14

Generation of Diagrams

In Section 5.4, we have developed a functional formalism to find all connected vacuum diagrams and their multiplicities. These serve as a basis for deriving all diagrams contributing to two- and four-point functions. This is done by removing from the connected vacuum diagrams one line and two lines, respectively. Instead of the two lines we may also remove a vertex, as pointed out in Section 5.5, where we also indicated a way to implement the recursive formalism for the generation of all graphs on a computer [1], and the reader is referred to the detailed original paper for a detailed explanation of the techniques.

In this chapter, we shall describe a somewhat shortened way of generating all diagrams which was used in the initial work on the five-loop determinations of the critical exponents [2]. Because of the special features of the regularization by minimal subtraction discussed in Section 11.6, we can restrict the generation to

- all 1PI four-point diagrams without tadpole parts and their weight factors for Z_g,
- all 1PI two-point diagrams without tadpole parts and their weight factors for Z_ϕ,
- all 1PI four-point diagrams without tadpole parts which result from differentiation with respect to m^2, and the weight factors of the corresponding two-point diagrams for Z_{m^2}.

The generation starts with all connected vacuum diagrams. For the generation of tadpole-free two- and four-point diagrams, only vacuum diagrams without cutvertices are needed. From these, the connected two- and four-point diagrams are constructed by cutting lines and vertices, respectively. These diagrams may now contain cutvertices. In the last step, we select the 1PI diagrams.

Since we shall eventually be interested only in the renormalization group functions of the theory, we can reduce the number of diagrams further by selecting only those two- and four-point 1PI diagrams without cutvertices. Recall that the pole part $\mathcal{K}\bar{R}G$ of a diagram G with a cutvertex contains only ε^{-n}-poles with $n \geq 2$, as explained on page 191. It contributes to the renormalization constants, but not to the renormalization group functions due to the cancellation of all higher ε-poles (see Section 10.3). Nevertheless, we find it useful to include diagrams with cutvertices at each stage of the calculations, since they permit us to check the correctness of the ε-poles via the necessary cancellations. They appear among the other diagrams in Appendix A, for example, the first two-, three-, and four-loop diagrams on page 416, or those with numbers 2 to 7 on page 417.

14.1 Algebraic Representation of Diagrams

When generating Feynman diagrams by the method described in Sections 5.4 and 5.5, an important problem is the identification of topologically inequivalent diagrams. For this purpose we set up an algebraic specification of diagrams. With each diagram G, we associate a *diagram matrix* $\mathbf{G}$. The row and column indices i and j of its matrix elements G^{ij} run from zero to the number of vertices p. Each element G^{ij} gives the number of lines joining the vertices i and j. In the ϕ^4-theory, the matrix elements all lie in the interval $0 \leq G^{ij} \leq 4$. The diagonal elements G^{ii} count the number of self connections of the i th vertex. External lines of a diagram are labeled as if they were connected to an additional dummy vertex with the number 0. The irrelevant matrix element G^{00} is set equal to zero. The matrix is symmetric and contains $(p+1)(p+2)/2-1$ independent elements. Unfortunately, the matrix is not unique. There are obviously several matrices characterizing each diagram, depending on the numbering of the vertices. All of them contain the same information on the numbers S, D, T or F of double, triple, or fourfold connection in a diagram, respectively.

The number S is given by the sum of the numbers on the diagonal, D by the number of times the value 2 appears above the diagonal, and T and F by the number of times the values 3 and 4 appear above the diagonal. For an amputated diagram in which the external lines are not connected to labeled vertices, these numbers include multiple connections of the external lines to an imagined additional dummy vertex (labeled 0 in the matrices). Note that this definition of S, D, T or F differs from the one on page 46 for diagrams whose external lines are connected to labeled external positions.

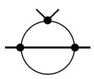
What does the matrix tell us about the number of identical vertex permutations $N_{\rm IVP}$ which appears in the multiplicity and weight formulas (3.14)–(3.17)? If the diagram is amputated, as it is the case for the diagrams under consideration, $N_{\rm IVP}$ includes all vertex permutations, also those that involve vertices with external lines. The number $N_{\rm IVP}$ is the number of all permutations of lines or rows which leave the matrix unchanged. The dummy vertex 0 is omitted from the permutations. This implies that the number of identical vertex permutations is not equal to that of vacuum diagrams obtained by connecting the external lines to the dummy vertex 0. Given the numbers $N_{\rm IVP}$, S, D, T, the total weight of an amputated Feynman diagram is given by

$$W_G = \frac{n!}{2^{S+D} 3!^T 4!^F \, N_{\rm IVP}} \,. \tag{14.1}$$

This formula yields the same number as formula (3.17), where we differentiated distinguished between internal and external lines.

A simple example is shown in Fig. 14.1. Table 14.1 lists explicitly the matrix elements and the numbers $N_{\rm IVP}$ for the vacuum diagrams up to eight loops without cutvertices.

The matrix G^{ij} contains some more information about the properties of a diagram. First, the *connectedness of a diagram* may be checked by inspection. If a diagram is not connected, the matrix G^{ij} is a block matrix for a certain vertex numbering. Second, the matrix $G^{(ij)}$ contains information on the presence of cutvertices. Recall their definition on page 188 and the description of its properties in Section 11.3 that if a connected diagram falls into two pieces after cutting a vertex, this vertex is called a cutvertex. If a diagram contains a tadpole part or a self-connection it necessarily contains a cutvertex. For vacuum diagrams, the opposite is also true. In the matrix G^{ij}, a cutvertex with the number i manifests itself by making the G^{ij} approximately a block matrix. The blocks overlap on the diagonal at (i,i), so that the matrix takes block form if the ith row and column are eliminated. Third, the matrix $G^{(ij)}$ helps us to recognize a cutline, a line whose cutting makes a connected diagram fall into two pieces. Recall the definition on page 55. Such a line is identified by cutting it and testing the matrix G^{ij} associated with the resulting modified diagram for connectedness. A connected diagram without cutlines is 1PI. In the normal phase of the ϕ^4-theory, vacuum diagrams do not contain any cutlines.



(a)

	0	1	2	3
0	-	2	1	1
1	2	0	1	1
2	1	1	0	2
3	1	1	2	0

201101120

(b1)

	0	1	2	3
0	-	1	2	1
1	1	0	1	2
2	2	1	0	1
3	1	2	1	0

102101210

(b2)

	0	1	2	3
0	-	1	1	2
1	1	0	2	1
2	1	2	0	1
3	2	1	1	0

101202110

(b3)

FIGURE 14.1 The diagram in (a) can be represented by the three matrices (b1), (b2), and (b3). The labeling of the upper vertex is 1 in (b1), 2 in (b2), and 3 in (b3). The number $\hat{G} = G^{10}G^{11}G^{20}G^{21}G^{22}G^{30}G^{31}G^{32}G^{33}$ underneath the matrices in b1)–b3) is constructed as explained in the text. The smallest number is read off from representation (b3) which is chosen as the unique representation of the diagram. The diagram has four external lines ($n = 4$), and two double connections ($D = 2$) including the one of the external lines to the additional dummy vertex 0. The number of identical vertex permutations $N_{\rm IVP}$ is 2, since two of the vertices are symmetric to each other. Hence $W_G = 4!/(2^2\ N_{\rm IVP}) = 24/(4 \cdot 2) = 3$. The weight is therefore $W_G = 3$, as in the second line of the diagrammatic equation (3.25), and in the third entry in the table in Appendix B.1.

Since the vertex numbering is arbitrary, the matrix representation of a diagram is, so far, not unique. Examples for such equivalent matrices are shown in Fig. 14.1. A unique representation is found by associating with each matrix a number whose digits are composed of the matrix elements G^{ij} $(0 \leq j \leq i \leq p,\ j \neq 0)$. More explicitly, we form the following number with $[(p+1)(p+2)/2-1]$- digits:

$$\hat{G} = G^{10}G^{11}G^{20}G^{21}G^{22}G^{30}G^{31}G^{32}G^{33} \ldots G^{pp}. \tag{14.2}$$

The matrix associated with the smallest number is chosen as the unique representation of the diagrams. In Fig. 14.1 it is the matrix (b3).

It will further be useful to introduce a more condensed notation for diagrams with no self-connections, where all numbers G^{ii} are omitted since they are zero. Then we are left with a reduced number

$$\hat{G}_r = G^{10}G^{20}G^{21}G^{30}G^{31}G^{32} \ldots G^{p+1,p}. \tag{14.3}$$

If we consider vacuum diagrams, there are no external lines such that $j \neq 0$. Therefore, vacuum diagrams without self-connections may be characterized uniquely by the matrix with the lowest number

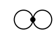
$$\hat{G}_V = G^{21}G^{31}G^{32} \ldots G^{p+1,p}. \tag{14.4}$$

All vacuum diagrams without self-connections are displayed in Fig. 14.3. The numbers (14.4) are listed in Table 14.1 , which shows also the number $N_{\rm IVP}$ of identical vertex permutations and the resulting weight factor. For example, diagram No. 13 with seven vertices has two double connections and two triple connections and there are two identical vertex permutations such that the weight factor is $[(2!)^2 \times (3!)^2 \times 2]^{-1} = 1/288$.

