4

Diagrams in Momentum Space

At the end of Section 2.3, we have observed that all correlation functions of the local field theory under consideration are invariant under spatial translations. This invariance has the consequence that Feynman integrals are greatly simplified by Fourier transformation. It is therefore useful to set up rules for composing Feynman integrals directly in momentum space.

4.1 Fourier Transformation

The Fourier transform of a function $F(\mathbf{x})$ is defined by

$$F(\mathbf{k}) \equiv \int d^D x \; e^{-i\,\mathbf{k}\cdot\mathbf{x}} \; F(\mathbf{x}). \tag{4.1}$$

The original function $F(\mathbf{x})$ is retrieved from $F(\mathbf{k})$ by the inverse Fourier transformation

$$F(\mathbf{x}) \equiv \int \frac{d^D k}{(2\pi)^D} \; e^{i\,\mathbf{k}\cdot\mathbf{x}} \; F(\mathbf{k}). \tag{4.2}$$

By applying a Fourier transformation to all arguments of the n-point functions $G^{(n)}(\mathbf{x}_1, \dots, \mathbf{x}_n)$, we obtain the n-point functions in momentum space $G^{(n)}(\mathbf{k}_1, \dots, \mathbf{k}_n)$.

4.1.1 Free Two-Point Function

The Fourier transform of the free two-point function $G_0(\mathbf{x}, \mathbf{x}') = G_0^{(2)}(\mathbf{x}, \mathbf{x}')$ reads

$$G_0(\mathbf{k}, \mathbf{k}') = \int d^D x \, d^D x' \; e^{-i\,(\mathbf{k}\cdot\mathbf{x}+\mathbf{k}'\cdot\mathbf{x}')} \; G_0(\mathbf{x}, \mathbf{x}'). \tag{4.3}$$

As observed after Eq. (2.34), the free two-point function depends only on the difference of its spatial variables, due to translational invariance. This simplifies the momentum space representation, which becomes

$$\begin{aligned} G_0(\mathbf{k}, \mathbf{k}') &= \int d^D x' \; e^{-i\,(\mathbf{k}'+\mathbf{k})\cdot\mathbf{x}'} \int d^D x \; e^{-i\,\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} G_0(\mathbf{x}-\mathbf{x}') \\ &= (2\pi)^D \; \delta^{(D)}(\mathbf{k}+\mathbf{k}') \; G_0(\mathbf{k}). \end{aligned} \tag{4.4}$$

Thus the Fourier components $G_0(\mathbf{k}, \mathbf{k}')$ depend only on one momentum variable $\mathbf{k}$, and the function $G_0(\mathbf{k})$ is simply the Fourier transform of the free $\mathbf{x}$-space Green function of Eq. (2.40) with a single argument $G_0(\mathbf{x}-\mathbf{x}')$, which has the Fourier representation

$$G_0(\mathbf{x}, \mathbf{x}') = G_0(\mathbf{x}-\mathbf{x}') = \int \frac{d^D k}{(2\pi)^D} \, e^{i\,\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \; G_0(\mathbf{k}). \tag{4.5}$$

In momentum space, the free propagator has a simple algebraic expression which follows directly from Eq. (2.34) and the definition of the functional inverse in (2.29). The free propagator satisfies the integral equation:

$$\int d^D x' \, D(\mathbf{x}, \mathbf{x}') G_0(\mathbf{x}', \mathbf{x}'') = \delta^{(D)}(\mathbf{x} - \mathbf{x}''). \tag{4.6}$$

Inserting the expression (2.16) for $D(\mathbf{x}, \mathbf{x}')$, and the Fourier representation (4.5) for $G_0(\mathbf{x}', \mathbf{x}'')$, we obtain

$$\begin{aligned} &\int d^D x' \, \delta^{(D)}(\mathbf{x} - \mathbf{x}')(-\partial_{\mathbf{x}}^2 + m^2) \int \frac{d^D k}{(2\pi)^D} \, e^{i\,\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}'')} \, G_0(\mathbf{k}) \\ &\quad = \int \frac{d^D k}{(2\pi)^D} \, (\mathbf{k}^2 + m^2) \, G_0(\mathbf{k}) \, e^{i\,\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}'')} = \delta^{(D)}(\mathbf{x} - \mathbf{x}''). \end{aligned} \tag{4.7}$$

