18

Cubic Anisotropy

In most magnetic systems existing in nature, the O(N) symmetry is broken by the crystal structure of the materials. The magnetization points in certain preferred directions, for example along the edges or the diagonals of a cubic lattice. In order to describe such situations in general, we generalize the O(N)-symmetric model, and extend it by an interaction which prefers these directions on a hypercube in the N-dimensional field space. The general form of the ϕ^4-interaction was given in Eq. (6.4). An interaction term $\lambda_2 \sum_{\alpha=1}^N \phi_\alpha^4$ accounting for the *cubic anisotropy* is added to the O(N)-symmetric interaction $\lambda_1(\sum_{\alpha=1}^N \phi_\alpha^2)^2$. The extended theory interpolates between an O(N)-symmetric and a system with increasing *cubic anisotropy*. The tensor associated with the two ϕ^4-interactions was introduced in Subsection 6.4.2, where the relevant changes in the perturbation expansions were discussed. We are now prepared to calculate the critical properties of the field fluctuations.

18.1 Basic Properties

The tensor in the ϕ^4-interaction of mixed O(N)-symmetric and cubic symmetry was written down in Eq. (6.47). It reads

$$T^{\rm cub}_{\alpha\beta\gamma\delta} = g_1\, T^{(1)}_{\alpha\beta\gamma\delta} + g_2\, T^{(2)}_{\alpha\beta\gamma\delta}, \tag{18.1}$$

where $T^{(1)}_{\alpha\beta\gamma\delta}$ and $T^{(2)}_{\alpha\beta\gamma\delta}$ have the symmetrized form:

$$T^{(1)}_{\alpha\beta\gamma\delta} = \frac{1}{3}(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) \sim T^{O(N)}_{\alpha\beta\gamma\delta}, \tag{18.2}$$

$$T^{(2)}_{\alpha\beta\gamma\delta} = \delta_{\alpha\beta\gamma\delta}. \tag{18.3}$$

For the definition of $\delta_{\alpha\beta\gamma\delta}$ see Eq. (6.48). The tensor $T^{\rm cub}$ fulfills the conditions (10.208) and (10.209) stated in Section 10.13, which ensure that the theory has only one length scale. For $N = 2, 3$, the combinations of $T^{(1)}_{\alpha\beta\gamma\delta}$ and $T^{(2)}_{\alpha\beta\gamma\delta}$ exhaust all tensors of rank 4 for which the theory has that property [1], i.e., for which Eq. (10.208) holds. For $N \geq 4$, more tensors are admissible without introducing new length scales [2].

The energy calculated with (18.1) is no longer invariant under rotations of the field at constant magnitude. It has minima for an axial order of the ground state of the field with an expectation value $\mathbf{\Phi} = \langle \boldsymbol{\phi}(\mathbf{x})\rangle = (1, 0, \dots, 0)$, or with a diagonal expectation value $\mathbf{\Phi} = \langle \boldsymbol{\phi}(\mathbf{x})\rangle = (1/\sqrt{N}, 1/\sqrt{N}, \dots, 1/\sqrt{N})$ in the N-dimensional hypercube (compare Section 1.1). For the axial order, the magnitude of the interaction is $(g_1+g_2)|\mathbf{\Phi}|^4$; for the diagonal ordering it assumes $(g_1+g_2/N)|\mathbf{\Phi}|^4$, implying that for positive g_2 and $N > 1$ diagonals are favored, whereas for negative g_2 and $N > 1$ the edges are favored. The energy is thus positive definite within the *mean-field stability wedge* bounded by the straight lines $g_1 + g_2 > 0$ and $Ng_1 + g_2 > 0$, as shown in Fig. 18.1. This will be changed drastically by the fluctuations, as indicated in Fig. 18.2.

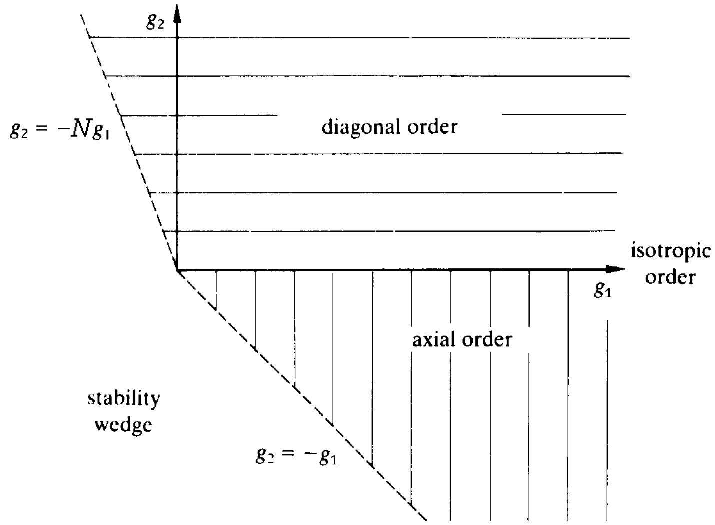

FIGURE 18.1 Stability wedge in mean-field approximation. There exist three stable ordered states below the critical temperature, depending on the two coupling constants g_1 and g_2. The left-hand side of the dashed lines is unstable.

Inside the mean-field domain of stability we have two special cases. For $g_1 = 0,\ g_2 \neq 0$, the field components decouple and we have a simple Ising-type system with an N-dependent overall factor. For $g_1 \neq 0,\ g_2 = 0$ we recover the O(N)-symmetric system.

The system turns out to have four fixed points. The Gaussian fixed point at the origin; the *Ising* fixed point with $g_1^I = 0,\ g_2^I \neq 0$; the *O(N)-symmetric* or *Heisenberg* fixed point with $g_1^H \neq 0,\ g_2^H = 0$; and the *cubic* fixed point with $g_1^C \neq 0,\ g_2^C \neq 0$. The cubic fixed point is the only new one; the others are known from the results in Chapter 17. The first-order results are

$$g_1^H(\varepsilon) = \frac{3\,\varepsilon}{8+N}, \qquad g_2^H(\varepsilon) = 0; \tag{18.4}$$

$$g_1^I(\varepsilon) = 0, \qquad g_2^I(\varepsilon) = \frac{\varepsilon}{3}; \tag{18.5}$$

$$g_1^C(\varepsilon) = \frac{\varepsilon}{N}, \qquad g_2^C(\varepsilon) = \frac{(N-4)\,\varepsilon}{3\,N}. \tag{18.6}$$

The location of the fixed points depends on N. For $N = 1$, there is only one coupling constant $g_1 + g_2$, and the O(N)-symmetric and the Ising fixed points are equivalent, as are the Gaussian and the cubic fixed points. For $N = 2$, a special symmetry comes into play. A rotation in the space of the fields through $\pi/4$:

$$\phi_1' = \frac{\phi_1 + \phi_2}{\sqrt{2}}, \qquad \phi_2' = \frac{\phi_1 - \phi_2}{\sqrt{2}}, \tag{18.7}$$

leaves the fourth-order interaction term invariant if the coupling constants are transformed with

$$g_1' = g_1 + \frac{3}{2}\,g_2, \qquad g_2' = -g_2. \tag{18.8}$$

These equations transform the Ising fixed point into the cubic fixed point for $N = 2$, as can be seen from Eqs. (18.5) and (18.6).

For increasing N, the cubic fixed point moves from the lower half-plane $g_2 < 0$ into the upper one with $g_2 > 0$. For a certain $N = N_c$, the O(N)-symmetric and the cubic fixed point coalesce and interchange stability. For $N < N_c$, the O(N)-symmetric fixed point is stable; for $N > N_c$, the cubic fixed point is stable (see Fig. 18.2). For $N < N_c$, there is symmetry restoration.

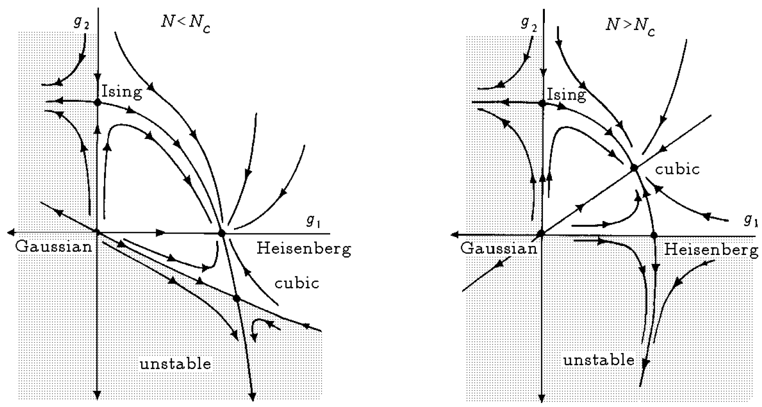

FIGURE 18.2 Stability of the fixed points in ϕ^4-theory with mixed O(N)-symmetric and cubic coupling for $N < N_c$ and $N > N_c$. The results of our analysis are compatible with $N_c = 3$.

Although the O(N) symmetry is broken at the mean field level by the second interaction, the symmetry is restored by the large fluctuations near the critical point. The physically most interesting value of N is 3, where the O(N)-symmetric fixed point characterizes the critical behavior of the classical Heisenberg model of magnetism.

Early work based on three-loop calculations in $D = 4 - \varepsilon$ estimated N_c to lie somewhere between 3 and 4 [3, 4, 5]. This implied that all magnetic systems with cubic symmetry occurring in nature show an O(3)-symmetric critical behavior. In contrast, the five-loop results suggest that the critical value N_c lies *below* $N = 3$, so that the cubic fixed point should govern the critical behavior of magnetic transitions in cubic crystals. A similar result was also found in a four-loop calculation in $D = 3$ dimensions [6]. The five-loop calculations to be presented below will favor this result, although not conclusively. The difficulty of deciding which is the correct fixed point has its parallel in the experimental difficulty of distinguishing the critical exponents of the two fixed points. On account of the vicinity of the Heisenberg fixed point, the difference between the critical exponents of the two fixed points is too small to be measured with present techniques. The cubic universality class for $N = 3$, if it exists, is practically indistinguishable from the O(3)-symmetric Heisenberg class.

