6

Diagrams for Multicomponent Fields

So far, we have considered only a single real field ϕ with a ϕ^4-interaction. The theory can, however, easily be extended to a set of N identical real fields ϕ_α with $\alpha = 1, \dots, N$. This extension does not produce any new Feynman diagrams. Additional work arises from the fact that the coupling constant g becomes now a tensor $g_{\alpha\beta\gamma\delta}$, and each momentum integral in a Feynman diagram is accompanied by a corresponding sum over indices.

If the coupling tensor is decomposed into basis tensors which satisfy certain symmetry and completeness properties, the result of each index contraction can again be decomposed into these basis tensors, with invariant factors called *symmetry factors* S_G. In this chapter we show how these are calculated.

6.1 Interactions with O(N) and Cubic Symmetry

In Section 1.1 we have explained that many physical systems are described by an O(N)-symmetric ϕ^4-theory with N identical fields. The simplest example is superfluid ^{4}He, whose local fluctuations are described by a complex field $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$, with an energy functional

$$E[\phi] = \int d^D x \left\{ \frac{1}{2} |\partial_i \phi(\mathbf{x})|^2 + \frac{m^2}{2} |\phi(\mathbf{x})|^2 + \frac{\lambda}{4!} \left(|\phi(\mathbf{x})|^2 \right)^2 \right\}. \tag{6.1}$$

This may be written as an O(2)-symmetric functional of a two-component vector field $\boldsymbol{\phi}(\mathbf{x}) = (\phi_1(\mathbf{x}), \phi_1(\mathbf{x}))$, of the form (1.83):

$$E[\boldsymbol{\phi}] = \int d^D x \left\{ \frac{1}{2} \partial_i \boldsymbol{\phi}(\mathbf{x}) \partial_i \boldsymbol{\phi}(\mathbf{x}) + \frac{m^2}{2} \boldsymbol{\phi}^2(\mathbf{x}) + \frac{\lambda}{4!} [\boldsymbol{\phi}^2(\mathbf{x})]^2 \right\}. \tag{6.2}$$

As explained in Chapter 1, the same energy functional with a three-component vector field $(\phi_1,\ \phi_2,\ \phi_3)$ describes the ferromagnetic phase transition, in which case the mass and interaction terms are proportional to

$$\begin{aligned} \boldsymbol{\phi}^2 &\to \phi_1^2 + \phi_2^2 + \phi_3^2, \\ |\boldsymbol{\phi}^2|^2 &\to \left(\phi_1^2 + \phi_2^2 + \phi_3^2 \right)^2. \end{aligned} \tag{6.3}$$

Beside the O(N)-symmetric interaction term, another fourth-order term is frequently encountered in physical systems with cubic crystalline structure, which will also be treated in this text. All our results up to five loops will be derived for an interaction energy with a mixture of an O(N)-symmetric and a cubic-symmetric interaction energy with two coupling constants, where the fourth-order term in (6.2) is replaced by

$$E_{\rm int}[\boldsymbol{\phi}] = \int d^D x \left[\lambda_1 \left(\sum_{\alpha=1}^{N} \phi_\alpha^2 \right)^2 + \lambda_2 \sum_{\alpha=1}^{N} \phi_\alpha^4 \right]. \tag{6.4}$$

This energy exhibits a broken O(N) symmetry, with physical consequences to be investigated in Chapter 18. In particular, we shall answer the important physical question as to the circumstances under which the broken O(N) symmetry can be restored by the violent fluctuations near the phase transition. In this case a system with a cubic-symmetric interaction energy has the same critical exponents as an isotropic system.

6.2 Free Generating Functional for N Fields

The energy functional for N free real fields is

$$E_0[\boldsymbol{\phi}] \;=\; \frac{1}{2}\int d^Dx \sum_{\alpha=1}^{N}\left\{[\partial_{\mathbf{x}}\phi_\alpha(\mathbf{x})]^2 \;+\; m^2\phi_\alpha^2(\mathbf{x})\right\}. \tag{6.5}$$

The equal mass for all field components makes the energy invariant under O(N)-rotations. The path integral for the free partition function factorizes into a product of N identical path integrals [see (2.7)]:

$$Z_0^{\text{phys}} = \prod_{\alpha=1}^{N}\left[\int \mathcal{D}\phi_\alpha \exp\left(\int d^Dx\left\{\frac{1}{2}[\partial\phi_\alpha(\mathbf{x})]^2 + \frac{1}{2}\,m^2\phi_\alpha^2(\mathbf{x})\right\}\right)\right]. \tag{6.6}$$

When adding an O(N)-symmetric linear source term

$$E_{\text{source}} = -\int d^Dx \sum_{\alpha=1}^{N}\phi_\alpha(\mathbf{x})j_\alpha(\mathbf{x}) \tag{6.7}$$

to the energy functional, the factorization property of (6.6) remains unchanged, and we obtain, with the normalization (2.9),

$$Z_0[\mathbf{j}] = \prod_{\alpha=1}^{N}\exp\left\{\frac{1}{2}\int d^Dx\, d^Dy\, j_\alpha(\mathbf{x})G_0(\mathbf{x},\mathbf{y})j_\alpha(\mathbf{y})\right\}. \tag{6.8}$$

The basic functional differentiation with respect to the current contains now an additional Kronecker symbol $\delta_{\alpha\beta}^{(N)}$ for the N field indices:

$$\frac{\delta}{\delta j_\alpha(\mathbf{x})}\int d^Dz\, j_\beta(\mathbf{z}) = \delta_{\alpha\beta}^{(N)}\int d^Dz\,\delta^{(D)}(\mathbf{x}-\mathbf{z})\,. \tag{6.9}$$

The same Kronecker symbol appears now as a factor in the propagator of the free theory:

$$G_{0;\alpha\beta}^{(2)}(\mathbf{x}_1,\mathbf{x}_2) \;\equiv \langle\phi_\alpha(\mathbf{x}_1)\phi_\beta(\mathbf{x}_2)\rangle_0, \tag{6.10}$$

which reads

$$\begin{aligned}
G_{0;\alpha\beta}^{(2)}(\mathbf{x},\mathbf{y}) \;&=\; \frac{\delta}{\delta j_\alpha(\mathbf{x})}\frac{\delta}{\delta j_\beta(\mathbf{y})}Z_0[\mathbf{j}]\Big|_{\mathbf{j}=0}\\
&=\; \frac{\delta}{\delta j_\alpha(\mathbf{x})}\frac{\delta}{\delta j_\beta(\mathbf{y})}\prod_{\gamma=1}^{N}\exp\left\{\frac{1}{2}\int d^Dx_1\, d^Dx_2\; j_\gamma(\mathbf{x}_1)G_0(\mathbf{x}_1,\mathbf{x}_2)j_\gamma(\mathbf{x}_2)\right\}\Big|_{\mathbf{j}=0}\\
&=\; \delta_{\alpha\beta}^{(N)}\,G_0(\mathbf{x},\mathbf{y}),
\end{aligned} \tag{6.11}$$

where $G_0(\mathbf{x}, \mathbf{y})$ is the propagator of the scalar field without labels α. The Fourier transformation yields the momentum space representation of the propagator

$$G^{(2)}_{0;\alpha\beta}(\mathbf{p}_1, \mathbf{p}_2) \;=\; (2\pi)^D \delta^{(D)}(\mathbf{p}_1 + \mathbf{p}_2)\, G_{0;\alpha\beta}(\mathbf{p}_1), \tag{6.12}$$

with

$$G_{0;\alpha\beta}(\mathbf{p}) \;=\; \delta^{(N)}_{\alpha\beta}\; G_0(\mathbf{p}) \;=\; \delta^{(N)}_{\alpha\beta} \frac{1}{\mathbf{p}^2 + m^2}. \tag{6.13}$$

By Wick's theorem, the n-point correlation functions of the free theory

$$G^{(2n)}_{0;\alpha_1\ldots\alpha_{2n}}(\mathbf{x}_1, \ldots, \mathbf{x}_{2n})$$

are sums over products of n two-point functions corresponding to all $(2n-1)!!$ pair contractions. Each contraction involves a free propagator $G_0(\mathbf{p})$ and a Kronecker symbol $\delta^{(N)}_{\alpha\beta}$.

