17

Critical Exponents of O(N)-Symmetric Theory

We shall now use the results of the previous chapters to derive explicitly the critical exponents of the O(N)-symmetric ϕ^4-theory. The unique tensor multiplying the product of fields $\phi_\alpha\phi_\beta\phi_\gamma\phi_\delta$ in the interaction is [recall (6.25)]

$$\lambda^{O(N)}_{\alpha\beta\gamma\delta} \equiv \frac{\lambda}{3}\left[\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}\right]. \tag{17.1}$$

As discussed in Chapter 1, different choices of N allow us to calculate the critical behavior of various physical systems. For $N = 1$, the field theory lies in the same universality class as a magnetic system with two preferred orientations, which can also be described by the Ising-model. For $N = 2$, the field theory matches the critical behavior of a magnet with preferred orientations in a plane, which can also be described by an XY-model. The same critical behavior is found in the superfluid transition of liquid helium near the λ-point. For $N = 3$, we obtain the critical behavior of a rotationally invariant ferromagnet, also described by the Heisenberg-model. Finally, the case $N = 0$ explains the critical behavior of polymers [1].

This tensor (17.1) appears in all vertex functions obtained from Feynman diagrams. This property is essential for the renormalizability of the theory (recall the discussion in Sections 6.4.4. and 10.13).

17.1 Series Expansions for Renormalization Group Functions

The renormalization constants are calculated via Eqs. (15.1)–(15.3). All required terms, the pole terms of the integrals, the weight factors, and the symmetry factors for O(N) symmetry are displayed in the tables of Appendix B. Only the simple poles of the renormalization constants contribute to the β-function and the anomalous dimensions γ_m and γ as we saw in Eqs. (10.46) of Chapter 10:

$$\gamma(g) = -\frac{g}{2}\frac{\partial Z_{\phi,1}}{\partial g}, \tag{17.2}$$

$$\gamma_m(g) = \frac{g}{2}\frac{\partial Z_{m^2,1}}{\partial g} + \gamma, \tag{17.3}$$

$$\beta(g) = -\varepsilon\, g + g^2\frac{\partial Z_{g,1}}{\partial g} + 4\,g\,\gamma \equiv -\varepsilon\, g - 2\,g^2\gamma_g + 4\,g\,\gamma\,. \tag{17.4}$$

The expansions in g up to 5th order are [2]

$$\beta(g) = -\varepsilon\, g + \tfrac{g^2}{3}(N+8) - \tfrac{g^3}{3}(3N+14)$$
$$+ \tfrac{g^4}{216}\left[33N^2 + 922N + 2960 + \zeta(3)\cdot 96(5N+22)\right]$$

$$
\begin{aligned}
&- \tfrac{g^5}{3888}\,[-5N^3 + 6320N^2 + 80456N + 196648\\
&\qquad + \zeta(3)\cdot 96(63N^2 + 764N + 2332)\\
&\qquad - \zeta(4)\cdot 288(5N + 22)(N + 8)\\
&\qquad + \zeta(5)\cdot 1920(2N^2 + 55N + 186)]\\
&+ \tfrac{g^6}{62208}\,[13N^4 + 12578N^3 + 808496N^2 + 6646336N + 13177344\\
&\qquad + \zeta(3)\cdot 16(-9N^4 + 1248N^3 + 67640N^2 + 552280N + 1314336)\\
&\qquad + \zeta^2(3)\cdot 768(-6N^3 - 59N^2 + 446N + 3264)\\
&\qquad - \zeta(4)\cdot 288(63N^3 + 1388N^2 + 9532N + 21120)\\
&\qquad + \zeta(5)\cdot 256(305N^3 + 7466N^2 + 66986N + 165084)\\
&\qquad - \zeta(6)(N + 8)\cdot 9600(2N^2 + 55N + 186)\\
&\qquad + \zeta(7)\cdot 112896(14N^2 + 189N + 526)]\ ,
\end{aligned}
\tag{17.5}
$$

$$
\begin{aligned}
\gamma(g) \;=\;& \tfrac{g^2}{36}(N+2) \;-\; \tfrac{g^3}{432}(N+2)[N+8]\\
&+ \tfrac{g^4}{5184}(N+2)\quad [5(-N^2 + 18N + 100)]\\
&- \tfrac{g^5}{186624}(N+2)\,[39N^3 + 296N^2 + 22752N + 77056\\
&\qquad - \zeta(3)\cdot 48(N^3 - 6N^2 + 64N + 184)\\
&\qquad + \zeta(4)\cdot 1152(5N + 22)]\ ,
\end{aligned}
\tag{17.6}
$$

