7

Scale Transformations of Fields and Correlation Functions

We now turn to the properties of ϕ^4-field theories which form the mathematical basis of the phenomena observed in second-order phase transitions. These phenomena are a consequence of a nontrivial behavior of fields and correlation functions under scale transformations, to be discussed in this chapter.

7.1 Free Massless Fields

Consider first a free massless scalar field theory, with an energy functional in D-dimensions:

$$E_0[\phi] = \int d^D x \frac{1}{2}[\partial\phi(\mathbf{x})]^2, \tag{7.1}$$

This is invariant under *scale transformations*, which change the coordinates by a scale factor

$$\mathbf{x} \to \mathbf{x}' = e^{\alpha}\mathbf{x}, \tag{7.2}$$

and transform the fields simultaneously as follows:

$$\phi(\mathbf{x}) \to \phi'_\alpha(\mathbf{x}) = e^{d^0_\phi \alpha}\phi(e^\alpha \mathbf{x}). \tag{7.3}$$

From the point of view of representation theory of Lie groups, the prefactor d^0_ϕ of the parameter α plays the role of a *generator* of the scale transformations on the field ϕ. Its value is

$$d^0_\phi = \frac{D}{2} - 1. \tag{7.4}$$

Under the scale transformations (7.2) and (7.3), the energy (7.1) is invariant:

$$E_0[\phi'_\alpha] = \int d^D x \frac{1}{2}[\partial\phi'_\alpha(\mathbf{x})]^2 = \int d^D x\, e^{2d^0_\phi \alpha}\frac{1}{2}[\partial\phi(e^\alpha\mathbf{x})]^2 = \int d^D x' \frac{1}{2}[\partial'\phi(\mathbf{x}')]^2 = E_0[\phi]. \tag{7.5}$$

The number d^0_ϕ is called the *field dimension* of the free field $\phi(\mathbf{x})$. Its value (7.4) is a direct consequence of the naive dimensional properties of $\mathbf{x}$ and ϕ. In the units used in this text, in which we have set $k_BT = 1$ [recall the convention stated before Eq. (2.6)], the exponential in the partition function is $e^{-E[\phi]}$, so that $E[\phi]$ is a dimensionless quantity. The coordinates have the dimension of a length. This property is expressed by the equation $[\mathbf{x}] = L$. The field in (7.5) has then a *naive dimension* (also called *engineering* or *technical dimension*) $[\phi] = L^{-d^0_\phi}$. To establish contact with the field theories of elementary particle physics, we shall use further natural units in which $c = \hbar = 1$. Then the length L is equal to an inverse mass μ^{-1} (more precisely, L is the Compton wavelength $L = \hbar/mc$ associated with the mass m) . It is conventional to specify the dimension of every quantity in *units of mass* μ, rewriting $[\mathbf{x}] = L$ and $[\phi] = L^{-(D/2-1)}$ as $[\mathbf{x}] = \mu^{-1}$ and $[\phi] = \mu^{d_\phi^0}$. Hereafter, we shall refer to the power d_ϕ^0 of μ as the *free-field dimension*, rather than the power. This will shorten many statements without danger of confusion.

The trivial scale invariance (7.5) of the free-field theory is the reason for a trivial power behavior of the free-field correlation functions. In momentum space, the two-point function reads [recall (4.9)]:

$$G(\mathbf{k}) = \frac{1}{\mathbf{k}^2}, \tag{7.6}$$

which becomes in $\mathbf{x}$-space

$$G(\mathbf{x}) = \int \frac{d^D k}{(2\pi)^D} \frac{e^{i\mathbf{k}\mathbf{x}}}{\mathbf{k}^2} = \frac{\Gamma(D/2-1)}{(4\pi)^{D/2}} \frac{2^{D-2}}{|\mathbf{x}|^{D-2}}. \tag{7.7}$$

It is instructive to rederive this power behavior as a consequence of scale invariance (7.5) of the energy functional. This implies that the correlation function of the transformed fields $\phi'_\alpha(\mathbf{x})$ in (7.3) must be the same as those of the initial fields $\phi(\mathbf{x})$:

$$\langle \phi'_\alpha(\mathbf{x}) \phi'_\alpha(\mathbf{y}) \rangle = \langle \phi(\mathbf{x}) \phi(\mathbf{y}) \rangle. \tag{7.8}$$

Inserting on the left-hand side the transformed fields from (7.3), we see that

$$e^{2d_\phi^0 \alpha} \langle \phi(e^\alpha \mathbf{x}) \phi(e^\alpha \mathbf{y}) \rangle = \langle \phi(\mathbf{x}) \phi(\mathbf{y}) \rangle. \tag{7.9}$$