18.2 Series Expansions for RG Functions

The general equations for the renormalization group functions of ϕ^4-interactions with two coupling constants were given in Eqs. (10.211)–(10.213). In dimensional regularization, they are given by the simple poles of the renormalization constants [recall Eq. (10.219)]:

$$\begin{aligned}
\beta_i(g_1, g_2) &= -\varepsilon g_i - 2\, g_i \Big[-\frac{g_1}{2}\frac{\partial Z_{g_i,1}}{\partial g_1} - \frac{g_2}{2}\frac{\partial Z_{g_i,1}}{\partial g_2} - 2\gamma(g_1, g_2)\Big] \\
&\equiv -\varepsilon g_i - 2 g_i \gamma_{g_i} + 4 g_i \gamma(g_1, g_2)\,, \qquad i = 1, 2, \qquad (18.9)\\
\gamma(g_1, g_2) &= -\frac{g_1}{2}\frac{\partial Z_{\phi^2,1}}{\partial g_1} - \frac{g_2}{2}\frac{\partial Z_{\phi^2,1}}{\partial g_2}\,, \qquad (18.10)
\end{aligned}$$

$$\gamma_m(g_1,g_2) \;=\; \frac{g_1}{2}\frac{\partial Z_{m^2,1}}{\partial g_1}+\frac{g_2}{2}\frac{\partial Z_{m^2,1}}{\partial g_2}+\gamma\,. \tag{18.11}$$

There are now two renormalization constants for the coupling constants: Z_{g_1} and Z_{g_2}. The four renormalization constants are calculated using the general diagrammatic sums (15.1)–(15.3). All expressions occuring in these sums, the pole terms of the integrals, the weight factors, and the symmetry factors for the cubic symmetry have been listed in the tables of Appendix B.

The resulting expansions for the renormalization group functions up to five loops are [7]:

$$\begin{aligned}
\beta_1(g_1,g_2) =& -\varepsilon\, g_1+g_1{}^2\,\tfrac{N+8}{3}+g_1\,g_2\;2+g_1{}^3\left[-N-\tfrac{14}{3}\right]-g_1{}^2\,g_2\,\tfrac{22}{3}-g_1\,g_2{}^2\,\tfrac{5}{3}\\
&+g_1{}^4\left[N^2\,\tfrac{11}{72}+N\left(\tfrac{461}{108}+\tfrac{20\,\zeta(3)}{9}\right)+\tfrac{370}{27}+\tfrac{88\,\zeta(3)}{9}\right]+g_1{}^3\,g_2\left[\tfrac{79\,N}{36}+\tfrac{659}{18}+\tfrac{64\,\zeta(3)}{3}\right]\\
&+g_1{}^2\,g_2{}^2\left[\tfrac{N}{24}+\tfrac{107}{4}+8\,\zeta(3)\right]+g_1\,g_2{}^3\;7\\
&+g_1{}^5\left[N^3\,\tfrac{5}{3888}+N^2\left(-\tfrac{395}{243}-\tfrac{14\,\zeta(3)}{9}+\tfrac{10\,\zeta(4)}{27}-\tfrac{80\,\zeta(5)}{81}\right)+N\left(-\tfrac{10057}{486}-\tfrac{1528\,\zeta(3)}{81}+\tfrac{124\,\zeta(4)}{27}-\tfrac{2200\,\zeta(5)}{81}\right)\right.\\
&\qquad\left.-\tfrac{24581}{486}-\tfrac{4664\,\zeta(3)}{81}+\tfrac{352\,\zeta(4)}{27}-\tfrac{2480\,\zeta(5)}{27}\right]\\
&+g_1{}^4\,g_2\left[N^2\left(\tfrac{7}{81}-\tfrac{\zeta(3)}{9}\right)+N\left(-\tfrac{1319}{54}-\tfrac{184\,\zeta(3)}{9}+\tfrac{38\,\zeta(4)}{9}-\tfrac{400\,\zeta(5)}{27}\right)-\tfrac{15967}{81}-\tfrac{4856\,\zeta(3)}{27}+\tfrac{340\,\zeta(4)}{9}-\tfrac{2560\,\zeta(5)}{9}\right]\\
&+g_1{}^3\,g_2{}^2\left[N\left(-\tfrac{301}{72}-\tfrac{35\,\zeta(3)}{9}\right)-\tfrac{13433}{54}-\tfrac{1456\,\zeta(3)}{9}+\tfrac{64\,\zeta(4)}{3}-\tfrac{2000\,\zeta(5)}{9}\right]\\
&\quad+g_1{}^2\,g_2{}^3\left[N\left(-\tfrac{25}{36}+\tfrac{\zeta(3)}{3}\right)-\tfrac{4867}{36}-50\,\zeta(3)-8\,\zeta(4)-\tfrac{160\,\zeta(5)}{3}\right]+g_1\,g_2{}^4\left[-\tfrac{477}{16}-3\,\zeta(3)-6\,\zeta(4)\right]\\
&+g_1{}^6\left[N^4\left(\tfrac{13}{62208}-\tfrac{\zeta(3)}{432}\right)+N^3\left(\tfrac{6289}{31104}+\tfrac{26\,\zeta(3)}{81}-\tfrac{2\,\zeta^2(3)}{27}-\tfrac{7\,\zeta(4)}{24}+\tfrac{305\,\zeta(5)}{243}-\tfrac{25\,\zeta(6)}{81}\right)\right.\\
&\qquad+N^2\left(\tfrac{50531}{3888}+\tfrac{8455\,\zeta(3)}{486}-\tfrac{59\,\zeta^2(3)}{81}-\tfrac{347\,\zeta(4)}{54}+\tfrac{7466\,\zeta(5)}{243}-\tfrac{1775\,\zeta(6)}{162}+\tfrac{686\,\zeta(7)}{27}\right)\\
&\qquad+N\left(\tfrac{103849}{972}+\tfrac{69035\,\zeta(3)}{486}+\tfrac{446\,\zeta^2(3)}{81}-\tfrac{2383\,\zeta(4)}{54}+\tfrac{66986\,\zeta(5)}{243}-\tfrac{7825\,\zeta(6)}{81}+343\,\zeta(7)\right)\\
&\qquad\left.+\tfrac{17158}{81}+\tfrac{27382\,\zeta(3)}{81}+\tfrac{1088\,\zeta^2(3)}{27}-\tfrac{880\,\zeta(4)}{9}+\tfrac{55028\,\zeta(5)}{81}-\tfrac{6200\,\zeta(6)}{27}+\tfrac{25774\,\zeta(7)}{27}\right]\\
&+g_1{}^5\,g_2\left[N^3\left(\tfrac{161}{10368}-\tfrac{17\,\zeta(3)}{648}-\tfrac{\zeta(4)}{36}\right)+N^2\left(\tfrac{59675}{15552}+\tfrac{170\,\zeta(3)}{27}-\tfrac{4\,\zeta^2(3)}{3}-\tfrac{19\,\zeta(4)}{4}+\tfrac{602\,\zeta(5)}{27}-\tfrac{50\,\zeta(6)}{9}\right)\right.\\
&\qquad+N\left(\tfrac{5723}{27}+\tfrac{21560\,\zeta(3)}{81}-\tfrac{190\,\zeta^2(3)}{27}-\tfrac{2339\,\zeta(4)}{27}+\tfrac{4046\,\zeta(5)}{9}-\tfrac{4075\,\zeta(6)}{27}+\tfrac{1274\,\zeta(7)}{3}\right)\\
&\qquad\left.+\tfrac{537437}{486}+\tfrac{116759\,\zeta(3)}{81}+\tfrac{3148\,\zeta^2(3)}{27}-\tfrac{10177\,\zeta(4)}{27}+\tfrac{75236\,\zeta(5)}{27}-\tfrac{24050\,\zeta(6)}{27}+\tfrac{11564\,\zeta(7)}{3}\right]\\
&+g_1{}^4\,g_2{}^2\left[N^2\left(-\tfrac{1921}{10368}+\tfrac{763\,\zeta(3)}{648}-\tfrac{17\,\zeta(4)}{36}+\tfrac{5\,\zeta(5)}{9}\right)\right.\\
&\qquad+N\left(\tfrac{270749}{2592}+\tfrac{9230\,\zeta(3)}{81}-\tfrac{232\,\zeta^2(3)}{27}-\tfrac{4841\,\zeta(4)}{108}+\tfrac{2045\,\zeta(5)}{9}-\tfrac{1450\,\zeta(6)}{27}+\tfrac{245\,\zeta(7)}{3}\right)\\
&\qquad\left.+\tfrac{1314497}{648}+\tfrac{171533\,\zeta(3)}{81}+\tfrac{1384\,\zeta^2(3)}{27}-\tfrac{23105\,\zeta(4)}{54}+\tfrac{96794\,\zeta(5)}{27}-\tfrac{25400\,\zeta(6)}{27}+\tfrac{14210\,\zeta(7)}{3}\right]\\
&+g_1{}^3\,g_2{}^3\left[N\left(\tfrac{30277}{1296}+\tfrac{344\,\zeta(3)}{27}-\tfrac{25\,\zeta(4)}{6}+\tfrac{208\,\zeta(5)}{9}\right)\right.\\
&\qquad\left.+\tfrac{2281727}{1296}+\tfrac{37789\,\zeta(3)}{27}-\tfrac{544\,\zeta^2(3)}{9}-\tfrac{337\,\zeta(4)}{3}+\tfrac{17444\,\zeta(5)}{9}-\tfrac{1600\,\zeta(6)}{9}+2352\,\zeta(7)\right]\\
&+g_1{}^2\,g_2{}^4\left[N\left(\tfrac{26171}{6912}-\tfrac{77\,\zeta(3)}{48}+\tfrac{7\,\zeta(4)}{8}-\tfrac{4\,\zeta(5)}{3}\right)+\tfrac{1336801}{1728}+\tfrac{5495\,\zeta(3)}{12}-\tfrac{190\,\zeta^2(3)}{3}\right.\\
&\qquad\left.+\tfrac{141\,\zeta(4)}{2}+\tfrac{1145\,\zeta(5)}{3}+\tfrac{575\,\zeta(6)}{3}+441\,\zeta(7)\right]\\
&+g_1\;\,g_2{}^5\left[\tfrac{158849}{1152}+\tfrac{1519\,\zeta(3)}{24}-18\,\zeta^2(3)+\tfrac{65\,\zeta(4)}{2}+2\,\zeta(5)+75\,\zeta(6)\right]\;,
\end{aligned} \tag{18.12}$$

