5

Structural Properties of Perturbation Theory

The structural properties of all diagrammatic expansions developed so far can be analyzed systematically with the help of functional equations.

5.1 Generating Functionals

In Chapter 3 we have seen that the correlation functions obtained from the functional derivatives of $Z[j]$ via relation (2.14), and the generating functional itself, contain many disconnected parts. Ultimately, however, we shall be interested only in the connected parts of $Z[j]$. Remember that a meaningful description of a very large thermodynamic system can only be given in terms of the free energy which is directly proportional to the total volume. In the limit of an infinite volume, also called *thermodynamic limit*, one has then a well-defined free energy density. The partition function, on the other hand, has no proper infinite-volume limit. We can observe this property directly in the diagrammatic expansion of $Z[j]$. Each component of a disconnected diagram is integrated over the entire space, thus contributing a volume factor. The expansion of $Z[j]$ therefore diverges at an infinite volume. In thermodynamics, we form the free energy from the logarithm of the partition function, which carries only a single overall volume factor and contains only connected diagrams.

Therefore we expect the logarithm of $Z[j]$ to provide us with the desired generating functional $W[j]$:

$$W[j] = \log Z[j]. \tag{5.1}$$

In this chapter we shall see that the functional derivatives of $W[j]$ produce, indeed, precisely the connected parts of the Feynman diagrams in each correlation function.

Consider the connected correlation functions $G_{\rm c}^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ defined by the functional derivatives

$$G_{\rm c}^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \frac{\delta}{\delta j(\mathbf{x}_1)} \cdots \frac{\delta}{\delta j(\mathbf{x}_n)}.W[j] \tag{5.2}$$

At the end, we shall be interested only in those functions at zero external current, where they reduce to the physical quantities (2.46) that vanish for odd n in the normal phase under study here. For the general development in this chapter, however, we shall consider them as functionals of $j(\mathbf{x})$, and go over to $j = 0$ only at the very end. The diagrammatic representation of these correlation functions contains only connected diagrams defined in Section 3.3. Moreover, the connected correlation functions $G_{\rm c}^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ collect *all* connected diagrams of the full correlation functions $G^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$, which then can be recovered via simple composition laws from the connected ones. In order to see this clearly, we shall derive the general relationship between the two types of correlation functions in Section 5.3. First, we shall prove the connectedness property of the derivatives (5.2).

5.2 Connectedness Structure of Correlation Functions

In this section, we shall prove that the generating functional $W[j]$ collects *only* connected diagrams in its Taylor coefficients $\delta^n W/\delta j(x_1)\dots\delta j(x_n)$. Later, after Eq. (5.26), we shall see that *all* connected diagrams of $G_{\rm c}^{(n)}(x_1,\dots,x_n)$ occur in $G^{(n)}(x_1,\dots,x_n)$.

The basis for the following considerations is the fact that the functional integral (2.13) for the generating functional $Z[j]$ satisfies an elementary identity

$$\int \mathcal{D}\phi \, \frac{\delta}{\delta\phi(\mathbf{x})} e^{-E[\phi,j]} = 0, \tag{5.3}$$

which follows from the vanishing of the Boltzmann factor $e^{-E[\phi,j]}$ at infinite field strength. After performing the functional derivative, we have

$$\int \mathcal{D}\phi \, \frac{\delta E[\phi,j]}{\delta\phi(\mathbf{x})} e^{-E[\phi,j]} = 0. \tag{5.4}$$

Inserting (2.12) for the functional $E[\phi,j]$, this reads

$$\int \mathcal{D}\phi \, \left[G_0^{-1}\phi(\mathbf{x}) + \frac{\lambda}{3!}\phi^3(\mathbf{x}) - j(\mathbf{x}) \right] e^{-E[\phi,j]} = 0. \tag{5.5}$$

Expressing the fields $\phi(\mathbf{x})$ as functional derivatives with respect to the source current $j(\mathbf{x})$, the brackets can be taken out of the integral, and we obtain the functional differential equation for the generating functional $Z[j]$:

$$\left\{ G_0^{-1}\frac{\delta}{\delta j(\mathbf{x})} + \frac{\lambda}{3!}\left[\frac{\delta}{\delta j(\mathbf{x})}\right]^3 - j(\mathbf{x}) \right\} Z[j] = 0. \tag{5.6}$$

With the short-hand notation

$$Z_{j(\mathbf{x}_1)j(\mathbf{x}_2)\dots j(\mathbf{x}_n)}[j] \equiv \frac{\delta}{\delta j(\mathbf{x}_1)}\frac{\delta}{\delta j(\mathbf{x}_2)}\cdots\frac{\delta}{\delta j(\mathbf{x}_n)} Z[j], \tag{5.7}$$

where the arguments of the currents will eventually be suppressed, this can be written as

$$G_0^{-1} Z_{j(\mathbf{x})} + \frac{\lambda}{3!} Z_{j(\mathbf{x})j(\mathbf{x})j(\mathbf{x})} - j(\mathbf{x}) Z[j] = 0. \tag{5.8}$$

Inserting here (5.1), we obtain a functional differential equation for $W[j]$:

$$G_0^{-1} W_j + \frac{\lambda}{3!}\left(W_{jjj} + 3W_{jj}W_j + W_j^3 \right) - j = 0. \tag{5.9}$$

We have employed the same short-hand notation for the functional derivatives of $W[j]$ as in (5.7):

$$W_{j(\mathbf{x}_1)j(\mathbf{x}_2)\dots j(\mathbf{x}_n)}[j] \equiv \frac{\delta}{\delta j(\mathbf{x}_1)}\frac{\delta}{\delta j(\mathbf{x}_2)}\cdots\frac{\delta}{\delta j(\mathbf{x}_n)} W[j], \tag{5.10}$$

suppressing the arguments $\mathbf{x}_1,\dots,\mathbf{x}_n$ of the currents, for brevity. Multiplying (5.9) functionally by G_0 gives

$$W_j = -\frac{\lambda}{3!} G_0 \left(W_{jjj} + 3W_{jj}W_j + W_j^3 \right) + G_0\, j. \tag{5.11}$$

We have omitted the integral over the intermediate space argument, for brevity. More specifically, we have written $G_0\, j$ for $\int d^D y\, G_0(\mathbf{x}, \mathbf{y}) j(\mathbf{y})$. Similar expressions abbreviate all functional products. This corresponds to a functional version of *Einstein's summation convention*.

Equation (5.11) may now be expressed in terms of the one-point correlation function

$$G_{\rm c}^{(1)} = W_j(x), \tag{5.12}$$

defined in (5.2), as

$$G_{\rm c}^{(1)} = -\frac{\lambda}{3!} G_0 \left\{ G_{{\rm c}\, jj}^{(1)} + 3 G_{{\rm c}\, j}^{(1)} G_{\rm c}^{(1)} + \left[G_{\rm c}^{(1)} \right]^3 \right\} + G_0\, j. \tag{5.13}$$

The solution to this equation is conveniently found by a diagrammatic procedure displayed in Fig. 5.1. To lowest, zeroth, order in λ we have