$$
\begin{aligned}
\gamma_m(g) \;=\;& \tfrac{g}{6}(N+2) \;-\; \tfrac{g^2}{36}(N+2)[5] \;+\; \tfrac{g^3}{72}(N+2)[5N+37]\\
&- \tfrac{g^4}{15552}(N+2)\,[-N^2 + 7578N + 31060\\
&\qquad + \zeta(3)\cdot 48(3N^2 + 10N + 68)\\
&\qquad + \zeta(4)\cdot 288(5N + 22)]\\
&+ \tfrac{g^5}{373248}(N+2)\,[21N^3 + 45254N^2 + 1077120N + 3166528\\
&\qquad + \zeta(3)\cdot 48(17N^3 + 940N^2 + 8208N + 31848)\\
&\qquad - \zeta^2(3)\cdot 768(2N^2 + 145N + 582)\\
&\qquad + \zeta(4)\cdot 288(-3N^3 + 29N^2 + 816N + 2668)\\
&\qquad + \zeta(5)\cdot 768(-5N^2 + 14N + 72)\\
&\qquad + \zeta(6)\cdot 9600(2N^2 + 55N + 186)]\ .
\end{aligned}
\tag{17.7}
$$

To have an idea of the growth behavior of the expansion coefficients, we write down the series numerically for $N = 1$:

$$\beta(g) = -\varepsilon\, g + 3.0\, g^2 - 5.67\, g^3 + 32.55\, g^4 - 271.6\, g^5 + 2848.57\, g^6\ , \tag{17.8}$$

$$\gamma(g) = 0.0833\, g^2 - 0.0625\, g^3 + 0.3385\, g^4 - 1.9256\, g^5\ , \tag{17.9}$$

$$\gamma_m(g) = 0.5\, g - 0.42\, g^2 + 1.75\, g^3 - 9.98\, g^4 + 75.38\, g^5\ . \tag{17.10}$$

17.2 Fixed Point and Critical Exponents

Beside the Gaussian fixed point at $g^* = 0$ (recall Section 10.5), there is a nontrivial fixed point called Heisenberg or isotropic fixed point. Its ε-expansion reads

$$
\begin{aligned}
g^*(\varepsilon) = \; & \tfrac{3\,\varepsilon}{8+N} + \tfrac{9\,\varepsilon^2}{(8+N)^3}\,[14 + 3\,N] \\
& + \tfrac{\varepsilon^3}{(8+N)^5}\big[\tfrac{3}{8}\,\left(4544 + 1760\,N + 110\,N^2 - 33\,N^3\right) - \zeta(3)\cdot 36\,\left(8+N\right)\left(22+5\,N\right)\big] \\
& + \tfrac{\varepsilon^4}{(8+N)^7}\big[\tfrac{1}{16}(529792 + 309312\,N + 52784\,N^2 - 5584\,N^3 - 2670\,N^4 - 5\,N^5) \\
& \qquad + \zeta(3)(8+N)\cdot 6\,(-9064 - 3796\,N - 82\,N^2 + 63\,N^3) \\
& \qquad - \zeta(4)(8+N)^3\cdot 18\,(22+5\,N) \\
& \qquad + \zeta(5)(8+N)^2\cdot 120\,(186 + 55\,N + 2\,N^2)\big] \\
& + \tfrac{\varepsilon^5}{(8+N)^9}\big[\tfrac{3}{256}(-21159936 - 8425472\,N + 3595520\,N^2 + 758144\,N^3 \\
& \qquad -625104\,N^4 - 179408\,N^5 - 1262\,N^6 - 13\,N^7) \\
& \qquad + \zeta(3)(8+N)\cdot\tfrac{3}{16}(-15131136 - 8873728\,N - 890208\,N^2 \\
& \qquad\qquad +310248\,N^3 + 45592\,N^4 - 1104\,N^5 + 9\,N^6) \\
& \qquad + \zeta(3)^2(8+N)^2\cdot 9\,(43584 + 24848\,N + 3626\,N^2 + 107\,N^3 + 6\,N^4) \\
& \qquad + \zeta(4)(8+N)^3\cdot\tfrac{27}{8}\,(-8448 - 3524\,N - 52\,N^2 + 63\,N^3) \\
& \qquad + \zeta(5)(8+N)^2\cdot 3\,(554208 + 255188\,N + 12246\,N^2 - 5586\,N^3 - 305\,N^4) \\
& \qquad + \zeta(6)(8+N)^4\cdot\tfrac{225}{2}\,(186 + 55\,N + 2\,N^2) \\
& \qquad - \zeta(7)(8+N)^3\cdot 1323\,(526 + 189\,N + 14\,N^2)\big].
\end{aligned} \tag{17.11}
$$

The critical exponents η, ν, and ω are obtained by evaluating the renormalization group functions at the fixed point:

$$
\eta = 2\gamma^*\ , \quad \nu^{-1} = 2(1-\gamma_m^*)\ , \quad \omega = \beta'(g^*)\ . \tag{17.12}
$$