Translational invariance makes this a function of $\mathbf{x} - \mathbf{y}$ only, which has then the property

$$e^{2d_\phi^0 \alpha} G(e^\alpha (\mathbf{x} - \mathbf{y})) = G(\mathbf{x} - \mathbf{y}). \tag{7.10}$$

Since $G(\mathbf{x} - \mathbf{y})$ is also rotationally invariant, Eq. (7.10) implies the power behavior

$$G(\mathbf{x} - \mathbf{y}) = \text{const} \times |\mathbf{x} - \mathbf{y}|^{-2d_\phi^0}, \tag{7.11}$$

which is precisely the form (7.7). Note that this power behavior is of the general type (1.11) observed in critical phenomena, with a trivial critical exponent $\eta = 0$.

In order to study the consequences of a continuous symmetry and its violations, it is convenient to first consider infinitesimal transformations. Then the defining transformations (7.2) and (7.3) read:

$$\delta \mathbf{x} = \alpha \mathbf{x}, \tag{7.12}$$

$$\delta \phi(\mathbf{x}) = \alpha (d_\phi^0 + \mathbf{x} \partial_\mathbf{x}) \phi(\mathbf{x}), \tag{7.13}$$

and the invariance law (7.9) takes the form of a differential equation:

$$(d_\phi^0 + \mathbf{x} \partial_\mathbf{x}) \langle \phi(\mathbf{x}) \phi(\mathbf{y}) \rangle + (d_\phi^0 + \mathbf{y} \partial_\mathbf{y}) \langle \phi(\mathbf{x}) \phi(\mathbf{y}) \rangle = 0. \tag{7.14}$$

For an n-point function, this reads

$$\sum_{i=1}^{n} (d_\phi^0 + \mathbf{x}_i \partial_{\mathbf{x}_i}) \langle \phi(\mathbf{x}_1) \cdots \phi(\mathbf{x}_n) \rangle = 0. \tag{7.15}$$

In the notation (2.10) for the correlation functions, this may be written as

$$\sum_{i=1}^{n} \mathbf{x}_i \partial_{\mathbf{x}_i} G^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = -n d_\phi^0 \, G^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n). \tag{7.16}$$

The differential operator on the left-hand side,

$$H_{\mathbf{x}} \equiv \sum_{i=1}^{n} \mathbf{x}_i \partial_{\mathbf{x}_i}, \tag{7.17}$$

measures the degree of homogeneity in the spatial variables $\mathbf{x}_i$, which is $-nd_\phi^0$. The power behavior (7.11) shows this directly.

If we multiply Eq. (7.16) by $e^{-i\Sigma_i \mathbf{k}_i \mathbf{x}_i}$, and integrate over all spatial coordinates [recall (4.10)], we find the momentum-space equation

$$\left(\sum_{i=1}^{n} \mathbf{k}_i \partial_{\mathbf{k}_i}\right) G^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n) = (nd_\phi^0 - D)\, G^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n). \tag{7.18}$$

The differential operator on the left-hand side,

$$H_{\mathbf{k}} \equiv \sum_{i=1}^{n} \mathbf{k}_i \partial_{\mathbf{k}_i}, \tag{7.19}$$

measures the degree of homogeneity in the momentum variables $\mathbf{x}_i$, which is $H_{\mathbf{k}} = -H_{\mathbf{x}} - D = nd_\phi^0 - D$.

Turning to the connected one-particle irreducible Green functions $G_{\rm c}^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n)$, and further to the vertex functions $\bar{\Gamma}^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n)$, by removing an overall δ-function and multiplying each leg by a factor $G^{-1}(\mathbf{k}_i)$, [recall Eq. (4.20)], we find for the vertex functions $\bar{\Gamma}^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n)$ the differential equation

$$\left(\sum_{i=1}^{n} \mathbf{k}_i \partial_{\mathbf{k}_i}\right) \bar{\Gamma}^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n) = (-nd_\phi^0 + D)\, \bar{\Gamma}^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n). \tag{7.20}$$

For the two-point vertex function, this implies the homogeneity property

$$\bar{\Gamma}^{(2)}(\mathbf{k}) = \text{const} \times \mathbf{k}^2, \tag{7.21}$$

which is, of course, satisfied by the explicit form (7.6) [recall (4.34)].