14.2 Generation Procedure

We now describe the shortened recursive graphical scheme starting with the generation of vacuum diagrams. The shortening is possible by observing that in the full graphical recursion relation in Fig. 5.3, the diagrams produced by the last, nonlinear term are no new diagrams, but change only the weights of those generated by the first two linear terms. Since we possess a simple counting formula (3.17) for the weights, we do not have to solve the full recursion relation, but generate the diagrams by a graphical recursion based only on the first two operations in Fig. 5.3, and ignoring the weights. The two- and four-point diagrams are then obtained by cutting one line and a vertex, respectively, as described in Section 5.5.

14.2.1 Vacuum Diagrams

The generation of all diagrams of the theory begins with  , the only vacuum diagram with one vertex. The addition of a further vertex to a diagram with p vertices is performed in two ways, both illustrated in Fig. 14.2. First, we cut a line and insert a new vertex with a

(a) (b)

FIGURE 14.2 Generation of diagrams by cutting one (a) or two (b) lines, and joining the ends in a new vertex.

self-connection. This is done for all lines in each vacuum diagram with p vertices. Cutting topologically equivalent lines generates equivalent diagrams, and only one of such equivalent lines needs to be cut. Next, we cut two lines at a time and join the four ends in a new vertex. The matrices G^{ij} of each newly generated diagram and their associated numbers of Eq. (14.2) have to be identified, and compared to those of previously obtained diagrams to avoid repeated generation of topologically equivalent diagrams. The occurrence of cutvertices during the generation of the vacuum diagrams is registered for later use.

The vacuum diagrams obtained in this way are automatically connected. For the calculation of the renormalization constants, we need only diagrams without tadpole parts which are generated by vacuum diagrams without cutvertices. All vacuum diagrams without cutvertices up to seven vertices are pictured in Fig. 14.3.

14.2.2 Two-Point Diagrams

A connected two-point diagram can be generated from a vacuum diagram with the same number of vertices by cutting a line. All connected two-point diagrams may be found in this way. Repeated generation is again avoided by taking into account symmetries among the lines, i.e. by selecting only one out of a set of topologically equivalent lines as illustrated in Fig. 14.4. All generated diagrams are connected because the ϕ^4-theory does not admit vacuum diagrams with cutlines. They are, however, not necessarily 1PI since they may have cutlines.

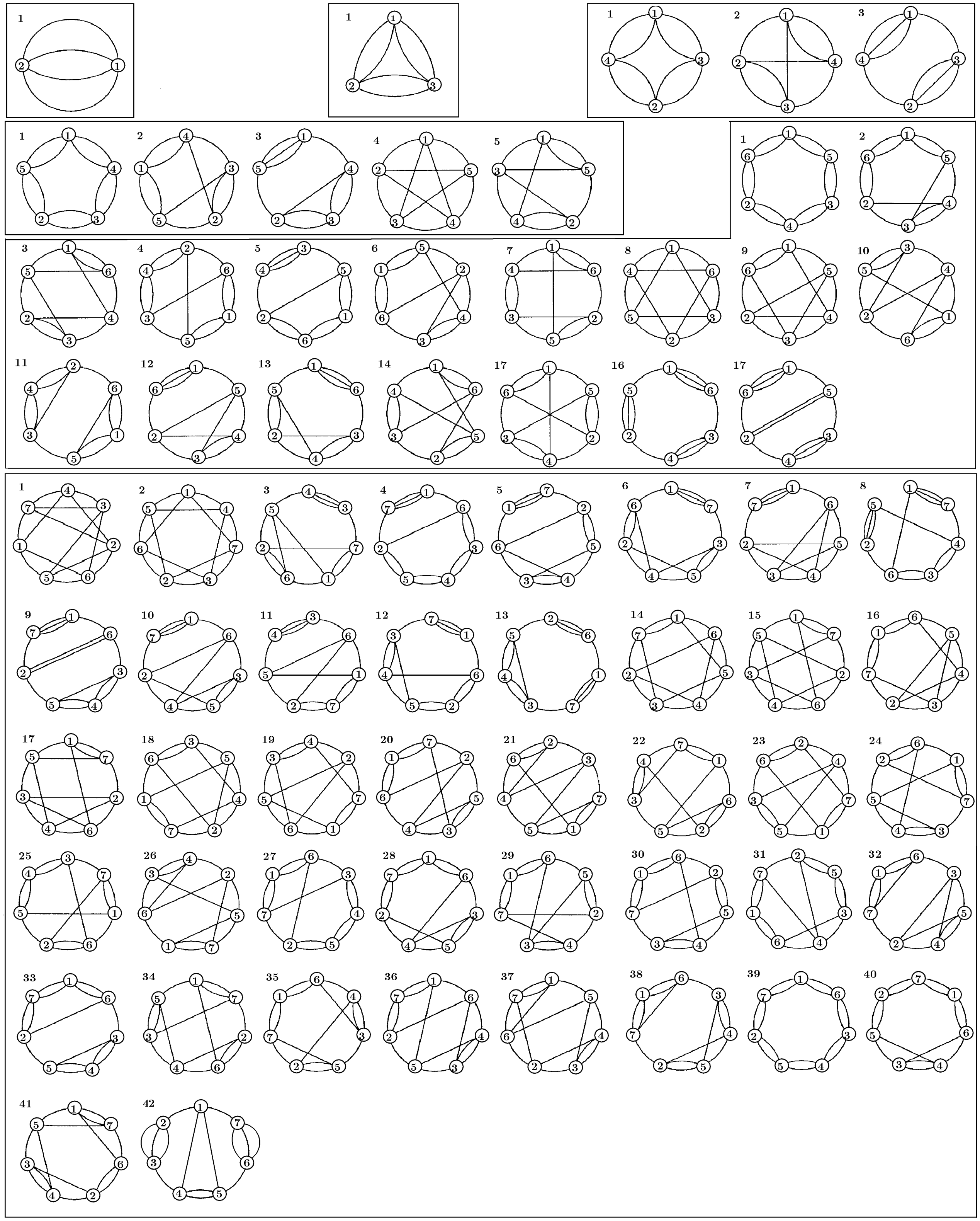


FIGURE 14.3 Vacuum diagrams without cutvertices up to seven vertices. The displayed diagrams allow the generation of all two- and four-point diagrams without tadpole parts (but with cutvertices) up to five loops. The numbers in the upper left corners are running labels, generally denoted by the symbol n_L^{Vac} . Each vacuum diagram possesses a unique classification number $\hat{G}_V = G^{21}G^{31}G^{32}G^{41}G^{42}G^{43}\ldots$ listed in Table 14.1 .

Table 14.1 Matrix representation for all vacuum diagrams of Fig. 14.3 specified by the numbers $\hat{G}_V = G^{21}G^{31}G^{32}G^{41}G^{42}G^{43}\dots$ of Eq. (14.4). To guide the eye, we have inserted bars separating the rows of the triangular part of the matrix below the diagonal. The vertices are labeled as in Fig. 14.3, with the order chosen to achieve the smallest number $\hat{G}$. The first column gives the running number of the diagrams in Fig. 14.3. For each pair $i > j$ in the two top rows, the number G^{ij} underneath gives the number of lines connecting the vertices i and j. The second last column lists the number of single, double, triple, and quadruple connections and the number of identical vertex permutations (S,D,T,Q,$N_{\rm IVP}$). The last column contains the weight factor W_G calculated from Eq. (3.6). For the corresponding list for the two- and four-point diagrams, see Table 14.3 .

2 vertices, 3 loops: 1 diagram				
j	2			
i	1			
No.	G^{ij}	$(S,D,T,Q;N_{\rm IVP})$	M_G	W_G
2-1	4	(0,0,0,1;2)	24	1/48

3 vertices, 4 loops: 1 diagram					
j	2	33			
i	1	12			
No.	G^{ij}		$(S,D,T,Q;N_{\rm IVP})$	M_G	W_G
3-1	2	22	(0,3,0,0;6)	1728	1/48

4 vertices, 5 loops: 3 diagrams						
j	2	33	444			
i	1	12	123			
No.	G^{ij}			$(S,D,T,Q;N_{\rm IVP})$	M_G	W_G
4-1	0	22	220	(0,4,0,0;8)	62208	1/128
4-2	1	12	211	(4,2,0,0;8)	248832	1/32
4-3	0	13	310	(2,0,2,0;4)	55296	1/144

5 vertices, 6 loops: 5 diagrams							
j	2	33	444	5555			
i	1	12	123	1234			
No.	G^{ij}				$(S,D,T,Q;N_{\rm IVP})$	M_G	W_G
5-1	0	02	202	2200	(0,5,0,0;10)	2985984	1/320
5-2	0	02	211	2110	(4,3,0,0;4)	29859840	1/32
5-3	0	02	112	3100	(3,2,1,0;2)	19906560	1/48
5-4	1	11	111	1111	(10,0,0,0,120)	7962624	1/120
5-5	0	11	121	2110	(6,2,0,0;4)	59719680	1/16