$$\begin{aligned}
\beta_2(g_1,g_2) =& -\varepsilon\, g_2+3\,g_2{}^2+4\,g_1\,g_2-g_2{}^3\,\tfrac{17}{3}-g_1\,g_2{}^2\,\tfrac{46}{3}-g_1{}^2\,g_2\,\tfrac{5\,N+82}{9}+g_2{}^4\left[\tfrac{145}{8}+12\,\zeta(3)\right]\\
&+g_1\,g_2{}^3\left[\tfrac{131}{2}+48\,\zeta(3)\right]+g_1{}^2\,g_2{}^2\left[\tfrac{17\,N}{24}+\tfrac{325}{4}+64\,\zeta(3)\right]+g_1{}^3\,g_2\left[-N^2\,\tfrac{13}{108}+N\left(\tfrac{92}{27}+\tfrac{16\,\zeta(3)}{9}\right)+\tfrac{821}{27}+\tfrac{224\,\zeta(3)}{9}\right]\\
&+g_2{}^5\left[-\tfrac{3499}{48}-78\,\zeta(3)+18\,\zeta(4)-120\,\zeta(5)\right]+g_1\,g_2{}^4\left[-\tfrac{1004}{3}-387\,\zeta(3)+96\,\zeta(4)-600\,\zeta(5)\right]\\
&+g_1{}^2\,g_2{}^3\left[N\left(-\tfrac{19}{24}-\tfrac{19\,\zeta(3)}{3}+4\,\zeta(4)\right)-\tfrac{10661}{18}-724\,\zeta(3)+184\,\zeta(4)-\tfrac{3440\,\zeta(5)}{3}\right]\\
&+g_1{}^3\,g_2{}^2\left[N^2\left(\tfrac{1}{6}+\tfrac{\zeta(3)}{9}\right)+N\left(-\tfrac{508}{27}-\tfrac{218\,\zeta(3)}{9}+12\,\zeta(4)-\tfrac{160\,\zeta(5)}{9}\right)-\tfrac{12349}{27}-\tfrac{5312\,\zeta(3)}{9}+\tfrac{440\,\zeta(4)}{3}-960\,\zeta(5)\right]\\
&+g_1{}^4\,g_2\left[N^3\left(-\tfrac{29}{1296}+\tfrac{\zeta(3)}{27}\right)+N^2\left(-\tfrac{7}{162}-\tfrac{8\,\zeta(3)}{9}+\tfrac{4\,\zeta(4)}{9}\right)\right.\\
&\qquad\left.+N\left(-\tfrac{3479}{162}-\tfrac{560\,\zeta(3)}{27}+\tfrac{68\,\zeta(4)}{9}-\tfrac{280\,\zeta(5)}{9}\right)-\tfrac{19679}{162}-168\,\zeta(3)+40\,\zeta(4)-\tfrac{7280\,\zeta(5)}{27}\right]
\end{aligned}$$

$$
\begin{aligned}
&+g_2{}^6\left[\tfrac{764621}{2304}+\tfrac{7965\,\zeta(3)}{16}+45\,\zeta^2(3)-\tfrac{1189\,\zeta(4)}{8}+987\,\zeta(5)-\tfrac{675\,\zeta(6)}{2}+1323\,\zeta(7)\right]\\
&+g_1\,g_2{}^5\left[\tfrac{1067507}{576}+\tfrac{35083\,\zeta(3)}{12}+288\,\zeta^2(3)-\tfrac{3697\,\zeta(4)}{4}+5920\,\zeta(5)-2100\,\zeta(6)+7938\,\zeta(7)\right]\\
&+g_1{}^2\,g_2{}^4\left[N\left(-\tfrac{16223}{3456}+\tfrac{2947\,\zeta(3)}{72}-17\,\zeta^2(3)-\tfrac{151\,\zeta(4)}{4}+\tfrac{290\,\zeta(5)}{3}-\tfrac{125\,\zeta(6)}{2}\right)\right.\\
&\qquad\left.+\tfrac{3633377}{864}+\tfrac{125459\,\zeta(3)}{18}+\tfrac{2266\,\zeta^2(3)}{3}-2263\,\zeta(4)+14328\,\zeta(5)-\tfrac{15575\,\zeta(6)}{3}+19404\,\zeta(7)\right]\\
&+g_1{}^3\,g_2{}^3\left[N^2\left(\tfrac{8213}{15552}-\tfrac{35\,\zeta(3)}{108}-\tfrac{2\,\zeta(4)}{3}+\tfrac{14\,\zeta(5)}{9}\right)\right.\\
&\qquad+N\left(\tfrac{496159}{7776}+\tfrac{1309\,\zeta(3)}{6}-\tfrac{452\,\zeta^2(3)}{9}-\tfrac{4076\,\zeta(4)}{27}+478\,\zeta(5)-\tfrac{2450\,\zeta(6)}{9}+196\,\zeta(7)\right)\\
&\qquad\left.+\tfrac{9309907}{1944}+\tfrac{224804\,\zeta(3)}{27}+\tfrac{3032\,\zeta^2(3)}{3}-\tfrac{73018\,\zeta(4)}{27}+\tfrac{155692\,\zeta(5)}{9}-6300\,\zeta(6)+23912\,\zeta(7)\right]\\
&+g_1{}^4\,g_2{}^2\left[N^3\left(\tfrac{127}{20736}-\tfrac{91\,\zeta(3)}{1296}+\tfrac{\zeta(4)}{18}\right)+N^2\left(-\tfrac{43295}{31104}+\zeta(3)-\tfrac{4\,\zeta^2(3)}{3}-\tfrac{121\,\zeta(4)}{24}+\tfrac{364\,\zeta(5)}{27}-\tfrac{50\,\zeta(6)}{9}\right)\right.\\
&\qquad+N\left(\tfrac{11495}{54}+\tfrac{31598\,\zeta(3)}{81}-\tfrac{1045\,\zeta^2(3)}{27}-\tfrac{10729\,\zeta(4)}{54}+\tfrac{20917\,\zeta(5)}{27}-\tfrac{21425\,\zeta(6)}{54}+\tfrac{1960\,\zeta(7)}{3}\right)\\
&\qquad\left.+\tfrac{1279979}{486}+\tfrac{784621\,\zeta(3)}{162}+\tfrac{18154\,\zeta^2(3)}{27}-\tfrac{83837\,\zeta(4)}{54}+\tfrac{275510\,\zeta(5)}{27}-\tfrac{98975\,\zeta(6)}{27}+\tfrac{43120\,\zeta(7)}{3}\right]\\
&+g_1{}^5\,g_2\left[N^4\left(-\tfrac{61}{15552}-\tfrac{5\,\zeta(3)}{972}+\tfrac{\zeta(4)}{108}\right)+N^3\left(-\tfrac{3557}{46656}-\tfrac{151\,\zeta(3)}{972}-\tfrac{4\,\zeta(4)}{27}+\tfrac{8\,\zeta(5)}{81}\right)\right.\\
&\qquad+N^2\left(\tfrac{111217}{23328}+\tfrac{2785\,\zeta(3)}{243}-\tfrac{92\,\zeta^2(3)}{81}-\tfrac{1055\,\zeta(4)}{162}+\tfrac{530\,\zeta(5)}{27}-\tfrac{950\,\zeta(6)}{81}+\tfrac{98\,\zeta(7)}{9}\right)\\
&\qquad+N\left(\tfrac{95588}{729}+\tfrac{15742\,\zeta(3)}{81}-\tfrac{92\,\zeta^2(3)}{81}-\tfrac{6592\,\zeta(4)}{81}+\tfrac{34460\,\zeta(5)}{81}-\tfrac{14750\,\zeta(6)}{81}+490\,\zeta(7)\right)\\
&\qquad\left.+\tfrac{389095}{729}+\tfrac{259358\,\zeta(3)}{243}+\tfrac{13288\,\zeta^2(3)}{81}-\tfrac{27166\,\zeta(4)}{81}+\tfrac{179696\,\zeta(5)}{81}-\tfrac{63500\,\zeta(6)}{81}+\tfrac{28420\,\zeta(7)}{9}\right]\,,
\end{aligned}
\tag{18.13}
$$