6.3 Perturbation Expansion for N Fields and Symmetry Factors

The interaction energy (6.4) is a special case of the most general local fourth-order expression

$$E_{\rm int}[\boldsymbol{\phi}] \;=\; \frac{1}{4!} \int d^D x \sum_{\alpha,\beta,\gamma,\delta=1}^{N} \lambda_{\alpha\beta\gamma\delta}\; \phi_\alpha(\mathbf{x}) \phi_\beta(\mathbf{x}) \phi_\gamma(\mathbf{x}) \phi_\delta(\mathbf{x})\,, \tag{6.14}$$

where $\lambda_{\alpha\beta\gamma\delta}$ is some combination of basis tensors:

$$\lambda_{\alpha\beta\gamma\delta} = \sum_i \lambda_i\, T^{(i)}_{\alpha\beta\gamma\delta}\,. \tag{6.15}$$

The basis tensors may be chosen to be symmetric in all indices. This greatly reduces their number. If they are not initially, we may replace them by

$$T^{(i)}_{\alpha\beta\gamma\delta} \to \frac{1}{24} \left[T^{(i)}_{\alpha\beta\gamma\delta} + \;\left\{ 23 \text{ perm.} \right\} \right]. \tag{6.16}$$

For systems with a symmetry, there are further limitations, which are analyzed systematically in the literature [1]. Further limitations come from the structure of the theory. First, the perturbative corrections to the free part of the energy functional should maintain the initial O(N) symmetry. This is only possible if the contraction of any two indices of all tensors $T^{(i)}_{\alpha\beta\gamma\delta}$ produces a Kronecker symbol $\delta^{(N)}_{\alpha\beta}$. Second, the perturbative corrections to the interaction energy should not produce new interactions that are not contained in the initial energy functional. The tensors $T^{(i)}_{\alpha\beta\gamma\delta}$ must therefore be complete in the sense that the contraction in two indices, of a product of any two linear combinations of $T^{(i)}_{\alpha\beta\gamma\delta}$ must yield again a linear combination of $T^{(i)}_{\alpha\beta\gamma\delta}$. This property will be important for the renormalizability of the theory. Tensors $T^{(i)}_{\alpha\beta\gamma\delta}$ with these properties will be referred to as *symmetry tensors*.

The expansion of the generating functional in powers of g reads

$$\begin{aligned} Z[\mathbf{j}] \;&=\; \sum_{p=0}^{\infty} \frac{1}{p!} \left(\frac{-1}{4!} \right)^p \int \mathcal{D}\boldsymbol{\phi}(\mathbf{x}) \int d^D z_1 \cdots d^D z_p \\ &\quad \times \prod_{i=1}^{p} \sum_{\alpha_1^{(i)},\ldots,\alpha_4^{(i)}=1}^{N} \lambda_{\alpha_1^{(i)}\alpha_2^{(i)}\alpha_3^{(i)}\alpha_4^{(i)}} \phi_{\alpha_1^{(i)}}(\mathbf{z}_i) \cdots \phi_{\alpha_4^{(i)}}(\mathbf{z}_i). \end{aligned} \tag{6.17}$$

The n-point correlation functions of the theory with interaction are obtained from the higher functional derivatives

$$G^{(n)}_{0;\alpha_1\ldots\alpha_n}(\mathbf{x}_1,\ldots,\mathbf{x}_n) \equiv \langle\phi_{\alpha_1}(\mathbf{x}_1)\cdots\phi_{\alpha_n}(\mathbf{x}_n)\rangle = Z^{-1}\left[\int\frac{\delta}{\delta j_{\alpha_1}(\mathbf{x}_1)}\cdots\frac{\delta}{\delta j_{\alpha_n}(\mathbf{x}_n)}Z[j]\right]_{\mathbf{j}\equiv 0}. \tag{6.18}$$

The pth order term of the perturbation expansion for the n-point function reads, by analogy with Eq. (3.2):

$$G^{(n)}_{p\,\alpha_1\ldots\alpha_n}(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \left(\frac{-1}{4!}\right)^p\frac{1}{p!}\frac{1}{(2p+n/2)!\,2^{2p+n/2}}\frac{\delta}{\delta j_{\alpha_1}(\mathbf{x}_1)}\cdots\frac{\delta}{\delta j_{\alpha_n}(\mathbf{x}_n)}$$
$$\times\prod_{i=1}^{p}\left[\int d^Dz_i\sum_{\gamma_1^{(i)},\ldots,\gamma_4^{(i)}}^{N}\lambda_{\gamma_1^{(i)}\gamma_2^{(i)}\gamma_3^{(i)}\gamma_4^{(i)}}\frac{\delta}{\delta j_{\gamma_1^{(i)}}(\mathbf{z}_i)}\frac{\delta}{\delta j_{\gamma_2^{(i)}}(\mathbf{z}_i)}\frac{\delta}{\delta j_{\gamma_3^{(i)}}(\mathbf{z}_i)}\frac{\delta}{\delta j_{\gamma_4^{(i)}}(\mathbf{z}_i)}\right]$$
$$\times\left[\sum_{\beta=1}^{N}\int d^Dz\,d^Dz'\,j_\beta(\mathbf{z})G_0(\mathbf{z},\mathbf{z}')j_\beta(\mathbf{z}')\right]^{2p+n/2}. \tag{6.19}$$

As before, this yields a sum over $(4p+n-1)!!$ terms. We write them down in the notation of Eq. (3.3), in which the tensor indices associated with different line ends are first distinguished, and their contractions with the tensor indices of the vertices are enforced by additional Kronecker symbols:

$$G^{(n)}_{p\,\alpha_1\ldots\alpha_n}(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \left(\frac{-1}{4!}\right)^p\frac{1}{p!}\int d^Dz_1\cdots d^Dz_p\int d^Dy_1\cdots d^Dy_{4p+n}$$
$$\times\prod_{l=1}^{n}\left[\delta^{(D)}(\mathbf{y}_{4p+l}-\mathbf{x}_l)\right]\prod_{k=1}^{p}\left[\delta^{(D)}(\mathbf{y}_{4k-3}-\mathbf{z}_k)\delta^{(D)}(\mathbf{y}_{4k-2}-\mathbf{z}_k)\delta^{(D)}(\mathbf{y}_{4k-1}-\mathbf{z}_k)\delta^{(D)}(\mathbf{y}_{4k}-\mathbf{z}_k)\right]$$
$$\times\sum_{\gamma_1^{(1)},\ldots,\gamma_4^{(p)},\beta_1,\ldots,\beta_{4p+n}=1}^{N}\delta^{(N)}_{\beta_1\gamma_1^{(1)}}\cdots\delta^{(N)}_{\beta_{4p}\gamma_4^{(p)}}\cdot\delta^{(N)}_{\beta_{4p+1}\alpha_1}\cdots\delta^{(N)}_{\beta_{4p+n}\alpha_n}\cdot\lambda_{\beta_1\beta_2\beta_3\beta_4}\cdots\lambda_{\beta_{4p-3}\ldots\beta_{4p}}$$
$$\times\sum_{i=1}^{(4p+n-1)!!}G_0(\mathbf{y}_{\pi_i(1)},\mathbf{y}_{\pi_i(2)})\cdots G_0(\mathbf{y}_{\pi_i(4p+n-1)},\mathbf{y}_{\pi_i(4p+n)})\,\delta_{\beta_{\pi_i(1)}\beta_{\pi_i(2)}}\cdots\delta_{\beta_{\pi_i(4p+n-1)}\beta_{\pi_i(4p+n)}}. \tag{6.20}$$

6.4 Symmetry Factors

The last sum over the index i in Eq. (6.20) runs over the relevant index permutations π_i defined for $N=1$ in Eq. (3.3). These expansion terms differ from those for $N=1$ only by a factor arising from a contraction of p symmetric tensors $\lambda_{\beta_1\beta_2\beta_3\beta_4}\cdots\lambda_{\beta_{4p-3}\ldots\beta_{4p}}$. This product of symmetry tensors will be called the *tensor factor* F_G of a diagram.