7.2 Free Massive Fields

We now introduce a mass term to the energy functional, and consider a field ϕ whose fluctuations are governed by

$$E[\phi] = \int d^D x \left[\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2\right]. \tag{7.22}$$

In the presence of m, the Fourier transform of the two-point function is [recall (4.9)]

$$G(\mathbf{k}) = \frac{1}{\mathbf{k}^2 + m^2}, \tag{7.23}$$

which becomes, in $\mathbf{x}$-space,

$$G(\mathbf{x}) = \int \frac{d^D k}{(2\pi)^D} \frac{e^{i\mathbf{k}\mathbf{x}}}{\mathbf{k}^2 + m^2} = \frac{1}{(4\pi)^{D/2}} \frac{2^{D-2}}{|\mathbf{x}|^{D-2}} \times \frac{1}{2^{D/2-2}} |m\mathbf{x}|^{D/2-1} K_{D/2-1}(|m\mathbf{x}|). \tag{7.24}$$

This possesses the scaling form of the general phenomenologically observed type (1.8), with a length scale $\xi = 1/m$. Since m^2 is proportional to the reduced temperature variable $t = T/T_c - 1$, the expression (7.24) has also the general scaling form (1.28), with a critical exponent $\nu = 1/2$.

In the presence of a mass term, the energy functional (7.22) is no longer invariant under the scale transformations (7.2) and (7.3). Whereas the free part $E_0[\phi]$ of the energy is invariant as in (7.5), the energy of the mass term of the transformed field $\phi'(\mathbf{x})$ is now related to that of the original field by

$$E_m[\phi'_\alpha] = \int d^Dx \frac{m^2}{2}\phi'^2_\alpha(\mathbf{x}) = e^{2d^0_\phi\alpha}\int d^Dx \frac{m^2}{2}\phi^2(e^\alpha\mathbf{x}) = e^{(2d^0_\phi - D)\alpha}\int d^Dx \frac{m^2}{2}\phi^2(\mathbf{x}) = e^{-2\alpha}E_m[\phi]. \tag{7.25}$$

7.3 Interacting Fields

Let us now add a ϕ^4-interaction to a massless energy functional $E_0[\phi]$, and consider the energy

$$E_{0,\lambda}[\phi] = \int d^Dx \left[\frac{1}{2}(\partial\phi)^2 + \frac{\lambda}{4!}\phi^4\right]. \tag{7.26}$$

This remains invariant under the scale transformations (7.2) and (7.3) only for $D = 4$. In less than four dimensions, where the critical phenomena in the laboratory take place, the invariance is broken by the ϕ^4-term. The interaction energy behaves under scale transformations (7.2) and (7.3) as follows:

$$E_\lambda[\phi'_\alpha] = \int d^Dx \frac{\lambda}{4!}\phi'^4_\alpha(\mathbf{x}) = e^{4d^0_\phi\alpha}\int d^Dx \frac{\lambda}{4!}\phi^4(e^\alpha\mathbf{x}) = e^{(4d^0_\phi - D)\alpha}\int d^Dx \frac{\lambda}{4!}\phi^4(\mathbf{x}) = e^{(D-4)\alpha}E_\lambda[\phi]. \tag{7.27}$$

This symmetry breakdown may be viewed as a consequence of the fact that, in $D \neq 4$ dimensions, the coupling constant is not dimensionless, but has the naive dimension $[\lambda] = \mu^{4-D}$. With a new dimensionless coupling constant

$$\hat{\lambda} \equiv \mu^{D-4}\lambda, \tag{7.28}$$

this dimension is made explicit by re-expressing the energy functional (7.26) as

$$E_{0,\lambda}[\phi] = \int d^Dx \left[\frac{1}{2}(\partial\phi)^2 + \frac{\mu^{4-D}\hat{\lambda}}{4!}\phi^4\right] \tag{7.29}$$

with some mass parameter μ.

The theory we want to investigate contains both mass and interaction terms in the energy functional:

$$E[\phi] = \int d^Dx \left[\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\mu^{4-D}\hat{\lambda}}{4!}\phi^4\right]. \tag{7.30}$$

The scale invariance of the free-field term is broken in four dimensions by the mass term, and in less than four dimensions by the mass and interaction terms.

Although the complete energy functional (7.30) is no longer invariant under the scale transformations (7.2) and (7.3), one can derive consequences from the nontrivial transformation behavior of its parts (7.25) and (7.27). These consequences may be formulated in the form of Ward identities characterizing the breakdown of scale invariance.