By comparing this with the Fourier representation of the δ-distribution,

$$\int \frac{d^D k}{(2\pi)^D} \, e^{i\,\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}'')} \quad = \quad \delta^{(D)}(\mathbf{x} - \mathbf{x}''), \tag{4.8}$$

we find the momentum space representation of the free propagator

$$G_0(\mathbf{k}) \quad = \quad \frac{1}{\mathbf{k}^2 \, + \, m^2}. \tag{4.9}$$

The Fourier transform of the n-point function is, of course, defined by

$$G^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n) = \int d^D x_1 \cdots d^D x_n \, e^{-i\,(\mathbf{k}_1\cdot\mathbf{x}_1 + \ldots + \mathbf{k}_n\cdot\mathbf{x}_n)} G^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n). \tag{4.10}$$

4.1.2 Connected n-Point Function

Since the correlation function associated with a disconnected Feynman diagram factorizes into those of the connected parts, the same is true for its Fourier transform. It will therefore be sufficient to set up the desired Feynman rules for connected diagrams in momentum space. We shall denote the external and internal points by the symbols $\mathbf{x}_k$ ($k = 1, \ldots, n$) and $\mathbf{z}_i$ ($i = 1, \ldots, p$). Each momentum can be represented by a line. Depending on the endpoints, we distinguish external lines ($k = 1, \ldots, n$) and internal lines $\mathbf{p}_i$ ($i = 1, \ldots, I$). The number of external lines is obviously n. The number of internal lines I is determined by n and the number of vertices p as follows:

$$I = (4p - n)/2. \tag{4.11}$$

This number is a direct consequence of each line having two ends, coinciding either with one of the n external points $\mathbf{x}_k$ or with one of the p internal points $\mathbf{z}_i$ at which four lines meet. Subtracting from the resulting total number $(4p + n)/2$ of lines the number n of external lines, we obtain (4.11).

Let $G_{\mathrm{c}}^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ stand for the Feynman integral symbolized by a connected diagram in $G_p^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$. Omitting for a moment the coupling and weight factors $(-g)^p$ and W_G, the integral involves a product of free propagators:

$$G_{\mathrm{c}}^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \int d^D z_1 \cdots d^D z_p \prod_{i=1}^{n} G_0(\mathbf{x}_i - \mathbf{z}_i) \times \prod_{i=1}^{I} G_0(\mathbf{z}_j - \mathbf{z}_j). \tag{4.12}$$

The Fourier transform of $G_{\rm c}^{(n)}(\mathbf{x}_1,\dots,\mathbf{x}_n)$ is defined as in (4.10):

$$G_{\rm c}^{(n)}(\mathbf{k}_1,\dots,\mathbf{k}_n)=\int d^Dx_1\cdots d^Dx_n\, e^{-i(\mathbf{k}_1\cdot\mathbf{x}_1+\ldots+\mathbf{k}_n\cdot\mathbf{x}_n)}G_{\rm c}^{(n)}(\mathbf{x}_1,\dots,\mathbf{x}_n). \tag{4.13}$$

It contains n factors $e^{-i\mathbf{k}_k\cdot\mathbf{x}_k}$, whose momenta $\mathbf{k}_k$ are represented by an external line. Each external point $\mathbf{x}_k$ appears in a free Green function, which has the Fourier representation

$$G_0(\mathbf{x}_k-\mathbf{z}_i)=\int\frac{d^Dk_k}{(2\pi)^D}\, e^{i\,\mathbf{k}_k\cdot(\mathbf{x}_k-\mathbf{z}_i)}\, G_0(\mathbf{k}_k). \tag{4.14}$$

These contribute to the integral (4.12) an exponential factor $e^{i\mathbf{k}_k\cdot(\mathbf{x}_k-\mathbf{z}_i)}$. Each pair of internal points $\mathbf{z}_i$ appears in a free Green function

$$G_0(\mathbf{z}_i-\mathbf{z}_j)=\int\frac{d^Dp}{(2\pi)^D}\, e^{i\,\mathbf{p}\cdot(\mathbf{z}_i-\mathbf{z}_j)}\, G_0(\mathbf{p}), \tag{4.15}$$

contributing an exponential factor $e^{i\mathbf{p}\cdot(\mathbf{z}_i-\mathbf{z}_j)}$. As each of the external points $\mathbf{x}_i$ appears twice in those phase factors, the integrals over $\mathbf{x}_i$ produce a factor $(2\pi)^D\delta^{(D)}(\mathbf{k}-\mathbf{p})$ for each external line. The subsequent n integrals over the corresponding momenta $\mathbf{p}$ can all be done, leaving us with $(4p-n)/2$ nontrivial momentum integrals, one for each internal line.