$$
\begin{aligned}
\gamma_2(g_1,g_2) = \;& g_1{}^2\,\tfrac{N+2}{36}+g_1\,g_2\,\tfrac{1}{6}+g_2{}^2\,\tfrac{1}{12}-g_1{}^3\left[\tfrac{N^2}{432}+\tfrac{5\,N}{216}+\tfrac{1}{27}\right]-g_1{}^2\,g_2\left[\tfrac{N}{48}+\tfrac{1}{6}\right]-g_1\,g_2{}^2\,\tfrac{3}{16}-g_2{}^3\,\tfrac{1}{16}\\
&+g_1{}^4\left[-\tfrac{5\,N^3}{5184}+\tfrac{5\,N^2}{324}+\tfrac{85\,N}{648}+\tfrac{125}{648}\right]+g_1{}^3\,g_2\left[-\tfrac{5\,N^2}{432}+\tfrac{5\,N}{24}+\tfrac{125}{108}\right]+g_1{}^2\,g_2{}^2\left[\tfrac{5\,N}{288}+\tfrac{145}{72}\right]+g_1\,g_2{}^3\,\tfrac{65}{48}+g_2{}^4\,\tfrac{65}{192}\\
&+g_1{}^5\left[N^4\left(-\tfrac{13}{62208}+\tfrac{\zeta(3)}{3888}\right)+N^3\left(-\tfrac{187}{93312}-\tfrac{\zeta(3)}{972}\right)+N^2\left(-\tfrac{1459}{11664}+\tfrac{13\,\zeta(3)}{972}-\tfrac{5\,\zeta(4)}{162}\right)\right.\\
&\qquad\left.+N\left(-\tfrac{1915}{2916}+\tfrac{13\,\zeta(3)}{162}-\tfrac{16\,\zeta(4)}{81}\right)-\tfrac{602}{729}+\tfrac{23\,\zeta(3)}{243}-\tfrac{22\,\zeta(4)}{81}\right]\\
&+g_1{}^4\,g_2\left[N^3\left(-\tfrac{65}{20736}+\tfrac{5\,\zeta(3)}{1296}\right)+N^2\left(-\tfrac{185}{7776}-\tfrac{5\,\zeta(3)}{216}\right)\right.\\
&\qquad\left.+N\left(-\tfrac{395}{216}+\tfrac{20\,\zeta(3)}{81}-\tfrac{25\,\zeta(4)}{54}\right)-\tfrac{1505}{243}+\tfrac{115\,\zeta(3)}{162}-\tfrac{55\,\zeta(4)}{27}\right]\\
&+g_1{}^3\,g_2{}^2\left[\tfrac{325\,N^2}{31104}+N\left(-\tfrac{4453}{3888}+\tfrac{23\,\zeta(3)}{216}-\tfrac{5\,\zeta(4)}{27}\right)-\tfrac{58177}{3888}+\tfrac{191\,\zeta(3)}{108}-\tfrac{130\,\zeta(4)}{27}\right]\\
&+g_1{}^2\,g_2{}^3\left[N\left(-\tfrac{671}{3456}+\tfrac{\zeta(3)}{72}\right)-\tfrac{13741}{864}+\tfrac{67\,\zeta(3)}{36}-5\,\zeta(4)\right]+g_1\,g_2{}^4\left[-\tfrac{18545}{2304}+\tfrac{15\,\zeta(3)}{16}-\tfrac{5\,\zeta(4)}{2}\right]\\
&+g_2{}^5\left[-\tfrac{3709}{2304}+\tfrac{3\,\zeta(3)}{16}-\tfrac{\zeta(4)}{2}\right]\,,
\end{aligned}
\tag{18.14}
$$

$$
\begin{aligned}
\gamma_m(g_1,g_2) = \;& g_1\,\tfrac{2+N}{6}+g_2\,\tfrac{1}{2}-g_1{}^2\,\tfrac{5\,N+10}{36}-g_1\,g_2\,\tfrac{5}{6}-g_2{}^2\,\tfrac{5}{12}\\
&+g_1{}^3\left[\tfrac{5\,N^2}{72}+\tfrac{47\,N}{72}+\tfrac{37}{36}\right]+g_1{}^2\,g_2\,\tfrac{37+5\,N}{8}+g_1\,g_2{}^2\,\tfrac{251+N}{48}+g_2{}^3\,\tfrac{7}{4}\\
&+g_1{}^4\left[N^3\left(\tfrac{1}{15552}-\tfrac{\zeta(3)}{108}\right)+N^2\left(-\tfrac{947}{1944}-\tfrac{4\,\zeta(3)}{81}-\tfrac{5\,\zeta(4)}{54}\right)\right.\\
&\qquad\left.+N\left(-\tfrac{5777}{1944}-\tfrac{22\,\zeta(3)}{81}-\tfrac{16\,\zeta(4)}{27}\right)-\tfrac{7765}{1944}-\tfrac{34\,\zeta(3)}{81}-\tfrac{22\,\zeta(4)}{27}\right]\\
&+g_1{}^3\,g_2\left[N^2\left(\tfrac{1}{1296}-\tfrac{\zeta(3)}{9}\right)+N\left(-\tfrac{421}{72}-\tfrac{10\,\zeta(3)}{27}-\tfrac{10\,\zeta(4)}{9}\right)-\tfrac{7765}{324}-\tfrac{68\,\zeta(3)}{27}-\tfrac{44\,\zeta(4)}{9}\right]\\
&+g_1{}^2\,g_2{}^2\left[N\left(-\tfrac{1841}{864}-\tfrac{\zeta(3)}{6}-\tfrac{\zeta(4)}{3}\right)-\tfrac{9199}{216}-\tfrac{13\,\zeta(3)}{3}-\tfrac{26\,\zeta(4)}{3}\right]\\
&+g_1\,g_2{}^3\left[N\left(-\tfrac{25}{72}+\tfrac{\zeta(3)}{6}\right)-\tfrac{4243}{144}-\tfrac{19\,\zeta(3)}{6}-6\,\zeta(4)\right]+g_2{}^4\left[-\tfrac{477}{64}-\tfrac{3\,\zeta(3)}{4}-\tfrac{3\,\zeta(4)}{2}\right]\\
&+g_1{}^5\left[N^4\left(\tfrac{7}{124416}+\tfrac{17\,\zeta(3)}{7776}-\tfrac{\zeta(4)}{432}\right)+N^3\left(\tfrac{2831}{23328}+\tfrac{487\,\zeta(3)}{3888}-\tfrac{\zeta^2(3)}{243}+\tfrac{23\,\zeta(4)}{1296}-\tfrac{5\,\zeta(5)}{486}+\tfrac{25\,\zeta(6)}{486}\right)\right.\\
&\qquad+N^2\left(\tfrac{291907}{93312}+\tfrac{1261\,\zeta(3)}{972}-\tfrac{149\,\zeta^2(3)}{486}+\tfrac{437\,\zeta(4)}{648}+\tfrac{2\,\zeta(5)}{243}+\tfrac{1475\,\zeta(6)}{972}\right)\\
&\qquad+N\left(\tfrac{83137}{5832}+\tfrac{2011\,\zeta(3)}{324}-\tfrac{436\,\zeta^2(3)}{243}+\tfrac{1075\,\zeta(4)}{324}+\tfrac{50\,\zeta(5)}{243}+\tfrac{1850\,\zeta(6)}{243}\right)\\
&\qquad\left.+\tfrac{49477}{2916}+\tfrac{1327\,\zeta(3)}{162}-\tfrac{194\,\zeta^2(3)}{81}+\tfrac{667\,\zeta(4)}{162}+\tfrac{8\,\zeta(5)}{27}+\tfrac{775\,\zeta(6)}{81}\right]\\
&+g_1{}^4\,g_2\left[N^3\left(\tfrac{35}{41472}+\tfrac{85\,\zeta(3)}{2592}-\tfrac{5\,\zeta(4)}{144}\right)+N^2\left(\tfrac{113135}{62208}+\tfrac{1175\,\zeta(3)}{648}-\tfrac{5\,\zeta^2(3)}{81}+\tfrac{145\,\zeta(4)}{432}-\tfrac{25\,\zeta(5)}{162}+\tfrac{125\,\zeta(6)}{162}\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&\quad +N\left(\tfrac{4675}{108}+\tfrac{95\,\zeta(3)}{6}-\tfrac{725\,\zeta^2(3)}{162}+\tfrac{85\,\zeta(4)}{9}+\tfrac{35\,\zeta(5)}{81}+\tfrac{6875\,\zeta(6)}{324}\right)\\
&\quad +\tfrac{247385}{1944}+\tfrac{6635\,\zeta(3)}{108}-\tfrac{485\,\zeta^2(3)}{27}+\tfrac{3335\,\zeta(4)}{108}+\tfrac{20\,\zeta(5)}{9}+\tfrac{3875\,\zeta(6)}{54}\Big]\\
&+{g_1}^3\,{g_2}^2\Big[N^2\left(-\tfrac{2045}{20736}+\tfrac{785\,\zeta(3)}{1296}-\tfrac{7\,\zeta(4)}{72}-\tfrac{7\,\zeta(5)}{54}\right)\\
&\quad +N\left(\tfrac{362281}{10368}+\tfrac{3743\,\zeta(3)}{324}-\tfrac{61\,\zeta^2(3)}{27}+\tfrac{1493\,\zeta(4)}{216}-\tfrac{11\,\zeta(5)}{18}+\tfrac{775\,\zeta(6)}{54}\right)\\
&\quad +\tfrac{267737}{864}+\tfrac{47327\,\zeta(3)}{324}-\tfrac{1154\,\zeta^2(3)}{27}+\tfrac{8039\,\zeta(4)}{108}+\tfrac{155\,\zeta(5)}{27}+\tfrac{4675\,\zeta(6)}{27}\Big]\\
&+{g_1}^2\,{g_2}^3\Big[N\left(\tfrac{26173}{2304}+\tfrac{71\,\zeta(3)}{48}-\tfrac{2\,\zeta^2(3)}{9}+\tfrac{25\,\zeta(4)}{12}-\tfrac{14\,\zeta(5)}{9}+\tfrac{25\,\zeta(6)}{9}\right)\\
&\quad +\tfrac{32003}{96}+\tfrac{627\,\zeta(3)}{4}-\tfrac{403\,\zeta^2(3)}{9}+\tfrac{475\,\zeta(4)}{6}+\tfrac{59\,\zeta(5)}{9}+\tfrac{3325\,\zeta(6)}{18}\Big]\\
&+g_1\,{g_2}^4\Big[N\left(\tfrac{26171}{13824}-\tfrac{77\,\zeta(3)}{96}+\tfrac{7\,\zeta(4)}{16}-\tfrac{2\,\zeta(5)}{3}\right)+\tfrac{589141}{3456}+\tfrac{959\,\zeta(3)}{12}-\tfrac{45\,\zeta^2(3)}{2}+\tfrac{643\,\zeta(4)}{16}+\tfrac{19\,\zeta(5)}{6}+\tfrac{375\,\zeta(6)}{4}\Big]\\
&+{g_2}^5\Big[\tfrac{158849}{4608}+\tfrac{1519\,\zeta(3)}{96}-\tfrac{9\,\zeta^2(3)}{2}+\tfrac{65\,\zeta(4)}{8}+\tfrac{\zeta(5)}{2}+\tfrac{75\,\zeta(6)}{4}\Big]\,. \qquad (18.15)
\end{aligned}
$$