Inserting the ε-expansion of the fixed point (17.11) into (17.6), (17.7), and the derivative $d\beta/dg(g)$ of Eq. (17.5), we obtain the ε-expansions for the critical exponents up to ε^5 [2]

$$
\begin{aligned}
\eta(\varepsilon) \;=\; & \tfrac{(N+2)\varepsilon^2}{2(N+8)^2}\ \Big\{1 + \tfrac{\varepsilon}{4(N+8)^2}[-N^2 + 56N + 272] \\
& - \tfrac{\varepsilon^2}{16(N+8)^4}[5N^4 + 230N^3 - 1124N^2 - 17920N - 46144 - \zeta(3)(N+8)384(5N+22)] \\
& - \tfrac{\varepsilon^3}{64(N+8)^6}[13N^6 + 946N^5 + 27620N^4 + 121472N^3 - 262528N^2 - 2912768N - 5655552 \\
& \qquad - \zeta(3)(N+8)16\,(N^5 + 10N^4 + 1220N^3 - 1136N^2 - 68672N - 171264) \\
& \qquad + \zeta(4)(N+8)^3 1152(5N+22)\ -\ \zeta(5)(N+8)^2\cdot 5120(2N^2 + 55N + 186)]\Big\},
\end{aligned} \tag{17.13}
$$

$$
\begin{aligned}
\nu^{-1}(\varepsilon) \;=\; & 2 + \tfrac{(N+2)\varepsilon}{N+8}\Big\{-1 - \tfrac{\varepsilon}{2(N+8)^2}[13N + 44] \\
& + \tfrac{\varepsilon^2}{8(N+8)^4}\ [3N^3 - 452N^2 - 2672N - 5312 \\
& \qquad + \zeta(3)(N+8)\cdot 96(5N+22)] \\
& + \tfrac{\varepsilon^3}{32(N+8)^6}\ [3N^5 + 398N^4 - 12900N^3 - 81552N^2 - 219968N - 357120 \\
& \qquad + \zeta(3)(N+8)\cdot 16(3N^4 - 194N^3 + 148N^2 + 9472N + 19488) \\
& \qquad + \zeta(4)(N+8)^3\cdot 288(5N+22) - \zeta(5)(N+8)^2\cdot 1280(2N^2 + 55N + 186)] \\
& + \tfrac{\varepsilon^4}{128(N+8)^8}[3N^7 - 1198N^6 - 27484N^5 - 1055344N^4 \\
& \qquad -5242112N^3 - 5256704N^2 + 6999040N - 626688 \\
& \qquad - \zeta(3)(N+8)\cdot 16(13N^6 - 310N^5 + 19004N^4 + 102400N^3 \\
& \qquad\qquad -381536N^2 - 2792576N - 4240640) \\
& \qquad - \zeta^2(3)(N+8)^2\cdot 1024(2N^4 + 18N^3 + 981N^2 + 6994N + 11688)
\end{aligned}
$$

$$
\begin{aligned}
&+\zeta(4)(N+8)^3\cdot 48(3N^4-194N^3+148N^2+9472N+19488)\\
&+\zeta(5)(N+8)^2\cdot 256(155N^4+3026N^3+989N^2-66018N-130608)\\
&-\zeta(6)(N+8)^4\cdot 6400(2N^2+55N+186)\\
&+\zeta(7)(N+8)^3\cdot 56448(14N^2+189N+526)]\Big\}\ , \qquad (17.14)
\end{aligned}
$$

$$
\begin{aligned}
\omega(\varepsilon) \;=\; \varepsilon - \tfrac{3\varepsilon^2}{(N+8)^2}\; & [3N+14]\\
+\tfrac{\varepsilon^3}{4(N+8)^4}\; & [33N^3+538N^2+4288N+9568+\zeta(3)(N+8)\cdot 96(5N+22)]\\
+\tfrac{\varepsilon^4}{16(N+8)^6}\; & [5N^5-1488N^4-46616N^3-419528N^2-1750080N-2599552\\
& -\zeta(3)(N+8)\cdot 96(63N^3+548N^2+1916N+3872)\\
& +\zeta(4)(N+8)^3\cdot 288(5N+22)\\
& -\zeta(5)(N+8)^2\cdot 1920(2N^2+55N+186)]\\
+\tfrac{\varepsilon^5}{64(N+8)^8}\; & [13N^7+7196N^6+240328N^5+3760776N^4\\
& +38877056N^3+223778048N^2+660389888N+752420864\\
& -\zeta(3)(N+8)\cdot 16(9N^6-1104N^5-11648N^4-243864N^3\\
& \qquad -2413248N^2-9603328N-14734080)\\
& -\zeta^2(3)(N+8)^2\cdot 768(6N^4+107N^3+1826N^2+9008N+8736)\\
& -\zeta(4)(N+8)^3\cdot 288(63N^3+548N^2+1916N+3872)\\
& +\zeta(5)(N+8)^2\cdot 256(305N^4+7386N^3+45654N^2+143212N+226992)\\
& -\zeta(6)(N+8)^4\cdot 9600(2N^2+55N+186)\\
& +\zeta(7)(N+8)^3\cdot 112896(14N^2+189N+526)]\ . \qquad (17.15)
\end{aligned}
$$