6 vertices, 7 loops: 17 diagrams								
j	2	33	444	5555	66666			
i	1	12	123	1234	12345			
No.	G^{ij}					$(S,D,T,Q;N_{\rm IVP})$	M_G	W_G
6-1	0	00	022	2020	22000	(0,6,0,0;12)	$1.7916\,10^{8}$	1/768
6-2	0	01	012	2011	22000	(4,4,0,0;4)	$2.1499\,10^{9}$	1/64
6-3	0	02	111	1110	20011	(8,2,0,0;16)	$2.1499\,10^{9}$	1/64
6-4	0	00	022	2110	21100	(4,4,0,0;8)	$1.0750\,10^{9}$	1/128
6-5	0	00	013	2110	22000	(3,3,1,0;2)	$1.4333\,10^{9}$	1/96
6-6	0	01	012	2101	21100	(6,3,0,0;2)	$8.5996\,10^{9}$	1/16
6-7	0	01	102	1210	21010	(6,3,0,0;4)	$4.2998\,10^{9}$	1/32
6-8	0	11	110	1111	11110	(12,0,0,0;48)	$2.8665\,10^{9}$	1/48
6-9	0	01	111	1111	21100	(10,1,0,0;8)	$8.5996\,10^{9}$	1/16
6-10	0	01	111	1120	21010	(8,2,0,0;2)	$1.7199\,10^{10}$	1/8
6-11	0	01	022	2010	21001	(4,4,0,0;4)	$2.1499\,10^{9}$	1/64
6-12	0	01	012	1111	31000	(7,1,1,0;4)	$2.8665\,10^{9}$	1/48
6-13	0	01	012	1201	30100	(5,2,1,0,;2)	$2.8665\,10^{9}$	1/48
6-14	0	01	102	1201	21100	(6,3,0,0;12)	$1.4333\,10^{9}$	1/96
6-15	0	00	112	1210	21100	(6,3,0,0;12)	$1.4333\,10^{9}$	1/96
6-16	0	00	013	1300	30100	(3,0,3,0;6)	$1.0617\,10^{8}$	1/1296
6-17	0	00	013	1210	31000	(4,1,2,0;4)	$4.7776\,10^{8}$	1/288

7 vertices, 8 loops: 41 diagrams									
j	2	33	444	5555	66666	777777			
i	1	12	123	1234	12345	123456			
No.	G^{ij}						$(S,D,T,Q;N_{\text{IVP}})$	M_G	W_G
7-1	0	00	111	1110	11101	111100	(14,0,0,0;48)	$4.8158\,10^{11}$	1/48
7-2	0	01	101	1101	11101	111100	(14,0,0,0;14)	$1.6511\,10^{12}$	1/14
7-3	0	00	003	1101	12001	211000	(7,2,1,0;1)	$9.6316\,10^{11}$	1/24
7-4	0	00	002	0202	11200	310000	(3,4,1,0;2)	$1.2039\,10^{11}$	1/192
7-5	0	00	002	0211	11110	310000	(7,2,1,0;2)	$4.8158\,10^{11}$	1/48
7-6	0	00	011	0121	12010	301000	(7,2,1,0;2)	$4.8158\,10^{11}$	1/48
7-7	0	01	011	0111	10111	310000	(11,0,1,0;12)	$3.2105\,10^{11}$	1/72
7-8	0	00	002	0301	11200	300100	(4,2,2,0;4)	$4.0132\,10^{10}$	1/576
7-9	0	00	002	0112	12100	310000	(5,3,1,0;2)	$2.4079\,10^{11}$	1/96
7-10	0	00	011	0121	11110	310000	(9,1,1,0;2)	$9.6316\,10^{11}$	1/24
7-11	0	00	003	1101	11101	220000	(7,2,1,0;4)	$2.4079\,10^{11}$	1/96
7-12	0	00	002	0211	12010	301000	(5,3,1,0;1)	$4.8158\,10^{11}$	1/48
7-13	0	00	002	0112	13000	301000	(4,2,2,0;2)	$8.0263\,10^{10}$	1/288
7-14	0	01	011	1011	11011	211000	(12,1,0,0;4)	$2.8895\,10^{12}$	1/8
7-15	0	00	011	1111	11110	211000	(12,1,0,0;8)	$1.4447\,10^{12}$	1/16
7-16	0	01	011	0111	20011	211000	(10,2,0,0;8)	$7.2237\,10^{11}$	1/32
7-17	0	01	011	1011	11110	210010	(12,1,0,0;4)	$2.8895\,10^{12}$	1/8
7-18	0	00	011	1111	11200	210100	(10,2,0,0;2)	$2.8895\,10^{12}$	1/8
7-19	0	00	012	1110	11101	210100	(10,2,0,0;4)	$1.4447\,10^{12}$	1/16
7-20	0	00	011	0121	21010	211000	(8,3,0,0;1)	$2.8895\,10^{12}$	1/8
7-21	0	01	011	1011	12010	201010	(10,2,0,0;2)	$2.8895\,10^{12}$	1/8
7-22	0	00	012	1110	12001	201100	(8,3,0,0;1)	$2.8895\,10^{12}$	1/8
7-23	0	00	011	1021	12100	210100	(8,3,0,0;2)	$1.4447\,10^{12}$	1/16
7-24	0	00	002	1111	12010	211000	(8,3,0,0;2)	$1.4447\,10^{12}$	1/16
7-25	0	00	002	1102	12100	211000	(6,4,0,0;4)	$3.6118\,10^{11}$	1/64
7-26	0	00	012	1110	11110	210010	(10,2,0,0;2)	$2.8895\,10^{12}$	1/8
7-27	0	00	002	0202	21100	211000	(4,5,0,0;4)	$1.8059\,10^{11}$	1/128
7-28	0	00	011	0121	20110	220000	(6,4,0,0;2)	$7.2237\,10^{11}$	1/32
7-29	0	00	012	0210	20101	210100	(6,4,0,0;4)	$3.6118\,10^{11}$	1/64
7-30	0	00	002	0211	21010	211000	(6,4,0,0;2)	$7.2237\,10^{11}$	1/32
7-31	0	00	011	0220	20110	210100	(6,4,0,0;4)	$3.6118\,10^{11}$	1/64
7-32	0	01	011	0112	20100	210001	(8,3,0,0;4)	$7.2237\,10^{11}$	1/32
7-33	0	00	002	0112	21100	220000	(4,5,0,0;2)	$3.6118\,10^{11}$	1/64
7-34	0	00	011	1021	12010	211000	(8,3,0,0;4)	$7.2237\,10^{11}$	1/32
7-35	0	00	012	0210	20110	210010	(6,4,0,0;4)	$3.6118\,10^{11}$	1/64
7-36	0	00	002	1111	11110	220000	(8,3,0,0;8)	$3.6118\,10^{11}$	1/64
7-37	0	01	012	1011	11001	210001	(10,2,0,0;4)	$1.4447\,10^{12}$	1/16
7-38	0	00	012	0211	20100	210001	(6,4,0,0;2)	$7.2237\,10^{11}$	1/32
7-39	0	00	002	0202	20200	220000	(0,7,0,0;14)	$1.2899\,10^{10}$	1/1792
7-40	0	00	002	0211	20110	220000	(4,5,0,0;4)	$1.8059\,10^{11}$	1/128
7-41	0	01	012	1011	12000	200011	(8,3,0,0;8)	$3.6118\,10^{11}$	1/64

To exclude diagrams with cutlines each line is taken away, and the modified diagram is checked for connectedness. In this way, all 1PI two-point diagrams can be obtained.

Two-point diagrams with tadpole parts are generated from vacuum diagrams with cutvertices, whereas vacuum diagrams without cutvertices lead to tadpole-free two-point diagrams.

(a) (b1) (b2) (b3)

FIGURE 14.4 Identical vertex permutations transform all lines of the vacuum diagram (a) into one of the lines 1, 2, or 3. The three diagrams (b1,b2,b3) of the connected two-point function $G_c^{(2)}$ are generated by cutting the line 1, 2, or 3, respectively.

All 1PI two-point diagrams, with and without tadpole parts, contribute to the mass renormalization. They are shown in Fig. 14.7. The weight factors and the unique matrix representations for all 1PI two-point diagrams are listed on page 266 in Table 14.3 . These diagrams can also be generated from the four-point diagrams as explained in Subsection 14.2.4. For the field renormalization, on the other hand, we need only the two-point diagrams without tadpoles. These diagrams can of course also be generated from the four-point diagrams, but they can more easily be generated directly from the vacuum diagrams without cutvertices listed in Fig 14.3. The resulting two-point diagrams are then automatically tadpole-free. The weight factors and the unique matrix representations for all tadpole-free 1PI two-point diagrams are listed in Table 14.3 .

A two-point diagram with L loops is generated out of a vacuum diagram with $L+1$ loops, such that for the five-loop calculation the vacuum diagrams up to six loops are needed. The only two-point diagram with one loop is the tadpole diagram itself. The lowest-order vacuum diagram leading to a non-trivial two-point function has therefore two loops. In Table 14.2 we list the numbers of diagrams up to six and eight loops.

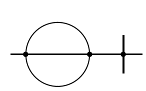
L	Field	Coupling	Mass
1	-	1	1
2	1	2	2
3	1	8	6
4	4	26	21
5	11	124	90
6	50	627	444

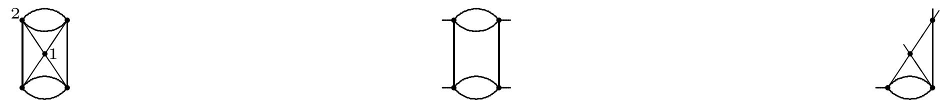
TABLE 14.2 Number of two- and four-point diagrams with L loops contributing to counterterms. The field renormalization requires all 1PI two-point diagrams without tadpole parts. The coupling constant requires renormalization of all 1PI four-point diagrams without tadpoles. Out of the latter, those with only three external vertices contribute to the mass renormalization.

