9

Renormalization

All Feynman integrals discussed in Chapter 8 are infinite in four dimensions. After dimensional regularization in $4-\varepsilon$ dimensions, they diverge in a specific way for $\varepsilon \to 0$. For an increasing number of loops L, the Feynman integrals possess singularities of the type $1/\varepsilon^i$ $(i=1,\ldots,L)$. These divergences turn out to contain all the information on the critical exponents of the theory in $4-\varepsilon$ dimensions.

If we want to find these divergences, we do not have to calculate the full Feynman integral. Expanding a Feynman integral of any 1PI diagram into a power series in the external momenta, we observe the following general property: except for the lowest expansion coefficient in the four-point function, and the lowest two coefficients in the two-point function, all higher expansion coefficients are convergent for $\varepsilon \to 0$. The lowest expansion coefficient is a constant. Due to the rotational invariance of the theory, the first is proportional to $\mathbf{q}^2$. Precisely such terms are found in a perturbation expansion of a theory with a modified energy functional $E[\phi]$ in Eq. (2.1), which contains additional terms of the form $\int d^Dx\,\phi^4(x)$, $\int d^Dx\,\phi^2$, and $\int d^Dx\,(\partial\phi)^2$, respectively. These terms are of the same form as those in the original energy functional, and it is this property which makes the theory *renormalizable*. Indeed, as anticipated in Section 7.4, finite observables can be obtained by multiplying, in each correlation function, the fields, the mass, and the coupling constant by compensating factors, the renormalization constants Z_ϕ, Z_{m^2}, and Z_g. If we use dimensional regularization in evaluating the Feynman integrals, we shall be able to give these factors the generic form

$$Z = 1 + \sum_{k=1}^{L} g^k \sum_{i=1}^{k} \frac{c_i^k}{\varepsilon^i}, \tag{9.1}$$

in which the coefficients a_k and c_i^k are pure c-numbers. The renormalization constants convert the initial objects in the energy functional, the *bare fields*, *bare mass*, and *bare coupling constant* introduced in Eqs. (7.56)–(7.58), into finite *renormalized fields*, *renormalized mass*, and *renormalized coupling constant*. The compensation takes place order by order in g. The expansion coefficients c_i are determined for each order to cancel the above divergences. If done consistently, all observables become finite in the limit $\varepsilon \to 0$ [1]. In Sections 9.2 and 9.3, we shall demonstrate in detail how this works up to second order in g.

The renormalization procedure may be performed basically in two different ways, which differ in their emphasis on the bare versus the renormalized quantities in the energy functional. Either way will be illustrated up to two loops in Sections 9.2 and 9.3. The more efficient method works with renormalized quantities. Starting out with an energy functional containing immediately the renormalized field, mass, and coupling strength, one determines at each order in g certain divergent counterterms to be added to the field energy to remove the divergences. Completely analogous to this *counterterm method* is the *recursive subtraction method* developed by Bogoliubov and Parasiuk [2]. It proceeds diagram by diagram, a fact which is essential for performing calculations up to five loops. The equivalence between the two methods is nontrivial because of the multiplicities of the diagrams and their symmetry factors. We shall study this equivalence up to two loops in Subsection 9.3.3 as an introduction to the recursive subtraction method which will be the main topic of Chapter 11.

The first conjecture on the renormalizability of quantum field theories was put forward by Dyson [3], stimulating Weinberg [4] to prove an important convergence theorem by which the renormalization program was completed. The recursive procedure of Bogoliubov and Parasiuk gave an independent proof and opened the way to the practical feasibility of higher-order calculations. An error in their work was corrected by K. Hepp [5], and for this reason the proof of the renormalizability for a large class of field theories is commonly referred to as the *H-theorem*.

9.1 Superficial Degree of Divergence

In order to localize the UV-divergence of a diagram, naive power counting is used. According to the Feynman rules in Subsection 4.1.2, a Feynman integral I_G of a diagram G with p vertices contains one integration per loop. A diagram with I internal lines contains

$$L = I - p + 1 \tag{9.2}$$

loop integrations and thus $DL = D(I - p + 1)$ powers of momentum in the numerator. Each of the I internal lines is associated with a propagator, thus contributing $2I$ powers of momentum in the denominator. Thus there are altogether

$$\omega(G) \ = \ DL - 2I \ = \ (D-2)I + D - Dp \tag{9.3}$$

powers of momentum in a Feynman integral. The behavior of the integral at large momenta can be characterized by rescaling all internal momenta as $\mathbf{p} \to \lambda\mathbf{p}$, and observing a power behavior

$$I_G \propto \lambda^{\omega(G)} \ \text{ for } \ \lambda \to \infty\,. \tag{9.4}$$

The power $\omega(G)$ is called the *superficial degree of divergence* of the diagram G. For $\omega(G) \geq 0$, a diagram G is said to be *superficially divergent*. For $\omega(G) = 0, 2, \ldots$, the superficial divergence of a diagram is *logarithmic, quadratic, ...* , respectively (see page 103). The superficial divergence arises from regions in momentum space of the Feynman integral where all loop momenta become simultaneously large.

A diagram is said to have *subdivergences* if it contains a superficially divergent *subdiagram*, i.e., a subdiagram γ with $\omega(\gamma) \geq 0$. A subdiagram is any subset of lines and vertices of G which form a ϕ^4-diagram of lower order in the perturbation expansion. Subdivergences come from regions in momentum space where the loop momenta of subdiagrams γ become large. If a diagram G has no subdivergences but if $\omega(G) \geq 0$, the superficial divergence is the only divergence of the integral, and the associated Feynman integral is convergent if one of the loop integrals is omitted, which corresponds to cutting one of the lines. A negative superficial degree of divergence $\omega(G)$, on the other hand, implies convergence only if no subdivergences are present. Some examples are shown in Fig. 9.1.

A Feynman diagram G is absolutely convergent if the superficial degree of divergence $\omega(G)$ is negative, and if the superficial degrees of divergence $\omega(\gamma)$ of *all* subdiagrams γ are negative as well. This is part of the famous *power counting theorem* of Dyson, whose proof was completed by Weinberg [4]. This theorem is also called the *Weinberg-Dyson convergence theorem*.

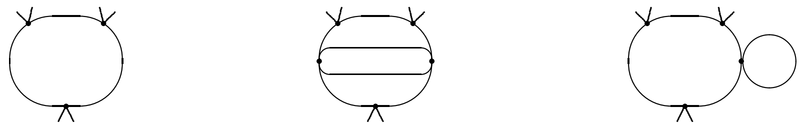


FIGURE 9.1 Three superficially convergent diagrams with $\omega(G) = -2$. The first has no subdivergences. Such a diagram is referred to as a *skeleton diagram*. The second diagram has a logarithmically divergent subdiagram $\gamma =$ $[\omega(\gamma) = 0]$. The third diagram has a quadratically divergent subdiagram $\gamma =$ $[\omega(\gamma) = 2]$.

A more elementary proof was given later by Hahn and Zimmermann [6]. It will not be repeated here since it can be found in standard textbooks [7, 8]. An essential part of the theorem is the elimination of possible extra *overlapping divergences*, which can in principle occur in sets of subdiagrams which have common loop momenta, as shown in Fig. 9.2. In Appendix 9A, the content of the theorem is illustrated by showing explicitly, in a diagram without subdivergences, that no extra divergences are created by overlapping integrations, as stated by the above theorem. All divergences come exclusively from superficial divergences of subdiagrams, and from the superficial divergence of the final integral, but not from overlapping divergences.

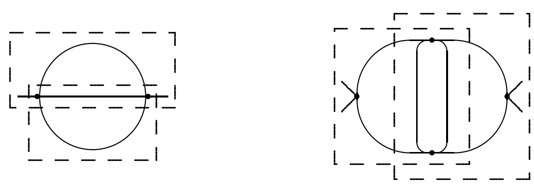


FIGURE 9.2 Two examples for overlapping divergences in ϕ^4-theory. The overlapping subdiagrams are enclosed by dashed boxes.

Historically, overlapping divergences were an obstacle to proving renormalizability of quantum electrodynamics. For the electron self-energy, the problem was solved with the help of the so-called *Ward identity* [9], which expresses the electron self-energy in terms of the vertex function, thereby eliminating all overlapping divergences. The convergence theorem is fundamental to renormalization theory since it enables us to replace all subdivergences by finite subtracted expressions.

A special outcome of the convergence theorem is that, after the subtraction of the divergences, a subdiagram γ behaves as a function of its external momentum like $\gamma(\lambda \mathbf{p}) = \lambda^{\omega(\gamma)} \log^k \lambda$ for $\lambda \to \infty$ with any k. Therefore, after the replacement of the superficially divergent subdiagrams by the corresponding finite subtracted expressions, power counting tells us that any superficially convergent diagram becomes finite. As an example, consider the diagram in Fig. 9.1. If the superficial divergences are subtracted from the subdiagram, it will depend on its external momentum like an ordinary vertex or like an ordinary propagator.

For ϕ^4-theories, the number of the internal lines I in a Feynman diagram may be expressed in terms of the number of vertices p and the number of external lines n as:

$$I \;=\; 2p \;-\; \frac{n}{2}. \tag{9.5}$$

The superficial degree of divergence of the associated integral becomes therefore

$$\omega(G) \;=\; D \;+ n\,(1 - D/2) \;+\; p(D-4). \tag{9.6}$$

In four dimensions, this simplifies to

$$\omega(G) \;=\; 4 \;-\; n, \tag{9.7}$$

implying that, in four dimensions, only two- and four-point 1PI diagrams are superficially divergent. Thus the only possible divergent subintegrations are those of two- and four-point subdiagrams. If the integrals of the two- and four-point functions are made finite by some mathematical procedure, any n-point function will be finite, as we know from the convergence theorem. A theory with these properties is said to be *renormalizable*. Hence the ϕ^4-theory is renormalizable in four dimensions.

In three and two dimensions, formula (9.6) yields $\omega(G) = 3 - n/2 - p$ and $\omega(G) = 2 - 2p$, respectively, implying that only a few low-order diagrams possess divergences. A theory with this property is said to be *superrenormalizable*.

In more than four dimensions, the last term $p(D-4)$ in formula (9.6) is positive, implying that new divergences appear at each higher order in perturbation theory. This property makes the theory *nonrenormalizable*. For a theory with an arbitrary power r of the field in the interaction ϕ^r, there exists the *upper critical dimension*

$$D_c = \frac{r}{r/2 - 1} \tag{9.8}$$

(recall the definition on page 20). For $D > D_c$, the superficial degree of divergence $\omega(G)$ becomes independent of the number p of vertices. In the case of ϕ^4-theory, the upper critical dimension is $D_c = 4$.

The smaller number of the UV-divergences in a superrenormalizable theory for $D < D_c$ makes the Feynman integrals more divergent in the infrared. This is seen as follows. If we let masses and momenta in a Feynman integral go to zero by rescaling them by a factor λ and letting $\lambda \to 0$, we observe a power behavior of the final integrand like $\lambda^{\omega(G)}$. The leading IR-behavior for zero mass is therefore determined by the same power of λ which governs the UV-behavior. The important difference between the two divergences is that the worst UV-divergence is given by the largest $\omega(G) > 0$, whereas the worst IR-divergence is associated with the smallest $\omega(G) < 0$. A nonrenormalizable ϕ^4-theory in $D > 4$ dimensions has, according to Eq. (9.6), the smallest $\omega(G)$ for $p = 0$, which is the case of a free theory, where the critical behavior is mean-field like.

For a superrenormalizable theory, the IR-divergences become worse for increasing order in perturbation theory. In the perturbation expansions, these divergences pile up to give rise to powers of masses in the critical regime. This way of deriving critical exponents will be discussed in detail in Chapters 20 and **??**.

Let us end this Section by noting that some authors no longer consider nonrenormalizability as a serious defect of fundamental quantum field theories (even though they did in their own earlier work). It is also possible to give simple calculation rules for extracting experimentally observable properties from such theories [10].

9.2 Normalization Conditions

In a renormalizable theory, Feynman integrals have superficial divergences, in any order of perturbation theory. On account of the power counting theorem, however, these divergences are limited. All superficial divergences are contained in the first coefficients of a Taylor series expansion of self-energy and vertex function around some chosen *normalization point* in the external momentum space. A differentiation with respect to the external momenta lowers the degree of divergence so that the higher coefficients do not contain superficial divergences. After removing all subdivergences of a diagram G, its superficial divergence is contained in the first coefficients of the expansion.