For $N=1$, these expansions reduce to

$$\eta = 0.0185\,\varepsilon^2+0.0187\,\varepsilon^3-0.0083\,\varepsilon^4+0.0257\,\varepsilon^5\ , \qquad (17.16)$$

$$\nu^{-1} = 2-0.333\,\varepsilon-0.117\,\varepsilon^2+0.124\,\varepsilon^3-0.307\,\varepsilon^4+0.951\,\varepsilon^5\ , \qquad (17.17)$$

$$\omega = \varepsilon-0.63\,\varepsilon^2+1.62\,\varepsilon^3-5.24\,\varepsilon^4+20.75\,\varepsilon^5\ . \qquad (17.18)$$

These ε-expansions are asymptotic series and, as explained in the last chapter, resummation techniques [3] have to be applied to obtain reliable estimates of the critical exponents. For this we need the large-order behavior of the ϵ-expansions, to be summarized in the next section.

17.3 Large-Order Behavior

At the tip of the left-hand cut in the complex coupling constant plane, i.e., for $g\to 0^-$, the O(N)-symmetric β-function has the behavior (16.7):

$$\beta(g)=\beta^p(g)-iC2^7 3\pi^4\left(-\frac{1}{\alpha g}\right)^{\beta+1}\exp\left(\frac{1}{\alpha g}\right)\ , \qquad (17.19)$$

where $\beta^p(g)$ denotes the perturbative part, the growth parameter β [not to be confused with the β-function of the renormalization group defined in Eq. (10.24)] is equal to $(N+2+D)/2$, and $1/\alpha g$ is the energy of the nontrivial solution of the field equation, the instanton [4]. In our normalization, $\alpha=1$. Application of the dispersion integral (16.5) to the imaginary part leads to the large-order behavior of the expansion coefficients β_k defined by $\beta(g)=\sum_{k=1}^{\infty}\beta_k g^k$ [recall (16.12)]:

$$\beta_k = \gamma\,(-\alpha)^k k^\beta k![1 + \mathcal{O}(1/k)]. \tag{17.20}$$

The other renormalization group functions $\gamma(g)$ and $\gamma_m(g)$ have the same general form for $g \to 0^-$, but with different prefactors and exponents β [5]:

$$\beta \;=\; \begin{cases} 3+N/2 \overset{N=3}{=} 9/2 & \text{for} \quad \beta(g), \\ 2+N/2 \overset{N=3}{=} 7/2 & \text{for} \quad \eta(g) = 2\gamma(g), \\ 3+N/2 \overset{N=3}{=} 9/2 & \text{for} \quad \nu^{-1}(g) = 2 - 2\gamma_m(g), \\ 4+N/2 \overset{N=3}{=} 11/2 & \text{for} \quad \omega(g) = \beta'(g). \end{cases} \tag{17.21}$$

From this we can easily derive the large-order behavior of the corresponding ε-series. The ε-expansion of the fixed point $g^*(\varepsilon)$ is defined by $\beta(g^*(\varepsilon)) = 0$. This implies that for negative ε, $g^*(\varepsilon)$ has an imaginary part, so that the cut of the β-function along the negative axis gives rise to a cut of $g^*(\varepsilon)$ along the negative ε-axis. At the tip of the cut, the imaginary part can be found by inserting a complex g into the full expression for β in (17.19) and separating the imaginary and real parts of the equation $\beta(g^*) = 0$. The real part of $g^*(\varepsilon)$ is given by the real part of the equation $\beta(g^*) = 0$, which is the perturbative part. The imaginary part of $[g^*(\varepsilon)]$ comes from the imaginary part of $\beta(g^*) = 0$, and reads

$$\text{Im } g^*(\varepsilon) = -C' 2^7 \pi^4 \left(-\frac{1}{\alpha\varepsilon}\right)^{(N+6+D)/2} \exp\left(\frac{1}{\alpha\varepsilon}\right). \tag{17.22}$$

Inserting this complex $g^*(\varepsilon)$ into $\gamma(g)$ and $\gamma_m(g)$ yields the imaginary parts of $\gamma(\varepsilon)$ and $\gamma_m(\varepsilon)$. Application of the dispersion integral leads again to a large order behavior of the type (17.20)

$$f(\varepsilon) = \sum_{k=0}^{\infty} f_k \varepsilon^k, \qquad \text{with} \qquad f_k = \gamma(-\alpha)^k k^\beta k![1 + O(1/k)], \tag{17.23}$$

where the growth parameter α is now equal to $3/(N+8)$, while the powers β are

$$\beta = \begin{cases} 4+N/2 \overset{N=3}{=} 11/2 & \text{for} \quad g^*(\varepsilon), \\ 3+N/2 \overset{N=3}{=} 9/2 & \text{for} \quad \eta(\varepsilon), \\ 4+N/2 \overset{N=3}{=} 11/2 & \text{for} \quad \nu^{-1}(\varepsilon), \\ 5+N/2 \overset{N=3}{=} 13/2 & \text{for} \quad \omega(\varepsilon). \end{cases} \tag{17.24}$$

The prefactor γ differs for each expansion, but will not be used in the sequel.