14.2.3 Four-Point Diagrams

A four-point diagram with p vertices is generated by removing a vertex from a vacuum diagram with $p+1$ vertices. The resulting four-point diagram is not connected if the removed vertex is either a cutvertex or a vertex with self-connections. The resulting four-point diagram will have no tadpole part if the vacuum diagram from which it is obtained has no cutvertex. All vacuum diagrams of this type up to seven vertices are listed in Fig. 14.3. Cutting vertices in these diagrams generates all connected four-point diagrams without tadpole parts. Removing vertices which participate in identical vertex permutations leads to equivalent diagrams. An example is drawn in Fig. 14.5. Choosing only one of these vertices avoids repeated generation of the same diagram [3].

Finally, the 1PI diagrams have to be selected. Each line of the four-point diagrams is taken away, and the modified diagram is checked for connectedness. For example, if in Fig. 14.3 the four-vertex vacuum diagram No. 3 is turned into a four-point diagram by deleting any of the vertices we obtain a connected diagram which is not 1PI:





(a) (b1) (b2)

FIGURE 14.5 The vertices at the four outer corners of the vacuum diagram (a) are transformed into one another by permutations which keep the figure unchanged. Only by removing the vertex 1 or 2 can we obtain different four-point diagrams shown in (b1) and (b2).

and therefore not connected after deleting the line connecting the vertex to the rest. A detailed list of the matrix representations and the weight factors of the 1PI four-point functions without tadpoles is given in Table 14.3 .

14.2.4 Four-Point Diagrams for Mass Renormalization

Recall that the diagrams relevant for the mass renormalization are a subset of the connected four-point diagrams with p vertices. They emerge by differentiating the connected two-point diagrams with $p-1$ vertices with respect to m^2. The differentiation generates a ϕ^2-vertex insertion, as explained in Section 2.4. The resulting four-point diagrams possess a vertex with two external lines carrying no momenta. Such four-point diagrams are found by inspecting the matrix representation. There must exist a double connection between an internal vertex and zero, which codes the external vertex. The matrix representation for the two-point diagram follows simply by deleting the double connection to the external vertex. The weight factor for the two-point diagram can then be read off from this matrix. Note that there is no one-to-one correspondence between two-point and four-point diagrams. The differentiation generates several four-point diagrams out of one two-point diagram. Differentiation with respect to the

(a) (b)

FIGURE 14.6 Weight factors of four-point diagrams relevant for coupling constant and for mass renormalization. (a) The weight factor for this four-point diagram is $W^g_{G^{(4)}} = 3/4$, the factor 3 coming from the symmetrization of the external lines. In the mass renormalization, the same four-point diagram is used with two external lines set to zero. The weight factor $W^{m^2}_{\partial_{m^2} G^{(4)}}$ of this diagram is related to the two-point diagram from which it emerges by differentiation as follows: the weight factor of the two-point diagram is 1/8. Differentiation with respect to the mass in the lines labeled by 1 generates a combinatorial factor 2 since the lines are topologically equivalent. The weight factor to be used for the mass renormalization together with the four-point diagram is $W^{m^2}_{\partial_{m^2} G^{(4)}} = 1/4$. (b) The weight factor of the two-point diagram is 1/4. Differentiation generates again a factor 2 even if the lines 1 and 2 are not topologically equivalent. Hence $W^{m^2}_{G^{(4)}} = 1/2$, but $W^g_{G^{(4)}} = 3/2$.

mass of topologically equivalent lines leads to equivalent four-point diagrams. The differentiation therefore generates a factor giving the number of lines in the two-point diagram which are topologically equivalent to the differentiated line. This factor is then combined with the weight factor of the two-point diagram $W^{m^2}_{G^{(2)}}$ to give the mass weight factor of the four-point diagram $W^{m^2}_{\partial_{m^2}G^{(2)}}$. Differentiating an n-fold connection gives the factor n. If a two-point diagram has only one external vertex, the two internal lines connected to it are topologically equivalent lines, and the differentiation of each of them leads to the same four-point diagram. Such a diagram always receives an extra factor 2. Two examples are shown in Fig. 14.6.

14.2.5 Check for Number of Connected Diagrams

The sum of the weight factors of all diagrams with a certain number of vertices and external lines is given by Wick's rule as discussed in Chapter 3. For checking the reduced number of selected 1PI diagrams without tadpole parts, however, the rule is of no use.

The sums of the weight factors of the diagrams appearing in the expansion coefficients of the partition function Z_p, the two-point function $G_p^{(2)}$, and the four-point function $G_p^{(4)}$ are called w_{Z_p}, $w_{G_p^{(2)}}$, and $w_{G_p^{(4)}}$. According to the Wick rule in Chapter 3, the sum of the weight factors for all diagrams with p vertices and n external lines is

$$w_{G_p^{(n)}} = \frac{(4p+n-1)!!}{(4!)^p p!}.$$

If only the connected diagrams in the cumulants $Z_{p\,\mathrm{c}}$, $G_{p\,\mathrm{c}}^{(2)}$, and $G_{p\,\mathrm{c}}^{(4)}$ are considered, the sums of the weight factors are called $w_{Z_{p\,\mathrm{c}}}$, $w_{G_{p\,\mathrm{c}}^{(2)}}$, and $w_{G_{p\,\mathrm{c}}^{(4)}}$. These were calculated from recursion relations on p. 67, but they may also be found by the technique explained in the previous sections. A relation between the weight factors for all diagrams and those for the connected diagrams is needed. It is obtained from the defining equation $Z[j] = e^{W[j]}$ of the generating functional $W[j]$ of all connected correlation functions [recall Eq. (3.19)]. For zero external source this equation relates all disconnected and connected vacuum diagrams with each other. Inserting $\sum_{p=0} w_{Z_p} x^p$ for the sum of the weights of all vacuum diagrams, and $\sum_{p=0} w_{Z_{p\,\mathrm{c}}} x^p$ for the sum of the weights of all connected vacuum diagrams, we find

$$\sum_{p=0}^{\infty} w_{Z_p} x^p = \sum_{m=0}^{\infty} \frac{1}{m!} \left[\sum_{p=0}^{\infty} w_{Z_{p\,\mathrm{c}}} x^p \right]^m . \tag{14.5}$$

Composing the coefficients of each order in x yields equations between $w_{Z_{p\,\mathrm{c}}}$ and w_{Z_p}. To calculate the weight factors of the two- and four-point diagrams, the results of Section 3.3.2 or 5.3 are used, especially Eqs. (3.21), (3.23), and (3.24). Replacing in Eq. (3.21) the sum over all n-point functions $\sum_{p=0} G_p^{(n)}$ by the sum over their weight factors $\sum_{p=0} w_{G_p^{(n)}} x^p$ and $\sum_{p=0} G_{p\,\mathrm{ext}}^{(n)}$ for $n = 2$ according to Eq. (3.23) by $\sum_{p=0} w_{G_{p\,\mathrm{c}}^{(2)}} x^p$, and for $n = 4$ according to Eq. (3.24) by $\sum_{p=0} w_{G_{p\,\mathrm{c}}^{(4)}} x^p + 3(\sum_{p=0} w_{G_{p\,\mathrm{c}}^{(2)}} x^p)^2$, where $w_{G_0^{(2)}} = 1,\ w_{G_0^{(2)}} = 3,\ w_{G_{0\,\mathrm{c}}^{(2)}} = 1,\ w_{G_{0\,\mathrm{c}}^{(4)}} = 0$, we find:

$$\sum_{p=0}^{\infty} w_{G_p^{(2)}} x^p = \left(\sum_{p=0}^{\infty} w_{G_p^{(0)}} x^p \right) \times \left(\sum_{p=0}^{\infty} w_{G_p^{(2)}} x^p \right), \tag{14.6}$$

$$\sum_{p=0}^{\infty} w_{G_p^{(4)}} x^p = \left(\sum_{p=0}^{\infty} w_{G_p^{(0)}} x^p \right) \times \left[3 \left(\sum_{p=0}^{\infty} w_{G_p^{(2)}} x^p \right)^2 + \sum_{p=0}^{\infty} w_{G_p^{(4)}} x^p \right]. \tag{14.7}$$

TABLE 14.3 Matrix representation for all diagrams contributing to Z_ϕ, Z_m and Z_g specified by the associated numbers $\hat{G} = G^{10}G^{11}G^{20}G^{21}G^{22}G^{30}G^{31}G^{32}G^{33}\dots$ of Eq. (14.2). They are ordered lexicographically in $\hat{G}$. The vertical lines separating groups of digits in $\hat{G}$ mark the successive rows of the below-diagonal part of the matrix G^{ij}. The entries for i and j show which matrix element G^{ij} is displayed below, The matrix elements specify the number of lines connecting the two vertices i and j. The tables display further the numbers of self-, double, and triple connections, and of irreducible vertex permutations $(S, D, T, N_{\mathrm{IVP}})$, the multiplicities M_G, and the weights W_G. The last column in the table for Z_g gives the running number of the diagrams used in the lists in Appendices A.1 and A.2. In these lists the two-point diagrams for the calculation of Z_m do not appear because Z_m is obtained from four-point diagrams. The two-point diagrams are therefore shown separately in Fig. 14.7 of this chapter.