Discussing now the divergences of the theory, we must now clearly distinguish whether we are dealing with *bare or renormalized quantities*. In the first method of renormalizing a field theory, we perform the perturbation starting from the bare energy functional (8.1) with the bare interaction (8.2). On the basis of power counting, we may verify that the superficial divergences are contained in $\Sigma_B(\mathbf{0})$, $\Sigma'_B(\mathbf{0})$, and $\bar{\Gamma}^{(4)}_B(\mathbf{0})$. Here and in what follows we denote the derivative with respect to $\mathbf{k}^2$ by a prime:

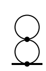
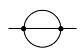
$$\Sigma'_B(\mathbf{k}) \equiv \frac{\partial}{\partial \mathbf{k}^2}\Sigma_B(\mathbf{k}). \tag{9.9}$$

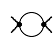
The Feynman integrals in $\Sigma_B(\mathbf{0})$ are quadratically divergent; those in $\Sigma'_B(\mathbf{0})$ and $\bar{\Gamma}^{(4)}_B(\mathbf{0})$ are logarithmically divergent. We separate the divergent parts from $\Sigma_B(\mathbf{k})$ and $\bar{\Gamma}^{(4)}_B(\mathbf{k}_i)$ by expanding these quantities in a power series. The superficially convergent parts are contained in the remainder, labeled by a subscript sc:

$$\Sigma_B(\mathbf{k}) = \Sigma_B(\mathbf{0}) + \Sigma'_B(\mathbf{0})\mathbf{k}^2 + \Sigma_{B\mathrm{sc}}(\mathbf{k}), \tag{9.10}$$

$$\bar{\Gamma}^{(4)}_B(\mathbf{k}_i) = \bar{\Gamma}^{(4)}_B(\mathbf{0}) + \bar{\Gamma}^{(4)}_{B\mathrm{sc}}(\mathbf{k}_i). \tag{9.11}$$

The diagrammatic expansion of these quantities up to two loops was given in Eqs. (8.53) and (8.54) [recalling that $\Sigma_B(\mathbf{k}) = \mathbf{k}^2 + m_B^2 - \bar{\Gamma}(\mathbf{k})$]. The analytic expressions associated with the individual diagrams will be denoted as follows:

$\Sigma(\mathbf{k}) = \frac{1}{2}$  $+ \frac{1}{4}$  $+ \frac{1}{6}$  $\equiv -\frac{g_B}{2}Q_1(m_B) + \frac{g_B^2}{4}Q_1(m_B)Q_2(m_B) + \frac{g_B^2}{6}Q_3(\mathbf{k}, m_B),$

$\bar{\Gamma}^{(4)}_B(\mathbf{k}_i) = -$  $- \frac{3}{2}$  $\equiv g_B - \frac{g_B^2}{2}[L_1(\mathbf{k}_1+\mathbf{k}_2, m_B) + 2\text{ perm}],$ (9.12)

where the notation "2 perm" indicates that the function $L_1(\mathbf{k}_1+\mathbf{k}_2, m_B)$ contributes also with the permuted arguments $\mathbf{k}_1+\mathbf{k}_3$ and $\mathbf{k}_1+\mathbf{k}_4$. The quadratically divergent integrals in $\Sigma_B(\mathbf{k})$ are called Q_i and the logarithmically divergent ones in $\bar{\Gamma}^{(4)}_B(\mathbf{k}_i)$ are called L_i. The following expressions collect all superficial divergences:

$$\Sigma_B(\mathbf{0}) = -\frac{g_B}{2}Q_1(m_B) + \frac{g_B^2}{4}Q_1(m_B)Q_2(m_B) + \frac{g_B^2}{6}Q_3(\mathbf{0}, m_B) + \mathcal{O}(g_B^3),$$

$$\Sigma'_B(\mathbf{0}) = \frac{g_B^2}{6}Q'_3(\mathbf{0}, m_B) + \mathcal{O}(g_B^3),$$

$$\bar{\Gamma}^{(4)}_B(\mathbf{0}) = g_B - g_B^2\frac{3}{2}L_1(\mathbf{0}, m_B) + \mathcal{O}(g_B^3).$$

At this point, the associated Feynman integrals may be made finite either by introducing a momentum cutoff, or by dimensional regularization. The remainders $\Sigma_{B\mathrm{sc}}(\mathbf{k})$ and $\bar{\Gamma}^{(4)}_{B\mathrm{sc}}(\mathbf{k}_i)$ are only superficially convergent and not, in general, finite for $\Lambda \to \infty$ or $\varepsilon \to 0$, due to the possible presence of subdivergences. These appear in second- and higher-order diagrams. We shall see immediately how these are removed, leading ultimately to finite physical results. The number of superficially divergent terms in the expansions (9.10) and (9.11) corresponds to the number of parameters of the theory. Together with the simple momentum dependence of the divergent terms, this fact makes it possible to absorb them into coupling constant, mass, and field normalization.

Let us explain this procedure in detail for the proper two-point vertex function $\bar{\Gamma}_B^{(2)}(\mathbf{k})$, which reads, according to Eq. (9.10),

$$\begin{aligned}\bar{\Gamma}_B^{(2)}(\mathbf{k}) &= \mathbf{k}^2 + m_B^2 - \Sigma_B(\mathbf{0}) - \Sigma'_B(\mathbf{0})\mathbf{k}^2 - \Sigma_{B\mathrm{sc}}(\mathbf{k}) \\ &= \mathbf{k}^2[1 - \Sigma'_B(\mathbf{0})] + [m_B^2 - \Sigma_B(\mathbf{0})] - \Sigma_{B\mathrm{sc}}(\mathbf{k}) \\ &= [1 - \Sigma'_B(\mathbf{0})]\left\{\mathbf{k}^2 + m_B^2 \frac{1 - \Sigma_B(\mathbf{0})/m_B^2}{1 - \Sigma'_B(\mathbf{0})} - \frac{\Sigma_{B\mathrm{sc}}(\mathbf{k})}{1 - \Sigma'_B(\mathbf{0})}\right\}. \qquad (9.13)\end{aligned}$$

We may now introduce the renormalized field and mass:

$$\phi \equiv [1 - \Sigma'_B(\mathbf{0})]^{1/2}\,\phi_B, \qquad (9.14)$$

$$m^2 \equiv \frac{1 - \Sigma_B(\mathbf{0})/m_B^2}{1 - \Sigma'_B(\mathbf{0})}\, m_B^2, \qquad (9.15)$$

and a finite renormalized self-energy

$$\Sigma(\mathbf{k}) \equiv \frac{\Sigma_{B\mathrm{sc}}(\mathbf{k})}{1 - \Sigma'_B(\mathbf{0})}, \qquad (9.16)$$

which starts out like $\mathcal{O}(\mathbf{k}^4)$ for small $\mathbf{k}$. Then the quantity in curly brackets in Eq. (9.13) constitutes a finite renormalized proper two-point vertex function

$$\bar{\Gamma}^{(2)}(\mathbf{k}) = \mathbf{k}^2 + m^2 - \Sigma(\mathbf{k}). \qquad (9.17)$$

The relation between bare and renormalized quantities is

$$\bar{\Gamma}_B^{(2)}(\mathbf{k}) = [1 - \Sigma'_B(\mathbf{0})]\bar{\Gamma}^{(2)}(\mathbf{k}). \qquad (9.18)$$

As announced at the beginning of this chapter, and earlier in the discussion of scale invariance in Eqs. (7.56)–(7.58), the proper two-point vertex functions can be made finite via a multiplicative renormalization employing three renormalization constants Z_ϕ, Z_{m^2}, and Z_g, to be called *field* or *wave function*, *mass*, and *coupling renormalization constant*, respectively:

$$\bar{\Gamma}_B^{(2)}(\mathbf{k}) = Z_\phi^{-1}\bar{\Gamma}^{(2)}(\mathbf{k}), \qquad (9.19)$$

$$m_B^2 = Z_{m^2} Z_\phi^{-1} m^2, \qquad (9.20)$$

$$g_B = Z_g Z_\phi^{-2} g. \qquad (9.21)$$

Comparing the first two equations with (9.14) and (9.15) yields

$$Z_\phi = [1 - \Sigma'_B(\mathbf{0})]^{-1}, \qquad Z_{m^2} = [1 - \Sigma_B(\mathbf{0})/m_B^2]^{-1}. \qquad (9.22)$$

The renormalized quantity $\bar{\Gamma}^{(2)}(\mathbf{k})$ satisfies the equations

$$\bar{\Gamma}^{(2)}(\mathbf{0}) = m^2, \qquad (9.23)$$

$$\frac{\partial}{\partial \mathbf{k}^2}\bar{\Gamma}^{(2)}(\mathbf{0}) = 1. \qquad (9.24)$$

The inverse of $\bar{\Gamma}^{(2)}(\mathbf{k})$ is the renormalized propagator of the renormalized field ϕ:

$$G(\mathbf{k}) = \frac{1}{\bar{\Gamma}^{(2)}(\mathbf{k})} = \frac{1}{\mathbf{k}^2 + m^2 - \Sigma(\mathbf{k})}, \qquad (9.25)$$

which is equal to the Fourier transformation of the correlation function of the renormalized fields defined in (9.14):

$$(2\pi)^D \, \delta^{(D)}(\mathbf{k}+\mathbf{k}')G(\mathbf{k}) = \int d^D x \, d^D x' \; e^{-i\,\mathbf{k}'\cdot\mathbf{x}'-i\,\mathbf{k}\cdot\mathbf{x}} \, \langle\phi(\mathbf{x})\phi(\mathbf{x}')\rangle. \tag{9.26}$$

The renormalized field is related to the bare field via the *wave function renormalization constant* Z_ϕ in Eq. (9.22):

$$\phi = Z_\phi^{-1/2}\phi_B. \tag{9.27}$$

As we shall see, the same field renormalization makes all n-point functions finite. In the present notation, the Green functions introduced in Eq. (2.10), but expressed in terms of the bare fields corresponding to the energy functionals (8.1) and (8.2), are the *bare Green functions*

$$G_B^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n) \;\; \equiv \;\; \langle\phi_B(\mathbf{x}_1)\cdots\phi_B(\mathbf{x}_n)\rangle. \tag{9.28}$$

In quantum field theories formulated in a continuous spacetime [11], physical observations are described with the help of renormalized Green functions, which are the correlation functions of the renormalized fields $\phi(\mathbf{x})$ in Eq. (9.71):

$$G^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n) \equiv \langle\phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_n)\rangle. \tag{9.29}$$

The two are related by a multiplicative renormalization:

$$G_B^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n) = Z_\phi^{n/2}G^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n). \tag{9.30}$$

The same relation holds between unrenormalized and renormalized Green functions in momentum space.

Remembering the relation (4.21) between correlation functions and proper vertex functions, and its generalization to an arbitrary number n, all proper vertex functions can be made finite by the multiplicative factors inverse to (9.30):

$$\bar\Gamma_B^{(n)}(\mathbf{k}_i) = Z_\phi^{-n/2}\bar\Gamma^{(n)}(\mathbf{k}_i). \tag{9.31}$$

This renormalization will be performed in detail for the four-point vertex function below, where we shall obtain a finite renormalized quantity $\bar\Gamma_B^{(4)}(\mathbf{k}_i)$ from the equation

$$\bar\Gamma_B^{(4)}(\mathbf{k}_i) = Z_\phi^{-2}\bar\Gamma^{(4)}(\mathbf{k}_i). \tag{9.32}$$

The finite value of $\bar\Gamma^{(4)}(\mathbf{k}_i)$ at $\mathbf{k}_i = 0$ will be defined as the renormalized coupling constant g:

$$\bar\Gamma^{(4)}(\mathbf{0}) = g. \tag{9.33}$$

The relation between the coupling constants g_B and g can be written precisely in the form (9.21), from which we identify the renormalization constant Z_g. Equations (9.23), (9.24), and (9.33) are called *normalization conditions.* More explicitly, we may write the relation (9.31) as follows:

$$\bar\Gamma_B^{(n)}(\mathbf{k}, m_B, g_B, \Lambda) = Z_\phi^{-n/2}\bar\Gamma^{(n)}(\mathbf{k}, m, g), \tag{9.34}$$

where we have added mass, coupling constant, and cutoff to the list of arguments of the proper vertex functions. The powers of the renormalization constant Z_ϕ determined by the renormalization of the two-point function absorbs all divergences that remain after having renormalized mass and coupling constant.