Since $\lambda_{\alpha\beta\gamma\delta}$ is symmetric in all its indices, the tensor factor F_G is invariant under all index permutations which merely relabel the lines of a vertex. It is also invariant under permutations which relabel the vertices, as this amounts only to a change of the order of the factors in the tensor product. As a consequence, the tensor contractions associated with a specific Feynman integral can also be represented by the same Feynman diagram, in which lines indicate index contractions rather than integrals. Such contractions yield sums of tensors $T^{(i)}_{\alpha\beta\gamma\delta}$ for $n=4$, or simply $\delta_{\alpha\beta}$ for $n = 2$, and a factor consisting of a sum over all possible distributions of coupling constants. For two coupling constants and an L-loop diagram with V vertices, we have as tensor factors:

$$F_G = \sum_{k=0}^{V} \lambda_1^{V-k} \lambda_2^{k} \left[S_{4_1;(V-k,k)}\, T^{(1)}_{\alpha\beta\gamma\delta} + S_{4_2;(k,V-k)}\, T^{(2)}_{\alpha\beta\gamma\delta} \right], \quad \text{for } n = 4, \tag{6.21}$$

$$F_G = \sum_{k=0}^{V} \lambda_1^{V-k} \lambda_2^{k} \left[S_{2\,;(V-k,k)} \delta_{\alpha\beta} \right], \quad \text{for } n = 2. \tag{6.22}$$

In these expressions we can replace V by L, since $V = L$ for a two-point diagram and $V = L + 1$ for a four-point diagram. This will be done in Appendix B where all symmetry factors up to five loops are listed and organized by the number of loops L. The factors $S_{2;(i,j)}$, $S_{4_1;(i,j)}$, and $S_{4_2;(i,j)}$ in (6.21) and (6.22) are called the *symmetry factors* of a diagram. The subscript records the number of external lines n, and the combination of coupling constants associated with the vertices. The label i is the power of λ_1, while j is the power of λ_2. Note that $S_{4_1;(0,V)} = S_{4_2;(V,0)} = 0$ for mixed $\mathrm{O}(N)$ and cubic symmetry, since contractions of $T^{(1)}$-type tensors cannot produce a $T^{(2)}$-type tensor, and vice versa. This fact will be important for the renormalizability of the theory with pure $\mathrm{O}(N)$ or cubic symmetry.

For one coupling constant and an L-loop diagram, the tensor factors reduce to:

$$F_G = \lambda^V\, S_4\, T^{(1)}_{\alpha\beta\gamma\delta}, \qquad \text{for } n = 4, \tag{6.23}$$

$$F_G = \lambda^V\, S_2\, \delta_{\alpha\beta}, \qquad \text{for } n = 2. \tag{6.24}$$

In this case the index of the symmetry factor consists only of the number of lines, since there is only one symmetry factor for each diagram. Since $T^{(i)}_{\alpha\beta\gamma\delta}$ and $\delta_{\alpha\beta}$ are normalized to unity for $N = 1$, the symmetry factors will all reduce to unity for $N = 1$. This is not true for more than one coupling constant. For two coupling constants, $S_{2;(V-k,k)}$ and $S_{4_1;(V-k,k)} + S_{4_2;(V-k,k)}$ reduce to $V!/(V-k)!k!$, the number of ways the coupling constants $\lambda_1^{V-k}\lambda_2^{k}$ can be arranged on the V vertices. We shall see this explicitly in Section 6.4.2 when calculating the symmetry factors for such a theory.

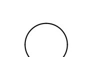
6.4.1 Symmetry Factors for O(*N*) Symmetry

The tensor $\lambda_{\alpha\beta\gamma\delta}$ introduced in Eq. (6.14) takes the following form for the $\mathrm{O}(N)$-symmetric interaction energy of Eq. (6.4):

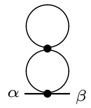
$$\lambda^{\mathrm{O}(N)}_{\alpha\beta\gamma\delta} = \frac{\lambda}{3} \left(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma} \right) = \lambda\, T^{(1)}_{\alpha\beta\gamma\delta}\,. \tag{6.25}$$

The tensor $T^{(1)}$ is one of the basis tensors which also appears in the mixed $\mathrm{O}(N)$-cubic-symmetric theory with two coupling constants.

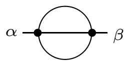
For the Feynman diagrams occurring in the perturbation expansions of the self-energy (4.27) and the vertex function (4.25) up to two loops, we have to form the corresponding tensor contractions. The symmetry factors for the $\mathrm{O}(N)$ symmetry will, in general, be denoted by an upper index $S^{\mathrm{O}(N)}$. Since we only consider $\mathrm{O}(N)$-symmetric tensors in this section, we shall omit this index.



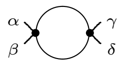
$$: \ \lambda^{\mathrm{O}(N)}_{\alpha\beta\sigma\sigma} \qquad = \lambda\, S_{\bigcirc}\, \delta_{\alpha\beta},$$



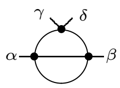
$$: \quad \lambda^{O(N)}_{\alpha\beta\sigma_1\sigma_2}\lambda^{O(N)}_{\sigma_1\sigma_2\sigma_3\sigma_3} = \lambda^2 S\, \delta_{\alpha\beta} = \lambda^2 S^2\, \delta_{\alpha\beta},$$



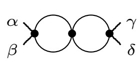
$$: \quad \lambda^{O(N)}_{\alpha\sigma_1\sigma_2\sigma_3}\lambda^{O(N)}_{\sigma_1\sigma_2\sigma_3\beta} = \lambda^2 S\, \delta_{\alpha\beta},$$



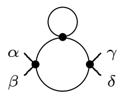
$$: \quad \lambda^{O(N)}_{\alpha\beta\sigma_1\sigma_2}\lambda^{O(N)}_{\sigma_1\sigma_2\gamma\delta}\Big|_{\rm sym} = \lambda^2 S\, T^{(1)}_{\alpha\beta\gamma\delta}, \tag{6.26}$$



$$: \quad \lambda^{O(N)}_{\alpha\sigma_1\sigma_2\sigma_3}\lambda^{O(N)}_{\beta\sigma_1\sigma_2\sigma_4}\lambda^{O(N)}_{\sigma_3\sigma_4\gamma\delta}\Big|_{\rm sym} = \lambda^3 S\, T^{(1)}_{\alpha\beta\gamma\delta},$$



$$: \quad \lambda^{O(N)}_{\alpha\beta\sigma_1\sigma_2}\lambda^{O(N)}_{\sigma_1\sigma_2\sigma_3\sigma_4}\lambda^{O(N)}_{\sigma_3\sigma_4\gamma\delta}\Big|_{\rm sym} = \lambda^3 S\, T^{(1)}_{\alpha\beta\gamma\delta},$$



$$: \quad \lambda^{O(N)}_{\alpha\beta\sigma_1\sigma_2}\lambda^{O(N)}_{\sigma_2\sigma_3\sigma_3\sigma_4}\lambda^{O(N)}_{\gamma\delta\sigma_1\sigma_4}\Big|_{\rm sym} = \lambda^3 S\, T^{(1)}_{\alpha\beta\gamma\delta} = \lambda^3 S\, S\, T^{(1)}_{\alpha\beta\gamma\delta}.$$

The external indices will always be labeled α, β, γ, and δ. The symbol $|_{\rm sym}$ denotes symmetrization of these indices, which replaces each product $\delta_{\alpha\beta}\delta_{\gamma\delta}$ by $T^{(1)}_{\alpha\beta\gamma\delta}$.