17.4 Resummation

The resummation techniques developed in Chapter 16 can be applied either to the expansions in g or to the expansions in ε. The most accurate results are obtained from the latter, which we shall use here, concentrating only upon the Janke-Kleinert algorithm described in Section 16.5. The growth parameters of the basis functions (16.59) are adjusted to those of the perturbation expansions given in the last section. By comparing (17.20) with (16.60), we identify σ of $I_n(g)$ with α, which in the present normalization of the coupling constant is equal to unity. The growth parameter β of $I_n(g)$ in (16.60) is given by Eqs. (17.24) for the different series and universality classes with O(N) symmetry. The resulting critical exponents are listed in Table 17.1 . The error in parentheses comes about as follows: for various values of the strong-coupling growth parameter s we plot the successive resummation results from the first, second, ... , fifth order of the ε expansions. This produces the curves shown in Fig. 17.1–17.4. The value of s where the curves just begin to oscillate makes the difference between the last two approximations as small as possible. The critical exponents associated with this s are listed in Table 17.1 . To see the typical systematic error, we vary s symmetrically around this optimal value over a range indicated in the figure headings. The resulting range of critical exponents is given on the top of each figure. For comparison, we list in Tables 17.3 and 17.2 the most recent results of Guida and Zinn-Justin [6].

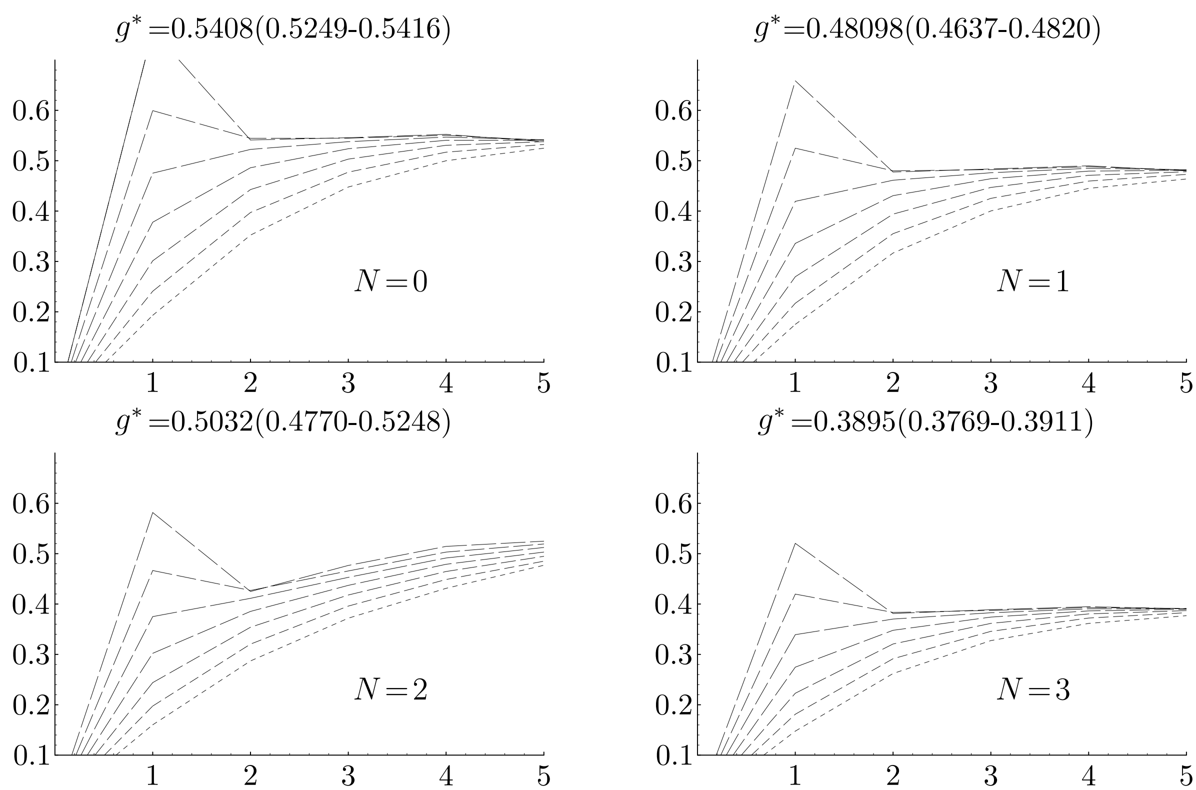

FIGURE 17.1 Resummed ε-expansion of g^* versus the number of loops for strong-coupling parameter s, which increases from 0.1 to 1.9 with increasing dash length. Further explanations are given in the text.