The presence of subdivergences complicates the renormalization procedure. Fortunately, the remaining divergences contained in the superficially convergent remainders $\Sigma_{B\text{sc}}(\mathbf{k})$ and $\bar{\Gamma}^{(4)}_{B\text{sc}}(\mathbf{k}_i)$ of Eqs. (9.10) and (9.11) are all removed when re-expressing the bare mass m_B and coupling constant g_B in terms of the renormalized quantities m and g. The renormalization constants of these quantities remove precisely all subdivergences. The normalization conditions (9.23), (9.24), and (9.33) may be used successively in each order of perturbation theory to calculate the renormalized quantities $\bar{\Gamma}^{(2)}(\mathbf{k})$ and $\bar{\Gamma}^{(4)}(\mathbf{k}_i)$, and from these the renormalized mass m and coupling constant g. At each order, the original variables in the energy functional can be re-expressed in terms of the renormalized ones. In this process, we eliminate order by order all subdivergences in $\Sigma_B(\mathbf{k})$, which therefore become finite for $\Lambda \to \infty$ or $\varepsilon \to 0$. We shall now calculate these finite expressions for one- and two-loop diagrams.

9.2.1 One-Loop Mass Renormalization

The one-loop approximation to $\bar{\Gamma}^{(2)}_B(\mathbf{k})$ is

$$\bar{\Gamma}^{(2)}_B(\mathbf{k}) = \mathbf{k}^2 + m_B^2 + \frac{g_B}{2}\, Q_1(m_B). \tag{9.35}$$

With condition (9.23), we find the (preliminary) renormalized mass

$$m_1^2 = m_B^2 + \frac{g_B}{2}\, Q_1(m_B). \tag{9.36}$$

Inverting this equation gives $m_B(m_1)$:

$$m_B^2 = m_1^2 - \frac{g_B}{2}\, Q_1(m_1), \tag{9.37}$$

where the first-order result $m_1^2 = m_B^2$ is inserted into the argument of Q_1, since the error committed in this way is of the order g_B^2. Reexpressing $\bar{\Gamma}^{(2)}_B$ in terms of the renormalized mass yields a finite expression up to the first order in g_B:

$$\bar{\Gamma}^{(2)}_B(\mathbf{k}) = \mathbf{k}^2 + m_1^2 + \mathcal{O}(g_B^2). \tag{9.38}$$

The bare mass m_B is now a function of g_B, m_1, and the cutoff Λ; if ε-regularization is used, it contains a pole term $1/\varepsilon$.

9.2.2 One-Loop Coupling Constant Renormalization

Consider now the one-loop expression for the proper four-point vertex function in Eq. (9.12). In $L_1(\mathbf{k}_1 + \mathbf{k}_2, m_B)$, the mass m_B can again be replaced by m_1 without creating additional terms of order g_B^2. So far there are no subdivergences, and the only divergence occurs in $L_1(\mathbf{0}, m_B)$. A regular expression for $\bar{\Gamma}^{(4)}(\mathbf{k}_i)$ is therefore found by defining

$$g_1 = g_B - \frac{3}{2} g_B^2 L_1(\mathbf{0}, m_1), \tag{9.39}$$

which is inverted to

$$g_B = g_1 + \frac{3}{2} g_1^2 L_1(\mathbf{0}, m_1). \tag{9.40}$$

The error committed in substituting g_B by g_1 in front of L_1 is of the order g_B^3. Expressing the four-point function in terms of g_1, we obtain

$$\bar{\Gamma}^{(4)}_B(\mathbf{k}_i) = g_1 - \frac{g_1^2}{2}[L_1(\mathbf{k}_1 + \mathbf{k}_2, m_1) + 2 \text{ perm} - 3\, L_1(\mathbf{0}, m_1)] + \mathcal{O}(g_1^3). \tag{9.41}$$

The subtracted expressions $L_1(\mathbf{k}, m_1) - L_1(\mathbf{0}, m_1)$ are finite for $\Lambda \to \infty$ or $\varepsilon \to 0$. This is seen explicitly by carrying out the subtraction in the integrand for each of the momenta permutations:

$$\frac{1}{(\mathbf{p}^2+m_1^2)[(\mathbf{p}-\mathbf{k})^2+m_1^2]} - \frac{1}{(\mathbf{p}^2+m_1^2)^2} = \frac{-\mathbf{k}^2+2\mathbf{pk}}{(\mathbf{p}^2+m_1^2)^2[(\mathbf{p}-\mathbf{k})^2+m_1^2]}, \tag{9.42}$$

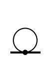
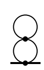
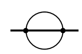
which lowers the degree of superficial divergence to a sum of Feynman integrals with $\omega = -1$ and $\omega = -2$.

Assuming g_B to be chosen such that g_1 is finite, the vertex function (9.41) is finite and can be identified directly with the renormalized quantity $\bar{\Gamma}^{(4)}(\mathbf{k}_i)$. The normalization condition (9.33) implies that the quantity g_1 constitutes the renormalized, finite coupling constant g. Thus, at the one-loop level, all divergences have been removed by a redefinition of the coupling constant and the mass.

At the two-loop level, new divergences will appear, and in particular subdivergences. The latter will, however, automatically disappear by the one-loop renormalization of the mass, whereas the superficial two-loop divergences will change further the renormalization constants of mass and coupling constant, and produce a first contribution to the renormalization constant of the field, which is of the order g_B^2.

9.2.3 Two-Loop Mass and Field Renormalization

The diagrammatical two-loop expansion is given by

$$\bar{\Gamma}_B^{(2)}(\mathbf{k}) = (\,—\,)^{-1} - \frac{1}{2}$$  $$- \frac{1}{4}$$  $$- \frac{1}{6}$$ 

$$= \mathbf{k}^2 + m_B^2 + \frac{g_B}{2} Q_1(m_B) - \frac{g_B^2}{4} Q_1(m_B) Q_2(m_B) - \frac{g_B^2}{6} Q_3(\mathbf{k}, m_B), \tag{9.43}$$

and the second-order renormalized mass m_2 is defined by the normalization condition (9.23), implying that the bare vertex function $\bar{\Gamma}_B^{(2)}(\mathbf{0})$ satisfies

$$\bar{\Gamma}_B^{(2)}(\mathbf{0}) = Z_\phi^{-1} m_2^2, \tag{9.44}$$

where Z_ϕ is yet to be determined.

Let us express m_B as a function of m_2 and g_1. For this we insert into (9.44) the renormalized mass m_1 of Eq. (9.37), and the renormalized coupling constant g_1 of Eq. (9.40). The term of order g_B then gives rise to additional terms of order g_1^2 which contribute at the two-loop level. Specifically, the one-loop integral is re-expanded as

$$\begin{aligned}
\frac{1}{2} g_B\, Q_1(m_B) &= \frac{1}{2} g_B \int \frac{d^D p}{(2\pi)^D} \frac{1}{\mathbf{p}^2 + m_B^2} \\
&= \frac{1}{2} \left[g_1 + \frac{3}{2} g_1^2\, L_1(\mathbf{0}, m_1) \right] \int \frac{d^D p}{(2\pi)^D} \frac{1}{\mathbf{p}^2 + m_1^2 - \frac{1}{2} g_1\, Q_1(m_1)} \\
&= \frac{1}{2} g_1\, Q_1(m_1) + \frac{1}{4} g_1^2\, Q_1(m_1) Q_2(m_1) + \frac{3}{4} g_1^2\, Q_1(m_1) L_1(\mathbf{0}, m_1) + \mathcal{O}(g_1^3).
\end{aligned} \tag{9.45}$$

Using the fact that $Q_2(m_1) = L_1(\mathbf{0}, m_1)$, we obtain

$$\frac{1}{2} g_B\, Q_1(m_B) = \frac{1}{2} g_1\, Q_1(m_1) + g_1^2\, Q_1(m_1) Q_2(m_1) + \mathcal{O}(g_1^3). \tag{9.46}$$

The term of order g_1^2 is a manifestation of the subtraction of the subdivergences by the previous first-order renormalizations of mass and coupling constant. Substituting (9.46) into (9.43), and replacing, in the terms quadratic in g_B, the coupling constant g_B and the mass m_B by g_1 and m_1, respectively, we find from (9.44) the equation for m_B^2 as a function of m_1^2:

$$m_B^2 = Z_\phi^{-1} m_2^2 - \frac{g_1}{2} Q_1(m_1) - \frac{3}{4} g_1^2\, Q_1(m_1) Q_2(m_1) + \frac{g_1^2}{6} Q_3(\mathbf{0}, m_1) + \mathcal{O}(g_1^3). \tag{9.47}$$

Thus we can rewrite (9.43) as

$$\bar{\Gamma}_B^{(2)}(\mathbf{k}) = \mathbf{k}^2 + Z_\phi^{-1} m_2^2 - \frac{g_2^2}{6} \left[Q_3(\mathbf{k}, m_2) - Q_3(\mathbf{0}, m_2)\right] + \mathcal{O}(g_2^3), \tag{9.48}$$

where the arguments m_1 and g_1 on the right-hand side have been replaced by m_2 and the second-order renormalized coupling constant g_2 without committing an error of order g_2^2. The first-order terms have disappeared.

The two-point function in Eq. (9.48) is not yet finite in the limit of an infinite cutoff $\Lambda \to \infty$, or $\varepsilon \to 0$, since the subtracted $Q_3(\mathbf{k}, m_2)$ has an integrand

$$\frac{-\mathbf{k}^2 + 2\mathbf{k}(\mathbf{p}+\mathbf{q})}{[(\mathbf{k}-\mathbf{p}-\mathbf{q})^2 + m_2^2](\mathbf{p}^2 + m_2^2)(\mathbf{q}^2 + m_2^2)[(\mathbf{p}+\mathbf{q})^2 + m_2^2]}, \tag{9.49}$$

leading to divergent integrals over $\mathbf{p}$ and $\mathbf{q}$. Thus we expand $Q_3(\mathbf{k}, m_2)$ into the sum of a quadratically and a logarithmically divergent integral, plus a finite subtracted part

$$Q_3(\mathbf{k}, m_2) = Q_3(0, m_2) + \mathbf{k}^2 Q_3'(\mathbf{0}, m_2) + Q_{3\,\mathrm{sc}}(\mathbf{k}, m_2)], \tag{9.50}$$

and rewrite (9.48) as

$$\bar{\Gamma}_B^{(2)}(\mathbf{k}) = \left[1 - \frac{g_2^2}{6} Q_3'(\mathbf{0}, m_2))\right] \mathbf{k}^2 + Z_\phi^{-1} m_2^2 - \frac{g_2^2}{6} Q_{\mathrm{sc}}(\mathbf{k}, m_2) + \mathcal{O}(g_2^3). \tag{9.51}$$

Here is the place where the field renormalization comes in. According to the normalization condition (9.24), the renormalized $\bar{\Gamma}^{(2)}(\mathbf{0})$ is supposed to have a unit coefficient of the $\mathbf{k}^2$ term, such that Eq. (9.31) gives for the unrenormalized expression the condition

$$\bar{\Gamma}_B^{(2)\prime}(\mathbf{k})\Big|_0 = Z_\phi^{-1}. \tag{9.52}$$

Thus we identify the wave function renormalization constant as being

$$Z_\phi^{-1} = \bar{\Gamma}_B^{(2)\prime}(\mathbf{0}) = 1 - \frac{g_2^2}{6}\, Q_3'(\mathbf{0}, m_2). \tag{9.53}$$

Inserting this into Eq. (9.47), and using Eq. (9.37) to replace m_B by m_1, we find that the renormalized mass m_2 of order g_2^2 differs from m_1 only by terms of order g_2^2:

$$m_2^2 = m_1^2 + \frac{3}{4} g_2^2\, Q_1(m_1) Q_2(m_1) - \frac{g_2^2}{6} Q_3(\mathbf{0}, m_1) + \frac{g_2^2}{6} m_1^2 Q_3'(\mathbf{0}, m_1) + \mathcal{O}(g_2^3). \tag{9.54}$$

We can therefore rewrite (9.47) directly as a relation between m_B^2 and m_2^2 as follows:

$$m_B^2 = m_2^2 - \frac{g_2}{2} Q_1(m_2) - \frac{3}{4} g_2^2\, Q_1(m_2) Q_2(m_2) + \frac{g_2^2}{6} Q_3(\mathbf{0}, m_2) - \frac{g_2^2}{6} m_2^2 Q_3'(\mathbf{0}, m_2) + \mathcal{O}(g_2^3). \tag{9.55}$$

Multiplying $\bar{\Gamma}^{(2)}_B(\mathbf{k})$ in Eq. (9.51) by Z_ϕ, we find the renormalized two-point vertex function

$$\bar{\Gamma}^{(2)}(\mathbf{k},m,g) = \mathbf{k}^2 + m_2^2 - \frac{g^2}{6}Q_{3\,\mathrm{sc}}(\mathbf{k},m) + \mathcal{O}(g^3), \tag{9.56}$$

where we have omitted the subscripts of the renormalized mass and coupling constant to the second-order m and g. The right-hand side has the properly normalized small-$\mathbf{k}$ expansion with unit coefficients of m^2 and $\mathbf{k}^2$:

$$\bar{\Gamma}^{(2)}(\mathbf{k},m,g) = \mathbf{k}^2 + m^2 + \mathcal{O}(\mathbf{k}^4). \tag{9.57}$$

This follows from $Q_{3\,\mathrm{sc}}(\mathbf{k},m) = \mathcal{O}(\mathbf{k}^4)$. Note that Z_ϕ also renormalizes the coupling constant, but not at the two-loop level.