In order to calculate the irreducible matrix elements S_n we observe that

$$\lambda^{O(N)}_{\alpha\beta\sigma\sigma} = \lambda\,\frac{N+2}{3}\,\delta_{\alpha\beta}, \tag{6.27}$$

$$\lambda^{O(N)}_{\alpha\sigma_1\sigma_2\sigma_3}\lambda^{O(N)}_{\sigma_1\sigma_2\sigma_3\beta} = \lambda^2\,\frac{N+2}{3}\,\delta_{\alpha\beta}, \tag{6.28}$$

so that

$$S = \frac{N+2}{3}, \tag{6.29}$$

$$S = \frac{N+2}{3}. \tag{6.30}$$

For the evaluation of S, we first note that

$$\lambda^{O(N)}_{\alpha\beta\sigma_1\sigma_2}\lambda^{O(N)}_{\sigma_1\sigma_2\gamma\delta} = \frac{\lambda^2}{9}\left[(N+4)\,\delta_{\alpha\beta}\delta_{\gamma\delta} + 2\delta_{\alpha\gamma}\delta_{\beta\delta} + 2\delta_{\alpha\delta}\delta_{\beta\gamma}\right]. \tag{6.31}$$

It is useful to abbreviate the three tensors in the brackets as $A_1,\ A_2,\ A_3$, i.e., we write

$$\lambda^{O(N)}_{\alpha\beta\sigma_1\sigma_2}\lambda^{O(N)}_{\sigma_1\sigma_2\gamma\delta} = \lambda^2\left[\frac{N+4}{9}A_1 + \frac{2}{9}A_2 + \frac{2}{9}A_3\right]_{\alpha\beta\gamma\delta}. \tag{6.32}$$

The symmetrization of the tensors $A_1,\ A_2,\ A_3$ gives $\lambda^{O(N)}_{\alpha\beta\gamma\delta}$ each time, so that

$$\lambda^{O(N)}_{\alpha\beta\sigma_1\sigma_2}\lambda^{O(N)}_{\sigma_1\sigma_2\gamma\delta}|_{\rm sym} = \lambda^2\,\frac{N+8}{9}\,T^{(1)}_{\alpha\beta\gamma\delta}, \tag{6.33}$$

and hence

$$S = \frac{N+8}{9}. \tag{6.34}$$

For the other products (6.26) we observe the multiplication rules

$$
\begin{aligned}
&(p_1 A_1 + p_2 A_2 + p_3 A_3)_{\alpha\beta\sigma_1\sigma_2} (q_1 A_1 + q_2 A_2 + q_3 A_3)_{\sigma_1\sigma_2\gamma\delta} \\
&= \{[p_1 (q_1 N + q_2 + q_3) + p_2 q_1 + p_3 q_1] A_1 + (p_2 q_2 + p_3 q_3) A_2 + (p_2 q_3 + p_3 q_2) A_3\}_{\alpha\beta\gamma\delta} .
\end{aligned}
\tag{6.35}
$$

If we want to form $\lambda^{O(N)}_{\alpha\beta\sigma_1\sigma_2}\lambda^{O(N)}_{\sigma_1\sigma_2\sigma_3\sigma_4}\lambda^{O(N)}_{\sigma_3\sigma_4\gamma\delta}$, we have to multiply $A_1+A_2+A_3$ by $\frac{N+4}{9}A_1+\frac{2}{9}A_2+\frac{2}{9}A_3$ and to symmetrize the result, which gives

$$
\lambda^3 \left[\frac{1}{3}\left(\frac{N+4}{9}N + \frac{4}{9}\right) + \frac{2}{3}\left(\frac{N+4}{9} + \frac{2}{9} + \frac{2}{9}\right)\right] T^{(1)}_{\alpha\beta\gamma\delta}
\tag{6.36}
$$

and hence

$$
S = \frac{1}{27}\left(N^2 + 6N + 20\right) .
\tag{6.37}
$$

The last product in (6.26) is found by using the general rule

$$
\begin{aligned}
(p_1 A_1 + p_2 A_2 + p_3 A_3)_{\alpha\beta\sigma_1\sigma_2} & (q_1 A_1 + q_2 A_2 + q_3 A_3)_{\sigma_1\gamma\sigma_2\delta} \\
&= \{[p_1 (q_1 + N q_2 + q_3) + (p_2 + p_3) q_2] A_1 \\
&\qquad (p_2 q_1 + p_3 q_3) A_2 + (p_3 q_1 + p_2 q_3) A_3\}_{\alpha\beta\gamma\delta} .
\end{aligned}
\tag{6.38}
$$

Applying this to $\frac{1}{3}(A_1 + A_2 + A_3)$ and $\frac{N+4}{9}A_1 + \frac{2}{9}A_2 + \frac{2}{9}A_3$ we find, after symmetrization,

$$
\begin{aligned}
S &= \frac{1}{3}\left(\frac{N+4}{9} + N\frac{2}{9} + \frac{2}{9}\right) + \left(\frac{1}{3} + \frac{1}{3}\right)\frac{2}{9} + 2\frac{1}{3}\left(\frac{N+4}{9} + \frac{2}{9}\right) \\
&= \frac{1}{27}(5N + 22) .
\end{aligned}
\tag{6.39}
$$

A summary of all reduced matrix elements up to two loops is given by the equations

$$
S = \frac{N+2}{3}, \tag{6.40}
$$

$$
S = \left(\frac{N+2}{3}\right)^2, \tag{6.41}
$$

$$
S = \frac{N+2}{3}, \tag{6.42}
$$

$$
S = \frac{N+8}{9}, \tag{6.43}
$$

$$
S = \frac{N^2 + 6N + 20}{27}, \tag{6.44}
$$

$$
S = \frac{N+2}{3}\frac{N+8}{9}, \tag{6.45}
$$

$$
S = \frac{5N+22}{27}. \tag{6.46}
$$

The results up to five loops are shown in Appendix B.3.

6.4.2 Symmetry Factors for Mixed O(N) and Cubic Symmetry

The tensor $\lambda_{\alpha\beta\gamma\delta}$ introduced in Eq. (6.14) takes the following form for the mixed O(N)-cubic-symmetric interaction energy of Eq. (6.4):

$$\begin{aligned}\lambda^{\text{cub}}_{\alpha\beta\gamma\delta} &= \frac{\lambda_1}{3}\left(\delta_{\alpha\beta}\delta_{\gamma\delta}+2\text{ perm}\right)+\lambda_2\,\delta_{\alpha\beta\gamma\delta}\\ &= \lambda_1 T^{(1)}_{\alpha\beta\gamma\delta}+\lambda_2 T^{(2)}_{\alpha\beta\gamma\delta}.\end{aligned}\tag{6.47}$$

The generalized δ-tensors are defined by:

$$\delta_{\alpha_1\dots\alpha_n}\equiv\begin{cases}1\,, & \alpha_1=\dots=\alpha_n,\\ 0\,, & \text{otherwise.}\end{cases}\tag{6.48}$$

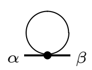
They satisfy the identities

$$\sum_\gamma \delta_{\alpha_1\cdots\alpha_n\gamma\gamma} = \delta_{\alpha_1\cdots\alpha_n},\tag{6.49}$$

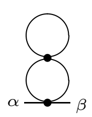
$$\sum_\gamma \delta_{\alpha_1\cdots\alpha_n\gamma}\delta_{\beta_1\cdots\beta_n\gamma} = \delta_{\alpha_1\cdots\alpha_n\beta_1\cdots\beta_n},\tag{6.50}$$

$$\sum_\gamma \delta_{\gamma\gamma} = N.\tag{6.51}$$

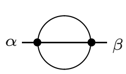
The tensor contractions for the first two orders in the perturbation expansion are [recall the diagrams in (3.24) and (3.24)]