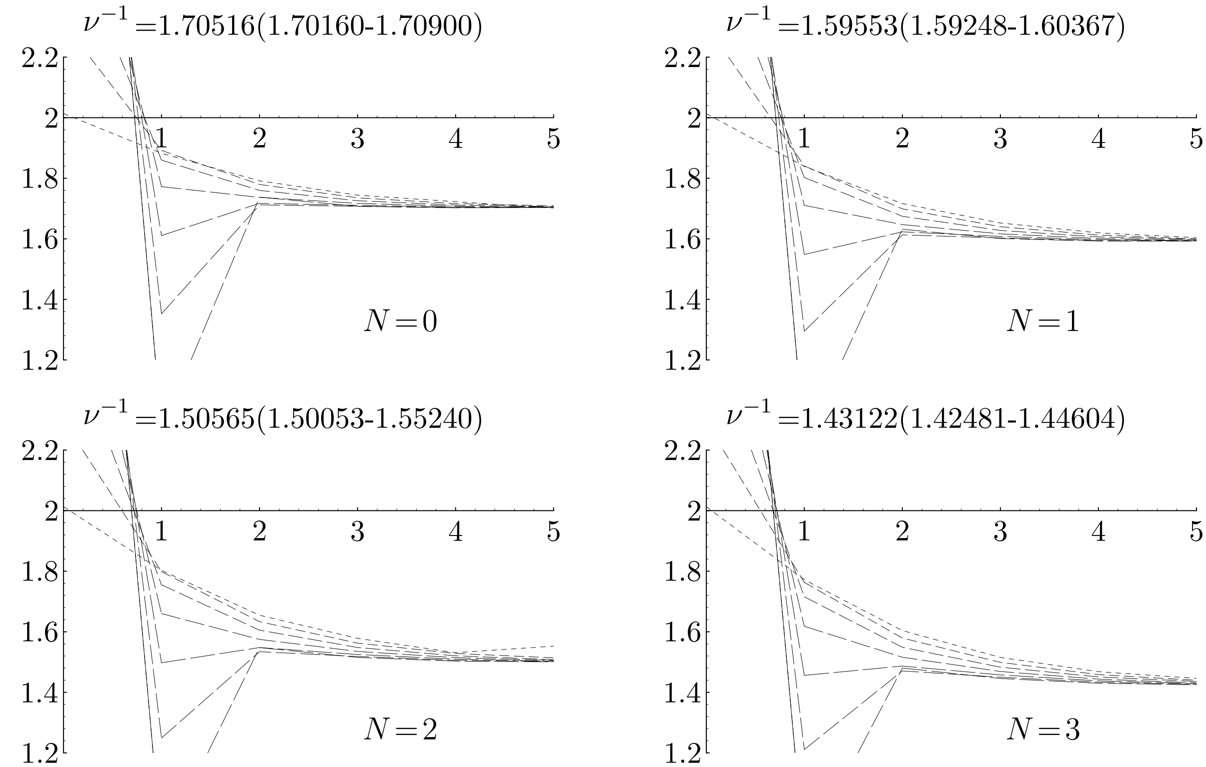

FIGURE 17.2 Resummed ε-expansion of ν^{-1} versus the number of loops. The strong-coupling parameters s changes from 0.0 to 1.2 with increasing dash length. Further explanations are given in the text.