At this place we must add a few remarks concerning the field-theoretic study of critical phenomena directly at the critical temperature T_c, i.e. for $m^2 = 0$. Then the normalization conditions (9.23)–(9.33) possess, in principle, additional IR-divergences. In $D = 4$ dimensions, however, these IR-divergences happen to be absent in logarithmically divergent integrals if the external momenta are *nonexceptional*. Nonexceptional means that none of the partial sums of external momenta vanishes. This will be explained in more detail in Sections 12.1 and 12.3. For quadratically divergent integrals, superficial IR-divergences are absent altogether in $D = 4$ dimensions. Thus at zero mass, the normalization conditions (9.23)–(9.33) have to be modified by employing nonzero external momenta in the two last equations, requiring instead

$$\bar{\Gamma}^{(2)}(\mathbf{0},0,g) = 0, \tag{9.58}$$

$$\left.\frac{\partial}{\partial \mathbf{k}^2}\bar{\Gamma}^{(2)}(\mathbf{k},0,g)\right|_{\mathbf{k}^2=\kappa^2} = 1, \tag{9.59}$$

$$\left.\bar{\Gamma}^{(4)}(\mathbf{k}_i,0,g)\right|_{\mathrm{SP}} = g. \tag{9.60}$$

The subscript SP denotes the symmetric point, where $\mathbf{k}_i \cdot \mathbf{k}_j = (4\delta_{ij} - 1)\kappa^2/4$. The momenta defined by this condition are always *nonexceptional*. In a more mathematical notation, the nonexceptionality means that $\sum_{i\in I}\mathbf{k}_i \neq 0$ for any subset I of the set of indices $\{1,\dots,n\}$ of the external momenta $\mathbf{k}_i$, $(i = 1,\dots,n)$.

The above renormalization procedure can be continued to any order in perturbation theory. Obviously, it will be quite difficult to keep track of all involved terms with the repeated re-expansions in terms of the lower-order renormalized quantities. Fortunately, work can be organized more efficiently by using the method of counterterms which will be explained in the next section, where it is carried out explicitly up to second order in the coupling strength. The procedure will be simplified even further by abandoning the normalization conditions and using the so-called minimal subtraction scheme to derive finite expressions for divergent ones.

9.3 Method of Counterterms and Minimal Subtraction

Depending on the regularization scheme, the Feynman integrals are seen to diverge with Λ^2 or $\log\Lambda$ for $\Lambda \to \infty$, or to have poles in ε for $\varepsilon \to 0$. These divergences can also be removed by working with renormalized fields, mass, and coupling constant right from the beginning. The renormalized quantities can be viewed as functions of the bare quantities and of Λ, or of ε. In this test we shall mainly work with dimensional regularization. The details will be explained in Subsection 9.3.2. An analogous treatment exists of course for a cutoff regularization, but this will not be considered here.

The renormalized theory is defined with the help of a renormalized energy functional

$$E[\phi] \;=\; E_0[\phi] + E_{\rm int}[\phi], \tag{9.61}$$

with a free part

$$E_0[\phi] \;=\; \int d^Dx \left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2\right] \tag{9.62}$$

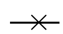
and an interaction part which is extended by additional quadratic terms, the so-called *counterterms*, to

$$E_{\rm int}[\phi] \;=\; \int d^Dx \left[\frac{\mu^\varepsilon g}{4!}\phi^4 + c_\phi\frac{1}{2}(\partial\phi)^2 + c_{m^2}\,\frac{1}{2}m^2\phi^2 + c_g\frac{\mu^\varepsilon g}{4!}\,\phi^4\right]. \tag{9.63}$$

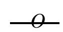
The additional terms are of the same type as the original ones, such that we can write

$$E[\phi] \;=\; \int d^Dx \left[(1+c_\phi)\frac{1}{2}(\partial\phi)^2 + (1+c_{m^2})\,\frac{1}{2}m^2\phi^2 + (1+c_g)\,\frac{\mu^\varepsilon g}{4!}\,\phi^4\right]. \tag{9.64}$$

The counterterms c_ϕ, c_{m^2}, and c_g produce additional vertices in the diagrammatic expansion. In momentum space, these have the form



$$= \;(-c_{m^2})\;m^2\,, \tag{9.65}$$



$$= \;(-c_\phi)\;\mathbf{k}^2\,, \tag{9.66}$$

$$= \;(-c_g)\;g\;\mu^\varepsilon\,. \tag{9.67}$$

The definition of the original vertex now includes the mass parameter μ, already introduced in Section 8.3 to make g dimensionless:

$$= \;(-g)\;\mu^\varepsilon\,. \tag{9.68}$$

The counterterms c_ϕ, c_{m^2}, and c_g are chosen in such a way that all divergent terms are subtracted and the Green functions are finite for $\varepsilon \to 0$, order by order in perturbation theory.

Now, dimensional analysis of Eq. (9.63) shows that the counterterms are dimensionless. In dimensional regularization, they can therefore only depend on the dimensionless coupling constant g or on dimensionless combinations like m^2/μ^2 or $\mathbf{k}^2/\mu^2$. It turns out that the combination $\mathbf{k}^2/\mu^2$ appears only at intermediate steps as $\log \mathbf{k}^2/\mu^2$. It is crucial for the renormalization program that the nonlocal terms all cancel in the final expressions for the counterterms. This implies that the counterterms c_ϕ, c_{m^2}, and c_g depend only on g, ε, and m^2/μ^2. In the minimal subtraction scheme introduced in the next subsection, the dependence on m^2/μ^2 also disappears. Then, the only dimensional dependence of the counterterm diagrams consists in the factors $\mathbf{k}^2$, m^2 and μ^ε in Eqs. (9.65)–(9.67). The cancellation of the logarithms will be observed explicitly in the calculation up to two loops on page 146.

The calculation of the weight factors proceeds as before, but when counting the identical vertex permutations, the different nature of the vertices has to be taken into account. The vertices with two legs require an extension of the previous rules. They carry a factor 1/2! by analogy with the factor 1/4! for vertices of degree 4.

The quantities ϕ, m, and g in Eq. (9.63) are the renormalized field, renormalized mass, and renormalized coupling constant. The original form of the theory is recovered by a *multiplicative renormalization*. We define the renormalization constants

$$Z_\phi \;\equiv\; 1 + c_\phi, \quad Z_{m^2} \;\equiv\; 1 + c_{m^2}, \quad Z_g \;\equiv\; 1 + c_g, \tag{9.69}$$

and the energy functional (9.64) becomes

$$E[\phi] \;=\; \int d^D x \left[\frac{1}{2} Z_\phi (\partial\phi)^2 + \frac{1}{2} m^2 Z_{m^2} \phi^2 + \frac{\mu^\varepsilon g}{4!} Z_g \phi^4\right], \tag{9.70}$$

with the ε-dependent coefficients. This energy functional still differs from the original one by the factor Z_ϕ in the gradient term. This may be removed by a renormalization of the field, defining the bare field

$$\phi_B \;\equiv\; Z_\phi^{1/2} \phi\,, \tag{9.71}$$

and bare mass and coupling constant

$$m_B^2 \;\equiv\; \frac{Z_{m^2}}{Z_\phi} m^2\,, \qquad g_B \;\equiv\; \frac{Z_g}{Z_\phi^2} \mu^\varepsilon g\,, \tag{9.72}$$

which brings (9.70) to the form:

$$E[\phi] = E[\phi_B] = \int d^D x \left[\frac{1}{2} (\partial\phi_B)^2 + \frac{1}{2} m_B^2 \phi_B^2 + \frac{g_B}{4!} \phi_B^4\right]. \tag{9.73}$$

This is precisely the initial energy functional in Eqs. (8.1) and (8.2) [or in Eq. (7.30), recalling that the quantities m, g, ϕ in that functional coincide with the presently used bare objects m_B, g_B, and ϕ_B; the subscript B was introduced afterwards in Section 7.4]. The bare quantities are functions of the renormalized quantities m, g, of the mass scale μ, and of $\varepsilon = 4 - D$.

9.3.1 Minimal Subtraction Scheme

The above normalization conditions (9.23), (9.24), and (9.33), [or (9.58)–(9.60)] can be used in connection with any regularization of the divergent integrals. However, if we decide to employ dimensional regularization, the most practical regularization is based on a simultaneous expansion of all Feynman integrals in powers of ε, followed by a subtraction of terms containing ε-poles. This procedure has an important advantage, especially for a study of the critical region in $D < 4$ dimensions. It can be used not only to remove the ultraviolet divergences of the four-dimensional theory, but also at finite small ε-values at the critical point, i.e. in less than four dimensions at zero mass. This is nontrivial, since for $D < 4$, massless diagrams develop IR-divergences, so that the zero-mass condition, Eq. (9.58), cannot be fulfilled, the left-hand side being infinite if the order of the perturbation expansion is sufficiently high. Dimensional analysis tells us that to nth order in the coupling constant g, the two-point vertex function diverges for small $\mathbf{k}$ like $\bar{\Gamma}^{(2)} \approx \mathbf{k}^2 \left|\mathbf{k}^{-n\varepsilon}\right| g^n$, so that the prefactor $\mathbf{k}^2$ will eventually turn into a negative power of $\mathbf{k}^2$. In dimensional regularization, this problem disappears since all quantities are expanded in powers of g and ε. Such an *ε-expansion* looks like [12]

$$\mathbf{k}^2 |\mathbf{k}|^{-n\varepsilon} = \mathbf{k}^2 \mu^{-n\varepsilon} \exp\left[-n\varepsilon \log(\mathbf{k}^2/\mu^2)\right] = \mathbf{k}^2 \mu^{-n\varepsilon} \left[1 - n\varepsilon \log(\mathbf{k}^2/\mu^2) + \dots\right], \tag{9.74}$$

and the condition (9.58) is fulfilled to all orders in g and ε. Fortunately, the $\log k$-terms are found to cancel in any renormalization scheme, which is necessary for renormalizability, since they could not be canceled by local counterterms in (9.63).

The normalization conditions (9.23)–(9.33) define counterterms which for $m^2 \neq 0$ depend on the mass. For the critical theory with zero mass, we may use the conditions (9.58)–(9.60). Then the counterterms will depend on the mass parameter κ^2 of the symmetry point.

An enormous simplification comes about by the existence of a regularization procedure in which the counterterms become *independent* of the mass m, except for a trivial overall factor m^2

in c_{m^2}. This is known as *minimal subtraction scheme* (MS). It was invented by G.'t Hooft [13] to renormalize nonabelian gauge theories of weak and electromagnetic interactions. In this scheme, the counterterms acquire the generic form (9.1), in which the coefficients of g^n consist of *pure* pole terms $1/\varepsilon^i$, with no finite parts for $\varepsilon \to 0$ [14]. The coefficients are c-numbers and do not contain the mass m or the mass parameter μ introduced in the process of dimensional regularization. In principle, these masses could have appeared in the form of a power series of the dimensionless ratios m^2/μ^2 or its logarithms. The absence of such logarithms is a highly nontrivial virtue of minimal subtraction.