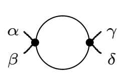
$$:\ \lambda^{\text{cub}}_{\alpha\beta\sigma\sigma} = \left(\lambda_1\,S^{\text{cub}}_{2;(1,0)} + \lambda_2\,S^{\text{cub}}_{2;(0,1)}\right)\delta_{\alpha\beta},\tag{6.52}$$



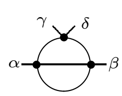
$$:\ \lambda^{\text{cub}}_{\alpha\beta\sigma_1\sigma_2}\lambda^{\text{cub}}_{\sigma_1\sigma_2\sigma_3\sigma_3} = \left(\lambda_1^2 S^{\text{cub}}_{2;(2,0)} + \lambda_1\lambda_2 S^{\text{cub}}_{2;(1,1)} + \lambda_2^2 S^{\text{cub}}_{2;(0,2)}\right)\delta_{\alpha\beta},\tag{6.53}$$



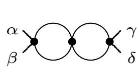
$$:\ \lambda^{\text{cub}}_{\alpha\sigma_1\sigma_2\sigma_3}\lambda^{\text{cub}}_{\sigma_1\sigma_2\sigma_3\beta} = \left(\lambda_1^2 S_{2;(2,0)} + \lambda_1\lambda_2 S_{2;(1,1)} + \lambda_2^2 S^{\text{cub}}_{2;(0,2)}\right)\delta_{\alpha\beta},\tag{6.54}$$



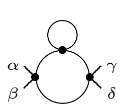
$$\begin{aligned}:\ \lambda^{\text{cub}}_{\alpha\beta\sigma_1\sigma_2}\lambda^{\text{cub}}_{\sigma_1\sigma_2\gamma\delta}\Big|_{\text{sym}} &= \left(\lambda_1 S^{\text{cub}}_{4_1;(2,0)} + \lambda_2 S^{\text{cub}}_{4_1;(1,1)}\right)\lambda_1\,T^{(1)}_{\alpha\beta\gamma\delta}\\ &+ \left(\lambda_1 S^{\text{cub}}_{4_2;(1,1)} + \lambda_2 S^{\text{cub}}_{4_2;(0,2)}\right)\lambda_2\,T^{(2)}_{\alpha\beta\gamma\delta},\end{aligned}\tag{6.55}$$



$$\begin{aligned}&:\ \lambda^{\text{cub}}_{\alpha\sigma_1\sigma_2\sigma_3}\lambda^{\text{cub}}_{\beta\sigma_1\sigma_2\sigma_4}\lambda^{\text{cub}}_{\sigma_3\sigma_4\gamma\delta}\Big|_{\text{sym}}\\ &= \left(\lambda_1^2 S^{\text{cub}}_{4_1;(3,0)} + \lambda_1\lambda_2 S^{\text{cub}}_{4_1;(2,1)} + \lambda_2^2 S^{\text{cub}}_{4_1;(1,2)}\right)\lambda_1\,T^{(1)}_{\alpha\beta\gamma\delta}\\ &+ \left(\lambda_1^2 S^{\text{cub}}_{4_2;(2,1)} + \lambda_1\lambda_2 S^{\text{cub}}_{4_2;(1,2)} + \lambda_2^2 S^{\text{cub}}_{4_2;(0,3)}\right)\lambda_2\,T^{(2)}_{\alpha\beta\gamma\delta},\end{aligned}\tag{6.56}$$



$$\begin{aligned}&:\ \lambda^{\text{cub}}_{\alpha\beta\sigma_1\sigma_2}\lambda^{\text{cub}}_{\sigma_1\sigma_2\sigma_3\sigma_4}\lambda^{\text{cub}}_{\sigma_3\sigma_4\gamma\delta}\Big|_{\text{sym}}\\ &= \left(\lambda_1^2 S^{\text{cub}}_{4_1;(3,0)} + \lambda_1\lambda_2 S^{\text{cub}}_{4_1;(2,1)} + \lambda_2^2 S^{\text{cub}}_{4_1;(1,2)}\right)\lambda_1\,T^{(1)}_{\alpha\beta\gamma\delta}\\ &+ \left(\lambda_1^2 S^{\text{cub}}_{4_2;(2,1)} + \lambda_1\lambda_2 S^{\text{cub}}_{4_2;(1,2)} + \lambda_2^2 S^{\text{cub}}_{4_2;(0,3)}\right)\lambda_2\,T^{(2)}_{\alpha\beta\gamma\delta},\end{aligned}\tag{6.57}$$



$$:\ \lambda^{\text{cub}}_{\alpha\beta\sigma_1\sigma_2}\lambda^{\text{cub}}_{\sigma_2\sigma_3\sigma_3\sigma_4}\lambda^{\text{cub}}_{\gamma\delta\sigma_1\sigma_4}\Big|_{\text{sym}}$$

$$
\begin{aligned}
&= \left(\lambda_1^2 S^{\text{cub}}_{4_1;(3,0)} + \lambda_1\lambda_2 S^{\text{cub}}_{4_1;(2,1)} + \lambda_2^2 S^{\text{cub}}_{4_1;(1,2)}\right) \lambda_1\, T^{(1)}_{\alpha\beta\gamma\delta} \\
&+ \left(\lambda_1^2 S^{\text{cub}}_{4_2;(2,1)} + \lambda_1\lambda_2 S^{\text{cub}}_{4_2;(1,2)} + \lambda_2^2 S^{\text{cub}}_{4_2;(0,3)}\right) \lambda_2\, T^{(2)}_{\alpha\beta\gamma\delta}.
\end{aligned} \tag{6.58}
$$

We shall now examine which symmetry factors arise from tensor contractions, without presenting the calculations in full detail as before. It will turn out that the symmetry factor $S^{\text{cub}}_{(V,0)}$ is obtained as a numerical factor resulting from a contraction of $T^{(1)}$-type tensors only. This makes it equal to the corresponding symmetry factor for a pure $\mathrm{O}(N)$ symmetry. The symmetry factor $S^{\text{cub}}_{(0,V)}$ is given by a contraction of $T^{(2)}$-type tensors only, yielding always unity.

The result for the first diagram in (6.52) is immediately obtained. Since

$$
\lambda^{\text{cub}}_{\alpha\beta\sigma\sigma} = \lambda_1 T^{(1)}_{\alpha\beta\sigma\sigma} + \lambda_2 T^{(2)}_{\alpha\beta\sigma\sigma} = \lambda_1 S^{\mathrm{O}(N)} \;\delta_{\alpha\beta} + \lambda_2\delta_{\alpha\beta} = \left(\lambda_1 \frac{N+2}{3} + \lambda_2\right)\delta_{\alpha\beta},
$$

we see directly that

$$
S^{\text{cub}}_{2;(1,0)} = \frac{N+2}{3}, \qquad S^{\text{cub}}_{2;(0,1)} = 1\,. \tag{6.59}
$$