18.3 Fixed Points and Critical Exponents

The only new fixed point is the cubic one (g_1^C,g_2^C), which represents a system with mixed O(N) and cubic symmetry. We have calculated the ε-expansion of the fixed-point couplings and critical exponents up to the order ε^5. For $N=2$, we verify the relation (18.8) between the cubic and the Ising fixed points:

$$
g_1^C = g_1^I + 3\,g_2^I/2, \qquad g_2^C = -g_2^I,
$$

as a consequence of the symmetry in Eq. (18.7). As N increases, the cubic fixed point approaches the Heisenberg fixed point from below, crossing it at $N=N_c$, where g_2^C changes its sign. For $N\to\infty$, the cubic fixed point moves towards the Ising fixed point.

The ε-expansions for the cubic fixed point are [8]

$$
\begin{aligned}
g_1^C(\varepsilon) \;=\;& \varepsilon\,\tfrac{1}{N}+\varepsilon^2\,\tfrac{(-106+125\,N-19\,N^2)}{27\,N^3}\\
&+\varepsilon^3\Big[\tfrac{22472}{729\,N^5}-\tfrac{45080}{729\,N^4}+\tfrac{38329}{972\,N^3}-\tfrac{41971}{5832\,N^2}-\tfrac{1955}{5832\,N}+\left(\tfrac{56}{9\,N^4}-\tfrac{28}{9\,N^3}-\tfrac{8}{3\,N^2}+\tfrac{8}{9\,N}\right)\zeta(3)\Big]\\
&+\varepsilon^4\Big[-\tfrac{5955080}{19683\,N^7}+\tfrac{5623300}{6561\,N^6}-\tfrac{5934115}{6561\,N^5}+\tfrac{8315992}{19683\,N^4}-\tfrac{3955061}{52488\,N^3}+\tfrac{113779}{104976\,N^2}-\tfrac{2987}{314928\,N}\\
&\quad+\left(-\tfrac{29680}{243\,N^6}+\tfrac{46696}{243\,N^5}-\tfrac{12764}{243\,N^4}-\tfrac{7934}{243\,N^3}+\tfrac{2744}{243\,N^2}+\tfrac{110}{243\,N}\right)\zeta(3)\\
&\quad+\left(\tfrac{32}{9\,N^4}-\tfrac{16}{9\,N^3}-\tfrac{16}{9\,N^2}+\tfrac{2}{3\,N}\right)\zeta(4)+\left(-\tfrac{80}{3\,N^5}+\tfrac{200}{27\,N^4}+\tfrac{80}{27\,N^3}+\tfrac{80}{9\,N^2}-\tfrac{80}{27\,N}\right)\zeta(5)\Big]\\
&+\varepsilon^5\Big[\tfrac{1767467744}{531441\,N^9}-\tfrac{6468340480}{531441\,N^8}+\tfrac{9496212881}{531441\,N^7}-\tfrac{14088835643}{1062882\,N^6}+\tfrac{43137004355}{8503056\,N^5}\\
&\quad-\tfrac{7400332843}{8503056\,N^4}+\tfrac{2080479877}{68024448\,N^3}+\tfrac{337198481}{136048896\,N^2}-\tfrac{5795035}{136048896\,N}\\
&\quad+\left(\tfrac{4404512}{2187\,N^8}-\tfrac{3678896}{729\,N^7}+\tfrac{9044242}{2187\,N^6}-\tfrac{1239931}{1458\,N^5}-\tfrac{1694161}{4374\,N^4}+\tfrac{653341}{4374\,N^3}-\tfrac{182483}{34992\,N^2}+\tfrac{13883}{34992\,N}\right)\zeta(3)\\
&\quad+\left(-\tfrac{18020}{243\,N^6}+\tfrac{56137}{486\,N^5}-\tfrac{2213}{81\,N^4}-\tfrac{47369}{1944\,N^3}+\tfrac{3929}{486\,N^2}+\tfrac{217}{648\,N}\right)\zeta(4)\\
&\quad+\left(\tfrac{16960}{27\,N^7}-\tfrac{579260}{729\,N^6}+\tfrac{107902}{729\,N^5}-\tfrac{508}{27\,N^4}+\tfrac{84818}{729\,N^3}-\tfrac{26296}{729\,N^2}-\tfrac{340}{243\,N}\right)\zeta(5)\\
&\quad+\left(-\tfrac{2225}{81\,N^5}+\tfrac{1525}{162\,N^4}+\tfrac{125}{81\,N^3}+\tfrac{850}{81\,N^2}-\tfrac{100}{27\,N}\right)\zeta(6)+\left(\tfrac{-1078}{9\,N^6}+\tfrac{1225}{3\,N^5}-\tfrac{2450}{9\,N^4}+\tfrac{539}{9\,N^3}-\tfrac{343}{9\,N^2}+\tfrac{98}{9\,N}\right)\zeta(7)\\
&\quad+\left(\tfrac{3136}{27\,N^7}-\tfrac{112}{N^6}-\tfrac{2834}{81\,N^5}+\tfrac{3089}{81\,N^4}+\tfrac{626}{81\,N^3}-\tfrac{296}{81\,N^2}-\tfrac{16}{27\,N}\right)\zeta^2(3)\Big]+\mathcal{O}(\varepsilon^6)\;, \qquad (18.16)
\end{aligned}
$$

$$
\begin{aligned}
g_2^C(\varepsilon) \;=\;& \varepsilon\,\tfrac{(N-4)}{3\,N}+\varepsilon^2\,\tfrac{(424-534\,N+93\,N^2+17\,N^3)}{81\,N^3}\\
&+\varepsilon^3\Big[-\tfrac{89888}{2187\,N^5}+\tfrac{187528}{2187\,N^4}-\tfrac{123707}{2187\,N^3}+\tfrac{90281}{8748\,N^2}+\tfrac{11713}{17496\,N}+\tfrac{709}{17496}+\left(-\tfrac{224}{27\,N^4}+\tfrac{16}{3\,N^3}+\tfrac{80}{27\,N^2}-\tfrac{32}{27\,N}-\tfrac{4}{27}\right)\zeta(3)\Big]\\
&+\varepsilon^4\Big[\tfrac{23820320}{59049\,N^7}-\tfrac{69389720}{59049\,N^6}+\tfrac{25018256}{19683\,N^5}-\tfrac{35478331}{59049\,N^4}+\tfrac{11944655}{118098\,N^3}+\tfrac{406721}{157464\,N^2}-\tfrac{511435}{944784\,N}+\tfrac{10909}{944784}\\
&\quad+\left(\tfrac{118720}{729\,N^6}-\tfrac{200768}{729\,N^5}+\tfrac{23752}{243\,N^4}+\tfrac{2704}{81\,N^3}-\tfrac{10450}{729\,N^2}-\tfrac{56}{81\,N}-\tfrac{106}{729}\right)\zeta(3)
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(-\tfrac{136}{27\,N^4}+\tfrac{28}{9\,N^3}+\tfrac{64}{27\,N^2}-\tfrac{28}{27\,N}-\tfrac{2}{27}\Big)\,\zeta(4)+\Big(\tfrac{320}{9\,N^5}-\tfrac{400}{27\,N^4}-\tfrac{440}{81\,N^3}-\tfrac{80}{9\,N^2}+\tfrac{280}{81\,N}+\tfrac{40}{81}\Big)\,\zeta(5)\Big]\\
+\varepsilon^5\Big[&-\tfrac{7069870976}{1594323\,N^9}+\tfrac{2937809504}{177147\,N^8}-\tfrac{4398801284}{177147\,N^7}+\tfrac{9923276525}{531441\,N^6}-\tfrac{5033294725}{708588\,N^5}+\tfrac{3132906331}{2834352\,N^4}+\tfrac{256333871}{17006112\,N^3}\\
&-\tfrac{264392957}{22674816\,N^2}+\tfrac{4069429}{45349632\,N}-\tfrac{321451}{408146688}\\
&+\Big(-\tfrac{17618048}{6561\,N^8}+\tfrac{46032704}{6561\,N^7}-\tfrac{40407016}{6561\,N^6}+\tfrac{10668718}{6561\,N^5}+\tfrac{4840987}{13122\,N^4}-\tfrac{1176529}{6561\,N^3}+\tfrac{64261}{26244\,N^2}+\tfrac{10361}{104976\,N}-\tfrac{11221}{104976}\Big)\zeta(3)\\
&+\Big(\tfrac{75472}{729\,N^6}-\tfrac{125462}{729\,N^5}+\tfrac{8347}{162\,N^4}+\tfrac{7747}{243\,N^3}-\tfrac{72941}{5832\,N^2}-\tfrac{323}{972\,N}-\tfrac{443}{5832}\Big)\,\zeta(4)\\
&+\Big(-\tfrac{67840}{81\,N^7}+\tfrac{845200}{729\,N^6}-\tfrac{202864}{729\,N^5}-\tfrac{10202}{729\,N^4}-\tfrac{83693}{729\,N^3}+\tfrac{29770}{729\,N^2}+\tfrac{1628}{729\,N}+\tfrac{373}{729}\Big)\,\zeta(5)\\
&+\Big(\tfrac{3100}{81\,N^5}-\tfrac{1525}{81\,N^4}-\tfrac{425}{162\,N^3}-\tfrac{1025}{81\,N^2}+\tfrac{275}{54\,N}+\tfrac{25}{54}\Big)\zeta(6)+\Big(\tfrac{4312}{27\,N^6}-\tfrac{14896}{27\,N^5}+\tfrac{3626}{9\,N^4}-\tfrac{2254}{27\,N^3}+\tfrac{980}{27\,N^2}-\tfrac{98}{9\,N}-\tfrac{49}{27}\Big)\zeta(7)\\
&+\Big(-\tfrac{12544}{81\,N^7}+\tfrac{13888}{81\,N^6}+\tfrac{712}{27\,N^5}-\tfrac{4354}{81\,N^4}-\tfrac{365}{81\,N^3}+\tfrac{10}{3\,N^2}+\tfrac{85}{81\,N}+\tfrac{11}{81}\Big)\,\zeta^2(3)\Big]+\mathcal{O}(\varepsilon^6)\ . \qquad (18.17)
\end{aligned}
$$