Z_ϕ, 2 vertices, 2 loops: 1 diagram						
i	11	222				
j	01	012				
#	G^{ij}		$(S,D,T;N_{\mathrm{IVP}})$	M_G	W_G	No.
2-1	10	130	(0,0,1;2)	192	1/6	1

Z_ϕ, 3 vertices, 3 loops: 1 diagram							
i	11	222	3333				
j	01	012	0123				
#	G^{ij}			$(S,D,T;N_{\mathrm{IVP}})$	M_G	W_G	No.
3-1	00	120	1210	(0,2,0;2)	20736	1/4	1

Z_ϕ, 4 vertices, 4 loops: 4 diagrams								
i	11	222	3333	44444				
j	01	012	0123	01234				
#	G^{ij}				$(S,D,T;N_{\mathrm{IVP}})$	M_G	W_G	No.
4-1	00	010	1120	12100	(0,2,0;2)	1990656	1/4	4
4-2	00	020	1020	12010	(0,3,0;2)	995328	1/8	1
4-3	00	020	1110	11110	(0,1,0;4)	1990656	1/4	3
4-4	00	030	1010	11020	(0,1,1;2)	663552	1/12	2

Z_ϕ, 5 vertices, 5 loops: 11 diagrams									
i	11	222	3333	44444	555555				
j	01	012	0123	01234	012345				
#	G^{ij}					$(S,D,T;N_{\mathrm{IVP}})$	M_G	W_G	No.
5-1	00	000	0030	12010	121000	(0,2,1;2)	39813120	1/24	3
5-2	00	000	0120	11110	121000	(0,2,0;1)	477757440	1/2	5
5-3	00	000	0120	11200	120100	(0,3,0;1)	238878720	1/4	6
5-4	00	000	0130	11100	120010	(0,1,1;1)	159252480	1/6	2
5-5	00	000	0220	10200	120010	(0,4,0;2)	59719680	1/16	1
5-6	00	000	0220	11100	111010	(0,2,0;4)	119439360	1/8	7
5-7	00	010	0110	10120	121000	(0,2,0;2)	238878720	1/4	9
5-8	00	010	0110	11110	111100	(0,0,0;12)	159252480	1/6	11
5-9	00	010	0120	10110	120010	(0,2,0;2)	238878720	1/4	10
5-10	00	010	0120	11010	111010	(0,1,0;2)	477757440	1/2	8
5-11	00	010	0220	10100	110020	(0,3,0;2)	119439360	1/8	4

Z_m, 2 vertices, 2 loops: 2 diagrams					
i	11	222			
j	01	012			
#	G^{ij}		$(S,D,T;N_{\mathrm{IVP}})$	M_G	$W_{\bar{G}}$
2-1	10	130	(0,0,1;2)	192	1/6
2-2	01	220	(1,2,0;1)	288	1/4

Z_m, 3 vertices, 3 loops: 5 diagrams						
i	11	222	3333			
j	01	012	0123			
#	G^{ij}			$(S,D,T;N_{\mathrm{IVP}})$	M_G	$W_{\bar{G}}$
3-1	00	030	2110	(0,1,1;2)	6912	1/12
3-2	00	120	1210	(0,2,0;2)	20736	1/4
3-3	01	110	1120	(1,1,0;2)	20736	1/4
3-4	00	021	2200	(1,3,0;1)	10368	1/8
3-5	01	011	2110	(2,1,0;2)	10368	1/8

Z_m, 4 vertices, 4 loops: 17 diagrams							
i	11	222	3333	44444			
j	01	012	0123	01234			
#	G^{ij}				$(S,D,T;N_{\mathrm{IVP}})$	M_G	$W_{\bar{G}}$
4-1	00	001	0220	22000	(1,4,0;1)	497664	1/16
4-2	00	001	0310	21100	(1,1,1;1)	663552	1/12
4-3	00	001	1210	12100	(1,2,0;2)	995328	1/8
4-4	00	010	0130	22000	(0,2,1;2)	331776	1/24
4-5	00	010	0220	21100	(0,3,0;2)	995328	1/8
4-6	00	010	1120	12100	(0,2,0;2)	1990656	1/4
4-7	00	010	1120	12100	(0,2,0;2)	1990656	1/4
4-8	00	011	0111	22000	(2,2,0;2)	497664	1/16
4-9	00	011	0201	21100	(2,2,0;1)	995328	1/8
4-10	00	011	0210	21010	(1,2,0;2)	995328	1/8
4-11	00	011	1110	12010	(1,1,0;1)	3981312	1/2
4-12	00	020	1020	12010	(0,3,0;2)	995328	1/8
4-13	00	021	1100	11020	(1,2,0;2)	995328	1/8
4-14	00	030	1010	11020	(0,1,1;2)	663552	1/12
4-15	01	001	0111	21100	(3,1,0;2)	497664	1/16
4-16	01	001	1110	11110	(2,0,0;4)	995328	1/8
4-17	01	011	1010	11020	(2,1,0;2)	995328	1/8

Z_m, 5 vertices, 5 loops: 68 diagrams								
i	11	222	3333	44444	555555			
j	01	012	0123	01234	012345			
#	G^{ij}					$(S,D,T;N_{\mathrm{IVP}})$	M_G	$W_{\bar{G}}$
5-1	00	000	0021	02200	220000	(1,5,0;1)	29859840	1/32
5-2	00	000	0021	03100	211000	(1,2,1;1)	39813120	1/24
5-3	00	000	0021	12100	121000	(1,3,0;2)	59719680	1/16
5-4	00	000	0030	02110	220000	(0,3,1;2)	19906560	1/48
5-5	00	000	0030	03010	211000	(0,1,2;2)	13271040	1/72
5-6	00	000	0030	12010	121000	(0,2,1;2)	39813120	1/24
5-7	00	000	0111	01300	220000	(1,2,1;1)	39813120	1/24
5-8	00	000	0111	11200	121000	(1,2,0;2)	119439360	1/8
5-9	00	000	0120	01210	220000	(0,4,0;2)	59719680	1/16
5-10	00	000	0120	02110	211000	(0,3,0;2)	119439360	1/8

Z_m, 5 vertices, 5 loops: 68 diagrams								
i	11	222	3333	44444	555555			
j	01	012	0123	01234	012345			
#	G^{ij}					$(S,D,T;N_{\text{IVP}})$	M_G	$W_{\tilde{G}}$
5-11	00	000	0120	02200	210100	(0,4,0;2)	59719680	1/16
5-12	00	000	0120	03100	201100	(0,2,1;2)	39813120	1/24
5-13	00	000	0120	11110	121000	(0,2,0;1)	477757440	1/2
5-14	00	000	0012	01120	0120100	(0,3,0;1)	238878720	1/4
5-15	00	000	0130	11100	120010	(0,1,1;1)	159252480	1/6
5-16	00	000	0220	01101	211000	(1,3,0;2)	59719680	1/16
5-17	00	000	0220	10200	120010	(0,4,0;2)	59719680	1/16
5-18	00	000	0220	11100	111010	(0,2,0;4)	119439360	1/8
5-19	00	001	0011	02110	220000	(2,3,0;2)	29859840	1/32
5-20	00	001	0011	03010	211000	(2,1,1;1)	39813120	1/24
5-21	00	001	0011	12010	121000	(2,2,0;2)	59719680	1/16
5-22	00	001	0101	01210	220000	(2,3,0;1)	59719680	1/16
5-23	00	001	0101	02110	211000	(2,1,0;1)	238878720	1/4
5-24	00	001	0101	02200	210100	(2,3,0;1)	59719680	1/16
5-25	00	001	0101	03100	201100	(2,1,1;2)	19906560	1/48
5-26	00	001	0110	01120	220000	(1,3,0;2)	59719680	1/16
5-27	00	001	0110	02020	211000	(1,3,0;1)	119439360	1/8
5-28	00	001	0110	02110	210100	(1,2,0;1)	238878720	1/4
5-29	00	001	0110	03010	201100	(1,1,1;2)	39813120	1/24
5-30	00	001	0110	11020	121000	(1,2,0;1)	238878720	1/4
5-31	00	001	0110	11110	120100	(1,1,0;1)	477757440	1/2
5-32	00	001	0111	01101	220000	(3,2,0;2)	29859840	1/32
5-33	00	001	0111	02001	211000	(3,2,0;1)	59719680	1/16
5-34	00	001	0111	02100	210010	(2,2,0;2)	59719680	1/16
5-35	00	001	0111	11100	120010	(2,1,0;1)	238878720	1/4
5-36	00	001	0120	02001	210100	(2,3,0;2)	29859840	1/32
5-37	00	001	0120	02010	210010	(1,3,0;2)	59719680	1/16
5-38	00	001	0120	11010	120010	(1,2,0;1)	238878720	1/4
5-39	00	001	0200	10120	121000	(1,3,0;2)	59719680	1/16
5-40	00	001	0200	11110	111100	(1,1,0;4)	119439360	1/8
5-41	00	001	0201	11100	111010	(2,1,0;2)	119439360	1/8
5-42	00	001	0210	10110	120010	(1,2,0;1)	238878720	1/4
5-43	00	001	0210	11010	111010	(1,1,0;1)	477757440	1/2
5-44	00	001	0220	11000	110020	(1,3,0;2)	59719680	1/16
5-45	00	001	0300	10110	111010	(1,0,1;2)	79626240	1/12
5-46	00	001	0310	10100	110020	(1,1,1;1)	79626240	1/12
5-47	00	010	0110	01120	211000	(0,2,0;4)	119439360	1/8
5-48	00	010	0110	11020	112000	(0,2,0;2)	238878720	1/4
5-49	00	010	0110	11110	111100	(0,0,0;12)	159252480	1/6
5-50	00	010	0111	01101	211000	(2,1,0;4)	59719680	1/16
5-51	00	010	0111	10200	120010	(1,2,0;2)	119439360	1/8
5-52	00	010	0111	11100	111010	(1,0,0;4)	238878720	1/4
5-53	00	010	0120	10110	120010	(0,2,0;2)	238878720	1/4
5-54	00	010	0120	11010	111010	(0,1,0;2)	477757440	1/2
5-55	00	010	0130	11000	110020	(0,1,1;4)	39813120	1/24
5-56	00	010	0220	10100	110020	(0,3,0;2)	119439360	1/8
5-57	00	011	0101	01011	211000	(3,1,0;2)	59719680	1/16
5-58	00	011	0101	02001	201100	(3,2,0;2)	29859840	1/32
5-59	00	011	0101	10110	120010	(2,1,0;2)	119439360	1/8
5-60	00	011	0101	11010	111010	(2,0,0;2)	238878720	1/4
5-61	00	011	0111	11000	110020	(2,1,0;4)	59719680	1/16
5-62	00	011	0201	10100	110020	(2,2,0;1)	119439360	1/8
5-63	00	020	0011	01101	210100	(2,2,0;1)	119439360	1/8
5-64	00	020	0111	10100	110020	(1,2,0;2)	119439360	1/8
5-65	01	001	0001	11110	111100	(3,0,0;12)	19906560	1/48
5-66	01	001	0011	01011	211000	(4,1,0;2)	29859840	1/32
5-67	01	001	0011	11010	110100	(3,0,0;2)	119439360	1/8
5-68	01	001	0111	10100	110020	(3,1,0;2)	59719680	1/16