7.3.1 Ward Identities for Broken Scale Invariance

There exists a functional formalism for deriving the invariance property (7.15) of arbitrary n-point functions from the scale invariance of the free-field theory. Consider the generating functional (2.13), normalized to $Z[0] = 1$:

$$Z[j] = (Z_0^{\text{phys}})^{-1} \int \mathcal{D}\phi\, e^{-E_0[\phi] + \int d^D x\, j\phi}, \tag{7.31}$$

and change the coordinates and the field in the functional integrand by the scale transformations (7.2) and (7.3). Since the free-field energy functional $E_0[\phi]$ is invariant, and since the measures of functional integration in numerator and denominator transform in the same way, we find the property

$$Z[j] = Z[j'_{-\alpha}], \tag{7.32}$$

where

$$j'_{-\alpha}(\mathbf{x}) = e^{(d_\phi^0 - D)\alpha} j(e^{-\alpha}\mathbf{x}) \tag{7.33}$$

is the scale-transformed current. Infinitesimally, the equality (7.32) becomes

$$Z[j] = Z[j - \alpha(D - d_0^\phi + \mathbf{x}\partial_\mathbf{x})j], \tag{7.34}$$

which can be written as

$$\int d^D x [(D - d_\phi^0 + \mathbf{x}\partial_\mathbf{x}) j(\mathbf{x})] \frac{\delta Z[j]}{\delta j(\mathbf{x})} = 0. \tag{7.35}$$

A partial integration brings this to the form

$$-\int d^D x j(\mathbf{x})(d_\phi^0 + \mathbf{x}\partial_\mathbf{x}) \frac{\delta Z[j]}{\delta j(\mathbf{x})} = 0. \tag{7.36}$$

Differentiating this functionally $n-1$ times with respect to the current $j(\mathbf{x})$ yields the invariance property (7.15) for the correlation functions.

Let us now see what happens if a mass term is added to the energy functional. According to (7.25), the massive energy functional transforms infinitesimally like

$$\delta E[\phi] = -2\alpha E_m[\phi]. \tag{7.37}$$

In the presence of a mass term, we consider now the extended generating functional

$$Z[j, K] = (Z_0^{\text{phys}})^{-1} \int \mathcal{D}\phi\, e^{-E_0[\phi] + \int dx (j\phi + K\phi^2/2)}. \tag{7.38}$$

The additional source $K(\mathbf{x})$ permits us to generate n-point functions with additional insertions of quadratic terms $\phi^2(\mathbf{x})$. Recall their introduction in Section 2.4. A mass term in the energy corresponds to a constant background source $K(\mathbf{x}) \equiv -m^2$.

Under an infinitesimal scale transformation (7.13), the generating functional (7.38) behaves like

$$Z[j, K] = Z[j - \alpha(D - d_\phi^0 + \mathbf{x}\partial_\mathbf{x})j, K - \alpha(D - 2d_\phi^0 + \mathbf{x}\partial_\mathbf{x})K], \tag{7.39}$$

implying the relation

$$\int d^D x \left\{ (D - d_\phi^0 + \mathbf{x}\partial_\mathbf{x}) j(\mathbf{x}) \frac{\delta Z[j, K]}{\delta j(\mathbf{x})} + (D - 2d_\phi^0 + \mathbf{x}\partial_\mathbf{x}) K(\mathbf{x}) \frac{\delta Z[j, K]}{\delta K(\mathbf{x})} \right\} = 0. \tag{7.40}$$

After a partial integration, this becomes

$$-\int d^D x \left\{ j(\mathbf{x})(d^0_\phi + \mathbf{x}\partial_{\mathbf{x}}) \frac{\delta Z[j,K]}{\delta j(\mathbf{x})} + K(\mathbf{x})(2d^0_\phi + \mathbf{x}\partial_{\mathbf{x}}) \frac{\delta Z[j,K]}{\delta K(\mathbf{x})} \right\} = 0. \tag{7.41}$$

Differentiating this n times with respect to $j(\mathbf{x})$ and setting $K = -m^2$, we find the *Ward identity*

$$\sum_{i=1}^{n} (d^0_\phi + \mathbf{x}_i\partial_{\mathbf{x}_i}) \langle \phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_n)\rangle - \frac{m^2}{2} \int d^D x (2d^0_\phi + \mathbf{x}\partial_{\mathbf{x}}) \langle \phi^2(\mathbf{x})\phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_n)\rangle = 0. \tag{7.42}$$

A partial integration brings this to the form

$$\sum_{i=1}^{n} (d^0_\phi + \mathbf{x}_i\partial_{\mathbf{x}_i}) \langle \phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_n)\rangle + m^2 \int d^D x \langle \phi^2(\mathbf{x})\phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_n)\rangle = 0. \tag{7.43}$$