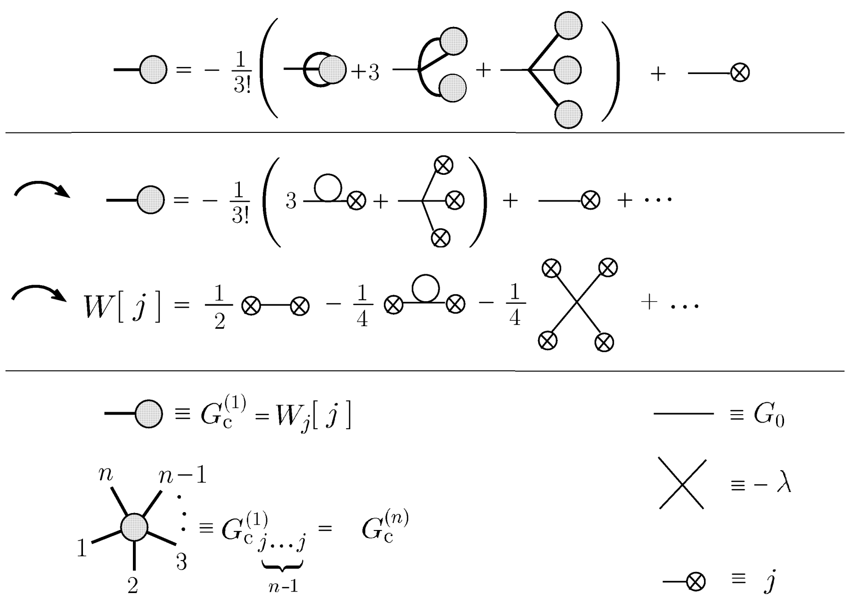


FIGURE 5.1 Diagrammatic solution of recursion relation (5.11) for the generating functional $W[j]$ of all connected correlation functions. First line represents Eq. (5.13), second (5.16), third (5.17). The remaining lines define the diagrammatic symbols.

$$G_{\rm c}^{(1)} = G_0\, j. \tag{5.14}$$

From this we find by functional integration the zeroth order generating functional $W[j]$

$$W_0[j] = \int \mathcal{D}j\, G_{\rm c}^{(1)} = \frac{1}{2} j G_0 j, \tag{5.15}$$

a result already known from (2.31) and (2.34). As in the perturbation expansions (2.47) of the correlation functions, subscripts of $W[j]$ indicate the order in the interaction strength λ.

Reinserting (5.14) on the right-hand side of (5.13) gives the first-order expression

$$G_{\rm c}^{(1)} = -G_0 \frac{\lambda}{3!} \left[3 G_0 G_0 j + (G_0 j)^3 \right] + G_0 j, \tag{5.16}$$

represented diagrammatically in the second line of Fig. 5.1. The expression (5.16) can be integrated functionally in j to obtain $W[j]$ up to first order in λ. Diagrammatically, this process amounts to multiplying the open line in each diagram by a current j, and dividing each term j^n by n. Thus we arrive at

$$W_0[j] + W_1[j] \;=\; \frac{1}{2} j G_0 j - \frac{\lambda}{4} G_0 \left(G_0 j\right)^2 - \frac{\lambda}{24} \left(G_0 j\right)^4 , \tag{5.17}$$

as illustrated in the third line of Fig. 5.1. This procedure can be continued to any order in λ.

This diagrammatic procedure allows us to prove that the generating functional $W[j]$ collects *only* connected diagrams in its Taylor coefficients $\delta^n W/\delta j(x_1)\dots\delta j(x_n)$. For the lowest two orders we can verify the connectedness by inspecting the third line in Fig. 5.1. The diagrammatic form of the recursion relation shows that this topological property remains true for all orders in λ, by induction. Indeed, if we suppose it to be true for some n, then all $G_{\rm c}^{(1)}$ inserted on the right-hand side are connected, and so are the diagrams constructed from these when forming $G_{\rm c}^{(1)}$ to the next, $(n+1)$st, order.

Note that this calculation is unable to recover the value of $W[j]$ at $j = 0$ since this is an unknown integration constant of the functional differential equation. For the purpose of generating correlation functions, this constant is irrelevant. We have seen in Section 3.2 that $W[0]$ consists of the sum of all connected vacuum diagrams contained in $Z[0]$.

5.3 Decomposition of Correlation Functions into Connected Correlation Functions

Using the logarithmic relation (5.1) between $W[j]$ and $Z[j]$ we can now derive general relations between the n-point functions and their connected parts. For the one-point function we find

$$G^{(1)}(\mathbf{x}) = Z^{-1}[j]\frac{\delta}{\delta j(\mathbf{x})} Z[j] = \frac{\delta}{\delta j(\mathbf{x})} W[j] = G_{\rm c}^{(1)}(\mathbf{x}). \tag{5.18}$$

This equation implies that the one-point function representing the ground state expectation value of the field is always connected:

$$\langle \phi(\mathbf{x})\rangle \equiv G^{(1)}(\mathbf{x}) = G_{\rm c}^{(1)}(\mathbf{x}) = \Phi. \tag{5.19}$$

Consider now the two-point function, which decomposes as follows:

$$\begin{aligned} G^{(2)}(\mathbf{x}_1,\mathbf{x}_2) &= Z^{-1}[j]\frac{\delta}{\delta j(\mathbf{x}_1)}\frac{\delta}{\delta j(\mathbf{x}_2)} Z[j] \\ &= Z^{-1}[j]\frac{\delta}{\delta j(\mathbf{x}_1)}\left\{\left(\frac{\delta}{\delta j(\mathbf{x}_2)} W[j]\right) Z[j]\right\} \\ &= Z^{-1}[j]\left\{W_{j(\mathbf{x}_1)j(\mathbf{x}_2)} + W_{j(\mathbf{x}_1)} W_{j(\mathbf{x}_2)}\right\} Z[j] \\ &= G_{\rm c}^{(2)}\left(\mathbf{x}_1,\mathbf{x}_2\right) + G_{\rm c}^{(1)}\left(\mathbf{x}_1\right) G_{\rm c}^{(1)}\left(\mathbf{x}_2\right). \end{aligned} \tag{5.20}$$

In addition to the connected diagrams with two ends there are two connected diagrams ending in a single line. These are absent in a ϕ^4-theory with positive m^2 at $j = 0$, in which case the system is in the *normal phase* (recall the discussion in Chapter 1). In that case, the two-point function is automatically connected, as we observed in Eq. (3.23).

For the three-point function we find

$$\begin{aligned}
G^{(3)}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) &= Z^{-1}[j]\frac{\delta}{\delta j(\mathbf{x}_1)}\frac{\delta}{\delta j(\mathbf{x}_2)}\frac{\delta}{\delta j(\mathbf{x}_3)}Z[j] \\
&= Z^{-1}[j]\frac{\delta}{\delta j(\mathbf{x}_1)}\frac{\delta}{\delta j(\mathbf{x}_2)}\left\{\left[\frac{\delta}{\delta j(\mathbf{x}_3)}W[j]\right]Z[j]\right\} \\
&= Z^{-1}[j]\frac{\delta}{\delta j(\mathbf{x}_1)}\left\{\left[W_{j(\mathbf{x}_3)j(\mathbf{x}_2)}+W_{j(\mathbf{x}_2)}W_{j(\mathbf{x}_3)}\right]Z[j]\right\} \qquad (5.21)\\
&= Z^{-1}[j]\left\{W_{j(\mathbf{x}_1)j(\mathbf{x}_2)j(\mathbf{x}_3)}+\left(W_{j(\mathbf{x}_1)}W_{j(\mathbf{x}_2)j(\mathbf{x}_3)}+W_{j(\mathbf{x}_2)}W_{j(\mathbf{x}_1)j(\mathbf{x}_3)}\right.\right. \\
&\qquad\left.\left.+W_{j(\mathbf{x}_3)}W_{j(\mathbf{x}_1)j(\mathbf{x}_2)}\right)+W_{j(\mathbf{x}_1)}W_{j(\mathbf{x}_2)}W_{j(\mathbf{x}_3)}\right\}Z[j] \\
&= G_{\rm c}^{(3)}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)+\left[G_{\rm c}^{(1)}(\mathbf{x}_1)G_{\rm c}^{(2)}(\mathbf{x}_2,\mathbf{x}_3)+2\text{ perm}\right]+G_{\rm c}^{(1)}(\mathbf{x}_1)G_{\rm c}^{(1)}(\mathbf{x}_2)G_{\rm c}^{(1)}(\mathbf{x}_3),
\end{aligned}$$