Z_g, 2 vertices, 1 loop: 1 diagram						
i	11	222				
j	01	012				
#	G^{ij}		$(S,D,T;N_{\mathrm{IVP}})$	M_G	$W_{\bar{G}}$	No.
2-1	20	220	(0,3,0;2)	1728	3/2	1

Z_g, 3 vertices, 2 loops: 2 diagrams							
i	11	222	3333				
j	01	012	0123				
#	G^{ij}			$e(S,D,T;N_{\mathrm{IVP}})$	M_G	$W_{\bar{G}}$	No.
3-1	00	220	2200	(0,4,0;2)	62208	3/4	1
3-2	10	120	2110	(0,2,0;2)	248832	3	2

Z_g, 4 vertices, 3 loops: 8 diagrams								
i	11	222	3333	44444				
j	01	012	0123	01234				
#	G^{ij}				$(S,D,T;N_{\mathrm{IVP}})$	M_G	$W_{\bar{G}}$	No.
4-1	00	020	2020	22000	(0,5,0;2)	2985984	3/8	1
4-2	00	020	2110	21100	(0,3,0;4)	5971968	3/4	8
4-3	00	030	2010	21010	(0,2,1;2)	3981312	1/2	3
4-4	00	110	1120	22000	(0,3,0;2)	11943936	3/2	2
4-5	00	110	1210	21100	(0,2,0;1)	47775744	6	7
4-6	00	120	1200	20110	(0,3,0;2)	11943936	3/2	5
4-7	10	100	1120	12100	(0,2,0;4)	11943936	3/2	6
4-8	10	110	1110	11110	(0,0,0;24)	7962624	1	4

Z_g, 5 vertices, 4 loops: 26 diagrams									
i	11	222	3333	44444	555555				
j	01	012	0123	01234	012345				
#	G^{ij}					$(S,D,T;N_{\mathrm{IVP}})$	M_G	$W_{\bar{G}}$	No.
5-1	00	000	0130	21100	220000	(0,3,1;1)	477757440	1/2	5
5-2	00	000	0220	20200	220000	(0,6,0;2)	179159040	3/16	1
5-3	00	000	0220	21100	211000	(0,4,0;4)	358318080	3/8	14
5-4	00	000	1120	11200	220000	(0,4,0;2)	716636160	3/4	4
5-5	00	000	1120	12100	211000	(0,3,0;2)	1433272320	3/2	22
5-6	00	010	0120	20110	220000	(0,4,0;2)	716636160	3/4	7
5-7	00	010	0120	21010	211000	(0,3,0;2)	1433272320	3/2	8
5-8	00	010	0220	20100	210010	(0,4,0;2)	716636160	3/4	15
5-9	00	010	1020	11110	220000	(0,3,0;1)	2866544640	3	6
5-10	00	010	1020	12010	211000	(0,3,0;2)	1433272320	3/2	20
5-11	00	010	1020	12100	210100	(0,3,0;1)	2866544640	3	23
5-12	00	010	1110	11110	211000	(0,1,0;4)	2866544640	3	21
5-13	00	010	1110	11200	2101 00	(0,2,0;1)	5733089280	6	26
5-14	00	020	1010	10120	220000	(0,4,0;2)	716636160	3/4	2
5-15	00	020	1010	11020	211000	(0,3,0;2)	1433272320	3/2	9
5-16	00	020	1010	11110	210100	(0,2,0;1)	5733089280	6	25
5-17	00	020	1010	12010	201100	(0,3,0;1)	2866544640	3	24
5-18	00	020	1020	12000	200110	(0,4,0;2)	716636160	3/4	18
5-19	00	020	1110	11100	200110	(0,2,0;4)	1433272320	3/2	17
5-20	00	030	1000	10120	210100	(0,2,1;1)	955514880	1	19
5-21	00	030	1010	11010	200110	(0,1,1;2)	955514880	1	16
5-22	00	100	1020	12010	121000	(0,3,0;2)	1433272320	3/2	13
5-23	00	100	1110	11110	121000	(0,1,0;2)	5733089280	6	11
5-24	00	100	1110	11200	120100	(0,2,0;1)	5733089280	6	12
5-25	00	110	1100	11020	112000	(0,2,0;8)	716636160	3/4	3
5-26	00	110	1100	11110	111100	(0,0,0;8)	2866544640	3	10

Z_g, 6 vertices, 5 loops: 124 diagrams										
i	11	222	3333	44444	555555	6666666				
j	01	012	0123	01234	012345	0123456				
#				G^{ij}			$(S,D,T;N_{\rm IVP})$	M_G	$W_{\bar{G}}$	No.
6-1	00	000	0020	01120	211000	2200000	(0,5,0;1)	103195607040	3/4	99
6-2	00	000	0020	02020	202000	2200000	(0,7,0;2)	12899450880	3/32	117
6-3	00	000	0020	02020	211000	2110000	(0,5,0;4)	25798901760	3/16	79
6-4	00	000	0020	02110	201100	2200000	(0,5,0;2)	51597803520	3/8	120
6-5	00	000	0020	02110	210100	2110000	(0,4,0;2)	103195607040	3/4	93
6-6	00	000	0020	03010	201100	2110000	(0,3,1;1)	68797071360	1/2	21
6-7	00	000	0020	03010	202000	2101000	(0,4,1;1)	34398535680	1/4	7
6-8	00	000	0020	11020	112000	2200000	(0,5,0;2)	51597803520	3/8	78
6-9	00	000	0020	11020	121000	2110000	(0,4,0;1)	206391214080	3/2	71
6-10	00	000	0020	11110	111100	2200000	(0,3,0;4)	103195607040	3/4	108
6-11	00	000	0020	11110	120100	2110000	(0,3,0;1)	412782428160	3	69
6-12	00	000	0020	12010	120100	2020000	(0,5,0;2)	51 597803520	3/8	80
6-13	00	000	0020	12010	121000	2011000	(0,4,0;2)	103195607040	3/4	92
6-14	00	000	0030	03000	200110	2110000	(0,2,2;4)	5733089280	1/24	14
6-15	00	000	0030	03010	201000	2100010	(0,2,2;2)	11466178560	1/12	23
6-16	00	000	0030	11010	111010	2200000	(0,2,1;2)	68797071360	1/2	18
6-17	00	000	0030	11010	120010	2110000	(0,2,1;1)	137594142720	1	6
6-18	00	000	0030	11010	121000	2100100	(0,2,1;1)	137594142720	1	4
6-19	00	000	0030	12000	120100	2010100	(0,3,1;1)	68797071360	1/2	20
6-20	00	000	0110	01120	202000	2200000	(0,5,0;4)	25798901760	3/16	121
6-21	00	000	0110	01120	211000	2110000	(0,3,0;8)	51597803520	3/8	109
6-22	00	000	0110	01210	201100	2200000	(0,4,0;1)	206391214080	3/2	84
6-23	00	000	0110	01210	210100	2110000	(0,3,0;1)	412782428160	3	47
6-24	00	000	0110	01300	200200	2200000	(0,4,1;2)	17199267840	1/8	8
6-25	00	000	0110	01300	210100	2101000	(0,2,1;4)	34398535680	1/4	19
6-26	00	000	0110	02200	200200	2110000	(0,5,0;2)	51597803520	3/8	81
6-27	00	000	0110	02200	201100	2101000	(0,4,0;2)	103195607040	3/4	95
6-28	00	000	0110	10120	112000	2200000	(0,4,0;1)	206391214080	3/2	90
6-29	00	000	0110	10120	121000	2110000	(0,3,0;1)	412782428160	3	70
6-30	00	000	0110	10210	111100	2200000	(0,3,0;1)	412782428160	3	46
6-31	00	000	0110	10210	120100	2110000	(0,3,0;2)	206391214080	3/2	102
6-32	00	000	0110	10210	121000	2101000	(0,3,0;1)	412782428160	3	63
6-33	00	000	0110	11110	111100	2110000	(0,1,0;4)	412782428160	3	29
6-34	00	000	0110	11110	112000	2101000	(0,2,0;1)	825564856320	6	38
6-35	00	000	0110	11200	121000	2002000	(0,4,0;2)	103195607040	3/4	72
6-36	00	000	0120	02100	200110	2110000	(0,4,0;4)	51597803520	3/8	73
6-37	00	000	0120	02100	201010	2101000	(0,4,0;4)	51597803520	3/8	88
6-38	00	000	0120	02100	201100	2100100	(0,4,0;4)	51597803520	3/8	106
6-39	00	000	0120	02110	201000	2100010	(0,4,0;2)	103195607040	3/4	116
6-40	00	000	0120	02200	200100	2100010	(0,5,0;2)	51597803520	3/8	100
6-41	00	000	0120	03100	200100	2010010	(0,3,1;2)	34398535680	1/4	15
6-42	00	000	0120	10110	111010	2200000	(0,3,0;1)	412782428160	3	48
6-43	00	000	0120	10110	120010	2110000	(0,3,0;1)	412782428160	3	65
6-44	00	000	0120	10110	121000	2100100	(0,3,0;1)	412782428160	3	101
6-45	00	000	0120	10200	110110	2200000	(0,4,0;1)	206391214080	3/2	94
6-46	00	000	0120	10200	120010	2101000	(0,4,0;1)	206391214080	3/2	86
6-47	00	000	0120	10200	120100	2100100	(0,4,0;1)	206391214080	3/2	104
6-48	00	000	0120	11010	111010	2110000	(0,2,0;1)	825564856320	6	37
6-49	00	000	0120	11010	112000	2100100	(0,2,0;1)	412782428160	3	50
6-50	00	000	0120	11010	120010	2020000	(0,4,0;1)	206391214080	3/2	83
6-51	00	000	0120	11010	121000	2010100	(0,3,0;1)	412782428160	3	64
6-52	00	000	0120	11100	111010	2101000	(0,2,0;2)	412782428160	3	43
6-53	00	000	0120	11100	111100	2100100	(0,2,0;1)	825564856320	6	74
6-54	00	000	0120	11100	120010	2011000	(0,3,0;1)	412782428160	3	62
6-55	00	000	0120	11100	120100	2010100	(0,3,0;1)	412782428160	3	57
6-56	00	000	0120	11100	121000	2001100	(0,3,0;1)	412782428160	3	67
6-57	00	000	0120	11110	120000	2010010	(0,3,0;1)	412782428160	3	59
6-58	00	000	0120	11200	120000	2001010	(0,4,0;1)	206391214080	3/2	89
6-59	00	000	0130	10100	110020	2200000	(0,3,1;1)	68797071360	1/2	22
6-60	00	000	0130	10100	120010	2100100	(0,2,1;1)	137594142720	1	5
6-61	00	000	0130	11000	110020	2110000	(0,2,1;2)	68797071360	1/2	9
6-62	00	000	0130	11000	111010	2100100	(0,1,1;1)	275188285440	2	16