It was shown in (2.58) that the integral over the $\phi^2/2$-insertion in an n-point function can be generated by differentiating an n-point function without insertion with respect to $-m^2$. Thus we arrive at the simple differential equation

$$\sum_{i=1}^{n} (d^0_\phi + \mathbf{x}_i\partial_{\mathbf{x}_i} - m\,\partial_m) \langle \phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_n)\rangle = 0. \tag{7.44}$$

By analogy with (7.16), this may be written as

$$\left(\sum_{i=1}^{n} \mathbf{x}_i\partial_{\mathbf{x}_i} - m\,\partial_m \right) G^{(n)}(\mathbf{x}_1,\dots,\mathbf{x}_n) = -nd^0_\phi\, G^{(n)}(\mathbf{x}_1,\dots,\mathbf{x}_n). \tag{7.45}$$

This equation implies that the correlation functions $G^{(n)}(\mathbf{x}_1,\dots,\mathbf{x}_n)$ are homogeneous of degree $-nd^0_\phi$ in $\mathbf{x}_i$ and $1/m$. They can therefore be written as

$$G^{(n)}(\mathbf{x}_1,\dots,\mathbf{x}_n) = m^{nd^0_\phi} f(m\mathbf{x}_i), \tag{7.46}$$

where $f(\mathbf{x}_i m)$ is some function of its dimensionless arguments. The two-point function (7.24) is an example of this general statement.

The interaction term can be included similarly by extending the generating functional $Z[j,K]$ in (7.38) to

$$Z[j,K,L] = (Z_0^{\rm phys})^{-1} \int \mathcal{D}\phi\, e^{-E_0[\phi] + \int d^D x (j\phi + K\phi^2/2 + L\phi^4/4!)}. \tag{7.47}$$

By going through the same derivations as before, we arrive at the differential equation

$$\left[\sum_{i=1}^{n} \mathbf{x}_i\partial_{\mathbf{x}_i} - m\,\partial_m - (4-D)\lambda\,\partial_\lambda \right] G^{(n)}(\mathbf{x}_1,\dots,\mathbf{x}_n) = -nd^0_\phi\, G^{(n)}(\mathbf{x}_1,\dots,\mathbf{x}_n), \tag{7.48}$$

implying the general homogeneity property of the correlation function

$$G^{(n)}(\mathbf{x}_1,\dots,\mathbf{x}_n) = m^{nd^0_\phi} f(m\mathbf{x}_i, m^{D-4}\lambda). \tag{7.49}$$

Upon multiplying Eq. (7.48) by $e^{-i\Sigma_i \mathbf{k}_i\mathbf{x}_i}$, and integrating over all spatial coordinates as in the treatment of Eq. (7.16), we find the momentum-space equation analogous to (7.18)

$$\left[\sum_{i=1}^{n} \mathbf{k}_i\partial_{\mathbf{k}_i} + m\,\partial_m + (4-D)\lambda\,\partial_\lambda \right] G^{(n)}(\mathbf{k}_1,\dots,\mathbf{k}_n) = n(d^0_\phi - D)\, G^{(n)}(\mathbf{k}_1,\dots,\mathbf{k}_n). \tag{7.50}$$

Taking $n = 2$ and removing an overall δ-function guaranteeing momentum conservation [recall (4.4)], we find the homogeneity of the Fourier-transformed correlation function $G(\mathbf{k})$:

$$[\mathbf{k}\partial_{\mathbf{k}} + m\,\partial_m + (4-D)\lambda\,\partial_\lambda]\,G(\mathbf{k}) = (2d_\phi^0 - D)\,G(\mathbf{k}) = -2\,G(\mathbf{k}). \tag{7.51}$$

For the vertex functions $\bar{\Gamma}(\mathbf{k}_1,\dots,\mathbf{k}_n)$ defined in (4.20), we obtain

$$\left[\sum_{i=1}^{n} \mathbf{k}_i\partial_{\mathbf{k}_i} + m\,\partial_m + (4-D)\lambda\,\partial_\lambda\right]\bar{\Gamma}^{(n)}(\mathbf{k}_1,\dots,\mathbf{k}_n) = (-nd_\phi^0 + D)\,\bar{\Gamma}^{(n)}(\mathbf{k}_1,\dots,\mathbf{k}_n), \tag{7.52}$$

implying the homogeneity property

$$\bar{\Gamma}^{(n)}(\mathbf{k}_1,\dots,\mathbf{k}_n) = m^{-nd_\phi^0+D} f(\mathbf{k}_i/m, m^{D-4}g). \tag{7.53}$$