Each vertex appears in four exponential factors $e^{i\,\mathbf{p}\cdot\mathbf{x}}$. The integrals over the internal vertex positions $\mathbf{z}_i$ yield p δ-distributions which express momentum conservation at each vertex. One of these can be chosen to contain the sum over all external momenta. It guarantees overall momentum conservation. The others can trivially be done, thereby removing $p-1$ integrations. We end up with

$$L=I-p+1 \tag{4.16}$$

nontrivial integrals over momentum variables $\mathbf{l}_i$, the *loop momenta*, which may be associated with the independent loops in the diagrams. The Fourier transform of $G_{\rm c}^{(n)}(\mathbf{x}_1,\dots,\mathbf{x}_n)$ has therefore the form

$$\begin{aligned}G_{\rm c}^{(n)}(\mathbf{k}_1,\dots,\mathbf{k}_n) &= G_0(\mathbf{k}_1)\,\cdots\, G_0(\mathbf{k}_n)\,(2\pi)^D\,\delta^{(D)}\left(\sum_{i=1}^n\mathbf{k}_i\right)\\ &\times\int\frac{d^Dl_1}{(2\pi)^D}\cdots\frac{d^Dl_L}{(2\pi)^D}G_0(\mathbf{p}_1(\mathbf{l},\mathbf{k}))\cdots G_0(\mathbf{p}_I(\mathbf{l},\mathbf{k})).\end{aligned} \tag{4.17}$$

Each line momentum is expressed by a combination of loop momenta $\mathbf{l}_i$ $(i=1,\dots,L)$ and external momenta $\mathbf{k}_i$ $(i=1,\dots,n)$, abbreviated by $(\mathbf{l},\mathbf{k})$.

The direction of a loop momentum is a consequence of momentum conservation. The external momenta $\mathbf{k}_k$ all flow out of each diagram into the external point.

Reintroducing the previously omitted factors $-g/4!$ and W_G, we see that a connected Feynman diagram in the $\mathbf{k}$-space contains:

1. a factor $G_0(\mathbf{k}_i)$ for each external line;

2. a factor $G_0(\mathbf{p}_j)$ with $j=1,\dots,I$ for each internal line, where each internal momentum $\mathbf{p}_j$ has an orientation and is expressed by a combination of the $L=I-p+1$ loop momenta and the n external momenta;

3. an integration over each independent loop momentum
$(1/2\pi)^D\int d^Dl_i,\ (i=1,\dots,L)$;

4. a factor $(2\pi)^D\,\delta^{(D)}(\mathbf{k}_1+\ldots+\mathbf{k}_n)$ to guarantee overall momentum conservation;
5. a factor $-\lambda/4!$ for each vertex; and
6. a weight factor W_G of the diagram.

From Eq. (4.17), we see that the full connected part of the propagator factorizes in the same way as the free propagator in Eq. (4.4):

$$G_{\rm c}^{(2)}(\mathbf{k},\mathbf{k}') \equiv G(\mathbf{k},\mathbf{k}') = (2\pi)^D\delta^{(D)}(\mathbf{k}+\mathbf{k}')G(\mathbf{k}), \tag{4.18}$$

thus defining a full propagator with a single momentum argument $G(\mathbf{k})$.

4.2 One-Particle Irreducible Diagrams and Proper Vertex Functions

Since the integrations associated with a Feynman diagram run only over the loop momenta, it is convenient to introduce a reduced diagram which represents precisely the loop integrations by removing all external lines. The result is a so-called *amputated diagram*. These diagrams are drawn with short, unlabeled external lines indicating the amputation points. The omission of the external lines removes any difference between diagrams with differently labeled external lines.