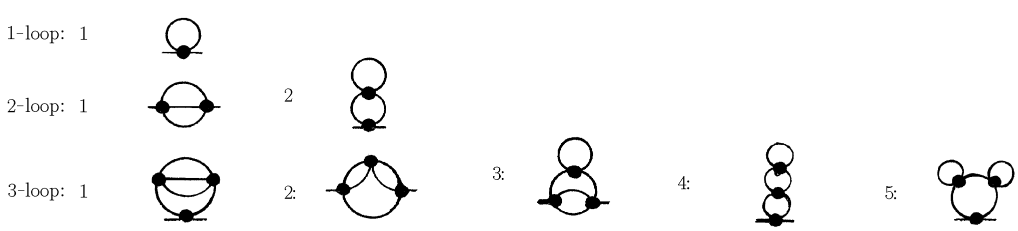
Z_g, 6 vertices, 5 loops: 124 diagrams										
i	11	222	3333	44444	555555	6666666				
j	01	012	0123	01234	012345	0123456				
#	G^{ij}						$(S,D,T;N_{\text{IVP}})$	M_G	$W_{\bar{G}}$	No.
6-63	00	000	0130	11000	120010	2010100	(0,2,1;1)	137594142720	1	12
6-64	00	000	0130	11100	120000	2000110	(0,2,1;1)	137594142720	1	10
6-65	00	000	0220	10100	101020	2200000	(0,5,0;2)	51597803520	3/8	119
6-66	0 0	000	0220	10100	110020	2110000	(0,4,0;2)	103195607040	3/4	91
6-67	00	000	0220	10100	111010	2100100	(0,3,0;1)	412782428160	3	51
6-68	00	000	0220	10100	120010	2010100	(0,4,0;1)	206391214080	3/2	82
6-69	00	000	0220	10200	120000	2000110	(0,5,0;2)	51597803520	3/8	118
6-70	00	000	0220	11100	111000	2000110	(0,3,0;4)	103195607040	3/4	107
6-71	00	000	1010	10120	121000	1210000	(0,3,0;4)	103195607040	3/4	110
6-72	00	000	1010	10210	120100	1210000	(0,3,0;1)	412782428160	3	45
6-73	00	000	1010	11020	112000	1210000	(0,3,0;2)	206391214080	3/2	66
6-74	00	000	1010	11110	111100	1210000	(0,1,0;2)	825564856320	6	27
6-75	00	000	1010	11110	112000	1201000	(0,2,0;1)	825564856320	6	39
6-76	00	000	1020	10200	120010	1201000	(0,4,0;4)	51597803520	3/8	96
6-77	00	000	1020	11100	111010	1201000	(0,2,0;4)	206391214080	3/2	41
6-78	00	000	1020	11100	111100	1200100	(0,2,0;2)	412782428160	3	52
6-79	00	000	1110	11100	111010	1111000	(0,0,0;16)	206391214080	3/2	2
6-80	00	010	0110	01110	200110	2110000	(0,2,0;8)	103195607040	3/4	32
6-81	00	010	0110	01120	201000	2100010	(0,3,0;4)	103195607040	3/4	98
6-82	00	010	0110	10020	102010	2200000	(0,4,0;2)	103195607040	3/4	87
6-83	00	010	0110	10020	111010	2110000	(0,2,0;2)	412782428160	3	42
6-84	00	010	0110	10020	112000	2100100	(0,3,0;1)	412782428160	3	56
6-85	00	010	0110	10110	101110	2200000	(0,2,0;4)	206391214080	3/2	31
6-86	00	010	0110	10110	110110	2110000	(0,1,0;2)	825564856320	6	26
6-87	00	010	0110	10110	111100	2100100	(0,1,0;2)	825564856320	6	36
6-88	00	010	0110	10110	120100	2010100	(0,2,0;1)	825564856320	6	54
6-89	00	010	0120	10010	101020	2200000	(0,4,0;2)	103195607040	3/4	105
6-90	00	010	0120	10010	110020	2110000	(0,3,0;1)	412782428160	3	61
6-91	00	010	0120	10010	111010	2100100	(0,2,0;1)	825564856320	6	75
6-92	00	010	0120	10110	120010	2010100	(0,3,0;1)	412782428160	1/2	58
6-93	00	010	0120	10110	110010	2100010	(0,2,0;2)	412782428160	3	113
6-94	00	010	0120	10110	120000	2000110	(0,3,0;2)	206391214080	3/2	123
6-95	00	010	0120	11 000	110020	2011000	(0,3,0;4)	103195607040	3/4	124
6-96	00	010	0120	11000	110110	2010100	(0,2,0;1)	825564856320	6	53
6-97	00	010	0120	11010	111000	2000110	(0,2,0;2)	412782428160	3	111
6-98	00	010	0220	10000	101020	2100100	(0,4,0;1)	206391214080	3/2	115
6-99	00	010	0220	10100	110010	2000110	(0,3,0;2)	206391214080	3/2	97
6-100	00	010	1000	10120	111100	1210000	(0,2,0;1)	825564856320	6	76
6-101	00	010	1000	10120	112000	1201000	(0,3,0;1)	412782428160	3	60
6-102	00	010	1000	10210	111100	1201000	(0,2,0;2)	412782428160	3	44
6-103	00	010	1000	11110	111100	1111000	(0,0,0;12)	275188285440	2	1
6-104	00	010	1010	10110	110110	1210000	(0,1,0;2)	825564856320	6	33
6-105	00	010	1010	10110	111010	1201000	(0,1,0;1)	1651129712640	12	24
6-106	00	010	1010	10120	111000	1200010	(0,2,0;2)	412782428160	3	114
6-107	00	010	1010	10200	110110	1201000	(0,2,0;2)	412782428160	3	40
6-108	00	010	1010	10200	110200	1200100	(0,3,0;2)	206391214080	3/2	68
6-109	00	010	1010	10210	110100	1200010	(0,2,0;2)	412782428160	3	77
6-110	00	010	1010	11010	111010	1111000	(0,0,0;2)	1651129712640	12	3
6-111	00	010	1010	11020	111000	1110010	(0,1,0;4)	412782428160	3	35
6-112	00	020	1000	10020	102010	1201000	(0,4,0;2)	103195607040	3/4	85
6-113	00	020	1000	10020	111010	1111000	(0,2,0;4)	206391214080	3/2	112
6-114	00	020	1000	10110	101110	1201000	(0,2,0;2)	412782428160	3	30
6-115	00	020	1000	10110	101200	1200100	(0,3,0;1)	412782428160	3	49
6-116	00	020	1000	10110	110110	1111000	(0,1,0;2)	825564856320	6	34
6-117	00	020	1000	10110	110200	1110100	(0,2,0;1)	825564856320	6	55
6-118	00	020	1010	10100	110020	1102000	(0,3,0;4)	103195607040	3/4	103
6-119	00	020	1010	10100	110110	1101100	(0,1,0;8)	206391214080	3/2	28
6-120	00	020	1010	10110	110010	1101010	(0,1,0;4)	412782428160	3	25
6-121	00	020	1010	10120	110000	1100020	(0,3,0;8)	51597803520	3/8	122
6-122	00	030	1000	10010	101020	1102000	(0,2,1;2)	68797071360	1/2	11
6-123	00	030	1000	10010	101110	1101100	(0,0,1;4)	137594142720	1	13
6-124	00	030	1 000	10020	101010	1101010	(0,1,1;2)	137594142720	1	17