This absence is the origin for another important property of the renormalization constants in the minimal subtraction scheme. They always have the same expansion in powers of the dimensionless coupling constant g, even if this were initially defined to carry an arbitrary analytic function $f(\varepsilon)$ with $f(0) = 1$ as a factor. In other words, if we were to redefine the coupling constant (8.58) to

$$g\mu^\varepsilon \longrightarrow g\mu^\varepsilon f(\varepsilon) = g\mu^\varepsilon \left(1 + f_1\varepsilon + f_2\varepsilon^2 + \dots\right), \tag{9.75}$$

we would find precisely the same expansions (9.1) for the renormalization constants Z_ϕ, Z_{m^2}, and Z_g in powers of the new g. The reason for this is rather obvious: we can always write $f(\varepsilon) = c^\varepsilon$ and absorb the factor c into the mass parameter μ. Since this mass parameter does not appear in the final expansions (9.1), the redefined mass parameter $\mu\, c$ cannot appear there as well. The mechanism for this cancellation will be illustrated once more at the end of this section, up to two loops.

Due to the invariance of the final expansions under a rescaling of $g \to gf(\varepsilon)$, there exists an infinite variety of subtraction schemes which may all be called minimal, depending on the choice of the function $f(\varepsilon)$. They all lead to the same counterterms and renormalization constants. In the strict version of MS, one expands *all* functions of ε in each regularized Feynman integral in powers of ε, e.g. the typical common factors $(4\pi)^{\varepsilon/2}$. This will be done in the next section. In a slight modification of this procedure, a common factor $f(\varepsilon) = (4\pi)^{\varepsilon/2}$ may be omitted from each power in g, since it can be thought of as having been absorbed into the irrelevant mass parameter μ. In the five-loop calculation to be presented later, a certain modification of MS will be used, which we shall call $\overline{MS}$*-scheme*, to be explained in more detail in Section 13.1.2 together with some other modified MS-schemes which have been used in the literature.

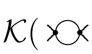
Formally, the MS-scheme is implemented with the help of an operator $\mathcal{K}$ defined to pick out the pure pole terms of the dimensionally regularized integral:

$$\mathcal{K} \sum_{n=-k}^{\infty} A_i\varepsilon^i = \sum_{n=-k}^{-1} A_i\varepsilon^i = \sum_{i=1}^{k} \frac{A_{-i}}{\varepsilon^i}\,. \tag{9.76}$$

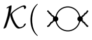
By definition, $\mathcal{K}$ is a projection operator since

$$\mathcal{K}^2 = \mathcal{K}\,. \tag{9.77}$$

Application of $\mathcal{K}$ to a diagram means application to the integral associated with the diagram. Take for example the divergent diagrams (8.62) and (8.68), whose pole terms are picked out as follows:

$\mathcal{K}($  $) = m^2 \left[\frac{g}{(4\pi)^2}\frac{2}{\varepsilon}\right], \qquad \mathcal{K}($  $) = \mu^\varepsilon g \left[\frac{g}{(4\pi)^2}\frac{2}{\varepsilon}\right].$ (9.78)

Both one-loop pole terms are local. They are proportional to m^2 for the quadratically divergent diagram, and to $\mu^\varepsilon g$ for the logarithmically divergent diagram. The latter is independent of the mass m. Note that the absence of the external momenta in the pole term on the right-hand side in (9.78) implies the important relation:

$\mathcal{K}($  $) = \mathcal{K}($  $),$ (9.79)

where the integral on the left-hand side emerges from the one on the right-hand side by setting the external momentum equal to zero. The pole term remains unchanged by this operation, as is obvious from Eq. (8.68). This kind of diagram will be important later on.

In the context of minimal subtraction, the absence of any nonlocal $\log \mathbf{k}$-terms in the counterterms has been shown to be an extension of the Dyson-Weinberg convergence theorem in Section 9.1. It can be shown, and we shall observe this explicitly below, that the pole part $\mathcal{K}G$ of any subdivergence-free diagram G is polynomial in its external momenta. This is not only the case in the ϕ^4-theories under study, but also in other renormalizable theories, where the residues of the ε-poles always contain the external momenta and masses as low-order polynomials [13, 14, 15, 16] The proof uses the following properties of the operator $\mathcal{K}$ and the differential operator with respect to the external momentum $\partial \equiv \partial/\partial k$ (omitting the component label):

1. A function vanishes after a finite number of momentum differentiations ∂ if and only if it is a polynomial in the momenta.

2. The $\mathcal{K}$-operation commutes with momentum differentiation, since the two operations act on different spaces.

3. The superficial degree of divergence of a diagram is reduced by one unit for each momentum derivative: $\omega(\partial^s G) = \omega(G) - s$. This is obvious on dimensional grounds, for example:
$$\frac{\partial}{\partial k_\mu} \frac{1}{(\mathbf{k}+\mathbf{p})^2 + m^2} = \frac{-2(k+p)_\mu}{[(\mathbf{k}+\mathbf{p})^2 + m^2]^2}.$$

The argument goes as follows: according to property 3, a subdivergence-free diagram G has $\omega(\partial^{\omega(G)+1} G) < 0$, where $\partial^{\omega(G)+1} G$ is also subdivergence-free [17]. Then the convergence theorem can be invoked stating that $\partial^{\omega(G)+1} G$ is absolutely convergent so that $\mathcal{K}(\partial^{\omega(G)+1} G) = 0$. Using property 2, we deduce that $\partial^{\omega(G)+1} \mathcal{K}G = 0$. Then property 1 implies that $\mathcal{K}G$ is a polynomial of degree lower or equal to $\omega(G)$ in the external momenta.

An analogous statement holds for derivatives with respect to the mass, $\partial_{m^2} G = \partial G/\partial m^2$. As mass and external momenta are the only dimensional parameters of the theory, the pole terms of the integrals are homogeneous polynomials of order $\omega(G)$ in these parameters. Therefore, $\mathcal{K}G$ is proportional to $\mathbf{k}^2$ or m^2 for quadratically divergent diagrams and independent of these dimensional parameters for logarithmically divergent diagrams.

This implies that all $\log(m^2/\mu^2)$-terms arising in the process of subtraction of the pole terms in ε can only survive in the finite parts, but not in the pole terms as required for renormalizability. This will be seen in the next section, where the counterterms for the ϕ^4-theory up to two loops are calculated explicitly. Furthermore , the result ensures that the counterterms can be chosen to be independent of the mass m^2, implying that the limit $m^2 \to 0$ does not produce IR-divergences in the counterterms.

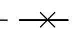
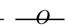
9.3.2 Renormalization in MS-Scheme

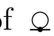
We shall now calculate finite two- and four-point vertex functions $\bar{\Gamma}^{(2)}(\mathbf{k}^2)$ and $\bar{\Gamma}^{(4)}(\mathbf{k}_i)$, starting from the renormalized energy functional (9.61):

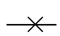
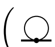
$$E[\phi] = \int d^D x \left[\frac{1}{2}(\partial\phi)^2 + c_\phi \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + c_{m^2}\frac{m^2}{2}\phi^2 + \frac{\mu^\varepsilon g}{4!}\phi^4 + c_g \frac{\mu^\varepsilon g}{4!}\phi^4 \right]. \tag{9.80}$$

One-Loop Calculation

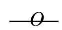
To first order in g, the counterterms which are necessary to make the two-point vertex function $\bar{\Gamma}^{(2)}(\mathbf{k})$ finite are also of first order in g. We may write the result in diagrammatical terms as

$$\bar{\Gamma}^{(2)}(\mathbf{k}) = \mathbf{k}^2 + m^2 - \Big(\frac{1}{2}$$  $$+$$  $$+$$  $$+ \mathcal{O}(g^2)\Big). \qquad (9.81)$$

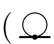
The cross and the small circle on a line indicate the contribution of the mass and field counterterms c_{m^2} and c_ϕ. The first is chosen to cancel precisely the pole term of  proportional to m^2, written down in Eq. (9.78), i.e., we set

 $$= -m^2 c^1_{m^2} = -\frac{1}{2}\mathcal{K}\Big($$  $$\Big) = -m^2 \frac{g}{(4\pi)^2}\frac{1}{\varepsilon}, \qquad (9.82)$$

where the superscript denotes the order of approximation. Since the counterterm (9.78) contains no contribution proportional to $\mathbf{k}^2$, there is no counterterm c_ϕ to first order in g:

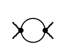
 $$= -\mathbf{k}^2 c^1_\phi = 0. \qquad (9.83)$$

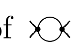
Choosing the counterterms in this way, the $1/\varepsilon$-pole in the one-loop diagram is canceled, and the renormalized two-point vertex function

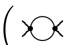
$$\bar{\Gamma}^{(2)}(\mathbf{k}) = \mathbf{k}^2 + m^2 - \Big[\frac{1}{2}$$  $$-\frac{1}{2}\mathcal{K}($$  $$)\Big] + \mathcal{O}(g^2) \qquad (9.84)$$

remains finite for $\varepsilon \to 0$ up to the first order in g.

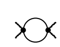
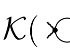
The first finite perturbative correction to the vertex function $\bar{\Gamma}^{(4)}(\mathbf{k}_i)$ is of the order g^2. The $1/\varepsilon$-pole in the Feynman integral is removed by a counterterm for the coupling constant, to be denoted by a fat dot:

$$\bar{\Gamma}^{(4)} = -\Big($$  $$+\frac{3}{2}$$  $$+$$  $$\Big) + \mathcal{O}(g^3). \qquad (9.85)$$

As in Eq. (9.41), the one-loop diagram contributes with three different momentum combinations $\mathbf{k}_1+\mathbf{k}_2$, $\mathbf{k}_1+\mathbf{k}_3$, and $\mathbf{k}_1+\mathbf{k}_4$, indicated by the prefactor 3. Choosing the pole term of  in Eq. (9.78) as a counterterm, we identify

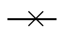
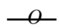
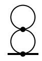
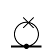
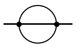
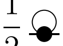
 $$= -\mu^\varepsilon g\, c^1_g = -\frac{3}{2}\mathcal{K}\Big($$  $$\Big) = -\mu^\varepsilon g \frac{3g}{(4\pi)^2}\frac{1}{\varepsilon}, \qquad (9.86)$$

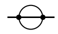
and obtain the finite vertex function

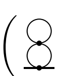
$$\bar{\Gamma}^{(4)} = -\Big[$$  $$+\frac{3}{2}$$  $$-\frac{3}{2}\mathcal{K}($$  $$)\Big] + \mathcal{O}(g^3). \qquad (9.87)$$

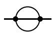
Two-Loop Calculation

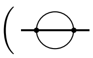
We now turn to the two-loop counterterms. At this stage the $\log m^2/\mu^2$- and $\log \mathbf{k}^2/\mu^2$-terms will enter at intermediate steps. First we calculate $\bar{\Gamma}^{(2)}(\mathbf{k})$ up to the order g^2. We have to form all previous second-order diagrams plus those which arise from the above-determined first-order counterterms:

$$\bar{\Gamma}^{(2)} = (\;—\;)^{-1} - \Big[\frac{1}{2}$$  $$+$$  $$+$$  $$+\frac{1}{4}$$  $$+\frac{1}{2}$$  $$+\frac{1}{6}$$  $$+\frac{1}{2}$$  $$\Big] + \mathcal{O}(g^3). \tag{9.88}$$

There are two second-order diagrams,  and , which contain only $g\phi^4$-interactions. The pole term of  is given in Eq. (8.72):

$$\frac{1}{4}\mathcal{K}\Big($$  $$\Big) = -\frac{m^2 g^2}{(4\pi)^4}\left[\frac{1}{\varepsilon^2} + \frac{\psi(1)+\psi(2)}{2\,\varepsilon} - \frac{1}{\varepsilon}\log\frac{m^2}{4\pi\mu^2}\right]. \tag{9.89}$$

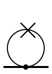
The pole term of  is found in Eq. (8.96):

$$\frac{1}{6}\mathcal{K}\Big($$  $$\Big) = -\frac{m^2 g^2}{(4\pi)^4}\left[\frac{1}{\varepsilon^2} + \frac{3}{2\,\varepsilon} + \frac{\psi(1)}{\varepsilon} - \frac{1}{\varepsilon}\log\frac{m^2}{4\pi\mu^2}\right] - \frac{g^2}{(4\pi)^4}\frac{\mathbf{k}^2}{12\varepsilon}. \tag{9.90}$$

The divergent term proportional to $\mathbf{k}^2$ will give the first contribution to the wave function counterterm c_ϕ.

