Conclusion

In this book we have explained the presently available calculational techniques to calculate the critical exponents of ϕ^4-theories in $D = 4 - \varepsilon$ dimensions. The calculations were organized with the help of Feynman diagrams, which were considered up to five loops. The analytic effort going into the evaluation of the associated Feynman integrals was considerable, and would have been unsurmountable without the use of computer-algebraic programs such as Reduce and Mathematica. The calculations yielded perturbation expansions for the critical exponents, which led to power series in ε to the order ε^5. These power series were calculated for interactions with O(N) symmetry and a combination of O(N) and cubic symmetry.

An important tool for deriving these series was the technique dimensional regularization with minimal subtraction developed by t'Hooft and Veltman.[1] This leads to ε-expansions for all observable critical quantities in less than $D=4$ dimensions of maximal simplicity.

The calculation of the renormalization constants was done most efficiently with the help of the R-operation developed by Bogoliubov and Shirkov.[2] This operation must be performed tediously diagram by diagram.

The most important simplification brought about by the minimal subtraction is the independence of the renormalization constants on the mass and the external momenta of the Feynman integrals. In this way, only massless integrals need to be calculated, with at most one external momentum. The external vertices of this momentum may be chosen quite freely, restricted only by the requirement that the infrared behavior of the integral remains unchanged (method of IR-rearrangement). This allowed us to use existing algorithms[3] for the reduction of the massless integrals to nested one-loop integrals, which can all be expressed in terms of Gamma-functions. Unfortunately, however, these algorithms are applicable to only some generic two- and three-loop diagrams which contain a subdiagram in triangle form. So far, no generalization has been found if a subdiagram has a square form.

A further class of diagrams becomes accessible when transforming the massless, dimensionally regularized integrals into their dual form by Fourier transformation. In **x**-space, the integrals are solvable or may be reduced to solvable integrals by applying the reduction algorithms in **x**-space (method of ideal index constellations).[4]

Most of the diagrams can be brought to one of the calculable forms by IR-rearrangement. In some cases, the IR-rearrangement produces artificial IR-divergences. In dimensional regularization such IR-divergences appear also in the form of poles in ε and must be removed by a procedure analogous to the R-operation, the R^*-operation.

In this way, all integrals up to five loops have been calculated algebraically, with only six exceptions, which require special methods based on partial integration, clever differentiation, and a final application of the R-operation. All calculations were done with the help of computer-

[1] G.'t Hooft and M. Veltman, Nuclear Physics B *44*, 189-213 (1972); G.'t Hooft, Nucl. Phys. B *61*, 455 (1973).

[2] N.N. Bogoliubov and D.V. Shirkov, *Introduction to the Theory of Quantized Fields*, Wiley Interscience, New York, 1958.

[3] K.G. Chetyrkin, A.L. Kataev, and F.V. Tkachov, Nucl. Phys. B *174*, 345-377 (1980).

[4] D.I. Kazakov, Phys. Lett. B *133*, 406-410 (1983); Theor. Math. Phys. *61*, 84-89 (1985).

algebraic programs to avoid trivial errors. The implementation of the subtraction procedures $\bar{R}$ and $\bar{R}^*$ on the computer has an obstacle in the ambiguities of IR-rearrangement. The choice of the external vertices selects different integrals to be calculated. Since it was not clear at the beginning which integrals would arise and which were calculable, the IR-rearrangement had to be done tediously by hand. For many diagrams there is only one arrangement of the external vertices which allowed us to calculate the corresponding integral.

At the six-loop level, new generic types of diagrams are encountered. A subdiagram of the square type cannot be avoided by IR-rearrangement. The above-described methods are therefore insufficient to calculate all six-loop diagrams, and new methods had to be developed.

In the case of fields with several indices and tensorial interactions, each Feynman diagram characterizes not only a momentum integral, but also an index sum. This sum leads to a symmetry factor for each diagram. In this work, a combination of O(N) and cubic symmetry was considered in the interaction. For $N=2,3$, the O(N) and the mixed $O(N)$-cubic symmetry cover all symmetries which lead to only one length scale. For $N=4$, there are several other symmetries with this property. In a two-loop calculation,[5] however, only two other stable fixed points were found in addition to the isotropic and the cubic one. In this text, we have extended these results to the five-loop level.

The five-loop expansions calculated in this text lead to critical exponents as power series ε up to ε^5 for critical phenomena of systems with a combination of O(N) and cubic symmetry. For the comparison with critical exponents measured by experiments in $D = 3$ dimensions, these series have to be evaluated at $\varepsilon=1$. This is not directly possible since the series are divergent and require special techniques for their resummation.

The most simple technique, the Padé approximation does not yield satisfactory results since it does not include other available information on the series. This comes from a knowledge of the behavior of all series at large orders. This knowledge can be exploited by re-expanding all series into functions with the same large-order behavior. This method has been applied to systems with O(N) and with a mixture of O(N) and cubic symmetry.

Two other resummation schemes have been developed. One is based on the fact that critical phenomena correspond to a strong-coupling limit of the divergent series when expanded in powers of the *bare* coupling constant.[6]

In $D = 3$ dimensions, six-loop calculations have been done a long time ago.[7] For the critical exponents η and η_m, even seven-loop expansions have recently become available.[8] Although the additional terms have large coefficients, their effect upon the critical exponents after resummation is very small.[9] These alternative expansions have been described and resummed in this text for comparison.

Other series for critical exponents have been derived from lattice models, the most extensive ones by Butera and Comi.[10] Also these have been discussed here in some detail.

An extension of the perturbation expansions in powers of ε in $4-\varepsilon$ dimensions to more than five loops seems prohibitively difficult at the present time. New theoretical ideas will

[5] J.-C. Toledano, L. Michel, P. Toledano, E. Brézin, Phys. Rev. B *31*, 7171 (1985).

[6] H. Kleinert, Phys. Rev. D **57**, 2264 (1998) (E-Print aps1997jun25_001); addendum (cond-mat/9803268); Phys. Rev. D (in press) (1999) (hep-th/9812197).

[7] B.G. Nickel, D.I. Meiron, and G.A. Baker, Jr., University of Guelph report, 1977 (unpublished).

[8] D.B. Murray and B.G. Nickel, unpublished.

[9] See R. Guida and J. Zinn-Justin, Saclay preprint 1098, cond-mat/9803240 and H. Kleinert, Phys. Rev. D (in press) (1999) (hep-th/9812197).

[10] See the references in the footnotes of Section 20.8.

probably be needed for such extensions. Recent knot-theoretic work[11] may be a first step in this direction.

[11]D.J. Broadhurst, Z. Phys. C *32*, 249 (1986); D. Kreimer, Phys. Lett. B *273*, 177 (1991); D.J. Broadhurst and D. Kreimer, Int. J. Mod. Physics C *6* 519 (1995); UTAS-HHYS-96-44 (hep-th/9609128).