and for the four-point function

$$\begin{aligned}
G^{(4)}(\mathbf{x}_1,\dots,\mathbf{x}_4) &= G_{\rm c}^{(4)}(\mathbf{x}_1,\dots,\mathbf{x}_4)+\left[G_{\rm c}^{(3)}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)\,G_{\rm c}^{(1)}(\mathbf{x}_4)+3\text{ perm}\right] \\
&+\left[G_{\rm c}^{(2)}(\mathbf{x}_1,\mathbf{x}_2)\,G_{\rm c}^{(2)}(\mathbf{x}_3,\mathbf{x}_4)+2\text{ perm}\right] \\
&+\left[G_{\rm c}^{(2)}(\mathbf{x}_1,\mathbf{x}_2)\,G_{\rm c}^{(1)}(\mathbf{x}_3)G_{\rm c}^{(1)}(\mathbf{x}_4)+5\text{ perm}\right] \\
&+\ G_{\rm c}^{(1)}(\mathbf{x}_1)\cdots G_{\rm c}^{(1)}(\mathbf{x}_4). \qquad (5.22)
\end{aligned}$$

In the pure ϕ^4-theory with positive m^2, i.e., in the *normal phase* of the system, there are no odd correlation functions and we are left with the decomposition (3.24), which was found in Chapter 3 diagrammatically up to second order in the coupling constant λ.

For the general correlation function $G^{(n)}$, the total number of terms is most easily retrieved by dropping all indices and differentiating with respect to j (the arguments $\mathbf{x}_1,\dots,\mathbf{x}_n$ of the currents are again suppressed):

$$\begin{aligned}
G^{(1)} &= e^{-W}\left(e^W\right)_j = W_j = G_{\rm c}^{(1)} \qquad (5.23)\\
G^{(2)} &= e^{-W}\left(e^W\right)_{jj} = W_{jj}+{W_j}^2 = G_{\rm c}^{(2)}+G_{\rm c}^{(1)2} \\
G^{(3)} &= e^{-W}\left(e^W\right)_{jjj} = W_{jjj}+3W_{jj}W_j+{W_j}^3 = G_{\rm c}^{(3)}+3G_{\rm c}^{(2)}G_{\rm c}^{(1)}+G_{\rm c}^{(1)3} \\
G^{(4)} &= e^{-W}\left(e^W\right)_{jjjj} = W_{jjjj}+4W_{jjj}W_j+3{W_{jj}}^2+6W_{jj}{W_j}^2+{W_j}^4 \\
&= G_{\rm c}^{(4)}+4G_{\rm c}^{(3)}G_{\rm c}^{(1)}+3G_{\rm c}^{(2)2}+6G_{\rm c}^{(2)}G_{\rm c}^{(1)2}+G_{\rm c}^{(1)4}.
\end{aligned}$$

All relations follow from the recursion relation

$$G^{(n)} = G_j^{(n-1)}+G^{(n-1)}G_{\rm c}^{(1)}, \qquad n\geq 2, \qquad (5.24)$$

if one uses $G_{{\rm c}\,j}^{(n-1)} = G_{\rm c}^{(n)}$ and the initial relation $G^{(1)} = G_{\rm c}^{(1)}$. By comparing the first four relations with the explicit forms (5.20)–(5.22) we see that the numerical factors on the right-hand side of (5.23) refer to the permutations of the arguments $\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\dots$ of otherwise equal expressions. Since there is no problem in reconstructing the explicit permutations we shall henceforth write all composition laws in the short-hand notation (5.23).

The formula (5.23) and its generalization is often referred to as *cluster decomposition*, or also as the *cumulant expansion*, of the correlation functions.

We can now prove that the connected correlation functions collect precisely all connected diagrams in the n-point functions. For this we observe that the decomposition rules can be inverted by repeatedly differentiating both sides of the equation $W[j] = \log Z[j]$ functionally with respect to the current j:

$$\begin{aligned} G_{\rm c}^{(1)} &= G^{(1)} \\ G_{\rm c}^{(2)} &= G^{(2)} - G^{(1)}G^{(1)} \\ G_{\rm c}^{(3)} &= G^{(3)} - 3G^{(2)}G^{(1)} + 2G^{(1)3} \\ G_{\rm c}^{(4)} &= G^{(4)} - 4G^{(3)}G^{(1)} + 12G^{(2)}G^{(1)2} - 3G^{(2)2} - 6G^{(1)4}. \end{aligned} \tag{5.25}$$

Each equation follows from the previous one by one more derivative with respect to j, and by replacing the derivatives on the right-hand side according to the rule

$$G_j^{(n)} = G^{(n+1)} - G^{(n)}G^{(1)}. \tag{5.26}$$

Again the numerical factors imply different permutations of the arguments and the subscript j denotes functional differentiations with respect to j.

Note that Eqs. (5.25) for the connected correlation functions are valid in the normal phase as well as in the phase with spontaneous symmetry breakdown. In the normal phase, the equations simplify, since all terms involving $G^{(1)} = \Phi = \langle\phi\rangle$ vanish.

It is obvious that any connected diagram contained in $G^{(n)}$ must also be contained in $G_{\rm c}^{(n)}$, since all the terms added or subtracted in (5.25) are products of $G_j^{(n)}$s, and thus necessarily disconnected. Together with the proof in Section 5.2 that the correlation functions $G_{\rm c}^{(n)}$ contain *only* the connected parts of $G^{(n)}$, we can now be sure that $G_{\rm c}^{(n)}$ contains precisely the connected parts of $G^{(n)}$.

5.4 Functional Generation of Vacuum Diagrams

The functional differential equation (5.11) for $W[j]$ contains all information on the connected correlation functions of the system. However, it does not tell us anything about the vacuum diagrams of the theory. These are contained in $W[0]$, which remains an undetermined constant of functional integration of these equations.

In order to gain information on the vacuum diagrams, we consider a modification of the generating functional (2.54), in which we set the external source j equal to zero, but generalize the source $K(\mathbf{x})$ to a bilocal form $K(\mathbf{x},\mathbf{y})$:

$$Z[K] \;=\; \int \mathcal{D}\phi \, e^{-E[\phi,K]}, \tag{5.27}$$

where $E[\phi, K]$ is the energy functional:

$$E[\phi,K] \equiv E_0[\phi] + E_{\rm int}[\phi] - \frac{1}{2}\int d^Dx \int d^Dy \, \phi(\mathbf{x})K(\mathbf{x},\mathbf{y})\phi(\mathbf{y}). \tag{5.28}$$

When forming the functional derivative with respect to $K(\mathbf{x},\mathbf{y})$ we obtain the correlation function in the presence of $K(\mathbf{x},\mathbf{y})$:

$$G^{(2)}(\mathbf{x},\mathbf{y}) = 2Z^{-1}[K]\frac{\delta Z}{\delta K(\mathbf{x},\mathbf{y})}. \tag{5.29}$$

At the end we shall set $K(\mathbf{x},\mathbf{y})=0$, just as previously the source j. When differentiating $Z[K]$ twice, we obtain the four-point function

$$G^{(4)}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4)=4Z^{-1}[K]\frac{\delta^2 Z}{\delta K(\mathbf{x}_1,\mathbf{x}_2)\delta K(\mathbf{x}_3,\mathbf{x}_4)}. \tag{5.30}$$

As before, we introduce the functional $W[K]\equiv \log Z[K]$. Inserting this into (5.29) and (5.30), we fin

$$G^{(2)}(\mathbf{x},\mathbf{y}) = 2\frac{\delta W}{\delta K(\mathbf{x},\mathbf{y})}, \tag{5.31}$$

$$G^{(4)}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) = 4\left[\frac{\delta^2 W}{\delta K(\mathbf{x}_1,\mathbf{x}_2)\delta K(\mathbf{x}_3,\mathbf{x}_4)}+\frac{\delta W}{\delta K(\mathbf{x}_1,\mathbf{x}_2)}\frac{\delta W}{\delta K(\mathbf{x}_3,\mathbf{x}_4)}\right]. \tag{5.32}$$