The weight factor for these unlabeled diagrams differs from those of the previous diagrams with labeled lines. This generally changes the weight factor by an additional factor $N_{\rm perm}$, which accounts for the permutations of the external lines and the additional permutations of the internal vertices, as discussed in the context of Eq. (3.16).

Amputated diagrams may contain lines without loop momentum, i.e., without additional factors of $G_0(\mathbf{k}_i)$. Such lines connect loop parts of a diagram, and the diagram falls apart when any of these lines is cut. These lines are called *cutlines*. An amputated diagram which possesses a cutline is said to be *one-particle reducible*. Amputated diagrams without cutlines are called *one-particle irreducible* (1PI) diagrams. They represent the smallest nontrivial Feynman diagrams which form the basic building blocks of all diagrams. The detailed laws of composition of these building blocks will be described in Chapter 5.

For each 1PI diagram with $n>2$ we introduce the product of loop integrals in Eq. (4.17) as the *proper vertex function* , apart from a minus-sign which is a matter of convention

$$\bar\Gamma^{(n)}(\mathbf{k}_1,\ldots,\mathbf{k}_n) \;\equiv -\int\frac{d^Dl_1}{(2\pi)^D}\cdots\frac{d^Dl_L}{(2\pi)^D}G_0(\mathbf{q}_1(\mathbf{l},\mathbf{k}))\cdots G_0(\mathbf{q}_I(\mathbf{l},\mathbf{k})), \qquad n>2. \tag{4.19}$$

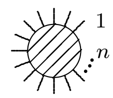
The 1PI part of a correlation function $G_{\rm c}^{(n)}(\mathbf{k}_1,\ldots,\mathbf{k}_n)$ is, of course, recovered from the proper vertex function by multiplication with a free two-point function for each external momentum, and with function $(2\pi)^D\delta^{(D)}(\sum_{i=1}^n\mathbf{k}_i)$ enforcing the total momentum conservation:

$$G_{\rm c}^{(n)}(\mathbf{k}_1,\ldots,\mathbf{k}_n)\Big|_{\rm 1PI} = -G_0(\mathbf{k}_1)\;\cdots\;G_0(\mathbf{k}_n)\,(2\pi)^D\,\delta^{(D)}\left(\sum_{i=1}^n\mathbf{k}_i\right)\bar\Gamma^{(n)}(\mathbf{k}_1,\ldots,\mathbf{k}_n), \qquad n>2. \tag{4.20}$$

Below, in Subsection 5.6, we shall see how to construct the missing one-particle reducible parts of the connected correlation functions $G_{\rm c}^{(n)}(\mathbf{k}_1,\ldots,\mathbf{k}_n)$ from the proper vertex functions $\bar\Gamma^{(n)}(\mathbf{k}_1,\ldots,\mathbf{k}_n)$. We shall find that for the ϕ^4-theory in the normal phase under study here, the four-point function has no one-particle-reducible parts, so that Eq. (4.20) happens to give the complete connected correlation function:

$$G_c^{(4)}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) = -G(\mathbf{k}_1)G(\mathbf{k}_2)G(\mathbf{k}_3)G(\mathbf{k}_4)(2\pi)^D\delta^{(D)}\left(\sum_{i=1}^{4}\mathbf{k}_i\right)\bar{\Gamma}^{(4)}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4). \quad (4.21)$$

Note that the proper vertex functions $\bar{\Gamma}^{(n)}(\mathbf{k}_1,\ldots,\mathbf{k}_n)$ are only defined for momenta $\mathbf{k}_i$ which add up to zero. They will be represented diagrammatically by

$$\bar{\Gamma}^{(n)}(\mathbf{k}_i) = -$$  . (4.22)

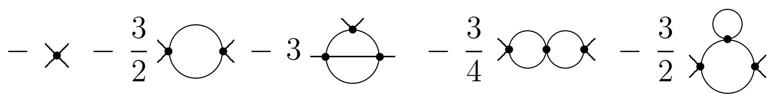
It is useful to introduce also the *proper correlation functions* $\bar{G}^{(n)}(\mathbf{k}_1,\ldots,\mathbf{k}_n)$ in which the δ-function for the overall momentum conservation is removed:

$$G^{(n)}(\mathbf{k}_1,\ldots,\mathbf{k}_n) \equiv (2\pi)^D\delta^{(D)}\left(\sum_{i=1}^{4}\mathbf{k}_i\right)\bar{G}^{(n)}(\mathbf{k}_1,\ldots,\mathbf{k}_n). \quad (4.23)$$