18.4 Stability

The stability of the fixed points is determined from the eigenvalues ω_1 and ω_2 of the matrix

$$
M_{ij}=\frac{\partial\beta_i(g_1,g_2)}{\partial g_j}\Big|_{g_1^*,g_2^*}\ . \qquad (18.18)
$$

For positive real parts of both eigenvalues, the corresponding fixed point is infrared stable. The Gaussian fixed point is doubly unstable. At the Ising fixed point, one eigenvalue ω_1 is negative. The Heisenberg and the cubic fixed points interchange stability for $N = N_c$, the former being stable for $N < N_c$ where $g_2^C < 0$. The stability wedges of the critical theory are visible in Fig. 18.1. They differ from the bare stability wedge. Outside these wedges, the transition is of first order [9].

To find the crucial number N_c determining which fixed point governs the critical behavior in $D = 3$ dimensions, we study the eigenvalue ω_2^C of the stability matrix as a function of N (the other eigenvalue ω_1^C remains positive and can be ignored). Its ε-expansion reads

$$
\begin{aligned}
\omega_2^C=\varepsilon\,\tfrac{N-4}{3\,N}+(N-1)\,\Big[&\varepsilon^2\,\frac{\left(-848+660\,N+72\,N^2-19\,N^3\right)}{81\,N^3\,(2+N)}\\
+\,\varepsilon^3\frac{\Sigma_{i=0}^{7}\,C_i^3\,N^i}{8748\,N^5\,(2+N)^3}&+\varepsilon^4\frac{\Sigma_{i=0}^{11}\,C_i^4\,N^i}{944784\,N^7\,(2+N)^5}+\varepsilon^5\frac{\Sigma_{i=0}^{15}\,C_i^5\,N^i}{102036672\,N^9\,(2+N)^7}\Big]+\mathcal{O}(\varepsilon^6)\ . \qquad (18.19)
\end{aligned}
$$

The coefficients C_i^j are listed in Table 18.1 . The condition of vanishing ω_2^C gives the ε-expansion of N_c:

$$
\begin{aligned}
N_c = 4-2\,\varepsilon+\varepsilon^2\left[-\tfrac{5}{12}+\tfrac{5\,\zeta(3)}{2}\right]&+\varepsilon^3\left[-\tfrac{1}{72}+\tfrac{5\,\zeta(3)}{8}+\tfrac{15\,\zeta(4)}{8}-\tfrac{25\,\zeta(5)}{3}\right]\\
+\ \varepsilon^4\left[-\frac{1}{384}+\frac{93\,\zeta(3)}{128}-\frac{229\,\zeta^2(3)}{144}+\frac{15\,\zeta(4)}{32}-\frac{3155\,\zeta(5)}{1728}-\frac{125\,\zeta(6)}{12}+\frac{11515\,\zeta(7)}{384}\right]&+\mathcal{O}(\varepsilon^5)\,.\,(18.20)
\end{aligned}
$$

The same expansion is found from the condition $g_2^C = 0$. This expansion is badly divergent, making it difficult to calculate the value of N_c at $\varepsilon = 1$. With the help of Padé approximants we obtain the following values:

Padé [1/1] :	N_c	=	3.128	Padé [2/2] :	N_c	=	2.958
Padé [2/1] :	N_c	=	2.792	Padé [1/2] :	N_c	=	2.893
Padé [3/1] :	N_c	=	3.068	Padé [1/3] :	N_c	=	2.972

The approximant Padé [1/3] is unreliable since the Padé denominator has a pole at positive ε,

C_0^3	2876416	C_2^4	$22268920832-5633556480\zeta(3)-1074954240\zeta(5)$
C_1^3	$-1740544+580608\zeta(3)$	C_3^4	$-5244000000+5418233856\zeta(3)+188116992\zeta(4)-2120048640\zeta(5)$
C_2^3	$-3188544+829440\zeta(3)$	C_4^4	$-21313343616+8161855488\zeta(3)+456855552\zeta(4)-2508226560\zeta(5)$
C_3^3	$2340592-62208\zeta(3)$	C_5^4	$11104506624-370593792\zeta(3)+295612416\zeta(4)-1425807360\zeta(5)$
C_4^3	$54656-404352\zeta(3)$	C_6^4	$1087015200-2657484288\zeta(3)-83980800\zeta(4)+432967680\zeta(5)$
C_5^3	$-162696-114048\zeta(3)$	C_7^4	$-1171729344-359023104\zeta(3)-173000448\zeta(4)+901393920\zeta(5)$
C_6^3	$15700+7776\zeta(3)$	C_8^4	$54071304+211455360\zeta(3)-67184640\zeta(4)+343388160\zeta(5)$
C_7^3	$-937+2592\zeta(3)$	C_9^4	$26339632+27133056\zeta(3)-5878656\zeta(4)+20062080\zeta(5)$
C_0^4	-12196003840	C_{10}^4	$-3320774-2164320\zeta(3)+1469664\zeta(4)-9797760\zeta(5)$
C_1^4	$5267640320-4923555840\zeta(3)$	C_{11}^4	$-24857+154224\zeta(3)+209952\zeta(4)-933120\zeta(5)$

C_0^5	57916383035392
C_1^5	$-14984240431104+35071472959488\,\zeta(3)$
C_2^5	$-143277741441024+39325680009216\,\zeta(3)+2022633897984\,\zeta^2(3)+10938734346240\,\zeta(5)$
C_3^5	$1211449212928-58923130945536\,\zeta(3)+5995664769024\,\zeta^2(3)$
	$-1595232092160\,\zeta(4)+22263974461440\,\zeta(5)-2085841207296\,\zeta(7)$
C_4^5	$162125071257600-92450858139648\,\zeta(3)+5046265184256\,\zeta^2(3)-3420504391680\,\zeta(4)$
	$+9499251179520\,\zeta(5)-580475289600\,\zeta(6)+3887249522688\,\zeta(7)$
C_5^5	$38426107425792+21006186430464\,\zeta(3)-2702434959360\,\zeta^2(3)-468572553216\,\zeta(4)$
	$-11373660831744\,\zeta(5)-1725301555200\,\zeta(6)+14600888451072\,\zeta(7)$
C_6^5	$-162464030196224+69899709652992\,\zeta(3)-6924425232384\,\zeta^2(3)+3943630872576\,\zeta(4)$
	$-19961756909568\,\zeta(5)-2644387430400\,\zeta(6)+13700184293376\,\zeta(7)$
C_7^5	$53925927185664+4311047245824\,\zeta(3)-3110702579712\,\zeta^2(3)+2963245731840\,\zeta(4)$
	$-11389462659072\,\zeta(5)-2410584883200\,\zeta(6)+5736063320064\,\zeta(7)$
C_8^5	$14422298978304-19789519380480\,\zeta(3)+993580204032\,\zeta^2(3)-319987003392\,\zeta(4)$
	$+1995616714752\,\zeta(5)-874744012800\,\zeta(6)+592568524800\,\zeta(7)$
C_9^5	$-7438725755776-2538882164736\,\zeta(3)+1199003959296\,\zeta^2(3)-1007366492160\,\zeta(4)$
	$+4815696457728\,\zeta(5)+528071270400\,\zeta(6)-1724374407168\,\zeta(7)$
C_{10}^5	$-147321611712+2076145468416\,\zeta(3)+251458670592\,\zeta^2(3)-263068176384\,\zeta(4)$
	$+1308067024896\,\zeta(5)+730632960000\,\zeta(6)-2005844456448\,\zeta(7)$
C_{11}^5	$258021138336+211760262144\,\zeta(3)-41520107520\,\zeta^2(3)+48221775360\,\zeta(4)$
	$-269590685184\,\zeta(5)+317951308800\,\zeta(6)-960701720832\,\zeta(7)$
C_{12}^5	$-8428414672-68664506112\,\zeta(3)-17716589568\,\zeta^2(3)+25217754624\,\zeta(4)$
	$-141247120896\,\zeta(5)+51900134400\,\zeta(6)-170733806208\,\zeta(7)$
C_{13}^5	$-2039889852-1386414144\,\zeta(3)-851565312\,\zeta^2(3)+1546506432\,\zeta(4)$
	$-7514135424\,\zeta(5)-3086294400\,\zeta(6)+12221725824\,\zeta(7)$
C_{14}^5	$-138029874+936160416\,\zeta(3)-25194240\,\zeta^2(3)-125341344\,\zeta(4)+1017287424\,\zeta(5)$
	$-1826582400\,\zeta(6)+6666395904\,\zeta(7)$
C_{15}^5	$-64327+12966480\,\zeta(3)-20155392\,\zeta^2(3)+12492144\,\zeta(4)-49268736\,\zeta(5)-125971200\,\zeta(6)+370355328\,\zeta(7)$

TABLE 18.1 Constants appearing in expansion (18.19) for ω_2^C.

where the exact result should be regular. The highest symmetric approximant is usually the most accurate one, from which we deduce the estimate:

$$N_c \approx 2.958. \tag{18.21}$$

Before the work of the present authors [7], the ε-expansion for N_c was known only up to the order ε^2 [10], so that only the Padé [1/1]-approximant was available which yielded $N_c \approx 3.128$. Estimates of N_c directly in $D = 3$ dimensions yielded $N_c = 3.4$ [11] and $N_c = 2.9$ [6]. Thus most of the previous results gave N_c-values larger than 3, implying that the critical behavior of magnetic systems with cubic symmetry is governed by the Heisenberg fixed point. The symmetric Padé approximant to our expansion suggests that N_c lies below three, so that the cubic fixed point is the relevant one.