With the same short notation as before, we shall use again a subscript K to denote functional differentiation with respect to K, and write

$$G^{(2)}=2W_K, \qquad G^{(4)}=4\left[W_{KK}+W_K W_K\right]=4\left[W_{KK}+G^{(2)}\,G^{(2)}\right]. \tag{5.33}$$

From Eq. (5.23) we know that in the absence of a source j and in the normal phase, $G^{(4)}$ has the connectedness structure

$$G^{(4)}=G^{(4)}_{\mathrm{c}}+3G^{(2)}_{\mathrm{c}}\,G^{(2)}_{\mathrm{c}}. \tag{5.34}$$

This shows that in contrast to W_{jjjj}, the derivative W_{KK} does not directly yield a connected four-point function, but two disconnected parts:

$$W_{KK}=G^{(4)}_{\mathrm{c}}+2G^{(2)}_{\mathrm{c}}\,G^{(2)}_{\mathrm{c}}, \tag{5.35}$$

the two-point functions being automatically connected in the normal phase. More explicitly

$$\frac{4\delta^2 W}{\delta K(\mathbf{x}_1,\mathbf{x}_2)\delta K(\mathbf{x}_3,\mathbf{x}_4)}=G^{(4)}_{\mathrm{c}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4)+G^{(2)}_{\mathrm{c}}(\mathbf{x}_1,\mathbf{x}_3)G^{(2)}_{\mathrm{c}}(\mathbf{x}_2,\mathbf{x}_4)+G^{(2)}_{\mathrm{c}}(\mathbf{x}_1,\mathbf{x}_4)G^{(2)}_{\mathrm{c}}(\mathbf{x}_2,\mathbf{x}_3). \tag{5.36}$$

Let us derive functional differential equations for $Z[K]$ and $W[K]$. By analogy with (5.3) we start out with the trivial functional differential equation

$$\int \mathcal{D}\phi\;\phi(\mathbf{x})\frac{\delta}{\delta\phi(\mathbf{y})}e^{-E[\phi,K]}=-\delta^{(D)}(\mathbf{x}-\mathbf{y})Z[K], \tag{5.37}$$

which is immediately verified by a functional integration by parts. By the chain rule of differentiation, this becomes

$$\int \mathcal{D}\phi\;\phi(\mathbf{x})\frac{\delta E[\phi,K]}{\delta\phi(\mathbf{y})}e^{-E[\phi,K]}=\delta^{(D)}(\mathbf{x}-\mathbf{y})Z[K]. \tag{5.38}$$

Performing the functional derivative and integrating over $\mathbf{y}$ yields

$$\int \mathcal{D}\phi\int d^D y\left\{\phi(\mathbf{x})G_0^{-1}(\mathbf{x},\mathbf{y})\phi(\mathbf{y})+\frac{\lambda}{3!}\phi(\mathbf{x})\phi^3(\mathbf{y})\right\}e^{-E[\phi,K]}=Z[K]. \tag{5.39}$$

For brevity, we have absorbed the source in the free-field correlation function G_0:

$$G_0^{-1}-K\to G_0^{-1}. \tag{5.40}$$

The left-hand side of (5.38) can obviously be expressed in terms of functional derivatives of $Z[K]$, and we obtain the functional differential equation whose short form reads

$$G_0^{-1} Z_K + \frac{\lambda}{3} Z_{KK} = \frac{1}{2} Z. \tag{5.41}$$

Inserting $Z[K] = e^{W[K]}$, this becomes

$$G_0^{-1} W_K + \frac{\lambda}{3}(W_{KK} + W_K W_K) = \frac{1}{2}. \tag{5.42}$$

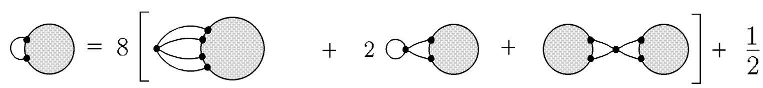
It is useful to reconsider the functional $W[K]$ as a functional $W[G_0]$. Then $\delta G_0/\delta K = G_0^2$, and the derivatives of $W[K]$ become

$$W_K = G_0^2\, W_{G_0}, \quad W_{KK} = 2G_0^3\, W_{G_0} + G_0^4\, W_{G_0 G_0}, \tag{5.43}$$

and (5.42) takes the form

$$G_0 W_{G_0} + \frac{\lambda}{3}(G_0^4 W_{G_0 G_0} + 2G_0^3 W_{G_0} + G_0^4 W_{G_0} W_{G_0}) = \frac{1}{2}. \tag{5.44}$$

This equation is represented diagrammatically in Fig. 5.2. The zeroth-order solution to this



$$G_0 W_{G_0} = 8\frac{-1}{4!}\left[\lambda G_0^4 W_{G_0 G_0} + 2G_0 \lambda G_0^2 W_{G_0} + W_{G_0} G_0^2 \lambda G_0^2 W_{G_0}\right] + \frac{1}{2}$$

FIGURE 5.2 Diagrammatic representation of functional differential equation (5.44). For the purpose of finding the multiplicities of the diagrams, it is convenient to represent here by a vertex the coupling strength $-\lambda/4!$, rather than $-\lambda$ as all other vertices in this book.

equation is obtained by setting $\lambda = 0$:

$$W^{(0)}[G_0] = \frac{1}{2}\text{Tr}\,\log(G_0). \tag{5.45}$$

This is precisely the exponent in the prefactor of the generating functional (2.31) of the free-field theory.

The corrections are found by iteration. For systematic treatment, we write $W[G_0]$ as a sum of a free and an interacting part,

$$W[G_0] = W^{(0)}[G_0] + W^{\text{int}}[G_0], \tag{5.46}$$

insert this into Eq. (5.44), and find the differential equation for the interacting part:

$$G_0 W^{\text{int}}_{G_0} + \frac{\lambda}{3}(G_0^4 W^{\text{int}}_{G_0 G_0} + 3G_0^3 W^{\text{int}}_{G_0} + G_0^4 W^{\text{int}}_{G_0} W^{\text{int}}_{G_0}) = 6\frac{-\lambda}{4!} G_0^2. \tag{5.47}$$

This equation is solved iteratively. Setting $W^{\text{int}}[G_0] = 0$ in all terms proportional to λ, we obtain the first-order contribution to $W^{\text{int}}[G_0]$:

$$W^{\text{int}}[G_0] = 3\frac{-\lambda}{4!} G_0^2. \tag{5.48}$$

This is precisely the contribution of the Feynman diagram. The number 3 is its multiplicity, as defined in Section 3.1.