The functions $\bar{G}^{(n)}(\mathbf{k}_1,\ldots,\mathbf{k}_n)$ are defined only for momentum arguments which add up to zero. Then we can invert relation (4.21) to

$$\bar{\Gamma}^{(4)}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) \equiv -G^{-1}(\mathbf{k}_1)\,G^{-1}(\mathbf{k}_2)\,G^{-1}(\mathbf{k}_3)\,G^{-1}(\mathbf{k}_4)\,\bar{G}_c^{(4)}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4). \quad (4.24)$$

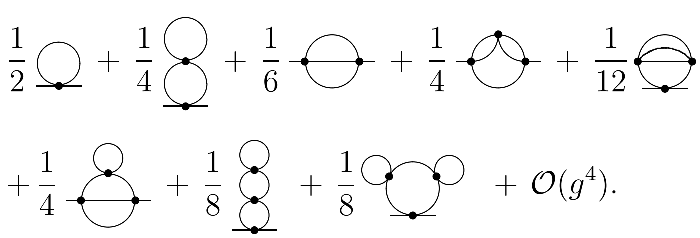
Up to two loops, the diagrammatic expansion of $\bar{\Gamma}^{(4)}$ is

$$\bar{\Gamma}^{(4)}(\mathbf{k}_i) =$$  $+\,\mathcal{O}(g^4)$. (4.25)

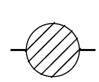
Note that with the notation (4.23), the relation (4.18) for the propagator implies the identity

$$G(\mathbf{k}) \equiv \bar{G}_c^{(2)}(\mathbf{k},-\mathbf{k}). \quad (4.26)$$

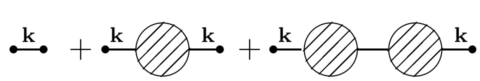
The decomposition of the diagrams in the two-point function is somewhat different. The sum of all 1PI diagrams in $G(\mathbf{k})$ without the zeroth-order diagram is defined as the *self-energy* $\Sigma(\mathbf{k})$. Up to third order in the coupling strength, the self-energy has the diagrammatic expansion

$$\Sigma(\mathbf{k}) =$$  $+\,\mathcal{O}(g^4)$. (4.27)

Given the self-energy, the full propagator $G(\mathbf{k})$ is found by forming a *chain* consisting of self-energies connected by single lines. If we denote the self-energy by a diagram

$$\Sigma(\mathbf{k}) =$$  , (4.28)

the sum over these chains looks as follows:

$$G(\mathbf{k}) =$$  $+\cdots$. (4.29)

Analytically, this sum reads

$$G(\mathbf{k}) = G_0(\mathbf{k}) + G_0(\mathbf{k})\Sigma(\mathbf{k})G_0(\mathbf{k}) + G_0(\mathbf{k})\Sigma(\mathbf{k})G_0(\mathbf{k})\Sigma(\mathbf{k})G_0(\mathbf{k}) + \ldots \; . \tag{4.30}$$

This is a geometric series, which is readily summed by

$$G(\mathbf{k}) = G_0(\mathbf{k})\sum_{l=0}^{\infty}[\Sigma(\mathbf{k})G_0(\mathbf{k})]^l = \left[G_0^{-1}(\mathbf{k}) - \Sigma(\mathbf{k})\right]^{-1} = \left[\mathbf{k}^2 + m^2 - \Sigma(\mathbf{k})\right]^{-1}. \tag{4.31}$$

Since the momenta in a proper vertex function must add up to zero, the proper vertex function for $n = 2$ has the arguments $\bar{\Gamma}^{(2)}(\mathbf{k}, -\mathbf{k})$. By analogy with the Fourier-transformed two-particle correlation function $G(\mathbf{k})$ which carries only one momentum argument, it will be useful to introduce a quantity

$$\bar{\Gamma}^{(2)}(\mathbf{k}) \equiv \bar{\Gamma}^{(2)}(\mathbf{k}, -\mathbf{k}). \tag{4.32}$$