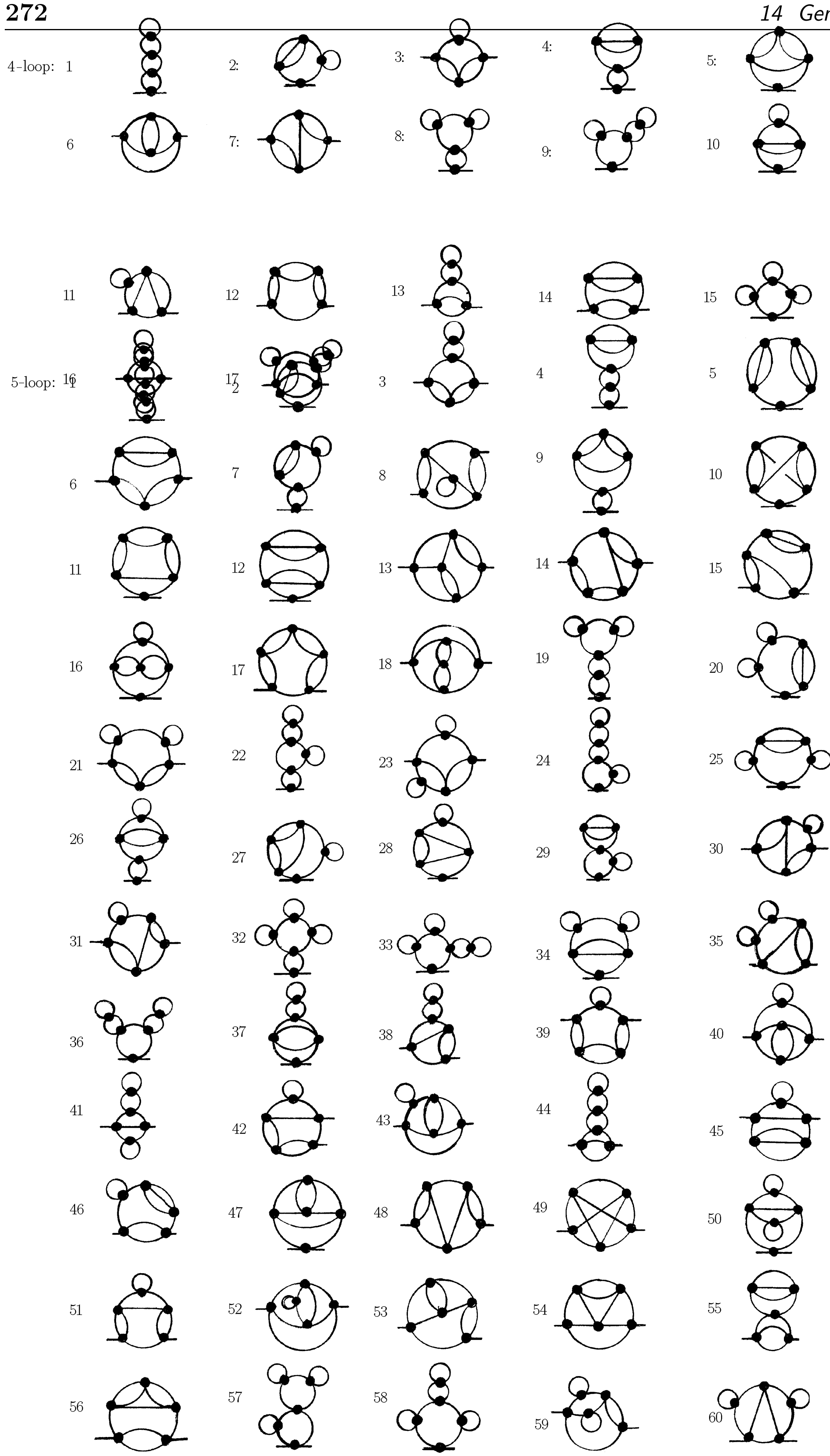
In Table 14.4 we list the number of the diagrams appearing in $Z_{p\,c}$, $G^{(2)}_{p\,c}$, and $G^{(4)}_{p\,c}$, and the sum of the corresponding weight factors.

TABLE 14.4 Number of connected diagrams with 0, 2 and 4 external lines and the sums of their weight factors.

p	$Z_{p\,c}$		$G^{(2)}_{p\,c}$		$G^{(4)}_{p\,c}$	
	number	$w_{Z_{p\,c}}$	number	$w_{G^{(2)}_{p\,c}}$	number	$w_{G^{(4)}_{p\,c}}$
1	1	$\frac{1}{8}$	1	$\frac{1}{2}$	1	1
2	2	$\frac{1}{12}$	3	$\frac{2}{3}$	2	$\frac{7}{2}$
3	4	$\frac{11}{96}$	8	$\frac{11}{8}$	8	$\frac{149}{12}$
4	10	$\frac{17}{72}$	30	$\frac{34}{9}$	37	$\frac{197}{4}$
5	28	$\frac{619}{960}$	118	$\frac{619}{48}$	181	$\frac{15905}{72}$
6	97	$\frac{709}{324}$	548	$\frac{1418}{27}$	1010	$\frac{107113}{96}$

FIGURE 14.7 One-particle irreducible two-point diagrams up to five loops contributing to the mass renormalization constant Z_m. The order is the same as in the list on page 266 in Table 14.3 which lists the weight factors and matrix representations of the diagrams. We show these diagrams here, because they will not appear in the calculations and accompanying tables. For the actual calculation of Z_m we will use the integrals of the four-point diagrams as explained in Section 11.7. These 1PI four-point diagrams, which are also needed when calculating Z_g, are shown in Appendix A. There, we also show some 1PI two-point diagrams again, but only those with no tadpoles, since they contribute to Z_ϕ, and this contribution has to be calculated from the two-point diagrams themselves.




4-loop: 1
2:
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4:
5:
6
7:
8:
9:
10
11
12
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14
15
5-loop: 1
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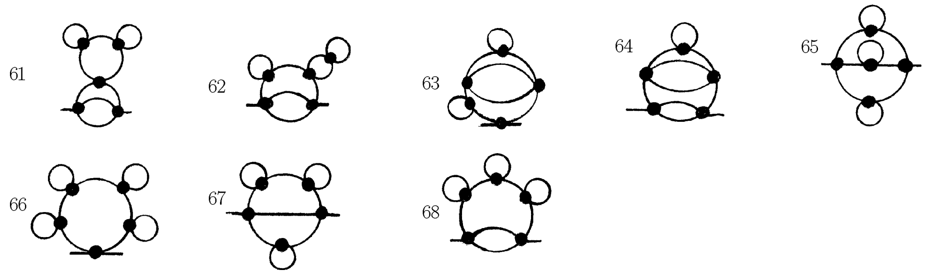

Notes and References

An overview on graph-theoretical notations can be found in
N. Nakanishi, *Graph Theory and Feynman Integrals*, Gordon and Breach, New York, 1971.

The individual citations in the text refer to:

[1] The construction of all diagrams and their multiplicities based on the recursion relation in Fig. 5.2 is described in
H. Kleinert, A. Pelster, B. Kastening, M. Bachmann, Phys. Rev. E **62**, 1537 (2000) (hep-th/9907168).
The method was developed in
H. Kleinert, Fortschr. Phys. **30**, 187 (1982); ibid. **30**, 351 (1982).
The Mathematica program and its output are available on the internet (www.physik.fu-berlin/~kleinert/294/programs/index.html#5).

The diagrams in the original five-loop paper [2] were generated by a computer-algebraic program written in our Berlin group in 1990 by J. Neu in his M.S. thesis.
There exist various other computer programs to generate diagrams and count multiplicites, for instance *FeynArts* by
J. Külbeck, M. Böhm, and A. Denner, Comp. Phys. Comm. **60**, 165 (1991);
T. Hahn, Nucl. Phys. Proc. Suppl. **89**, 231 (2000) (hep-ph/0005029).
The programs can be downloaded from http://www-itp.physik.uni-karlsruhe.de/feynarts.
Another procedure is *QGRAF* by
P. Nogueira, J. Comput. Phys. **105**, 279 (1993),
available from ftp://gtae2.ist.utl.pt/pub/qgraf. These programs are based on a combinatorial enumeration of all possible ways of connecting vertices by lines according to Feynman's rules.

Important early enumeration work was done by
B.R. Heap, J. Math. Phys. **7**, 1582 (1966); J.F. Nagle, J. Math. Phys. **7**, 1588 (1966).

[2] H. Kleinert, J. Neu, V. Schulte-Frohlinde, K.G. Chetyrkin, S.A. Larin, Phys. Lett. B **272**, 39 (1991) (hep-th/9503230); Erratum ibid. **319**, 545 (1993).

[3] Note that this would be hard to achieve if we were to generate all four-point diagrams by another procedure: by cutting two lines in a connected vacuum diagram with p vertices rather than by removing a vertex.