In order to see how the iteration of Eq. (5.47) may be solved systematically, let us ignore for the moment the functional nature of Eq. (5.47), and treat G_0 as an ordinary real variable rather than a functional matrix. We expand $W[G_0]$ in a Taylor series:

$$W^{\rm int}[G_0] = \sum_{p=1}^{\infty} \frac{1}{p!} W_p \left(\frac{-\lambda}{4!}\right)^p (G_0)^{2p}, \tag{5.49}$$

and find for the expansion coefficients the recursion relation

$$W_{p+1} = 4\left\{[2p\,(2p-1) + 3(2p)]\,W_p + \sum_{q=1}^{p-1} \binom{p}{q} 2q\,W_q \times 2(p-q)W_{p-q}\right\}. \tag{5.50}$$

Solving this with the initial number $W_1 = 3$, we obtain the multiplicities of the connected vacuum diagrams of pth order:

$$3, 96, 9504, 1880064, 616108032, 301093355520, 205062331760640, 185587468924354560,$$
$$215430701800551874560, 312052349085504377978880. \tag{5.51}$$

To check these numbers, we go over to $Z[G] = e^{W[G_0]}$, and find the expansion:

$$\begin{aligned} Z[G_0] &= \exp\left[\frac{1}{2}{\rm Tr}\,\log G_0 + \sum_{p=1}^{\infty} \frac{1}{p!} W_p \left(\frac{-\lambda}{4!}\right)^p (G_0)^{2p}\right] \\ &= {\rm Det}^{1/2}[G_0]\left[1 + \sum_{p=1}^{\infty} \frac{1}{p!} z_p \left(\frac{-\lambda}{4!}\right)^p (G_0)^{2p}\right] \end{aligned} \tag{5.52}$$

The expansion coefficients z_p count the total number of vacuum diagrams of order p. The exponentiation (5.52) yields $z_p = (4p-1)!!$, which is the correct number of Wick contractions of p interactions ϕ^4.

In fact, by comparing coefficients in the two expansions in (5.52), we may derive another recursion relation for W_p:

$$W_p + 3\binom{p-1}{1} W_{p-1} + 7\cdot 5\cdot 3\binom{p-1}{2} + \ldots + (4p-5)!!\binom{p-1}{p-1} = (4p-1)!!, \tag{5.53}$$

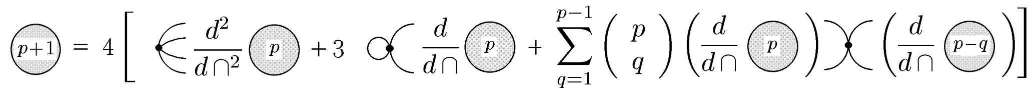
which is fulfilled by the solutions of (5.50).

In order to find the associated Feynman diagrams, we must perform the differentiations in Eq. (5.47) functionally. The numbers W_p become then a sum of diagrams, for which the recursion relation (5.50) reads

$$W_{p+1} = 4\left[\; G_0^4 \frac{d^2}{d\cap^2}\, W_p \;+ 3\cdot G_0^3\, \frac{d}{d\cap}\, W_p \;+ \sum_{q=1}^{p-1} \binom{p}{q} \left(\frac{d}{d\cap} W_q\right) G_0^2 \cdot G_0^2 \left(\frac{d}{d\cap} W_{p-q}\right)\right], \tag{5.54}$$

where the differentiation $d/d\cap$ removes one line connecting two vertices in all possible ways. This equation is solved diagrammatically, as shown in Fig. 5.3.

Starting the iteration with $W_1 = 3$ ∞ , we have $dW_p/d\cap = 6$ ◯ and $d^2W_p/d\cap^2 = 6$ ×. Proceeding to order five loops and going back to the usual vertex notation $-\lambda$, we find the



$$W_{p+1} = 4\left[\, G_0^4 \frac{d^2}{d\cap^2} W_p + 3\cdot G_0^3 \frac{d}{d\cap} W_p + \sum_{q=1}^{p-1} \binom{p}{q} \left(\frac{d}{d\cap} W_q\right) G_0^2 \cdot G_0^2 \left(\frac{d}{d\cap} W_{p-q}\right)\right]$$

FIGURE 5.3 Diagrammatic representation of functional differential equation (5.54). A vertex represents the coupling strength $-\lambda$.

vacuum diagrams with their weight factors as shown in Fig. 5.4. For more than five loops, the reader is referred to the paper quoted in Notes and References, and to the internet address from which Mathematica programs can be downloade,d which solve the recursion relations and plot all diagrams of $W[0]$ and the resulting two-and four-point functions.

In Section 14.2 we shall describe a somewhat shorter computer scheme for generating all diagrams used in this text.

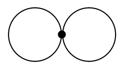
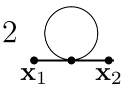
order	diagrams and multiplicities	number
g^1	3	3
g^2	$\frac{1}{2!}$ (24 72)	96
g^3	$\frac{1}{3!}$ (1728 3456 2592 1728)	9504
g^4	$\frac{1}{4!}$ (62208 66296 248832 497664 165888 248832 165888 124416 248832 62208)	1880064

FIGURE 5.4 Vacuum diagrams up to five loops and their multiplicities. In contrast to Fig. 5.3, and the usual diagrammatic notation in (3.5), a vertex stands here for $-\lambda/4!$ for brevity. For more than five loops see the tables on the internet (www.physik.fu-berlin/˜kleinert/294/programs).

5.5 Correlation Functions From Vacuum Diagrams

The vacuum diagrams contain information on all correlation functions of the theory. One may rightly say that the vacuum is the world. The two- and four-point functions are given by the functional derivatives (5.33) of the vacuum functional $W[K]$. Diagrammatically, a derivative with respect to K corresponds to cutting one line of a vacuum diagram in all possible ways. Thus, all diagrams of the two-point function $G^{(2)}$ can be derived from such cuts, multiplied by a factor 2. As an example, consider the first-order vacuum diagram of $W[K]$ in Table 5.4

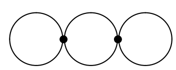
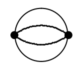
(compare also Fig. 3.5). Cutting one line, which is possible in two ways, and recalling that in Table 5.4 a vertex stands for $-\lambda/4!$ rather than $-\lambda$, as in the other diagrams, we find

$$W_1[0] = \frac{1}{8}$$  $$\longrightarrow \quad G_1^{(2)}(\mathbf{x}_1, \mathbf{x}_2) = 2 \times \frac{1}{8}\, 2$$ 

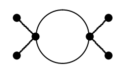
. (5.55)

The right-hand side is the correct first-order contribution to the two-point function [recall Eq. (3.23)].

The second equation in (5.33) tells us that all connected contributions to the four-point function $G^{(4)}$ may be obtained by cutting two lines in all combinations, and multiplying the result by a factor 4. As an example, take the second-order vacuum diagrams of $W[0]$ with the proper translation of vertices by a factor 4! (compare again Fig. 3.5), which are

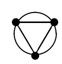
$$W_2[0] = \frac{1}{16}$$  $$+ \frac{1}{48}$$  . (5.56)

Cutting two lines in all possible ways yields the following contributions to the connected diagrams of the two-point function:

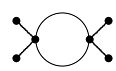
$$G^{(4)} = 4 \times \left(2 \cdot 1 \cdot \frac{1}{16} + 4 \cdot 3 \cdot \frac{1}{48}\right)$$  . (5.57)

This agrees with the first-order contribution calculated in Eq. (3.25).

It is also possible to find all diagrams of the four-point function from the vacuum diagrams by forming a derivative of $W[0]$ with respect to the coupling constant $-\lambda$, and multiplying the result by a factor 4!. This follows directly from the fact that this differentiation applied to $Z[0]$ yields the correlation function $\int d^D x \langle \phi^4 \rangle$. As an example, take the first diagram of order g^3 in Table 5.4 [with the vertex normalization (3.5)]:

$$W_2[0] = \frac{1}{48}$$  . (5.58)

Removing one vertex in the three possible ways and multiplying by a factor 4! yields

$$G^{(4)} = 4! \times \frac{1}{48}\, 3$$  . (5.59)

which agrees with the contribution of this diagram in Eq. (3.25).

These relations will be used in Chapter 14 to generate all diagrams by computer methods.