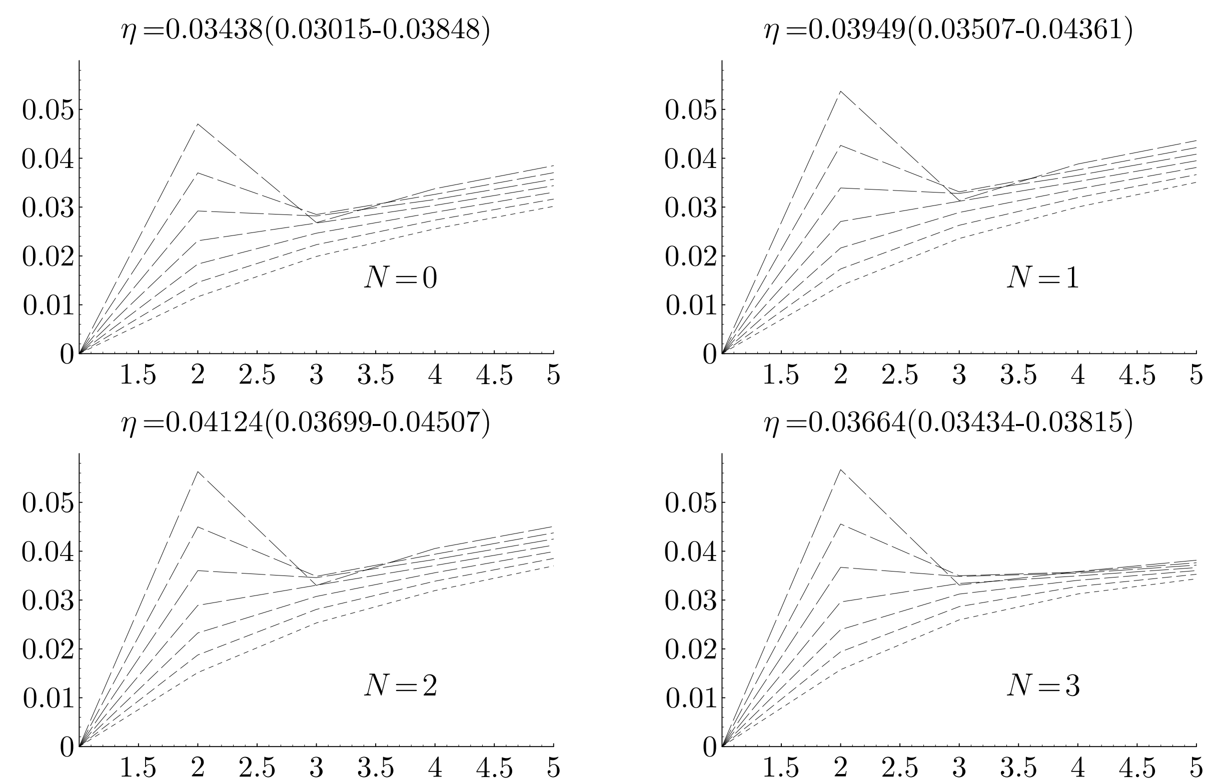

FIGURE 17.3 Resummed ε-expansion of η versus the number of loops. The strong-coupling parameter s changes from 1.6 to 3.4 with increasing dash length. Further explanations are given in the text.

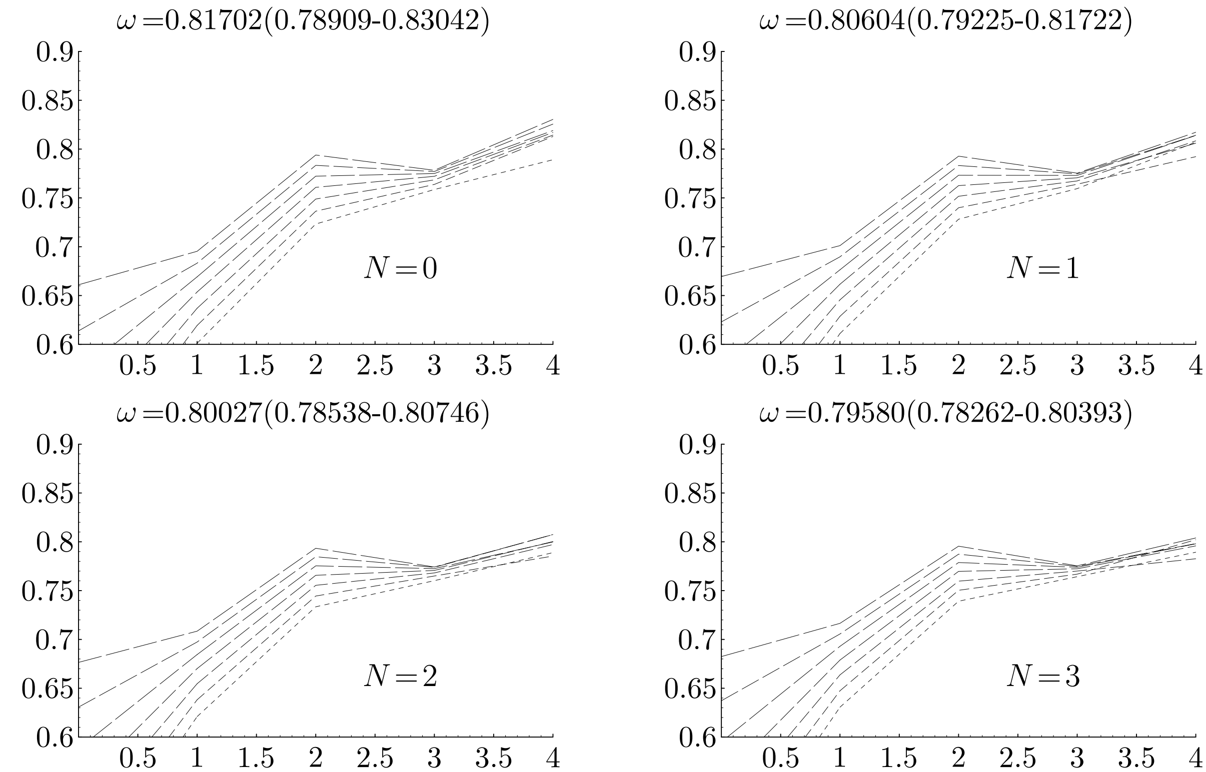

FIGURE 17.4 Resummed ε-expansion of ω versus the number of loops. The strong-coupling parameter s changes from -1.15 to -0.55 with increasing dash length. Further explanations are given in the text.

N	0	1	2	3
$g^*(\varepsilon)$	0.5408(83)	0.4810(91)	0.5032(239)	0.3895(71)
$\nu(\varepsilon)$	0.5865(13)	0.6268(22)	0.6642(111)	0.6987(51)
$\eta(\varepsilon)$	0.0344(42)	0.0395(43)	0.0412(41)	0.0366(20)
$\omega(\varepsilon)$	0.817(21)	0.806(13)	0.800(13)	0.796(11)

TABLE 17.1 Estimates of the critical exponents for $N = 0, 1, 2, 3$ and $\varepsilon = 1$. Resummation with the algorithm of Section 16.5 for the ε-expansions as shown in Figs. 17.1-17.4. The errors are estimated from the variation in dependence on s, as explained in the text.