This is set equal to

$$\bar{\Gamma}^{(2)}(\mathbf{k}) \equiv G_0^{-1}(\mathbf{k}) - \Sigma(\mathbf{k}) = \mathbf{k}^2 + m^2 - \Sigma(\mathbf{k}), \tag{4.33}$$

such that

$$G(\mathbf{k}) = \frac{1}{\bar{\Gamma}^{(2)}(\mathbf{k})}. \tag{4.34}$$

Inserted into (4.18), we obtain for the connected two-point function the relation

$$G_{\rm c}^{(2)}(\mathbf{k}, \mathbf{k}') = (2\pi)^D \, \delta^{(D)}(\mathbf{k} + \mathbf{k}') \left[\bar{\Gamma}^{(2)}(\mathbf{k})\right]^{-1}. \tag{4.35}$$

This plays the role of the relation (4.20) for $n = 2$. By analogy with (4.24) we may also write this as

$$\bar{\Gamma}^{(2)}(\mathbf{k}) = G^{-1}(\mathbf{k})\bar{G}^{(2)}(\mathbf{k}, -\mathbf{k})_{\rm c}G^{-1}(-\mathbf{k}). \tag{4.36}$$

Using these composition formulas, the further development will require only the calculation of 1PI diagrams of two- and four-point functions.

4.3 Composite Fields

A Fourier transformation of the correlation functions $G^{(1,n)}(\mathbf{x}; \mathbf{x}_1, \ldots, \mathbf{x}_n)$ introduced in Section 3.5 yields

$$G^{(1,n)}(\mathbf{q}, \mathbf{k}_1, \ldots, \mathbf{k}_n) = \prod_{i=1}^{n}\left[\int d^D x_i \, \mathrm{e}^{-i\,\mathbf{k}_i\cdot\mathbf{x}_i}\right]\int d^D y \, \mathrm{e}^{-i\,\mathbf{q}\cdot\mathbf{y}} \, G^{(1,n)}(\mathbf{y}, \mathbf{x}_1, \ldots, \mathbf{x}_n).$$

We define the proper vertex functions $\bar{\Gamma}^{(1,n)}(\mathbf{q} = \mathbf{0}, \mathbf{k}_1, \ldots, \mathbf{k}_n)$ by analogy to $\bar{\Gamma}^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n)$ by selecting the connected 1PI diagrams in the correlation function $\bar{G}^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n)$, and by amputating the legs. In $\bar{G}^{(1,2)}(0, \mathbf{k}, \mathbf{k}')$ and $\bar{\Gamma}^{(1,2)}(0, \mathbf{k}, \mathbf{k}')$ we shall omit the second momentum argument $\mathbf{k}' = -\mathbf{k}$, just as in $\bar{\Gamma}^{(2)}(\mathbf{k})$ in Eq. (4.32). The proper vertex function $\bar{\Gamma}^{(1,2)}(0, \mathbf{k})$ is then obtained from $\bar{G}_{\rm c}^{(1,2)}(0, \mathbf{k})$, by analogy with Eq. (4.36), with the help of the relation

$$\bar{\Gamma}^{(1,2)}(0, \mathbf{k}) \equiv \bar{\Gamma}^{(1,2)}(0, \mathbf{k}, -\mathbf{k}) = G^{-1}(\mathbf{k})\bar{G}_{\rm c}^{(1,2)}(0, \mathbf{k}, -\mathbf{k})G^{-1}(-\mathbf{k}). \tag{4.37}$$

We now translate the relation (2.57) to momentum space, where it reads

$$G^{(1,n)}(\mathbf{q} = \mathbf{0}, \mathbf{k}_1, \ldots, \mathbf{k}_n) = -\frac{\partial}{\partial m^2}G^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n), \qquad n \geq 2, \tag{4.38}$$

yielding the relation

$$\frac{\partial}{\partial m^2} G^{-1}(\mathbf{k}) = G^{-1}(\mathbf{k}) \bar{G}_c^{(1,2)}(\mathbf{0}, \mathbf{k}, -\mathbf{k}) G^{-1}(-\mathbf{k}). \tag{4.39}$$