After expressing the correlation functions in terms of the dimensionless coupling $\hat{\lambda}$ of Eq. (7.28), the differential equation (7.52) looses its derivative term with respect to λ and becomes

$$\left[\sum_{i=1}^{n-1} \mathbf{k}_i\partial_{\mathbf{k}_i} + m\,\partial_m\right]\bar{\Gamma}^{(n)}(\mathbf{k}_1,\dots,\mathbf{k}_n)\Big|_{\hat{\lambda}} = (-nd_\phi^0 + D)\,\bar{\Gamma}^{(n)}(\mathbf{k}_1,\dots,\mathbf{k}_n). \tag{7.54}$$

Note that the homogeneity relations derived in this way are actually a trivial consequence of the invariance of the theory under *trivial scale transformations* (also referred to as *engineering* or *technical scale transformations*) in which one changes $\mathbf{x}$ as in (7.2), $\phi(\mathbf{x})$ as in (7.3), and simultaneously

$$m \to e^{-\alpha}m, \qquad \lambda \to e^{-(4-D)\alpha}\lambda. \tag{7.55}$$

Every physical quantity changes by a phase factor $e^{-\alpha}$ for each of its mass dimension found by a naive dimensional analysis. These mass dimensions of correlation and vertex functions can be read off directly from Eqs. (7.49) and (7.53). These trivial scale transformations are *not* the origin of the physical scaling properties of a system near a second-order phase transition.

7.4 Anomaly in the Ward Identities

The above results would be correct if all correlation functions were finite. In a fluctuating field system, however, the infinitely many degrees of freedom give rise to infinities in the Feynman diagrams, which require a regularization. This leads to a correction term in the Ward identities (7.48) and (7.52).

Consider the case of four dimensions, where the interaction is scale invariant under (7.2) and (7.3). The Feynman integrals can be made finite only with the help of regularization procedures. These will be introduced in the next chapter. In the present context we use only the simplest of these based on a restriction of all momentum space integrals to a sphere of radius Λ. This is called a *cutoff regularization*. With a cutoff, all correlation functions depend on Λ in addition to $\mathbf{k}_i, m$, and λ. Now, a field theory such as the ϕ^4-theory under study has an important property which will be discussed in detail in Chapter 9: it is *renormalizable*. This will allow us to multiply the parameters of the theory m^2 and λ and all vertex functions $\bar{\Gamma}^{(n)}(\mathbf{k}_1,\dots,\mathbf{k}_n)$ by Λ-dependent *renormalization constants*, leading to new renormalized quantities which remain finite in the limit of an infinite cutoff Λ. To describe this procedure quantitatively, we shall call all quantities in the original energy density more specifically *bare quantities*, and emphasize this by attaching a subscript B to them. The quantities without this subscript will henceforth denote renormalized quantities. The multiplicative renormalization establishes the following relations between bare and renormalized quantities:

$$\bar{\Gamma}_B^{(2)}(\mathbf{0}) = Z_\phi^{-1}(\lambda, m, \Lambda)\bar{\Gamma}^{(2)}(\mathbf{0}), \tag{7.56}$$

$$m_B^2 = \frac{Z_{m^2}(\lambda, m, \Lambda)}{Z_\phi(\lambda, m, \Lambda)} m^2, \tag{7.57}$$

$$m_B^{D-4}\lambda_B = \frac{Z_\lambda(\lambda, m, \Lambda)}{Z_\phi^2(\lambda, m, \Lambda)} m^{D-4}\lambda. \tag{7.58}$$

The renormalizability of the ϕ^4-theory implies that the renormalized quantities $\bar{\Gamma}^{(2)}(\mathbf{0}), m^2, \lambda$ have a finite value in the limit of an infinite cutoff Λ.

Equation (7.58) contains factors of mass to the power $D-4$ which is zero in four dimensions. However, as we have announced in Section 4.4, it will be important to study the theory in a continuous number of dimensions, in particular in the neighborhood of $D = 4$. The equations (7.56)–(7.58) are then applicable for any D close to four.