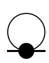
Both pole terms (9.89) and (9.90) contain logarithms of the form $\log(m^2/4\pi\mu^2)$. These arise from subdivergences as follows: a regular $\log(m^2/4\pi\mu^2)$ term of one of the loop integrals is multiplied by an ε-pole of the other, and vice versa. The fact that the argument of the logarithm is always $m^2/4\pi\mu^2$ can easily be understood. For dimensional reasons, a two-point diagram is proportional to $m^{2-L\varepsilon}$, where L is the number of loops. Furthermore, each power of g carries a factor μ^ε and each loop integration generates a factor $1/(4\pi)^{2-\varepsilon/2}$. Since the number of loops equals the number of coupling constants, we always run into the combination $(m^2/4\pi\mu^2)^\varepsilon$ whose ε-expansion yields the above logarithms.

The expansion in (9.88) contains in addition two diagrams,  and , arising from the first-order counterterms, to be called *counterterm diagrams*. They are calculated by replacing the coupling constant in the corresponding ϕ^4-diagram by the counterterm. For , we replace one of the coupling constants $-\mu^\varepsilon g$ in  $\Big|_{\mathbf{k}^2=0} =$  by $-m^2\, c^1_{m^2}$ and find

$$\frac{1}{2}$$  $$= -\frac{-m^2\, c^1_{m^2} g}{(4\pi)^2}\left[\frac{1}{\varepsilon} + \frac{1}{2}\psi(1) - \frac{1}{2}\int_0^1 d\alpha\,\log\frac{m^2+\mathbf{k}^2\alpha(1-\alpha)}{4\pi\mu^2} + \mathcal{O}(\varepsilon)\right]\Bigg|_{\mathbf{k}^2=0} \tag{9.91}$$

$$= \frac{m^2 g^2}{(4\pi)^4}\left[\frac{1}{\varepsilon^2} + \frac{1}{2\varepsilon}\psi(1) - \frac{1}{2\,\varepsilon}\,\log\frac{m^2}{4\pi\mu^2} + \mathcal{O}(\varepsilon^0)\right]. \tag{9.92}$$

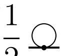
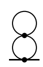
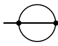
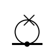
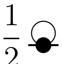
The calculation for  merely requires replacing the coupling constant $-\mu^\varepsilon g$ in  by $-\mu^\varepsilon g\, c^1_g$:

$$\frac{1}{2}$$  $$= -\frac{-m^2\, g\, c^1_g}{(4\pi)^2}\left[\frac{1}{\varepsilon} + \frac{\psi(2)}{2} - \frac{1}{2}\log\frac{m^2}{4\pi\mu^2} + \mathcal{O}(\varepsilon)\right]$$

$$= \frac{3\, m^2\, g^2}{(4\pi)^4}\left[\frac{1}{\varepsilon^2} + \frac{1}{2\varepsilon}\,\psi(2) - \frac{1}{2\varepsilon}\,\log\frac{m^2}{4\pi\mu^2} + \mathcal{O}(\varepsilon^0)\right]. \tag{9.93}$$

There are again terms of the form $\log(m^2/4\pi\mu^2)$. But this time they come with a prefactor $1/2$ because they are generated only by the single loop integral. The counterterm c_g^1 is free of logarithms and supplies only a factor proportional to $1/\varepsilon$ [see (9.86)]. This is crucial for the ultimate cancellation of all logarithmic terms. The cancellation mechanism will be illustrated in more detail at the end of this section.

Collecting all terms contributing to Eq. (9.88), and using the relation $\psi(n+1)-\psi(n) = 1/n$, we find the contributions to the counterterms c_{m^2} and c_ϕ up to the second order in g:

 $= -\mathcal{K}\Big[\frac{1}{2}$  $+\frac{1}{4}$  $+\frac{1}{6}$  $+\frac{1}{2}$  $+\frac{1}{2}$  $\Big]$

$$= -\left[\frac{g}{(4\pi)^2}\frac{m^2}{\varepsilon} + \frac{g^2}{(4\pi)^4}\left(\frac{2m^2}{\varepsilon^2} - \frac{m^2}{2\varepsilon} - \frac{\mathbf{k}^2}{12\varepsilon}\right)\right]. \tag{9.94}$$

The whole expression is polynomial in m^2 and $\mathbf{k}^2$. The pole terms proportional to m^2 extend the mass counterterms as follows:

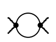
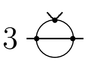
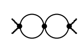
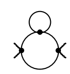
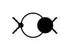
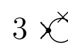
$$m^2\left(c_{m^2}^1 + c_{m^2}^2\right) = m^2\left[\frac{g}{(4\pi)^2}\frac{1}{\varepsilon} + \frac{g^2}{(4\pi)^4}\left(\frac{2}{\varepsilon^2} - \frac{1}{2\varepsilon}\right)\right]. \tag{9.95}$$

The pole terms proportional to $\mathbf{k}^2$ give the second-order counterterm of the field renormalization

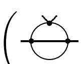
$\mathbf{k}^2 c_\phi^2 = \frac{1}{6}\mathcal{K}\Big($  $\Big)\Big|_{m=0} = -\mathbf{k}^2\frac{g^2}{(4\pi)^4}\frac{1}{12\,\varepsilon}.$ (9.96)

As noted before, c_ϕ possesses no first-order term in g.

We now turn to the two-loop renormalization of the four-point vertex function, whose diagrammatic expansion reads

$\bar{\Gamma}^{(4)} = -\Big[$  $+\frac{3}{2}$  $+$  $+3$  $+\frac{3}{4}$  $+\frac{3}{2}$  $+3$  $+3$  $\Big].$ (9.97)

As in the one-loop diagram in Eqs. (9.41) and (9.85), we had to sum over all combinations of external momenta, resulting in the factors 3. The pole term in the first two-loop diagram was calculated in Eq. (8.114), yielding

$3\,\mathcal{K}\Big($  $\Big) = -\mu^\varepsilon g\frac{3g^2}{(4\pi)^4}\left\{\frac{2}{\varepsilon^2}+\frac{1}{\varepsilon}+\frac{2}{\varepsilon}\psi(1)-\frac{2}{\varepsilon}\int_0^1 d\alpha\,\log\left[\frac{m^2+\mathbf{k}^2\alpha(1-\alpha)}{4\pi\mu^2}\right]\right\},$ (9.98)

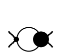
where $\mathbf{k}$ indicates either of the three different momentum combinations $\mathbf{k}_1+\mathbf{k}_2$, $\mathbf{k}_1+\mathbf{k}_3$, and $\mathbf{k}_1+\mathbf{k}_4$. The pole term in the second two-loop diagram was obtained in Eq. (8.99):

$\frac{3}{4}\mathcal{K}\Big($  $\Big) = -\mu^\varepsilon g\frac{3g^2}{(4\pi)^4}\left\{\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\psi(1)-\frac{1}{\varepsilon}\int_0^1 d\alpha\,\log\left[\frac{m^2+\mathbf{k}^2\alpha(1-\alpha)}{4\pi\mu^2}\right]\right\}.$ (9.99)

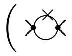
The pole term in the third integral was calculated with the help of the integral formula (8.69) and Eq. (8.62), and reads, according to Eq. (8.103),

$\frac{3}{2}\mathcal{K}\Big($  $\Big) = -\mu^\varepsilon g\frac{3g^2}{(4\pi)^4}\frac{1}{\varepsilon}\int_0^1 d\alpha\,\frac{m^2(1-\alpha)}{\mathbf{k}^2\alpha(1-\alpha)+m^2}.$ (9.100)

The pole term of the first counterterm diagram is calculated with the help of Eq. (8.68), where $-\mu^\varepsilon g$ is replaced by  $= -\mu^\varepsilon g\,3g/(4\pi)^2\varepsilon$:

$3\,\mathcal{K}\Big($  $\Big) = \mu^\varepsilon g\frac{3g^2}{(4\pi)^4}\left\{\frac{6}{\varepsilon^2}+\frac{3}{\varepsilon}\psi(1)-\frac{3}{\varepsilon}\int_0^1 d\alpha\,\log\left[\frac{m^2+\mathbf{k}^2\alpha(1-\alpha)}{4\pi\mu^2}\right]\right\}.$ (9.101)

The pole term of the second counterterm diagram is calculated as in (9.98), yielding

$$3\,\mathcal{K}\Big($$  $$\Big) = \mu^\varepsilon g\,\frac{3g^2}{(4\pi)^4}\frac{1}{\varepsilon}\int_0^1 d\alpha\,\frac{m^2\,(1-\alpha)}{\mathbf{k}^2\alpha(1-\alpha)+m^2}\,. \qquad (9.102)$$

All pole terms containing either $\psi(1)$ or the parameter integral cancel each other in the counterterm of the coupling constant, which becomes, up to the second order in g,

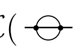
$$\mu^\varepsilon g\,\left(c_g^1 + c_g^2\right) = \mu^\varepsilon g\left[\frac{g}{(4\pi)^2}\frac{3}{\varepsilon} + \frac{g^2}{(4\pi)^4}\left(\frac{9}{\varepsilon^2}-\frac{3}{\varepsilon}\right)\right]. \qquad (9.103)$$

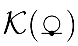
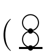
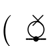
The counterterms in Eqs. (9.95), (9.96), and (9.103) have all the local form (9.65)–(9.67). No nonlocal proportional to $\log(\mathbf{k}^2/\mu^2)$ appear, which would have impeded the incorporation of the pole terms into the initial energy functional (9.80).

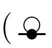
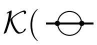
Up to the order g^2, we thus obtain finite correlation functions by starting out from the initial energy functional (9.80), written in the form:

$$E[\phi] = \int d^Dx\left[\frac{1}{2}Z_\phi(\partial\phi)^2 + \frac{m^2}{2}Z_{m^2}\phi^2 + \frac{\mu^\varepsilon g}{4!}Z_g\phi^4\right], \qquad (9.104)$$

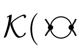
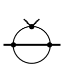
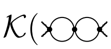
with the renormalization constants

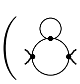
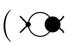
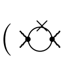
$$Z_\phi(g,\varepsilon^{-1}) = 1 + c_\phi = 1 + \frac{1}{\mathbf{k}^2}\frac{1}{6}\mathcal{K}($$  $$)\Big|_{m^2=0} = 1 - \frac{g^2}{(4\pi)^4}\frac{1}{12}\frac{1}{\varepsilon}, \qquad (9.105)$$

$$Z_{m^2}(g,\varepsilon^{-1}) = 1 + c_{m^2} = 1 + \frac{1}{m^2}\Big[\frac{1}{2}\mathcal{K}($$  $$) + \frac{1}{4}\mathcal{K}($$  $$) + \frac{1}{2}\mathcal{K}($$  $$)$$

$$+\frac{1}{2}\mathcal{K}($$  $$) + \frac{1}{6}\mathcal{K}($$  $$)\Big|_{\mathbf{k}^2=0}\Big]$$

$$= 1 + \frac{g}{(4\pi)^2}\frac{1}{\varepsilon} + \frac{g^2}{(4\pi)^4}\left(\frac{2}{\varepsilon^2} - \frac{1}{2\,\varepsilon}\right), \qquad (9.106)$$

$$Z_g(g,\varepsilon^{-1}) = 1 + c_g = 1 + \frac{1}{\mu^\varepsilon g}\Big[\ \frac{3}{2}\mathcal{K}($$  $$) + 3\,\mathcal{K}\Big($$  $$\Big) + \frac{3}{4}\mathcal{K}($$  $$)$$

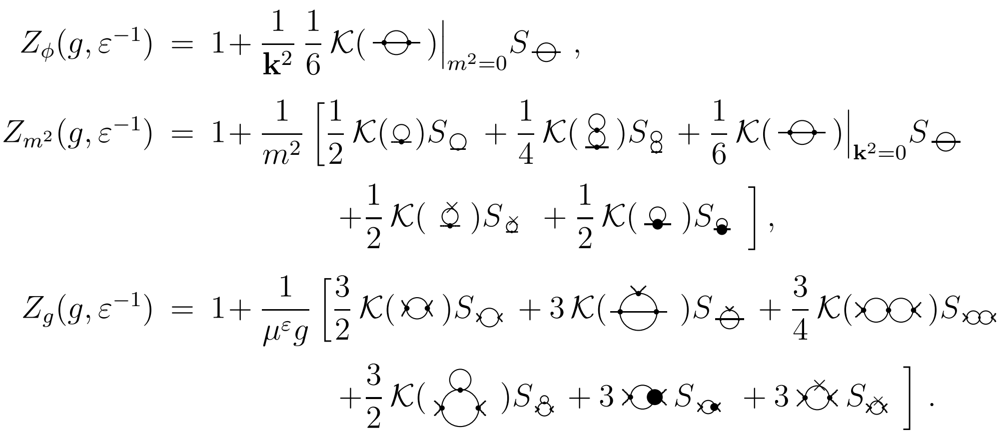
$$+\frac{3}{2}\mathcal{K}\Big($$  $$\Big) + 3\,\mathcal{K}($$  $$) + 3\,\mathcal{K}($$  $$)\Big]$$

$$= 1 + \frac{g}{(4\pi)^2}\frac{3}{\varepsilon} + \frac{g^2}{(4\pi)^4}\left(\frac{9}{\varepsilon^2} - \frac{3}{\varepsilon}\right). \qquad (9.107)$$

The renormalization constants are expansions in the dimensionless coupling constant g, with expansion coefficients containing only pole terms of the form $1/\varepsilon^i$, where i runs from 1 to n in the term of order g^n. This is precisely the form anticipated in Eq. (9.1).