For the first two-loop diagram in (6.53), we find:

$$
\begin{aligned}
&\lambda^{\text{cub}}_{\alpha\beta\sigma_1\sigma_2}\lambda^{\text{cub}}_{\sigma_1\sigma_2\sigma_3\sigma_3} \\
&\quad= \lambda_1^2\, T^{(1)}_{\alpha\beta\sigma_1\sigma_2} T^{(1)}_{\sigma_1\sigma_2\sigma_3\sigma_3} + \lambda_1\,\lambda_2 \left[T^{(1)}_{\alpha\beta\sigma_1\sigma_2} T^{(2)}_{\sigma_1\sigma_2\sigma_3\sigma_3} + T^{(2)}_{\alpha\beta\sigma_1\sigma_2} T^{(1)}_{\sigma_1\sigma_2\sigma_3\sigma_3}\right] + \lambda_2^2\, T^{(2)}_{\alpha\beta\sigma_1\sigma_2} T^{(2)}_{\sigma_1\sigma_2\sigma_3\sigma_3} \\
&\quad= \left[\lambda_1^2\, S_2^{\mathrm{O}(N)} + \lambda_1\lambda_2\, 2\left(\frac{N+2}{3}\right) + \lambda_2^2\right]\delta_{\alpha\beta},
\end{aligned}
$$

and we extract:

$$
S^{\text{cub}}_{2;(2,0)} = \left(\frac{N+2}{3}\right)^2, \quad S^{\text{cub}}_{2;(1,1)} = 2\left(\frac{N+2}{3}\right), \quad S^{\text{cub}}_{2;(0,2)} = 1. \tag{6.60}
$$

For the second two-loop diagram in (6.54), the calculation proceeds analogously:

$$
\begin{aligned}
&\lambda^{\text{cub}}_{\alpha\sigma_1\sigma_2\sigma_3}\lambda^{\text{cub}}_{\sigma_1\sigma_2\sigma_3\beta} \\
&\quad= \lambda_1^2\, T^{(1)}_{\alpha\sigma_1\sigma_2\sigma_3} T^{(1)}_{\sigma_1\sigma_2\sigma_3\beta} + \lambda_1\lambda_2 \left(T^{(1)}_{\alpha\sigma_1\sigma_2\sigma_3} T^{(2)}_{\sigma_1\sigma_2\sigma_3\beta} + T^{(2)}_{\alpha\sigma_1\sigma_2\sigma_3} T^{(1)}_{\sigma_1\sigma_2\sigma_3\beta}\right) + \lambda_2^2\, T^{(2)}_{\alpha\sigma_1\sigma_2\sigma_3} T^{(2)}_{\sigma_1\sigma_2\sigma_3\beta} \\
&\quad= \left[\lambda_1^2\, S_2^{\mathrm{O}(N)} + \lambda_1\lambda_2\, 2 + \lambda_2^2\,\right]\delta_{\alpha\beta},
\end{aligned}
$$

and we get

$$
S^{\text{cub}}_{2;(2,0)} = \frac{N+2}{3}, \qquad S^{\text{cub}}_{2;(1,1)} = 2, \qquad S^{\text{cub}}_{2;(0,2)} = 1. \tag{6.61}
$$

Now we turn to the four-point diagram in (6.55)

$$
\begin{aligned}
&\lambda^{\text{cub}}_{\alpha\beta\sigma_1\sigma_2}\lambda^{\text{cub}}_{\sigma_1\sigma_2\gamma\delta}\Big|_{\text{sym}} \\
&\quad= \lambda_1^2\, T^{(1)}_{\alpha\beta\sigma_1\sigma_2} T^{(1)}_{\sigma_1\sigma_2\gamma\delta}\Big|_{\text{sym}} + \lambda_1\lambda_2 \left(T^{(1)}_{\alpha\beta\sigma_1\sigma_2} T^{(2)}_{\sigma_1\sigma_2\gamma\delta}\Big|_{\text{sym}} + T^{(2)}_{\alpha\beta\sigma_1\sigma_2} T^{(1)}_{\sigma_1\sigma_2\gamma\delta}\Big|_{\text{sym}}\right) + \lambda_2^2\, T^{(2)}_{\alpha\beta\sigma_1\sigma_2} T^{(2)}_{\sigma_1\sigma_2\gamma\delta}\Big|_{\text{sym}} \\
&\quad= \lambda_1^2\, S_4^{\mathrm{O}(N)}\, T^{(1)}_{\alpha\beta\gamma\delta}, + \lambda_1\lambda_2 \left(\frac{2}{3}\, T^{(1)}_{\alpha\beta\gamma\delta} + \frac{4}{3}\, T^{(2)}_{\alpha\beta\gamma\delta}\right) + \lambda_2^2\, T^{(2)}_{\alpha\beta\gamma\delta},
\end{aligned}
$$

yielding the symmetry factors:

$$S^{\rm cub}_{4_1;(2,0)} = \frac{N+8}{9}, \quad S^{\rm cub}_{4_1;(1,1)} = \frac{2}{3}, \quad S^{\rm cub}_{4_2;(1,1)} = \frac{4}{3}, \quad S^{\rm cub}_{4_2;(0,2)} = 1. \tag{6.62}$$

The calculation of the symmetry factors of the four-point diagram with two loops in (6.56) produces many terms:

$$\begin{aligned}
&\lambda^{\rm cub}_{\alpha\sigma_1\sigma_2\sigma_3}\lambda^{\rm cub}_{\beta\sigma_1\sigma_2\sigma_4}\lambda^{\rm cub}_{\sigma_3\sigma_4\gamma\delta}\Big|_{\rm sym} \\
&= \lambda_1^3\, T^{(1)}_{\alpha\sigma_1\sigma_2\sigma_3}T^{(1)}_{\beta\sigma_1\sigma_2\sigma_4}T^{(1)}_{\sigma_3\sigma_4\gamma\delta}\Big|_{\rm sym} \\
&+ \lambda_1^2\,\lambda_2\left(T^{(1)}_{\alpha\sigma_1\sigma_2\sigma_3}T^{(1)}_{\beta\sigma_1\sigma_2\sigma_4}T^{(2)}_{\sigma_3\sigma_4\gamma\delta}\Big|_{\rm sym}+T^{(1)}_{\alpha\sigma_1\sigma_2\sigma_3}T^{(2)}_{\beta\sigma_1\sigma_2\sigma_4}T^{(1)}_{\sigma_3\sigma_4\gamma\delta}\Big|_{\rm sym}+T^{(2)}_{\alpha\sigma_1\sigma_2\sigma_3}T^{(1)}_{\beta\sigma_1\sigma_2\sigma_4}T^{(1)}_{\sigma_3\sigma_4\gamma\delta}\Big|_{\rm sym}\right) \\
&+ \lambda_1\lambda_2^2\left(T^{(1)}_{\alpha\sigma_1\sigma_2\sigma_3}T^{(2)}_{\beta\sigma_1\sigma_2\sigma_4}T^{(2)}_{\sigma_3\sigma_4\gamma\delta}\Big|_{\rm sym}+T^{(2)}_{\alpha\sigma_1\sigma_2\sigma_3}T^{(1)}_{\beta\sigma_1\sigma_2\sigma_4}T^{(2)}_{\sigma_3\sigma_4\gamma\delta}\Big|_{\rm sym}+T^{(2)}_{\alpha\sigma_1\sigma_2\sigma_3}T^{(2)}_{\beta\sigma_1\sigma_2\sigma_4}T^{(1)}_{\sigma_3\sigma_4\gamma\delta}\Big|_{\rm sym}\right) \\
&+ \lambda_2^3\, T^{(2)}_{\alpha\sigma_1\sigma_2\sigma_3}T^{(2)}_{\beta\sigma_1\sigma_2\sigma_4}T^{(2)}_{\sigma_3\sigma_4\gamma\delta}\Big|_{\rm sym} \\
&= \lambda_1^3\, S_4^{{\rm O}(N)}\, T^{(1)}_{\alpha\beta\gamma\delta} + \lambda_1^2\,\lambda_2\left(\frac{12}{9}T^{(1)}_{\alpha\beta\gamma\delta} + \frac{N+14}{9}T^{(2)}_{\alpha\beta\gamma\delta}\right) + \lambda_2\,\lambda_2^2\left(\frac{1}{3}T^{(1)}_{\alpha\beta\gamma\delta} + \frac{8}{3}T^{(2)}_{\alpha\beta\gamma\delta}\right) + \lambda_2^3\, T^{(2)}_{\alpha\beta\gamma\delta}\,.
\end{aligned}$$

From these we read off the following symmetry factors:

$$\begin{aligned}
S^{\rm cub}_{4_1;(3,0)} &= \frac{5N+22}{27}, & S^{\rm cub}_{4_1;(2,1)} &= \frac{12}{9}, & S^{\rm cub}_{4_1;(1,2)} &= \frac{1}{3}, \\
S^{\rm cub}_{4_2;(0,3)} &= 1, & S^{\rm cub}_{4_2;(2,1)} &= \frac{N+14}{9}, & S^{\rm cub}_{4_2;(1,2)} &= \frac{8}{3}.
\end{aligned} \tag{6.63}$$

The symmetry factors for the remaining two four-point diagrams (6.57) and (6.58) are calculated in the same way.