By inserting the expansions (18.16) and (18.17) into (18.14) and (18.15), we find the critical exponents η_C and ν_C of the cubic fixed point:

$$1/\nu^C = \; 2+(N-1)\left\{-\varepsilon\,\frac{2}{3\,N}+\varepsilon^2\left[\frac{1}{162\,N^3}\left(424-326\,N+19\,N^2\right)\right]\right.$$
$$+\varepsilon^3\left[\frac{1}{17496\,N^5}\left(-359552+573728\,N-264936\,N^2+28358\,N^3+937\,N^4+\frac{4\,(N+2)}{27\,N^4}\left(-14+11\,N-N^2\right)\zeta(3)\right)\right]$$

$$+\varepsilon^4\Big[\tfrac{1}{1889568\,N^7}\big(381125120-923268480\,N+798088608\,N^2-284926360\,N^3 +32693424\,N^4+768780\,N^5+24857\,N^6\big) +\tfrac{1}{1458\,N^6}\big(118720-152032\,N+29816\,N^2+17936\,N^3-4124\,N^4-119\,N^5\big)\,\zeta(3) +\tfrac{(2+N)}{9\,N^4}\big(-14+11\,N-N^2\big)\,\zeta(4)+\tfrac{40}{81\,N^5}\big(36-2\,N-4\,N^2-8\,N^3+N^4\big)\,\zeta(5)\Big]$$
$$+\varepsilon^5\Big[\tfrac{1}{204073344\,N^9}\big(-452471742464+1470211004416\,N-1869697955840\,N^2+1160186503168\,N^3 -350446218272\,N^4+41122747144\,N^5+144762448\,N^6-68383472\,N^7+64327\,N^8\big) +\tfrac{1}{52488\,N^8}\big(-70472192+153589248\,N-106996288\,N^2 +18129888\,N^3+6458072\,N^4-1726592\,N^5-8716\,N^6-3335\,N^7\big)\,\zeta(3) +\tfrac{1}{1944\,N^6}\big(118720-152032\,N+29816\,N^2+17936\,N^3-4124\,N^4-119\,N^5\big)\,\zeta(4) +\tfrac{1}{2187\,N^7}\big(-915840+897560\,N-53320\,N^2-2676\,N^3-86879\,N^4+20128\,N^5+528\,N^6\big)\,\zeta(5) +\tfrac{50}{81\,N^5}\big(36-2\,N-4\,N^2-8\,N^3+N^4\big)\,\zeta(6)+\tfrac{49}{27\,N^6}\big(44-132\,N+65\,N^2-9\,N^3+8\,N^4-N^5\big)\,\zeta(7) +\tfrac{8}{81\,N^7}\big(-784+588\,N+232\,N^2-163\,N^3-23\,N^4+5\,N^5+N^6\big)\,\zeta^2(3)\Big]\Big\}+\mathcal{O}(\varepsilon^6)\;. \tag{18.22}$$

$$\eta^C=(N-1)\Big\{\varepsilon^2\Big[\tfrac{(2+N)}{54\,N^2}\Big]+\varepsilon^3\Big[\tfrac{-1696+1728\,N-222\,N^2+109\,N^3}{5832\,N^4}\Big] +\varepsilon^4\Big[\tfrac{1}{629856\,N^6}\big(1797760-3566912\,N+2292328\,N^2-507952\,N^3+28832\,N^4+7217\,N^5\big) +\tfrac{4}{243\,N^5}\big(28-6\,N-16\,N^2+4\,N^3-N^4\big)\,\zeta(3)\Big] +\varepsilon^5\Big[\tfrac{1}{68024448\,N^8}\big(-2134300672+6125897728\,N-6643967232\,N^2+3326175872\,N^3 -731940728\,N^4+46139232\,N^5+1948700\,N^6+321511\,N^7\big) +\tfrac{1}{17496\,N^7}\big(-189952+266624\,N-31584\,N^2-80376\,N^3+29704\,N^4-2196\,N^5-329\,N^6\big)\,\zeta(3) +\tfrac{1}{81\,N^5}\big(28-6\,N-16\,N^2+4\,N^3-N^4\big)\,\zeta(4)+\tfrac{40}{729\,N^6}\big(-36+N^2+18\,N^3-5\,N^4+N^5\big)\,\zeta(5)\Big]\Big\}+\mathcal{O}(\varepsilon^6). \tag{18.23}$$

As anticipated, for $N=1$ the exponents for the cubic fixed point ($g_1^C=-g_2^C$) are degenerate with those of the Gaussian fixed point, taking free-field values. At $N=2$, the exponents take Ising values by virtue of the symmetry of the interaction under (18.7).

18.5 Resummation

Different resummation techniques are applied to get a better estimate for the cubic fixed point and the critical exponents.

18.5.1 Padé Approximations for Critical Exponents

The results of Padé approximations to the ε-expansions of the critical exponents are shown in Table 18.2 for $N=3$. The exponents for the symmetric and the cubic fixed point lie very close to each other. Most of the approximants contain poles at real positive ε, and are therefore useless. This is also true for the Padé approximants of the Borel-transformed series, where only the [4/1] approximant has no such unphysical property. For the Heisenberg fixed point, the approximants can be compared with results of resummations which include information from the large-order behavior.

The stability of the cubic fixed point for $N=3$ is given by the sign of $\omega^{(2)}$. The Padé approximants suggest that the Heisenberg fixed point is the only stable one, as predicted [5]. In contrast, the Borel-Padé method indicates that the sign might change upon resummation.

$[M/N]$	[2/2]	–	[4/1]	Borel[4/1]	Lit. [12]
η_H	0.0164		0.03706	0.03685	0.040(3)
η_C	0.0157		0.03713	0.03689	–
η_I	0.0127		0.03517	0.03493	–

$[M/N]$	[3/2]	[2/3]	[4/1]	Borel[4/1]	Lit. [12]
$1/\nu_H$	1.4249	1.4273	1.4251	1.4165	1.408(14)
$1/\nu_C$	1.4235	1.42745	1.4241	1.4139	–
$1/\nu_I$	1.5951	1.5994	1.5991	1.5903	–

$[M/N]$	[3/2]	–	[4/1]	Borel[4/1]	Lit. [12]
$\omega_H^{(1)}$	0.8003		0.8698	0.7641	0.79(4)
$\omega_C^{(1)}$	0.8185		0.8869	0.775	–
$\omega_I^{(1)}$	0.8058		0.9339	0.7925	–

$[M/N]$	[3/2]	–	[4/1]	Borel[4/1]	
$\omega_H^{(2)}$	0.0017		0.0111	−0.011	–
$\omega_C^{(2)}$	−0.0012		−0.0048	−0.0034	–
$\omega_I^{(2)}$	−0.1902		−0.1981	−0.18	–

TABLE 18.2 Padé approximations to ε-expansions of critical exponents for cubic symmetry with $N = 3$ and $\varepsilon = 1$. Only the approximants with no poles for $\varepsilon > 0$ are listed. The second-last column contains the result of a Borel-Padé transformation, in which the Borel sums are evaluated with the [4/1]-Padé approximant.

For $N = 4$ the situation is clear. The number $\omega_C^{(2)}$ is definitely positive, whereas $\omega_H^{(2)}$ is negative. Here the cubic fixed point is stable.

Instead of applying the Padé approximant to the series in ε, a two-variable Padé $[M, M]/[N, N]$ can be used to resum the series in g_1 and g_2 [13]. This leads to $N_c = 2.9$ [6], where a $[2, 2]/[2, 2]$-Padé is used to evaluate the Borel sum emerging from a four-loop calculation in fixed dimension. The $[3, 3]/[2, 2]$- and the $[4, 4]/[1, 1]$-Padé approximants of the five-loop expansions (18.12) and (18.13), as well as of the associated Borel sums contain poles at positive coupling constants, which renders them useless.

18.5.2 Resummations for Cubic Fixed Point

The cubic fixed point lies in the upper half of the coupling constant plane ($g_2 > 0$) only if $N > N_C$. Calculating the cubic fixed point for $N = 3$ as a zero of the resummed β-function, verifies whether $g_2^C > 0$ or, equivalently, $N_C < 3$.