By expressing the right-hand side in terms of $\bar{\Gamma}^{(1,2)}(\mathbf{0}, \mathbf{k})$ using (4.37), and the left-hand side in terms of $\bar{\Gamma}^{(2)}(\mathbf{k})$ using (4.34), we obtain the relation for the proper vertex functions:

$$\bar{\Gamma}^{(1,2)}(\mathbf{0}, \mathbf{k}) = \frac{\partial}{\partial m^2} \bar{\Gamma}^{(2)}(\mathbf{k}). \tag{4.40}$$

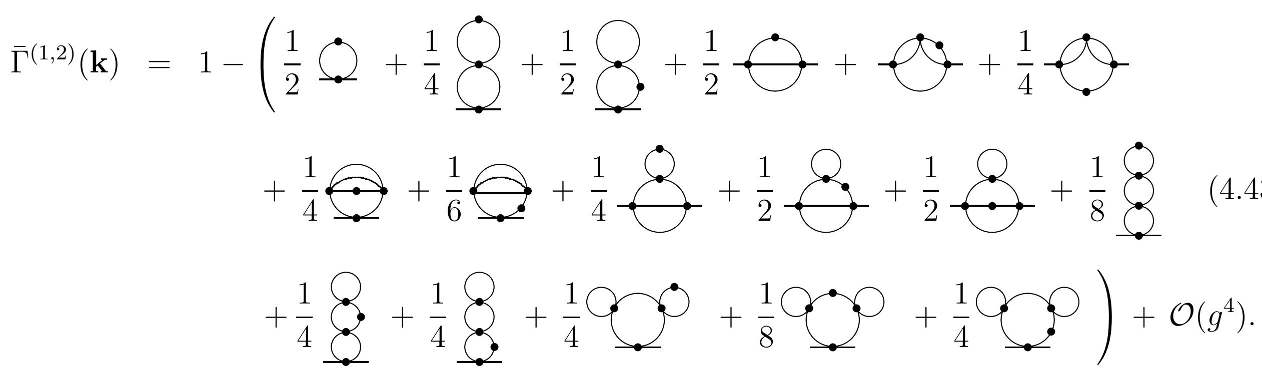
Inserting on the right-hand side the decomposition (4.33), we arrive at the formula

$$\bar{\Gamma}^{(1,2)}(\mathbf{0}, \mathbf{k}) = 1 - \frac{\partial}{\partial m^2} \Sigma(\mathbf{k}). \tag{4.41}$$

The derivative with respect to m^2 can be applied directly to each line in the diagrammatic expansion (4.27) of the self-energy. In Eq. (3.28) we indicated a differentiation with respect to m^2 diagrammatically by a fat dot on a line. Here we have the momentum-space version of this operation. For each line, the differentiation yields

$$\frac{\partial}{\partial m^2} \frac{1}{\mathbf{k}^2 + m^2} = -\frac{1}{(\mathbf{k}^2 + m^2)^2} \;\hat{=}\; \frac{\partial}{\partial m^2}$$ (4.42)

Using Eq. (4.41), we find from (4.27)



(4.43)

From a similar diagrammatic analysis it is obvious that there exists an analogous relation to (4.40) for any n-point proper vertex function:

$$\bar{\Gamma}^{(1,n)}(\mathbf{0}, \mathbf{k}_1, \ldots, \mathbf{k}_n) = \frac{\partial}{\partial m^2} \bar{\Gamma}^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n), \tag{4.44}$$

as will be proved in general in Section 5.8.

4.4 Theory in Continuous Dimension D

In all the foregoing development we have left the value D of the space dimension open. The phenomena we want to explain take place in $D = 3$ dimensions. For their theoretical explanation, it will be important to be able to define the theory for continuous values of D. In particular, we shall need to connect the theory for $D = 3$ with the theory for $D = 4$ in an analytic way. This will indeed be possible, and the specific mathematical prescription on how to do this will be given in Chapter 8.

Notes and References

See again the textbook cited at the end of last chapter, and
N. Nakanishi, *Graph Theory and Feynman Integrals*, Gordon and Breach, New York, 1971.