Let us collect the results for the reduced matrix elements up to two loops for the mixed O(N) and cubic symmetry:

$$\begin{aligned}
S^{\rm cub}_{2;(1,0)} &= \frac{N+2}{3}, & S^{\rm cub}_{2;(0,1)} &= 1, \\
S^{\rm cub}_{2;(2,0)} &= \left(\frac{N+2}{3}\right)^2, & S^{\rm cub}_{2;(1,1)} &= \frac{2(N+2)}{3}, & S^{\rm cub}_{2;(0,2)} &= 1, \\
S^{\rm cub}_{2;(2,0)} &= \frac{N+2}{3}, & S^{\rm cub}_{2;(1,1)} &= 2, & S^{\rm cub}_{2;(0,2)} &= 1, \\
S^{\rm cub}_{4_1;(2,0)} &= \frac{N+8}{9}, & S^{\rm cub}_{4_1;(1,1)} &= \frac{2}{3}, & S^{\rm cub}_{4_2;(1,1)} &= \frac{4}{3}, & S^{\rm cub}_{4_2;(0,2)} &= 1, \\
S^{\rm cub}_{4_1;(3,0)} &= \frac{5N+22}{27}, & S^{\rm cub}_{4_1;(2,1)} &= \frac{4}{3}, & S^{\rm cub}_{4_1;(1,2)} &= \frac{1}{3}, \\
S^{\rm cub}_{4_2;(0,3)} &= 1, & S^{\rm cub}_{4_2;(2,1)} &= \frac{N+14}{9}, & S^{\rm cub}_{4_2;(1,2)} &= \frac{8}{3}, \\
S^{\rm cub}_{4_1;(3,0)} &= \frac{N^2+6N+20}{27}, & S^{\rm cub}_{4_1;(2,1)} &= \frac{4+N}{3}, & S^{\rm cub}_{4_1;(1,2)} &= 1, \\
S^{\rm cub}_{4_2;(0,3)} &= 1, & S^{\rm cub}_{4_2;(2,1)} &= \frac{4}{3}, & S^{\rm cub}_{4_2;(1,2)} &= 2, \\
S^{\rm cub}_{4_1;(3,0)} &= \frac{N+2}{3}\frac{N+8}{9}, & S^{\rm cub}_{4_1;(2,1)} &= \frac{N+4}{3}, & S^{\rm cub}_{4_1;(1,2)} &= \frac{2}{3},
\end{aligned} \tag{6.64}$$

$$S^{\rm cub}_{4_2;(0,3)} = 1, \qquad S^{\rm cub}_{4_2;(2,1)} = \frac{4(N+2)}{9}, \quad S^{\rm cub}_{4_2;(1,2)} = \frac{N+6}{3}. \tag{6.65}$$

Note that for $N = 1$, $S_{2;(V,0)}$ and $S_{4_1;(V,0)}$ reduce to unity, whereas $S_{2;(V-k,k)}$ and $S_{4_1;(V-k,k)} + S_{4_2;(V-k,k)}$ reduce to $V!/(V-k)!k!$, which is the number of ways the coupling constants $\lambda_1^{V-k}\lambda_2^k$ can be arranged on V vertices.

A complete list of all symmetry factors up to five loops is found in Appendix B.4.

6.4.3 Other Symmetries

A further generalization sometimes encountered in the literature that is useful for applications is the case of n fields ϕ^i_α $(\alpha = 1, \ldots, n)$, each of which occurring in an O(q)-symmetric combination:

$$E = \int d^Dx \left\{ \frac{1}{2} \sum_{\alpha,i}^{n,q} \left[\left(\partial\phi^i_\alpha\right)^2 + m^2(\phi^i_\alpha)^2 \right] + \frac{1}{4!} \sum_{\alpha,\alpha'=1}^{n} \lambda_{\alpha\alpha'} \sum_{i=1}^{q} (\phi^i_\alpha)^2 \sum_{j=1}^{q} (\phi^j_{\alpha'})^2 \right\}. \tag{6.66}$$

Among these energies, the case of a cubic symmetry in the indices is of special interest. Its interacting part contains only two independent couplings

$$E_{\rm int} \;=\; \int d^Dx \left\{ \lambda_1 \left[\sum_{\alpha=1}^{n} \sum_{i=1}^{q} (\phi^i_\alpha)^2 \right]^2 + \lambda_2 \sum_{\alpha=1}^{n} \left[\sum_{i=1}^{q} \left(\phi^i_\alpha\right)^2 \right]^2 \right\}. \tag{6.67}$$

The special case of n complex fields, i.e., of fields ϕ^i_α with $i = 1, 2$, governs the statistical mechanics of ensembles of dislocation lines in a crystal [2]. Furthermore, the case of n fields with $q = 0$ belongs to the same universality class as the so-called random n-vector model [3]. The model with mixed O(N)-cubic-symmetric interactions studied before is a special case of (6.66) with $n = N$ and $q = 1$.

The interaction (6.67) corresponds to the tensor (6.25) being replaced by

$$\begin{aligned} \lambda_{\alpha i,\beta j,\gamma k,\delta l} &= \lambda_1 S_{\alpha i,\beta j,\gamma k,\delta l} + \lambda_2 F_{\alpha i,\beta j,\gamma k,\delta l} \\ &= \frac{\lambda_1}{3} \left(\delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{ij}\delta_{kl} + \delta_{\alpha\gamma}\delta_{\beta\delta}\delta_{ik}\delta_{jl} + \delta_{\alpha\delta}\delta_{\beta\gamma}\delta_{il}\delta_{jk} \right) \\ &\quad + \frac{\lambda_2}{3}\, \delta_{\alpha\beta\gamma\delta} \left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \right). \end{aligned} \tag{6.68}$$

We have changed the notation for the tensors from $T^{(1)}$ and $T^{(2)}$ to S and F, in order to avoid any confusion.

It is straightforward to generalize the previous results to the present interaction. Take, for example, the symmetry factor S arising from the contraction

$$T_{\alpha\beta\sigma_1\sigma_2} T_{\sigma_1\sigma_2\gamma\delta}\Big|_{\rm sym} = T_{\alpha\beta\gamma\delta} S\,. \tag{6.69}$$

Multiplying the tensors S, F with each other, we find

$$\begin{aligned} SS|_{\rm sym} &= \frac{nq+8}{9} S, \\ FF|_{\rm sym} &= \frac{q+8}{9} F, \\ FS|_{\rm sym} &= \frac{2}{3} F + \frac{q+2}{9} S, \end{aligned} \tag{6.70}$$

such that the result can again be decomposed into the tensors S and F.