Large-Order Behavior

From Padé approximants, we know that the cubic fixed point lies close to the isotropic fixed point with $|g_2| < 0.1$. A large-order calculation [14], designed for the region of weak anisotropy where $g_2 \to 0$, will therefore be used for the resummation. This calculation works with the two coupling constants $g = g_1 + g_2$ and $l = g_2$, $|l| < 0.1$. A continuous transition between $l = 0$ and $l \neq 0$ is assumed. The anisotropy is treated as a perturbation of the isotropic case whose large-order behavior is found by an isotropic solution of the field equation (instanton). The β-functions for g and l are given as expansions in l around the isotropic case. The $l = 0$ -term of β_g consists of the isotropic β-function. The same expansion is performed for the imaginary part of the β-functions for $g \to 0^-$. The general form of the expansion is

$$\beta_{g/l}(g,l) = \beta^p_{g/l} + i \sum_{n=0}^{\infty} l^n a_n^{g/l} \left(\frac{-1}{\sigma g}\right)^{\frac{D+4+N+4n}{2}} \exp\left(\frac{1}{\sigma g}\right) \tag{18.24}$$

$$= \sum_{n=0/1}^{\infty} l^n \left[\sum_{m=0}^{\infty} A^{g/l}_{n,m} g^m + i a_n^{g/l} \left(\frac{-1}{\sigma g}\right)^{\frac{D+4+N+4n}{2}} \exp\left(\frac{1}{\sigma g}\right)\right] \tag{18.25}$$

$$= \sum_{n=0/1}^{\infty} l^n \, B_n^{g/l}(g), \tag{18.26}$$

where $\beta^p_{g/l}$ is the perturbative part and $A^{g/l}_{n,m} g^m$ are the coefficients of l^n in the perturbative part. The parameter σg is again the numerical value of the instanton energy, where $\sigma = 1$ in this parametrization. The coefficients $a_n^{g/l}$ are rather complicated expressions.

There is a cut only along the negative real axis of the complex g-plane. For small l, the imaginary part of $\beta(g)$ for small $-g > 0$ governs the asymptotic series. The parameter l, being very small, is treated as an expansion parameter. The sum over all l^n-terms has a non-zero radius of convergence and the expansion is asymptotic in l. But the error of an asymptotic series decreases up to the order $N_{\min} \approx 1/\sigma l$. With $\sigma = 1$ and $l = 0.1$, we find $N_{\min} \approx 10$ and the error $\Delta \approx 4.54 \cdot 10^{-5}$, so that partial sums in l can yield acceptable approximations. Even for $N = 2$ and $l = 0.1$, the error is still small: $\Delta \approx 0.019$. For $N = 1$ and $l = 0.01$, the error is $\Delta \approx 0.009$.

A resummation is needed only for the expansions B_n in g. Application of the dispersion integral to the imaginary part of each $B_n(g)$ in (18.25) reveals [5] the asymptotic large-order behavior of the expansions $B_n(g) = \sum_k B_{n,k} g^k$:

$$B^g_{n,k} \stackrel{k\to\infty}{\to} \gamma_n^g (-\sigma)^k k! k^{\beta_n} [1 + \mathcal{O}(1/k)] , \quad \text{with } \beta_n = \frac{D+5+4n}{2}, \tag{18.27}$$

$$B^l_{n,k} \stackrel{k\to\infty}{\to} \gamma_n^l (-\sigma)^k k! k^{\beta_n} [1 + \mathcal{O}(1/k)] , \quad \text{with } \beta_n = \frac{D+5+4n}{2}, \tag{18.28}$$

and allows a resummation by the methods described in Section 16.5.

The structure of Eq. (18.25) does not permit us to find the imaginary part and therefore the large-order behavior of the ε-series for both coupling constants, $g^*(\varepsilon)$ and $l^*(\varepsilon)$. The different treatment of the coupling constants is incompatible with the idea of the ε-expansion, which deals with the two coupling constants at the same level by using one expansion parameter for both of them.

Cubic Fixed Point from Resummed β-Function

For $\varepsilon = 1$ and $N = 3$, each series $B_n(g)$ is resummed separately. The resummation of $B_n(g)$ is getting less accurate for increasing n since the corresponding series in g have fewer and fewer terms. But taking into account the smallness of the parameter, the expansion in l may be truncated. This is seen when using Padé approximants to resum $B_n(g)$ in Eqs. (18.29) and (18.30) and calculating the simultaneous zeros l^*, g^* of the β-functions as shown in Table 18.3 . The β-functions at the cubic fixed point will be approximated by

$$0 = \beta_g(g^*, l^*) \approx \sum_{n=0}^{5} l^{*n} \, \mathrm{res}[B_n^g(g^*)], \tag{18.29}$$

$n=0:$	$[3/2]B_0^g(g^*) = 0$	$\Rightarrow$	$g^* = 0.38081$
	$[3/2]B_0^l(g^*) = 0$		$l^* = 0.0$
$n=1:$	$[3/2]B_0^g(g^*) + l^*\,[3/2]B_1^g(g^*) = 0$	$\Rightarrow$	$g^* = 0.373716$
	$[3/2]B_0^l(g^*) + l^*\,[2/2]B_1^l(g^*) = 0$		$l^* = -0.01845$
$n=2:$	$[3/2]B_0^g(g^*) + \ldots + l^{*2}\,[2/2]B_2^g(g^*) = 0$	$\Rightarrow$	$g^* = 0.373835$
	$[3/2]B_0^l(g^*) + \ldots + l^{*2}\,[2/1]B_2^l(g^*) = 0$		$l^* = -0.0177435$
$n=3:$	$[3/2]B_0^g(g^*) + \ldots + l^{*3}\,[2/1]B_3^g(g^*) = 0$	$\Rightarrow$	$g^* = 0.37384$
	$[3/2]B_0^l(g^*) + \ldots + l^{*3}\,[1/1]B_3^l(g^*) = 0$		$l^* = -0.0177294$
$n=4:$	$[3/2]B_0^g(g^*) + \ldots + l^{*4}\,[1/1]B_4^g(g^*) = 0$	$\Rightarrow$	$g^* = 0.37377$
	$[3/2]B_0^l(g^*) + \ldots + l^{*4}B_4^l\,(g^*) = 0$		$l^* = -0.017903$

TABLE 18.3 Padé approximation in presence of anisotropic contributions to β-function. From zeroth to first order in l, the value for g^* decreases by about 1.9%. Including the second order in l changes g^* further by about 0.032%, whereas l^* changes by about 3.98%. The third order in l, changes this value for l^* only by about 0.079%. The main contribution to the difference between the Heisenberg and the cubic fixed points is contained in the first-order contribution. This gives an error of approximately 0.03% in g^*, and 4% in l^*.

$$0 \;=\; \beta_l(g^*, l^*) \;\;\approx\;\; l^* \sum_{n=0}^{5} l^{*n}\,\mathrm{res}[B_n^l(g^*)], \tag{18.30}$$

where res[...] indicates the resummed series. Neglecting all but the first two terms of each β-function leads to the following simple equation:

$$l_g^* \equiv \frac{\mathrm{res}[B_0^g(g^*)]}{\mathrm{res}[B_1^g(g^*)]} = \frac{\mathrm{res}[B_0^l(g^*)]}{\mathrm{res}[B_1^l(g^*)]} \equiv l_l^*\,. \tag{18.31}$$

The sign of l^* depends trivially on the sign of res$[B_n]$. Since the strong-coupling behavior of the β-function is unknown for the ϕ^4-theory in 4 dimensions, we have to use some trial value of α. This is chosen to acquire an optimal convergence of the series, as mentioned in Section 16.5. With $\varepsilon = 1$, the following four expansions have to be resummed:

$$\begin{aligned}
B_0^g &= -g+11/3g^2-23/3g^3+47.6514g^4-437.6456g^5+4998.6184g^6, && \beta=\tfrac{9}{2},\\
B_1^g &= -4/3g+44/9g^2-45.8922g^3+564.7871g^4-8113.7392g^5, && \beta=\tfrac{13}{2},\\
B_1^l &= -1+4g-97/9g^2+75.8751g^3-776.2604g^4+9707.3624g^5, && \beta=\tfrac{13}{2},\\
B_2^l &= -1+56/9g-67.3187g^2+944.0496g^3-15030.8879g^4, && \beta=\tfrac{17}{2}.
\end{aligned}$$

Here, β is again the growth parameter of the large-order behavior. The parameter α turns out to be different for each term. It is chosen as follows:

$$B_0^g : \alpha = 1.45, \quad B_1^g : \alpha = 0.35, \quad B_1^l : \alpha = 0.325, \quad B_2^l : \alpha = -1.4. \tag{18.32}$$

The result is displayed in Fig. 18.3 where l_g and l_l, defined in Eq. (18.31), are plotted against g. The resulting cubic fixed point is

$$g_C^* \approx 0.399 \qquad l_C^* \approx 0.01\,. \tag{18.33}$$

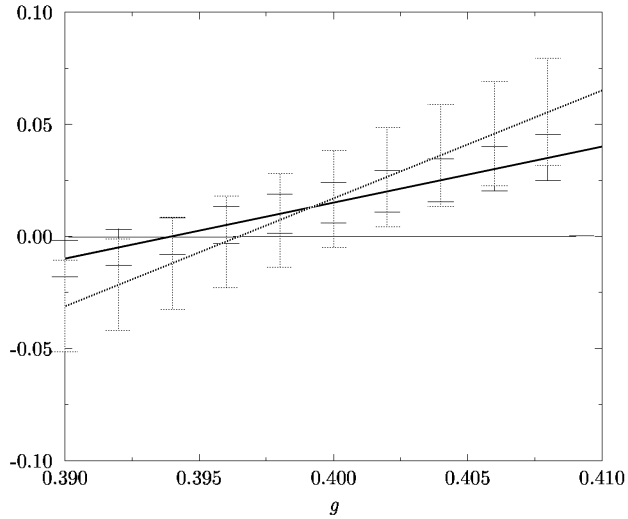

FIGURE 18.3 Determination of $l^* = g_2^*$ by resummation in $g = g_1 + g_2$. The quantities l_g and l_l defined in Eq. (18.31) are plotted against g. Their crossing point determines the values for l^* and g^*.

The errors resulting from variations in α are indicated in Fig. 18.3. They are too large to make a conclusive statement. A further improvement of the resummation of β-functions in two coupling constants is needed to find the correct answer. For more details see the original discussion in Ref. [14].

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