Since the cutoff has the same dimension as the mass m, the function f in (7.49), which we shall now call f_B to emphasize its bare nature, and the similar function f in (7.53) depend in general also on Λ/m_B. In the differential equations (7.48) and (7.52), this corresponds to an extra derivative term $\mp\Lambda\partial_\Lambda$, respectively. Then the latter equation reads

$$\left[\sum_{i=1}^{n} \mathbf{k}_i\partial_{\mathbf{k}_i} + m_B\partial_{m_B} + \Lambda\partial_\Lambda + (4-D)\lambda_B\partial_{\lambda_B}\right] \bar{\Gamma}_B^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n)$$
$$= (-nd_\phi^0 + D)\, \bar{\Gamma}_B^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n). \tag{7.59}$$

The extra term $\Lambda\partial_\Lambda$ ruins the naive Ward identities, and is therefore called an *anomaly.* The general solution of Eq. (7.59) is

$$\bar{\Gamma}_B^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n) = m_B^{-nd_\phi^0+D} f_B(\mathbf{k}_i/m_B, m_B^{D-4}\lambda_B, \Lambda/m_B). \tag{7.60}$$

The understanding of the scaling properties of the theory will make it necessary to find the dependence of bare quantities in (7.56)–(7.58) on the cutoff Λ.

We have stated above that the renormalized vertex functions $\bar{\Gamma}^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n)$ have a finite limit for $\Lambda \to \infty$. Using the same naive dimensional arguments by which we interpreted the functional form (7.53) derived from the Ward identity (7.52), we can now conclude that the original Ward identity (7.54) remains valid for the renormalized vertex functions, if the derivatives refer to renormalized masses and coupling constants.

The above statements are independent of the way the theory is regularized. Since we are going to define the theory for a continuous number of dimensions, another regularization will be possible, called *analytic regularization.* It will turn out that after an analytic continuation, the integrals which require a cutoff in four dimensions no longer need a cutoff in $D \leq 4$ dimensions. Nevertheless, the detailed specification of the infinities requires a mass parameter, which in a massive theory can be the physical mass m, but in a massless theory must be introduced separately, usually under the name μ. In this case, the derivative $\mu\partial_\mu$ replaces the term $\Lambda\partial_\Lambda$ in (7.59).

The anomaly in the Ward identity (7.59) will turn out to be the origin of the nontrivial critical exponents in the scaling laws observed in critical phenomena, as we shall see in Chapter 10. The reason is that in the limit $m \to 0$, the renormalization factors $Z(\lambda, m, \Lambda)$ in Eqs. (7.56)–(7.58) behave like powers $(m/\Lambda)^{\text{power}}$. Moreover, for small bare mass m_B, the renormalized quantity $m^{D-4}\lambda$ tends to a constant, as we shall see in Section 10.5, where this limit is denoted by g^*.

In order to illustrate the nontrivial scaling behavior more specifically, consider the two-point vertex function (7.53). Before renormalization, it has the general form

$$\bar{\Gamma}_B^{(2)}(\mathbf{k}) = m_B^2 f_B(\mathbf{k}/m_B, m_B^{D-4}\lambda_B, \Lambda/m_B). \tag{7.61}$$

At zero momentum this becomes

$$\bar{\Gamma}_B^{(2)}(\mathbf{0}) = m_B^2 f_B(m_B^{D-4}\lambda_B, \Lambda/m_B). \tag{7.62}$$

After renormalization, the Λ-dependence disappears for large Λ, so that the functional dependence becomes [see (7.53)]

$$\bar{\Gamma}^{(2)}(\mathbf{k}) = m^2 f(\mathbf{k}/m, m^{D-4}g). \tag{7.63}$$

Since the renormalized quantity $m^{D-4}\lambda$ tends to a constant g^* for $m_B \to 0$, the renormalized vertex function tends towards $m^2 f(\mathbf{k}/m)$. The function may be expanded in powers of $\mathbf{k}/m$. Rotational invariance leads to $f(\mathbf{k}/m) = c_1 + c_2\mathbf{k}^2/m^2 + \dots$. The renormalization constants Z_ϕ, Z_{m^2} may be chosen such that the constants c_1 and c_2 are both equal to 1, so that the vertex function starts out for small $\mathbf{k}$ like

$$\bar{\Gamma}^{(2)}(\mathbf{k}) = m^2 + \mathbf{k}^2 + \dots\ . \tag{7.64}$$

From these first two terms we identify the coherence length as being equal to the inverse mass:

$$\xi = 1/m. \tag{7.65}$$

In the critical regime, the renormalization constants show power behavior, so that the renormalization equations (7.56)—(7.58) read for small m:

$$\bar{\Gamma}_B^{(2)}(\mathbf{k}) \underset{m_B\approx 0}{\approx} \left(\frac{\Lambda}{m}\right)^{\eta} \bar{\Gamma}^{(2)}(\mathbf{k}), \tag{7.66}$$

$$m_B^2 \underset{m_B\approx 0}{\approx} \left(\frac{\Lambda}{m}\right)^{\eta_m} m^2, \tag{7.67}$$

$$\lambda_B \underset{m_B\approx 0}{\approx} \left(\frac{\Lambda}{m}\right)^{\beta+4-D} \lambda, \tag{7.68}$$

where η, η_m, and β are constants depending on g^*. The constancy of $m^{D-4}\lambda$ in the limit $m_B \to 0$ implies that β vanishes at g^*.