6.4.4 General Symmetry Factors

The formation of the symmetry factor for any diagram can be visualized by the modified Feynman rules:

$$\alpha,\mathbf{x}\bullet\!\!\longrightarrow\!\!\bullet\beta,\mathbf{y} \quad \hat{=} \quad G^{(2)}_{0;\alpha\beta}(\mathbf{x},\mathbf{y}) \;=\; \delta_{\alpha\beta}G^{(2)}_{0}(\mathbf{x},\mathbf{y})\;; \tag{6.71}$$

$${}^{\beta}_{\alpha}\!\times^{\delta}_{\gamma} \quad \hat{=} \quad -\sum_{\alpha,\beta,\gamma,\delta=1}^{N} T_{\alpha\beta\gamma\delta}\int d^Dz\;. \tag{6.72}$$

There are $4p$ summations for a total of $4p+n$ indices, two for each of the $(4p+n)/2$ lines. A subset of $(4p+n)/2$ index summations can be carried out immediately by contracting the indices of the tensors $T_{\beta_i\beta_j\beta_k\beta_l}$ with the Kronecker symbols $\delta_{\alpha\beta}$ for each line, as specified by a particular diagram. The number of the $(4p-n)/2$ remaining summations equals the number of internal lines $I=(4p-n)/2$ [recall (4.11)]. The symmetry tensor F_G is thus calculated as a sum over the indices of the internal lines of the product of p tensors T and depends on the indices of the external lines which are the only noncontracted indices.

In Section 4.2, we introduced the so-called amputated diagrams, in which the external lines are removed. The calculation of the symmetry tensor of these diagram in momentum space contains only $4p-n$ summations for $4p+n$ indices. It is not summed over the indices of the amputated lines, since their factors $\delta_{\alpha\beta}$ are now also absent. Only a subset of $(4p-n)/2$ summations is now carried out immediately, and the number of the nontrivial summations again equals the number of the internal lines. The symmetry factor is finally averaged in the indices of the amputated external lines.

As will be explained in Chapter 14, all two- and four-point diagrams are generated from vacuum diagrams. The symmetry factor S_0 of the vacuum diagram determines also the symmetry factor of the generated diagram. Since the process of generation of two- and four-point diagrams changes the number of loops L, the symmetry factors in this subsection have an additional upper index indicating the number of loops of the corresponding diagram.

The two-point diagrams with L loops are generated by cutting a line in a vacuum diagram with $L+1$ loops. Conversely, by connecting the two end points of a two-point diagram with L loops we obtain a vacuum diagram with $L+1$ loops. The symmetry factors of the two diagrams are therefore related by

$$\sum_{\alpha\beta} S_2^L\,\delta_{\alpha\beta} = S_0^{L+1}. \tag{6.73}$$

Since $\sum_{\alpha\beta}\delta_{\alpha\beta}\delta_{\alpha\beta}=N$, the relation becomes

$$S_2^L = \frac{S_0^{L+1}}{N}\,. \tag{6.74}$$

The symmetry factor is therefore the same for all two-point diagrams generated from a single vacuum diagram. This is true for all symmetries that have only one quadratic invariant $\delta_{\alpha\beta}$, which is the case in systems with $\mathrm{O}(N)$ symmetry and mixture of $\mathrm{O}(N)$ and cubic symmetry. If there are two coupling constants, the symmetry factors appear in a sum over all combinations of coupling constants, and the symmetry factors for the vacuum diagrams are related to the symmetry factors of the two-point diagrams as follows:

$$\sum_{k=0}^{V}\lambda_1^{V-k}\lambda_2^k\,S^{L+1}_{0;(V-k,k)} \;=\; \sum_{\alpha\beta}\sum_{k=0}^{V}\lambda_1^{V-k}\lambda_2^k\,S^{L}_{2;(V-k,k)}\,\delta_{\alpha\beta} = N\sum_{k=0}^{V}\lambda_1^{V-k}\lambda_2^k\,S^{L}_{2;(V-k,k)}. \tag{6.75}$$

The four-point diagrams are generated by cutting out a vertex. Closing a four-point diagram to a vacuum diagram with an additional vertex implies that the associated symmetry factor is multiplied by $T_{\alpha\beta\gamma\delta}$. In the case of O($N$) symmetry, this gives the following factor:

$$\delta_{\alpha\beta}\delta_{\gamma\delta}\,\lambda^{\mathrm{O}(N)}_{\alpha\beta\gamma\delta} = \delta_{\alpha\beta}\delta_{\gamma\delta}\,\frac{1}{3}(\delta_{\alpha\beta}\delta_{\gamma\delta}\ +\delta_{\alpha\gamma}\delta_{\beta\delta}\ +\delta_{\alpha\delta}\delta_{\beta\gamma}) = \frac{N(N+2)}{3}\,, \tag{6.76}$$

since the unsymmetrized four-point diagram is proportional to $\delta_{\alpha\beta}\delta_{\gamma\delta}$. For a vacuum diagram with $L+3$ loops and the symmetry factor S_0^{L+3}, the symmetry factor S_4^L of the generated four-point diagram with L loops is in O(N) symmetry given by

$$S_4^L = 3\frac{S_0^{L+3}}{N(N+2)}. \tag{6.77}$$

All four-point diagrams in O(N) symmetry derived from one vacuum diagram have the same symmetry factor.

Thus, the number of different symmetry factors equals the number of vacuum diagrams. By combining (6.74) with (6.77) we find a relation between the symmetry factors of the four-point diagrams with L loops S_4^L and the symmetry factors of the two-point diagrams with $L+2$ loops S_2^{L+2}:

$$S_2^{L+2} = \frac{N+2}{3}S_4^L\,. \tag{6.78}$$

The statements for the four-point diagrams are much more complicated in systems with mixed O(N) and cubic symmetry, since there are two quartic invariants. In this case, Eq. (6.76) becomes:

$$\sum_{k=0}^{V+1} S^{L+3}_{0;(V+1-k,k)}\lambda_1^{V+1-k}\lambda_2^k$$
$$= \sum_{k=0}^{V}\left(S^L_{4_1;(V-k,k)}\,\lambda_1^{V-k}\lambda_2^k\,\delta_{\alpha\beta}\delta_{\gamma\delta} + S^L_{4_2;(V-k,k)}\,\lambda_1^{V-k}\lambda_2^k\,\delta_{\alpha\beta\gamma\delta}\right)\left(\lambda_1 T^{(1)}_{\alpha\beta\gamma\delta} + \lambda_2 T^{(2)}_{\alpha\beta\gamma\delta}\right) \tag{6.79}$$
$$= \sum_{k=0}^{V}\left[S^L_{4_1;(V-k,k)}\,\lambda_1^{V-k}\,\lambda_2^k\left(\lambda_1\frac{N(N+2)}{3} + \lambda_2 N\right) + S^L_{4_2;(V-k,i)}\,\lambda_1^{V-k}\,\lambda_2^k\,(\lambda_1 N + \lambda_2 N)\right]. \tag{6.80}$$

Remember that $S_{4_1;(0,V)} = S_{4_2;(V,0)} = 0$. Again, the symmetry factors of all four-point diagrams which are derived from the same vacuum diagram are related to the symmetry factors of the vacuum diagram. But this time the relation holds only for the sum of $S^L_{4_1}$ and $S^L_{4_2}$. In fact, there are many more different symmetry factors $S^L_{4_1}$ and $S^L_{4_2}$ than vacuum diagrams. The reason is that it makes a difference whether a λ_1-vertex or a λ_2-vertex is cut out of a vacuum diagram. Not all combinations of λ_1- and λ_2-vertices result in the two quartic invariants, and thus contribute to $S^L_{4_1}$ and $S^L_{4_2}$. This may be verified by inspecting the tables in Appendix B.4.2 where all symmetry factors for the mixed O(N) and cubic symmetry are listed.

Note that for $\lambda_2 = 0$, Eq. (6.80) reduces to

$$S^{L+3}_{0;(V+1,0)} = S^L_{4_1;(V,0)}\,\frac{N(N+2)}{3}. \tag{6.81}$$

This is the same relation found for O(N) symmetry with only one coupling constant. The symmetry factors $S^L_{4_1;(V,0)}$ in the case of mixed O(N) and cubic symmetry equal the factors S_4^L in the pure O(N) case.

Notes and References

There are various articles containing discussions of the tensor structures of the interactions, for example,

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A detailed discussion of this and other interesting cases can be found in the article ny A. Aharony cited above.