = -0.0112 \pm 0.0021, \tag{21.71}$$

in best agreement with the experimental number (21.69).

A comparison between these numbers and the results of other authors in Fig. 21.7 shows that our results are among the more accurate ones.

Notes and References

The hyper-Borel transformation was first introduced in Ref. [1]. In the mathematical literature, this transformation has never been investigated, although it is contained in a class of quite general mathematical transformations introduced in the textbook of Hardy, *Divergent Series* (Oxford University Press, Oxford 1949 in the context of *moment constant methods*. These comprise transformation $B(y) = \sum f_k y^k/\mu_k$, where the μ_k are given by a Stieltjes integral $\mu_k = \int_0^\infty x^k d\chi(x)$ and χ is a bounded and increasing function of x guaranteeing the convergence of the Stieltjes integral. This definition includes our transformation for the somewhat complicated choice $d\chi(x) = \Gamma(\beta_0)x^{-s-1} \oint_C dt\, e^{t+x^\omega t^{1-\omega}} t^{s(1-1/\omega)-\beta_0}\, dx/2\pi i$.

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