TABLE 17.2 Critical exponents of the O(N)-models from ε-expansion [6].

N	0	1	2	3
γ (free)	1.1575 ± 0.0050	1.2360 ± 0.0040	1.3120 ± 0.0085	1.3830 ± 0.0135
γ (bc)	1.1571 ± 0.0030	1.2380 ± 0.0045	1.317	1.392
ν (free)	0.5875 ± 0.0018	0.6293 ± 0.0026	0.6685 ± 0.0040	0.7050 ± 0.0055
ν (bc)	0.5878 ± 0.0011	0.6304 ± 0.0021	0.671	0.708
η (free)	0.030 ± 0.006	0.0360 ± 0.006	0.0385 ± 0.0065	0.0380 ± 0.0060
η (bc)	0.0315 ± 0.0025	0.0365 ± 0.0045	0.0370	0.0355
β (free)	0.3024 ± 0.0024	0.3260 ± 0.0020	0.3472 ± 0.0015	0.3660 ± 0.0015
β (bc)	0.3032 ± 0.0011	0.3265 ± 0.0012		
ω	0.828 ± 0.023	0.814 ± 0.018	0.802 ± 0.018	0.794 ± 0.018
θ	0.486 ± 0.015	0.512 ± 0.013	0.536 ± 0.015	0.560 ± 0.017

TABLE 17.3 Critical exponents of the O(N)-models from $d = 3$ expansion [6].

N	0	1	2	3
g^*_{Ni}	1.413 ± 0.006	1.411 ± 0.004	1.403 ± 0.003	1.391 ± 0.004
g^*	26.63 ± 0.11	23.64 ± 0.07	21.16 ± 0.05	19.07 ± 0.05
γ	1.1598 ± 0.0020	1.2397 ± 0.0013	1.3169 ± 0.0020	1.3895 ± 0.0050
ν	0.5882 ± 0.0011	0.6304 ± 0.0013	0.6703 ± 0.0013	0.7073 ± 0.0030
η	0.0284 ± 0.0025	0.0335 ± 0.0025	0.0354 ± 0.0025	0.0355 ± 0.0025
β	0.3025 ± 0.0008	0.3258 ± 0.0014	0.3470 ± 0.0014	0.3662 ± 0.0025
α	0.235 ± 0.003	0.109 ± 0.004	-0.011 ± 0.004	-0.122 ± 0.009
ω	0.812 ± 0.016	0.799 ± 0.011	0.789 ± 0.011	0.782 ± 0.0013
$\theta = \omega\nu$	0.478 ± 0.010	0.504 ± 0.008	0.529 ± 0.008	0.553 ± 0.012

Notes and References

The ε-expansions for the critical exponents used here are from
H. Kleinert, J. Neu, V. Schulte-Frohlinde, K.G. Chetyrkin, S.A. Larin, Phys. Lett. B **272**, 39-44 (1991); ibid. **319**, 545 (1993) (Erratum).},
V. Schulte-Frohlinde, FU-Berlin Ph.D. Thesis, 1969.

The resummation described here is from
H. Kleinert, V. Schulte-Frohlinde, J. Phys. A **34** 1037 (2001) (cond-mat/9907214).
Among the most accurate resummation data in 4-ε dimensions available in the literature are those in Ref. [6].

The individual citations in the text refer to:

[1] P.G. de Gennes, Phys. Lett. A **38**, 339 (1972).

[2] The expansions can be downloaded from the internet address www.physik.fu-berlin.de/ ~kleinert/b8/programs/index.html#1 .

[3] D.I. Kazakov and D.V. Shirkov, Fortschr. Phys. **28**, 465 (1980); J. Zinn-Justin, Phys. Rep. **70**, 3 (1981).

[4] See for instance Sections 17.10 and 17.12 in
H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics and Polymer Physics,* World Scientific, Singapore 1995, Second extended edition (www.physik.fu-berlin.de/~kleinert/b5).

[5] D.I. Kazakov, O.V. Tarasov, D.V. Shirkov, Teo. Mat. Fiz. **38**, 15 (1979);
A.J. McKane and D.J. Wallace, J. Phys. A **11**, 2285 (1978);
I.T. Drummond and G.M. Shore, Ann. Phys. **121**, 204 (1979);
E. Brézin, J.C. Le Guillou, J. Zinn-Justin, Phys. Rev. D **15**, 1544, 1558 (1977).

[6] R. Guida, J. Zinn-Justin, *Critical Exponents of the N-Vector Model*, cond-mat 9803240, (1998).
They are improvements of earlier results in
J.C. Le Guillou, J. Zinn-Justin, Phys. Rev. B **21**, 3976 (1980); J. Physique Lett. **46**, 137 (1985).