For the functions f_B and f in Eqs. (7.61) and (7.63), equation (7.66) implies that

$$m_B^2 f_B(\mathbf{0}, m_B^{D-4}\lambda_B, \Lambda/m_B) \underset{m_B\approx 0}{\approx} \left(\frac{\Lambda}{m}\right)^{\eta} m^2 f(\mathbf{0}, g^*). \tag{7.69}$$

By assumption, the bare mass parameter in the initial energy density behaves near the critical temperature T_c like $m_B^2 \propto t$ where $t \equiv T/T_c - 1$. For the renormalized mass we find from (7.67)

$$m \propto t^{1/(2-\eta_m)}, \tag{7.70}$$

implying that the coherence length diverges like $t^{-1/(2-\eta_m)}$. This fixes the critical exponents ν defined in (1.10) as being

$$\nu = \frac{1}{2-\eta_m}. \tag{7.71}$$

The bare vertex function $\bar{\Gamma}_B^{(2)}(\mathbf{0})$ behaves therefore like $t^{(2-\eta)/(2-\eta_m)}$. This quantity is observable in magnetic susceptibility experiments:

$$\chi \propto G(\mathbf{0}) = \frac{1}{\bar{\Gamma}_B^{(2)}(\mathbf{0})} \propto m^{-(2-\eta)} \propto t^{-(2-\eta)/(2-\eta_m)}. \tag{7.72}$$

Recalling (1.17), we identify the associated critical exponent as

$$\gamma = \frac{2-\eta}{2-\eta_m} = \nu(2-\eta). \tag{7.73}$$

The existence of such power laws may be interpreted as a consequence of modified exact scale invariance, which are analogous to the free scaling equations (7.20), but hold now for the renormalized correlation functions and the corresponding vertex functions. In fact, in Section 10.1 we shall prove that at the critical point where $m = 0$ and $m^{D-4}g = g^*$, the renormalized vertex functions satisfy a scaling equation

$$\left(\sum_{i=1}^{n-1} \mathbf{k}_i \partial_{\mathbf{k}_i}\right) \bar{\Gamma}^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n) = \left[-n(d_\phi^0 + \eta/2) + D\right] \bar{\Gamma}^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n). \tag{7.74}$$

Thus we recover, at the critical point, the free massless scaling equations (7.20) with only a small modification: the *value* of the free-field dimension d_ϕ^0 on the right-hand side is changed to the new value

$$d_\phi = d_\phi^0 + \eta/2. \tag{7.75}$$

The number $\eta/2$ is called the *anomalous dimension* of the interacting field ϕ. Remembering the derivation of the differential equation (7.20) from the invariance (7.9) under the transformation (7.3), we see that the renormalized correlation functions of the interacting theory are obviously invariant under the modified scale transformations of the renormalized fields

$$\phi(\mathbf{x}) \to \phi'_\alpha(\mathbf{x}) = e^{d_\phi \alpha} \phi(e^\alpha \mathbf{x}). \tag{7.76}$$

Thus, at the critical point, the scale-breaking interactions in the energy functional do not lead to a destruction of the scale invariance of the free-field correlation functions, but to a new type of scale invariance with a different field dimension.

The behavior of the coherence length $\xi \propto t^{-\nu}$ may also be interpreted as a consequence of such modified transformation laws. Initially, the coherence length is given by the inverse bare mass $m_B^{-1} \propto t^{-1/2}$. After the interaction is turned on, the role is taken over by the inverse renormalized mass, whose dimension is m_B^ν rather than m_B.

These general considerations will be made specific in the subsequent chapters. Our goal is to calculate the properties of interacting ϕ^4-theories in the scaling regime. Since these theories are naively scale invariant only in $D = 4$ dimensions, it was realized by Wilson that the modified scaling laws in the physical dimension $D = 3$ become accessible by considering the correlation functions as analytic functions in D. By expanding all results around $D = 4$ in powers of $\epsilon = 4 - D$, the scale invariance can be maintained at every level of calculations. This is the approach to be followed in this text.

Notes and References

For supplementary reading see the introduction into four-dimensional dilation invariance of quantum field theory by

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