5.6 Generating Functional for Vertex Functions

Apart from the connectedness structure, the most important step in economizing the calculation of Feynman diagrams consists in the decomposition of higher connected correlation functions into 1PI vertex functions and 1PI two-particle correlation functions, as shown in Section 4.2. There is, in fact, a simple algorithm which supplies us in general with such a decomposition. For this purpose let us introduce a new generating functional $\Gamma[\Phi]$, to be called the *effective energy* of the theory. It is defined via a Legendre transformation of $W[j]$:

$$-\Gamma[\Phi] \equiv W[j] - W_j\, j. \tag{5.60}$$

Here and in the following, we use a short-hand notation for the functional multiplication, $W_j\, j = \int d^D x\, W_j(\mathbf{x}) j(\mathbf{x})$, which considers fields as vectors with a continuous index $\mathbf{x}$. The new variable Φ is the functional derivative of $W[j]$ with respect to $j(\mathbf{x})$ [recall (5.10)]:

$$\Phi(\mathbf{x}) \equiv \frac{\delta W[j]}{\delta j(\mathbf{x})} \equiv W_{j(\mathbf{x})} = \langle \phi \rangle_{j(\mathbf{x})}, \tag{5.61}$$

and thus gives the ground state expectation of the field operator in the presence of the current j. When rewriting (5.60) as

$$-\Gamma[\Phi] \equiv W[j] - \Phi\, j, \tag{5.62}$$

and functionally differentiating this with respect to Φ, we obtain the equation

$$\Gamma_\Phi[\Phi] = j. \tag{5.63}$$

This equation shows that the physical field expectation $\Phi(\mathbf{x}) = \langle \phi(\mathbf{x}) \rangle$, where the external current is zero, extremizes the effective energy:

$$\Gamma_\Phi[\Phi] = 0. \tag{5.64}$$

In this text, we shall only study physical systems whose ordered low-temperature phase has a uniform field expectation value $\Phi(\mathbf{x}) \equiv \Phi_0$. Thus we shall not consider systems such as cholesteric or smectic liquid crystals, which possess a space dependent $\Phi_0(\mathbf{x})$, although such systems can also be described by ϕ^4-theories by admitting more general types of gradient terms, for instance $\phi(\partial^2 - k_0^2)^2\phi$. The ensuing space dependence of $\Phi_0(\mathbf{x})$ may be crystal- or quasicrystal-like [1]. Thus we shall assume a constant

$$\Phi_0 = \langle \phi \rangle |_{j=0}, \tag{5.65}$$

which may be zero or non-zero, depending on the phase of the system.

Let us now demonstrate that the effective energy contains all the information on the proper vertex functions of the theory. These can be found directly from the functional derivatives:

$$\Gamma^{(n)}\left(\mathbf{x}_1, \ldots, \mathbf{x}_n\right) \equiv \frac{\delta}{\delta \Phi(\mathbf{x}_1)} \cdots \frac{\delta}{\delta \Phi(\mathbf{x}_n)} \Gamma[\Phi]\,. \tag{5.66}$$

We shall see that the proper vertex functions of Section 4.2 are obtained from these functions by a Fourier transform and a simple removal of an overall factor $(2\pi)^D \delta^{(D)}\left(\sum_{i=1}^n \mathbf{k}_i\right)$ to ensure momentum conservation. The functions $\Gamma^{(n)}\left(\mathbf{x}_1, \ldots, \mathbf{x}_n\right)$ will therefore be called *vertex functions*, without the adjective *proper* which indicates the absence of the δ-function. In particular, the Fourier transforms of the vertex functions $\Gamma^{(2)}\left(\mathbf{x}_1, \mathbf{x}_2\right)$ and $\Gamma^{(4)}\left(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\right)$ are related to their proper versions by

$$\Gamma^{(2)}(\mathbf{k}_1, \mathbf{k}_2) \;=\; (2\pi)^D \delta^{(D)}\left(\mathbf{k}_1 + \mathbf{k}_2\right) \bar{\Gamma}^{(2)}(\mathbf{k}_1), \tag{5.67}$$

$$\Gamma^{(4)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) \;=\; (2\pi)^D \delta^{(D)}\left(\sum_{i=1}^4 \mathbf{k}_i\right) \bar{\Gamma}^{(4)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4). \tag{5.68}$$

For the functional derivatives (5.66) we shall use the same short-hand notation as for the functional derivatives (5.10) of $W[j]$, setting

$$\Gamma_{\Phi(\mathbf{x}_1) \ldots \Phi(\mathbf{x}_n)} \equiv \frac{\delta}{\delta \Phi(\mathbf{x}_1)} \cdots \frac{\delta}{\delta \Phi(\mathbf{x}_n)} \Gamma[\Phi]\,. \tag{5.69}$$

The arguments $\mathbf{x}_1, \ldots, \mathbf{x}_n$ will usually be suppressed.

In order to derive relations between the derivatives of the effective energy and the connected correlation functions, we first observe that the connected one-point function $G_{\rm c}^{(1)}$ at a nonzero source j is simply the field expectation Φ [recall (5.19)]:

$$G_{\rm c}^{(1)} = \Phi. \tag{5.70}$$

Second, we see that the connected two-point function at a nonzero source j is given by

$$G_{\rm c}^{(2)} = G_j^{(1)} = W_{jj} = \frac{\delta \Phi}{\delta j} = \left(\frac{\delta j}{\delta \Phi}\right)^{-1} = \Gamma^{-1}_{\Phi\Phi}. \tag{5.71}$$

The inverse symbols on the right-hand side are to be understood in the functional sense, i.e., $\Gamma^{-1}_{\Phi\Phi}$ denotes the functional matrix:

$$\Gamma^{-1}_{\Phi(\mathbf{x})\Phi(\mathbf{y})} \equiv \left[\frac{\delta^2 \Gamma}{\delta\Phi(\mathbf{x})\delta\Phi(\mathbf{y})}\right]^{-1}, \tag{5.72}$$

which satisfies

$$\int d^D y \, \Gamma^{-1}_{\Phi(\mathbf{x})\Phi(\mathbf{y})} \Gamma_{\Phi(\mathbf{y})\Phi(\mathbf{z})} = \delta^{(D)}(\mathbf{x}-\mathbf{z}). \tag{5.73}$$

Relation (5.71) states that the second derivative of the effective energy determines directly the connected correlation function $G_{\rm c}^{(2)}(\mathbf{k})$ of the interacting theory in the presence of the external source j. Since j is an auxiliary quantity, which eventually be set equal to zero thus making Φ equal to Φ_0, the actual physical propagator is given by

$$G_{\rm c}^{(2)}\Big|_{j=0} = \Gamma^{-1}_{\Phi\Phi}\Big|_{\Phi=\Phi_0}. \tag{5.74}$$

By Fourier-transforming this relation and removing a δ-function for the overall momentum conservation, the propagator $G(\mathbf{k})$ in Eq. (4.18) is related to the vertex function $\Gamma^{(2)}(\mathbf{k})$, defined in (5.67) by

$$G(\mathbf{k}) \equiv \bar{G}^{(2)}(\mathbf{k}) = \frac{1}{\bar{\Gamma}^{(2)}(\mathbf{k})}, \tag{5.75}$$

as observed before on diagrammatic grounds in Eq. (4.34).

The third derivative of the generating functional $W[j]$ is obtained by functionally differentiating W_{jj} in Eq. (5.71) once more with respect to j, and applying the chain rule:

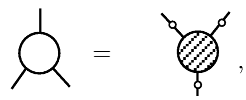
$$W_{jjj} = -\Gamma^{-2}_{\Phi\Phi}\Gamma_{\Phi\Phi\Phi}\frac{\delta\Phi}{\delta j} = -\Gamma^{-3}_{\Phi\Phi}\Gamma_{\Phi\Phi\Phi} = -G^3\Gamma_{\Phi\Phi\Phi}. \tag{5.76}$$

This equation has a simple physical meaning. The third derivative of $W[j]$ on the left-hand side is the full three-point function at a nonzero source j, so that

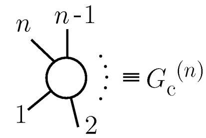
$$G_{\rm c}^{(3)} = W_{jjj} = -{G_c^{(2)}}^3\Gamma_{\Phi\Phi\Phi}. \tag{5.77}$$

This equation states that the full three point function arises from a third derivative of $\Gamma[\Phi]$ by attaching to each derivation a full propagator, apart from a minus sign. This structure was observed empirically in the low-order diagrammatic expansion (4.21) for the four-point function.