By writing the energy functional in the form (9.104), it is multiplicatively renormalized. A comparison with the bare energy functional (9.73) allows us to identify the bare field, mass, and coupling constant as in Eqs. (9.71) and (9.72). Note that since the divergences in the ϕ^4-theory come exclusively from the 2- and 4-point 1PI diagrams, all n-point vertex functions are finite for $\varepsilon \to 0$ up to this order in g, if the perturbation expansion proceeds from the energy functional (9.104).

For N field components, the two-loop results are extended by the symmetry factors introduced in Section 6.3 and listed in Eqs. (6.40)–(6.46). They multiply each diagram as follows:

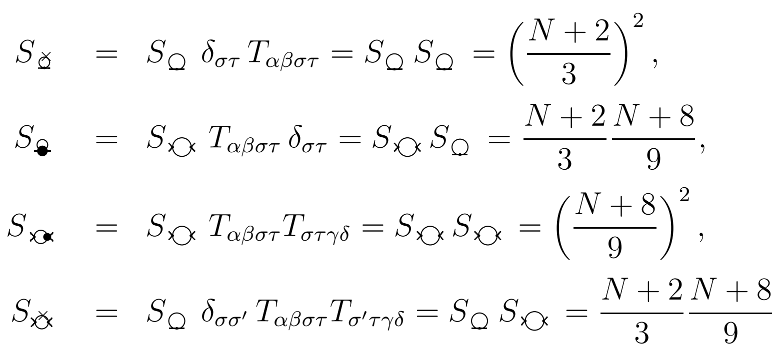


(9.108)

(9.109)

(9.110)

The symmetry factors associated with the counterterm diagrams must still be calculated, with the following results:



(9.111)

(9.112)

(9.113)

(9.114)

With these symmetry factors, the renormalization constants up to g^2 are

$$Z_\phi(g,\varepsilon^{-1}) = 1 - \frac{g^2}{(4\pi)^4}\frac{1}{12}\frac{1}{\varepsilon}\frac{N+2}{3}, \tag{9.115}$$

$$\begin{aligned} Z_{m^2}(g,\varepsilon^{-1}) = {}& 1 + \frac{g}{(4\pi)^2}\frac{1}{\varepsilon}\frac{N+2}{3} \\ &+ \frac{g^2}{(4\pi)^4}\Bigg[\left(-\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\log\frac{m^2}{4\pi\mu^2}-\frac{\psi(1)+\psi(2)}{2\varepsilon}\right)\left(\frac{N+2}{3}\right)^2 \\ &\quad + \left(-\frac{1}{\varepsilon^2}-\frac{3}{2\varepsilon}+\frac{1}{\varepsilon}\log\frac{m^2}{4\pi\mu^2}-\frac{\psi(1)}{\varepsilon}\right)\frac{N+2}{3} \\ &\quad + \left(\frac{1}{\varepsilon^2}-\frac{1}{2\varepsilon}\log\frac{m^2}{4\pi\mu^2}+\frac{\psi(1)}{2\varepsilon}\right)\left(\frac{N+2}{3}\right)^2 \\ &\quad + \left(\frac{3}{\varepsilon^2}-\frac{3}{2\varepsilon}\log\frac{m^2}{4\pi\mu^2}+\frac{3\psi(2)}{2\varepsilon}\right)\frac{N+2}{3}\frac{N+8}{9}\Bigg] \end{aligned} \tag{9.116}$$

$$= 1 + \frac{g}{(4\pi)^2}\frac{1}{\varepsilon}\frac{N+2}{3} + \frac{g^2}{(4\pi)^4}\left[-\frac{1}{\varepsilon}\frac{N+2}{6}+\frac{1}{\varepsilon^2}\frac{(N+2)(N+5)}{9}\right], \tag{9.117}$$

$$\begin{aligned} Z_g(g,\varepsilon^{-1}) = {}& 1 + \frac{g}{(4\pi)^2}\frac{3}{\varepsilon}\frac{N+8}{9} \\ &+ \frac{g^2}{(4\pi)^4}\left\{-\left[\frac{2}{\varepsilon^2}+\frac{1}{\varepsilon}+\frac{2\psi(1)}{\varepsilon}-\frac{2}{\varepsilon}\int_0^1 d\alpha\log\frac{m^2+\mathbf{k}^2\alpha(1-\alpha)}{4\pi\mu^2}\right]\frac{5N+22}{9}\right. \end{aligned}$$

$$
\begin{aligned}
&- \left[\frac{1}{\varepsilon^2} + \frac{\psi(1)}{\varepsilon} - \frac{1}{\varepsilon}\int_0^1 d\alpha \log \frac{m^2 + \mathbf{k}^2\alpha(1-\alpha)}{4\pi\mu^2}\right] \frac{N^2+6N+20}{9} \\
&- \left[\frac{1}{\varepsilon}\int_0^1 d\alpha \frac{m^2(1-\alpha)}{\mathbf{k}^2\alpha(1-\alpha)+m^2}\right] \frac{N^2+10N+16}{27} \\
&+ \left[\frac{6}{\varepsilon^2} + \frac{3\psi(1)}{\varepsilon} - \frac{3}{\varepsilon}\int_0^1 d\alpha \log \frac{m^2 + \mathbf{k}^2\alpha(1-\alpha)}{4\pi\mu^2}\right] \frac{(N+8)^2}{27} \\
&+ \left[\frac{1}{\varepsilon}\int_0^1 d\alpha \frac{m^2(1-\alpha)}{\mathbf{k}^2\alpha(1-\alpha)+m^2}\right] \frac{N^2+10N+16}{9}\Bigg\} \qquad (9.118)
\end{aligned}
$$

$$
= 1 + \frac{g}{(4\pi)^2}\frac{3}{\varepsilon}\frac{N+8}{9} + \frac{g^2}{(4\pi)^4}\left[\frac{1}{\varepsilon^2}\frac{(N+8)^2}{9} - \frac{1}{\varepsilon}\frac{5N+22}{9}\right]. \qquad (9.119)
$$

Note that, although the symmetry factors are different for the counterterm diagrams, the combinatorics involved in constructing these are just right to cancel conveniently all logarithms in the final expressions, and lead to local counterterms as necessary for the renormalizability of the theory.

As discussed before, the absence of the mass parameter μ in these expansions offers us the opportunity to redefine the coupling constant g with an arbitrary factor $f(\varepsilon)$ without changing these expansions. It is, however, important to realize that such a redefinition cannot be simply done in the final expressions (9.115), (9.117), (9.119). If we were to replace in these the coupling constant g to $gf(\varepsilon) = g(1 + f_1\varepsilon + f_2\varepsilon^2 + \dots)$ and delete all positive powers of ε, the coefficients of $1/\varepsilon$ in the g^2-terms would change. The invariance is a consequence of the special preparation of the counterterms, as pointed out earlier after Eq. (9.90). With the prescription for determining the counterterms by minimal subtraction, the redefinition does not modify the coupling constants in the counterterms of the diagrammatic expansion (9.108)–(9.110). The intermediate expressions (9.116) for Z_{m^2} can once more be used to demonstrate this invariance. The powers g^n of the coupling constants, which do not come from a counterterm subdiagram, are multiplied by $(1 + f_1\varepsilon + \dots + f_{n-1}\varepsilon^{n-1})^n$, exhibiting only relevant powers up to ε^n. This transforms the graphical expansion (9.109) for Z_{m^2} into

$$
\begin{aligned}
Z_{m^2}(g,\varepsilon^{-1}) = 1 + \frac{1}{m^2}\Bigg[&\frac{1}{2}\mathcal{K}(\;)S + \frac{(1+f_1\varepsilon)^2}{4}\mathcal{K}(\;)S + \frac{(1+f_1\varepsilon)^2}{6}\mathcal{K}(\;)\Big|_{\mathbf{k}^2=0}S \\
&+ \frac{(1+f_1\varepsilon)}{2}\mathcal{K}(\;)S + \frac{(1+f_1\varepsilon)}{2}\mathcal{K}(\;)S\Bigg]. \qquad (9.120)
\end{aligned}
$$

In the analytic expression (9.117), this replacement changes the logarithms as follows:

$$
\log\frac{m^2}{4\pi\mu^2} \longrightarrow \log\frac{m^2}{4\pi\mu^2} - 2f_1 = \log\frac{m^2}{4\pi(\mu e^{f_1})^2},
$$

thus multiplying the mass parameter μ by a factor e^{f_1} to this order in g. Since all logarithms disappear in the final renormalization constant (9.117), the redefinition of g leaves no trace in the renormalization constant Z_{m^2}.

9.3.3 Recursive Diagrammatic Subtraction

The subtraction of the ε-poles by the counterterm diagrams can be organized in a different manner. Each counterterm diagram can be associated with one or more ϕ^4-diagrams from which it subtracts the ε-poles coming from subdiagrams of these ϕ^4-diagrams. Only the pole term remaining after all these subtractions contributes to the counterterm in this order. It will be called *superficial pole term*. The main difficulty in these calculations is to find the correct combinatorial factors.

As an example, we calculate recursively the second order contribution to the three counterterms. We begin with c_{m^2} and c_ϕ. To first order in g, there are two counterterm diagrams, which can be written diagrammatically as $= -\frac{1}{2}\mathcal{K}(\;)$ and $= -\frac{3}{2}\mathcal{K}(\;)$. With these, the counterterm diagrams of second order in g may be expressed as follows:

$$\frac{1}{2} \quad = \quad -m^2 c_{m^2} * \frac{1}{2} \quad = -\frac{1}{2}\mathcal{K}(\;) * \frac{1}{2} \quad , \tag{9.121}$$

$$\frac{1}{2} \quad = \quad -\mu^\varepsilon g\, c_g * \frac{1}{2} \quad = -\frac{3}{2}\mathcal{K}(\;) * \frac{1}{2} \quad , \tag{9.122}$$

where the operation $*$ denotes the substitution of the counterterm $-m^2 c_{m^2}$ or $-\mu^\varepsilon g\, c_g$ for the dots in the diagrams or $\times$, respectively. If the counterterm does not depend on the momentum, this substitution leads simply to the multiplication of the counterterm by the remaining integral, which contains one coupling constant less than the initial diagram. The star operation is less trivial for the counterterms of wave function renormalization which contains a factor $\mathbf{k}^2$. This momentum-dependent factor must be included into the integrand of the remaining loop integrals, thereby complicating its evaluation.

The star operation is now used in the diagrammatic expansion of the sum of the counterterms c_{m^2} and c_ϕ in Eq. (9.94), by splitting the factors and inserting Eqs. (9.121), (9.122), and (9.79):

$$\begin{aligned} &\mathcal{K}\left[\frac{1}{4}\left(\quad + 2 \quad + \frac{2}{3} \quad\right) + \frac{1}{6}\left(\quad + 2 \quad\right)\right] \\ = \; &\mathcal{K}\left[\frac{1}{4}\left(\quad - \mathcal{K}(\;) * \quad - \mathcal{K}(\;) * \quad\right) + \frac{1}{6}\left(\quad - 3\,\mathcal{K}(\;) * \quad\right)\right] \\ = \; &\frac{m^2 g^2}{(4\pi)^4}\frac{1}{\varepsilon^2} + \frac{m^2 g^2}{(4\pi)^4}\left(\frac{1}{\varepsilon^2} - \frac{1}{2\varepsilon}\right) - \frac{\mathbf{k}^2 g^2}{(4\pi)^4}\frac{1}{12\varepsilon}. \end{aligned} \tag{9.123}$$

From the ϕ^4-diagrams and , all terms are subtracted in which the integration of a superficially divergent subdiagram , , or is replaced by its counterterm. This step will be referred to as the *diagrammatic subtraction of subdivergences*. As a result, only the pole terms of the superficial divergences of and remain, which are fewer than those in the full counterterms $\mathcal{K}(\;)$ and $\mathcal{K}(\;)$. They provide us with the second-order contributions to the counterterms c_{m^2} and c_ϕ.

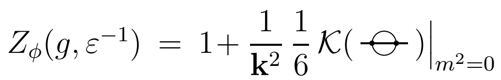
The counterterms associated with the vertex function (9.97) can be calculated similarly. After rewriting

$$3 \quad = -\frac{3}{2}\mathcal{K}\left(\quad\right) * 3 \quad \quad \text{and} \quad 3 \quad = -\frac{1}{2}\mathcal{K}\left(\quad\right) * 3 \quad , \tag{9.124}$$

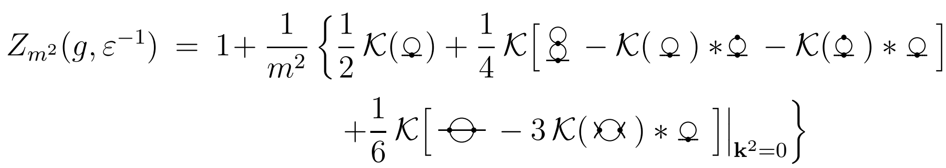
the right-hand side of (9.97) yields the following superficial pole terms

$$\begin{aligned} &3\mathcal{K}\left[\quad - \mathcal{K}\left(\quad\right) * \quad\right] + \frac{3}{4}\mathcal{K}\left[\quad - 2\mathcal{K}\left(\quad\right) * \quad\right] + \frac{3}{2}\mathcal{K}\left[\quad - \mathcal{K}\left(\quad\right) * \quad\right] \\ &= \frac{g^3}{(4\pi)^4}\left[\left(\frac{6}{\varepsilon^2} - \frac{3}{\varepsilon}\right) + \frac{3}{\varepsilon^2} + 0\right]. \end{aligned} \tag{9.125}$$

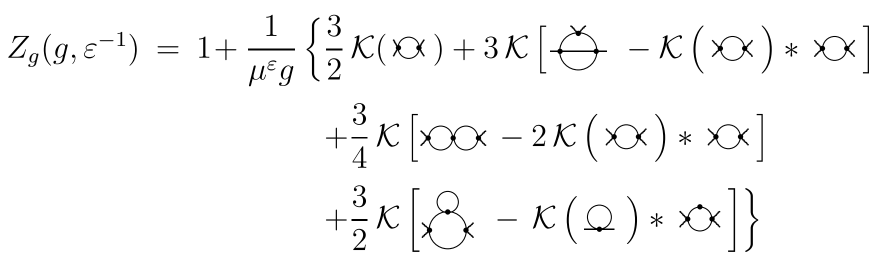
We have spaced the second line to better indicate the association with the terms above. Using this recursive procedure, Eqs. (9.105)–(9.107) can be calculated diagrammatically as follows:



$$= 1 - \frac{g^2}{(4\pi)^4} \frac{1}{12} \frac{1}{\varepsilon}, \tag{9.126}$$



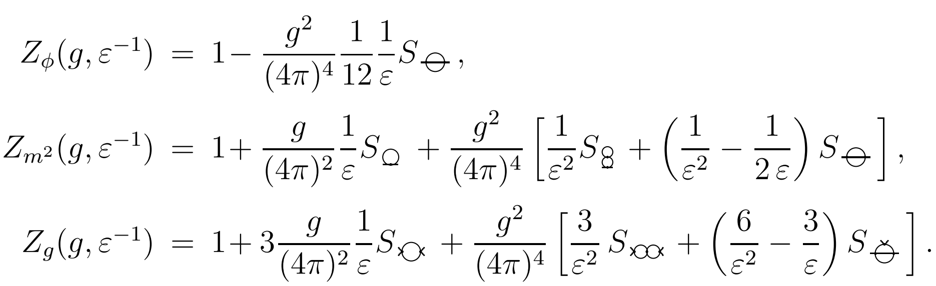
$$= 1 + \frac{g}{(4\pi)^2} \frac{1}{\varepsilon} + \frac{g^2}{(4\pi)^4} \left[\frac{1}{\varepsilon^2} + \left(\frac{1}{\varepsilon^2} - \frac{1}{2\varepsilon} \right) \right], \tag{9.127}$$



$$= 1 + \frac{g}{(4\pi)^2} \frac{3}{\varepsilon} + \frac{g^2}{(4\pi)^4} \left[\left(\frac{6}{\varepsilon^2} - \frac{3}{\varepsilon} \right) + \frac{3}{\varepsilon^2} + 0 \right]. \tag{9.128}$$

The result is, of course, the same as before.

For N field components, the recursive procedure offers an important advantage over the previous calculation scheme. It saves us from having to calculate the symmetry factors for the counterterm diagrams! This is due to the fact that all subtraction terms of a vertex diagram carry the same symmetry factor. Thus we extend the expansions (9.126)–(9.128) immediately to O(N)-symmetric ϕ^4-theory:



(9.129)

(9.130)

(9.131)

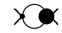
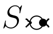
After inserting the symmetry factors of Eqs. (6.40)–(6.46), we obtain more explicitly

$$Z_\phi(g, \varepsilon^{-1}) = 1 - \frac{g^2}{(4\pi)^4} \frac{1}{12} \frac{1}{\varepsilon} \frac{N+2}{3}, \tag{9.132}$$

$$\begin{aligned} Z_{m^2}(g, \varepsilon^{-1}) &= 1 + \frac{g}{(4\pi)^2} \frac{1}{\varepsilon} \frac{N+2}{3} \\ &\quad + \frac{g^2}{(4\pi)^4} \left[\frac{1}{\varepsilon^2} \left(\frac{N+2}{3} \right)^2 + \left(\frac{1}{\varepsilon^2} - \frac{1}{2\,\varepsilon} \right) \frac{N+2}{3} \right], \end{aligned} \tag{9.133}$$

$$Z_g(g, \varepsilon^{-1}) = 1 + 3 \frac{g}{(4\pi)^2} \frac{1}{\varepsilon} \frac{N+8}{9} \tag{9.134}$$

$$+\frac{g^2}{(4\pi)^4}\left[\frac{3}{\varepsilon^2}\frac{N^2+6N+20}{27}+\left(\frac{6}{\varepsilon^2}-\frac{3}{\varepsilon}\right)\frac{5N+22}{27}\right],$$

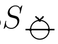
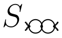
in agreement with Eqs. (9.115)–(9.119). The simplification brought about by the recursive procedure is nontrivial: it is based on the fact that the symmetry factor of the counterterm  picks up the symmetry factors of the counterterm diagrams  and  in a symmetrized form. For example, S $\neq S$ since  is symmetrized, whereas the subdiagram  in  is not.

Specifically, the proper symmetry factors in Eq. (9.130) for $Z_{m^2}(g,\varepsilon^{-1})$ are a result of the equality:

$\frac{1}{4}S$ $+\frac{1}{2}S$ $=\frac{3}{4}S$ ,

$$\frac{1}{4}\left(\frac{N+2}{3}\right)^2+\frac{1}{2}\frac{N+2}{3} = \frac{3}{4}\frac{N+2}{3}\frac{N+8}{9}.$$

In Eq. (9.131) for $Z_g(g,\varepsilon^{-1})$, the analogous equalities are

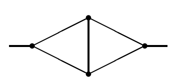
$3S$ $+\frac{3}{2}S$ $=\frac{9}{2}S$ ,

$$\frac{5N+22}{9}+\frac{1}{2}\frac{N^2+6N+20}{9} = \frac{1}{2}\frac{(N+8)^2}{9}.$$

The generation of all possible counterterm diagrams via a diagrammatic subtraction of subdivergences in this example can be developed into a systematic technique with the help of the so-called R-operation which will be introduced in the Chapter 11.

Appendix 9A Overlapping Divergences

An overlapping divergence could, in principle, arise from the integral over certain directions in the multidimensional space of all loop momenta, even though a diagram has $\omega(G)<0$ and all $\omega(\gamma)<0$. According to the convergence theorem, this cannot happen. As an example, consider the following Feynman integral for $D=4$:

 $\hat{=} \int_{\mathrm{IR}} \frac{d^4k\, d^4p}{\mathbf{p}^4(\mathbf{p}+\mathbf{k})^2\mathbf{k}^4}\,.$ (9A.1)

Since danger comes only from the large momentum-region, all masses and external momenta have been set equal to zero, for simplicity. The subscript IR on the integral indicates some cutoff at small momenta to prevent IR-divergences. Power counting shows that $\omega(G)=-2$, thus indicating a superficial convergence. Subdivergences are not present, as we can see also by naive power counting. Obviously, naive power counting fails to inform us whether the integral converges in the subspace with fixed $(\mathbf{k}+\mathbf{p})^2$. The above theorem implies that this cannot happen. To verify this, we consider the integral in the eight-dimensional space of the two loop momenta $\mathbf{k}$ and $\mathbf{p}$. A divergence could in principle appear, and this would not be caused by subdivergences, since there are none. Such a divergence could not be predicted by the power counting theorem. The danger of a divergence in the present example is eliminated by the following consideration: the eight-dimensional momentum space is divided into several regions in such a way that, in each region, one of the squared momenta in the denominator is smaller than the others. For the integral in (9A.1), one of the regions to be considered is

$U = \{\mathbf{k}|\mathbf{k}^2 \geq 1\}$, $V = \{\mathbf{p}|\mathbf{p}^2 \geq \mathbf{k}^2, (\mathbf{p}+\mathbf{k})^2 \geq \mathbf{k}^2\}$. The momenta are then rescaled by the inverse absolute value of the smallest momentum (here $\mathbf{k}$). In the example, this rescaling is $\mathbf{k} = \hat{\mathbf{k}}k$, where $\hat{\mathbf{k}}$ is a four vector of unit length, and $\mathbf{p} = \mathbf{p}'k$. The resulting integral is

$$\int_U dk\, k^3 \frac{1}{k^6} \int d\hat{\mathbf{k}} \int_{V'} d^4p' \, \frac{1}{\mathbf{p}'^4(\mathbf{p}'+\hat{\mathbf{k}})^2}, \tag{9A.2}$$

with $V' = \{\mathbf{p}'|\mathbf{p}'^2 \geq 1 = \hat{\mathbf{k}}^2, (\mathbf{p}'+\hat{\mathbf{k}})^2 \geq 1\}$. The integral $\int d\hat{\mathbf{k}}$ covers the surface S_4 of the sphere in four dimensions with unit radius. Now, the integration over $\mathbf{p}'$ is absolutely convergent for each fixed $\hat{\mathbf{k}}$, since (9A.1) is free of subdivergences. The remaining integration over the absolute value of $\mathbf{k}$ is governed by the degree of superficial divergence $\omega(G)$, and is convergent in this region. Similar arguments apply to the other regions. In general, the integrations over the larger momenta are always absolutely convergent if no subdivergences are present. After having carried out the large-momentum integrals, the final integration is always of the form:

$$\int_{\mathrm{U}} dk\, k^3 \, \frac{1}{|k|^{2I-4(L-1)}} \sim \int_1^\infty dk \, \frac{1}{|k|^{-\omega(G)+1}}\,, \tag{9A.3}$$

which is convergent as long as $\omega(G) < 0$.

Notes and References

The possibility to remove all infinities by multiplicative renormalization was first observed by Dyson for quantum electrodynamics in Ref. [3]. For general discussions of renormalization see the textbooks
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