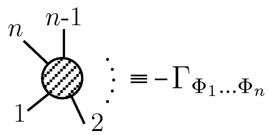
We shall express Eq. (5.77) diagrammatically as follows:



where



denotes the connected n-point function, and

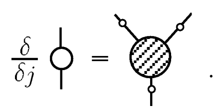


the negative n-point vertex function.

For the general analysis of the diagrammatic content of the effective energy, we observe that according to Eq. (5.76), the functional derivative of the correlation function G with respect to the current j satisfies

$$G_c^{(2)}{}_j = W_{jjj} = G_c^{(3)} = -{G_c^{(2)}}^3 \Gamma_{\Phi\Phi\Phi}. \tag{5.78}$$

This is pictured diagrammatically as follows:

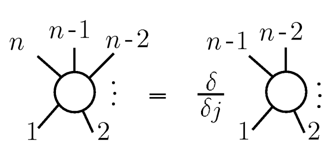


(5.79)

This equation may be differentiated further with respect to j in a diagrammatic way. From the definition (5.2) we deduce the trivial recursion relation

$$G_{\mathrm{c}}^{(n)}\left(\mathbf{x}_1, \dots, \mathbf{x}_n\right) = \frac{\delta}{\delta j(\mathbf{x}_n)} G_{\mathrm{c}}^{(n-1)}\left(\mathbf{x}_1, \dots, \mathbf{x}_{n-1}\right), \tag{5.80}$$

which is represented diagrammatically as

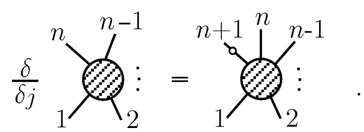


$n > 2$.

By applying $\delta/\delta j$ repeatedly to the left-hand side of Eq. (5.78), we generate all higher connected correlation functions. On the right-hand side of (5.78), the chain rule leads to a derivative of all correlation functions $G = G_c^{(2)}$ with respect to j, thereby changing a line into a line with an extra three-point vertex as indicated in the diagrammatic equation (5.79). On the other hand, the vertex function $\Gamma_{\Phi\Phi\Phi}$ must be differentiated with respect to j. Using the chain rule, we obtain for any n-point vertex function:

$$\Gamma_{\Phi\dots\Phi j} = \Gamma_{\Phi\dots\Phi\Phi} \frac{\delta\Phi}{\delta j} = \Gamma_{\Phi\dots\Phi\Phi} G_c^{(2)}, \tag{5.81}$$

which may be represented diagrammatically as



With these diagrammatic rules, we can differentiate (5.76) any number of times, and derive the diagrammatic structure of the connected correlation functions with an arbitrary number of external legs. The result up to $n = 5$ is shown in Fig. 5.5.

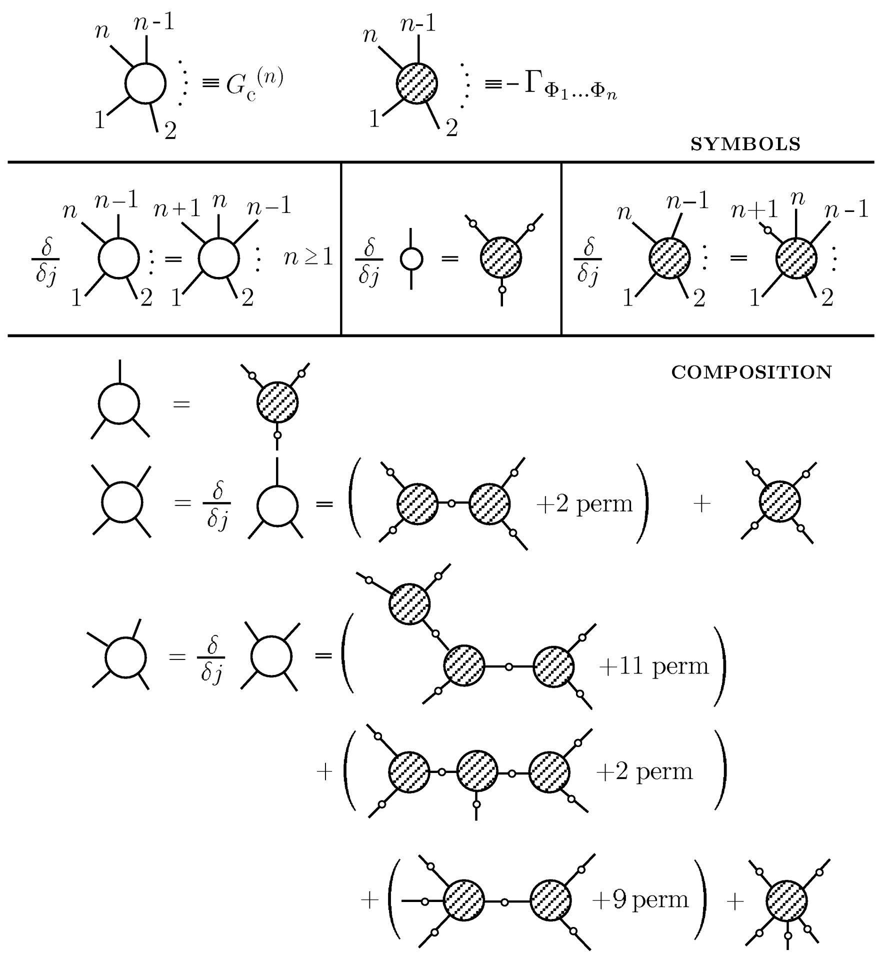


FIGURE 5.5 Diagrammatic differentiations for deriving the decomposition of connected correlation functions into trees of 1PI diagrams. The last term in each decomposition contains, after amputation and removal of an overall δ-function of momentum conservation, precisely all 1PI diagrams of Eqs. (4.19) and (4.22).

The diagrams generated in this way have a tree-like structure, and for this reason they are called *tree diagrams*. The tree decomposition reduces all diagrams to their one-particle irreducible contents. This proves our earlier statement that the vertex functions contain precisely the same Feynman diagrams as the proper vertex functions defined diagrammatically in Section 4.2, apart from the δ-function that ensures overall momentum conservation.

The effective energy $\Gamma[\Phi]$ can be used to prove an important composition theorem: The full propagator G can be expressed as a geometric series involving the so-called *self-energy*, a fact that was observed diagrammatically for low orders earlier in the explicit expansion (4.30). Let us decompose the vertex function as

$$\bar{\Gamma}^{(2)} = G_0^{-1} + \bar{\Gamma}^{\rm int}_{\Phi\Phi}, \tag{5.82}$$

such that the full propagator (5.74) can be rewritten as

$$G = \left(1 + G_0 \bar{\Gamma}^{\rm int}_{\Phi\Phi}\right)^{-1} G_0. \tag{5.83}$$

Expanding the denominator, this can also be expressed in the form of an integral equation:

$$G = G_0 - G_0 \bar{\Gamma}^{\rm int}_{\Phi\Phi} G_0 + G_0 \bar{\Gamma}^{\rm int}_{\Phi\Phi} G_0 \bar{\Gamma}^{\rm int}_{\Phi\Phi} G_0 - \ldots \ . \tag{5.84}$$

In this equation we identify the self-energy introduced diagrammatically in Eq. (4.30) as

$$\Sigma \equiv -\bar{\Gamma}^{\rm int}_{\Phi\Phi}, \tag{5.85}$$

i.e., the self-energy is given by the interacting part of the second functional derivative of the effective energy, except for an opposite sign.

Equation (5.84) is the analytic proof of the chain decomposition (4.30) of the full propagator G. All diagrams can be obtained from a repetition of self-energy diagrams connected by a single line. The corresponding Eq. (5.83) confirms the earlier observation, in diagrams of lower orders, that the full propagator can be expressed in terms of Σ as [recall Eq. (4.31)]:

$$G \equiv [G_0^{-1} - \Sigma]^{-1}. \tag{5.86}$$

This equation can, incidentally, be rewritten in the form of an integral equation for the correlation function G:

$$G = G_0 - G_0 \bar{\Gamma}^{\rm int}_{\Phi\Phi} G. \tag{5.87}$$

5.7 Landau Approximation to Generating Functional

Since the vertex functions are the functional derivatives of the effective energy [see (5.66)], we can expand the effective energy into a functional Taylor series

$$\Gamma[\Phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^D x_1 \ldots d^D x_n \Gamma^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n) \Phi(\mathbf{x}_1) \ldots \Phi(\mathbf{x}_n). \tag{5.88}$$

The expansion in the number of loops of the generating functional $\Gamma[\Phi]$ collects systematically the contributions of fluctuations. To zeroth order, all fluctuations are neglected, and the effective energy reduces to the initial energy, which is the Landau approximation to the Gibbs functional [2] described in Chapter 1. In fact, in the absence of loop diagrams, the vertex functions contain only the lowest-order terms in $\Gamma^{(2)}$ and $\Gamma^{(4)}$:

$$\Gamma_0^{(2)}(\mathbf{x}_1, \mathbf{x}_2) = \left(-\partial^2_{\mathbf{x}_1} + m^2\right) \delta^{(D)}(\mathbf{x}_1 - \mathbf{x}_2), \tag{5.89}$$

$$\Gamma_0^{(4)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \lambda\, \delta^{(D)}(\mathbf{x}_1 - \mathbf{x}_2) \delta^{(D)}(\mathbf{x}_1 - \mathbf{x}_3) \delta^{(D)}(\mathbf{x}_1 - \mathbf{x}_4). \tag{5.90}$$

Inserted into (5.88), this yields the zero-loop approximation to $\Gamma[\Phi]$:

$$\Gamma_0[\Phi] = \frac{1}{2!} \int d^D x\, [(\partial_{\mathbf{x}} \Phi)^2 + m^2 \Phi^2] + \frac{\lambda}{4!} \int d^D x\, \Phi^4. \tag{5.91}$$

This is precisely the original energy functional (2.1). By allowing $\Phi(\mathbf{x})$ to be a vector $\mathbf{\Phi}(\mathbf{x})$, we recover the original phenomenological Ginzburg-Landau energy functional (1.83). Upon replacing the fluctuating field $\phi(\mathbf{x})$ further by its constant expectation value Φ_0, and by identifying this with the magnetic order parameter $\mathbf{M}$, we find the Gibbs free energy (1.38) used by Landau to explain the magnetic phase transition in the mean-field approximation:

$$\Gamma_0[\mathbf{M}] = V\left(\frac{m^2}{2!}\mathbf{M}^2 + \frac{\lambda}{4!}\mathbf{M}^4\right). \tag{5.92}$$

5.8 Composite Fields

In Sections 2.4, 3.5, and 4.3, we encountered a correlation function in which two fields coincide at one point, to be denoted by

$$G^{(1,n)}(\mathbf{x}, \mathbf{x}_1, \ldots, \mathbf{x}_n) = \frac{1}{2}\langle \phi^2(\mathbf{x})\phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_n)\rangle. \tag{5.93}$$

If multiplied by a factor m^2, the composite operator $m^2\phi^2(\mathbf{x})/2$ is precisely the mass term in the energy functional (2.2). For this reason one speaks of a *mass insertion* into the correlation function $G^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$. The negative sign is chosen for convenience.

Actually, we shall never make use of the full correlation function (5.93), but only of the integral over $\mathbf{x}$ in (5.93). This can be obtained directly from the generating functional $Z[j]$ of all correlation functions by differentiation with respect to the square mass in addition to the source terms

$$\int d^D x\, G^{(1,n)}(\mathbf{x}, \mathbf{x}_1, \ldots, \mathbf{x}_n) = -\, Z^{-1}\frac{\partial}{\partial m^2}\frac{\delta}{\delta j(\mathbf{x}_1)}\cdots\frac{\delta}{\delta j(\mathbf{x}_n)}Z[j]\bigg|_{j=0}. \tag{5.94}$$

The desire to have a positive sign on the right-hand side was the reason for choosing a minus sign in the definition of the mass insertion (5.93). By going over to the generating functional $W[j]$, we obtain in a similar way the connected parts:

$$\int d^D x\, G_c^{(1,n)}(\mathbf{x}, \mathbf{x}_1, \ldots, \mathbf{x}_n) = -\frac{\partial}{\partial m^2}\frac{\delta}{\delta j(\mathbf{x}_1)}\cdots\frac{\delta}{\delta j(\mathbf{x}_n)}W[j]\bigg|_{j=0}. \tag{5.95}$$

The right-hand side can be rewritten as

$$\int d^D x\, G_c^{(1,n)}(\mathbf{x}, \mathbf{x}_1, \ldots, \mathbf{x}_n) = -\frac{\partial}{\partial m^2}G_c^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n). \tag{5.96}$$

The connected correlation functions $G_c^{(1,n)}(\mathbf{x}, \mathbf{x}_1, \ldots, \mathbf{x}_n)$ can be decomposed into tree diagrams consisting of lines and one-particle irreducible vertex functions $\Gamma^{(1,n)}(\mathbf{x}, \mathbf{x}_1, \ldots, \mathbf{x}_n)$. The integral over $\mathbf{x}$ of these diagrams is obtained from the Legendre transform (5.60) by a further differentiation with respect to m^2:

$$\int d^D x\, \Gamma^{(1,n)}(\mathbf{x}, \mathbf{x}_1, \ldots, \mathbf{x}_n) = -\frac{\partial}{\partial m^2}\frac{\delta}{\delta \Phi(\mathbf{x}_1)}\cdots\frac{\delta}{\delta \Phi(\mathbf{x}_n)}\Gamma[\Phi]\bigg|_{\Phi_0}, \tag{5.97}$$

implying the relation

$$\int d^D x\, \Gamma^{(1,n)}(\mathbf{x}, \mathbf{x}_1, \ldots, \mathbf{x}_n) = -\frac{\partial}{\partial m^2}\Gamma^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n), \tag{5.98}$$

which was derived by diagrammatic arguments for the proper vertex functions in Eq. (4.44), and which will be needed later in Section 10.1.

Notes and References

The derivation of the graphical recursion relation in Fig. 5.2 was given in
H. Kleinert, Fortschr. Phys. **30**, 187 (1982) (www.physik.fu-berlin/~kleinert/82); also in Fortschr. Phys. **30**, 351 (1982) (www.physik.fu-berlin/~kleinert/84).
Its evaluation is discussed in detail in
H. Kleinert, A. Pelster, B. Kastening, M. Bachmann, Phys. Rev. E **62**, 1537 (2000) (hep-th/9907168).
Diagrams beyond five loops can be found on the internet (www.physik.fu-berlin/~kleinert/294/programs).

The individual citations in the text refer to:

[1] See, for example,
H. Kleinert and K. Maki, Fortschr. Phys. **29**, 1 (1981);
H. Kleinert, Phys. Lett. A **90**, 259 (1982);
H. Kleinert and F. Langhammer, Phys. Rev. A **40**, 5988 (1989).
The 1981 paper was the first to investigate icosahedral quasicrystalline structures discovered later in aluminum.

[2] L.D. Landau, J.E.T.P. **7**, 627 (1937).