3

Feynman Diagrams

In the previous chapter we have derived perturbative expansion formulas for the partition function and the n-point correlation functions. The expansion coefficients are sums of multiple integrals whose number grows rapidly with increasing order. Their organization is simplified by means of diagrammatic techniques due to Feynman. All expansions will be performed in the normal phase of the system which possesses the symmetry of the energy functional [recall the remarks following Eq. (1.1)].

3.1 Diagrammatic Expansion of Correlation Functions

According to Eq. (2.47), the n-point correlation functions $G^{(n)}$ possesses an expansion in powers of the coupling constant λ as:

$$G^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n) = Z^{-1}\sum_{p=0}^{\infty} G_p^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n)\,, \tag{3.1}$$

where the expansion terms $G_p^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n)$ of order λ^p are given by the integrals (2.48). Using the generating formula (2.32), these can be obtained from the functional derivatives

$$\begin{aligned} G_p^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n) &= \left(\frac{-\lambda}{4!}\right)^p \frac{1}{p!}\,\frac{1}{(2p+n/2)!\,2^{2p+n/2}} \\ &\times\frac{\delta}{\delta j(\mathbf{x}_1)}\cdots\frac{\delta}{\delta j(\mathbf{x}_n)}\left[\prod_{i=1}^{p}\int d^D z_i\left(\frac{\delta}{\delta j(\mathbf{z}_i)}\right)^4\right]\left[\int d^D x\, d^D y\, j(\mathbf{x})G_0(\mathbf{x},\mathbf{y})j(\mathbf{y})\right]^{2p+n/2}. \end{aligned} \tag{3.2}$$

The $4p+n$ functional derivatives on the right-hand side produce a sum of $(4p+n)!$ terms consisting of products of free two-point correlation functions $G_0(\mathbf{x},\mathbf{x}')$. Among these, $(2p+n/2)!\;2^{2p+n/2}$ are identical because of the symmetry in the arguments and the commutativity of the associated propagators. This multiplicity factor cancels out the factorials in the denominators of (3.2) which had their origin in the Taylor expansion of the exponential (2.32). One is left with a sum of $(4p+n-1)!!$ terms, each a product of $(4p+n)/2$ free propagators.

The counting is clearest by writing each term in such a way that coinciding $\mathbf{z}$-arguments of the free propagators are initially distinguished. Their coincidence is enforced in an extended integral by means of δ-functions. In this way we obtain the expression

$$\begin{aligned} G_p^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n) &\equiv \frac{1}{p!}\left(\frac{-\lambda}{4!}\right)^p\int d^D z_1\cdots d^D z_p\int d^D y_1\cdots d^D y_{4p+n}\prod_{l=1}^{n}\left[\delta^{(D)}(\mathbf{y}_{4p+l}-\mathbf{x}_l)\right] \\ &\times\prod_{k=1}^{p}\left[\delta^{(D)}(\mathbf{y}_{4k-3}-\mathbf{z}_k)\delta^{(D)}(\mathbf{y}_{4k-2}-\mathbf{z}_k)\delta^{(D)}(\mathbf{y}_{4k-1}-\mathbf{z}_k)\delta^{(D)}(\mathbf{y}_{4k}-\mathbf{z}_k)\right] \end{aligned}$$

$$\times \sum_{i=1}^{(4p+n-1)!!} \prod_{j=1}^{\frac{4p+n}{2}} G_0(\mathbf{y}_{\pi_i^{(4p+n)}(2j-1)}, \mathbf{y}_{\pi_i^{(4p+n)}(2j)}). \tag{3.3}$$

We have given the pair indices $\pi_i(2j-1), \pi_i(2j)$ a superscript which indicates the total number of indices from which the pairs have been selected. The sum includes all possible permutations of the $\mathbf{y}_i$ variables, except for those which correspond only to an interchange of the spatial arguments within a propagator, or to an interchange of identical propagators as a whole. Those permutations have already been accounted for by factors $(2p+l)!\,2^{(2p+n/2)}$, which are subsequently canceled by the denominators of the Taylor expansion of the exponential (2.32). The remaining permutations in the sum (3.3) are called *relevant permutations.* This restriction of the sum has to be kept in mind when carrying out the $\mathbf{y}$- and $\mathbf{z}$-integrations.

Each product in the sum (3.3) is pictured by a *Feynman diagram* or *Feynman graph.* We will use $G_p^{(n)}$ to denote both, the integral representation and the diagram representation. The position variables $\mathbf{x}_l$ and $\mathbf{z}_k$ are represented by points, and the free propagators $G_0(\mathbf{y}, \mathbf{y}')$ by lines

$$\underset{\mathbf{y}\quad\;\; \mathbf{y}'}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet} \;\hat{=}\; G_0(\mathbf{y}, \mathbf{y}') \tag{3.4}$$

connecting the points $\mathbf{y}$ and $\mathbf{y}'$. The points $\mathbf{y}_{4p+1} = \mathbf{x}_1, \dots, \mathbf{y}_{4p+n} = \mathbf{x}_n$ are endpoints of a line. They are called *external points* and the corresponding lines are called *external lines.* The points $\mathbf{y}_1, \dots, \mathbf{y}_{4p}$ correspond to integration variables. They are called *internal points.* They carry δ-functions in (3.3) that enforce the coincidence of groups of four internal points. Such coinciding points with four emerging lines are called *vertices.* In this introductory discussion, they are represented by a small circle marked by the variable of integration. Later we shall always include the coupling constant into the definition of the vertex and omit specifying the associated integration variable. This will be indicated by a dot:

$$\times \;\hat{=}\; -\lambda \int d^D z\ . \tag{3.5}$$

The diagrams associated with an n-point function are called *n-point diagrams.* The sum in (3.3) runs over all diagrams which can be drawn for p vertices and n external points. Each diagram appears repeatedly with permuted line and vertex labels.

After carrying out the $\mathbf{y}$-integrations, many terms in the sum (3.3) coincide. As an example, consider the diagrammatic representation of the expansion coefficients $G_1^{(2)}$. According to Wick's rule, there are $(4\times 1+2-1)!! = 5\times 3\times 1 = 15$ contractions, and an equal number of diagrams. Four of them are pictured in Fig. 3.1. All other diagrams in the sum are either

FIGURE 3.1 Diagrams contributing to expansion of $G_1^{(2)}$.

of type (a) and (b), or of type (c) and (d), but with the indices of the $\mathbf{y}$-variables permuted.

Omitting the labels $\mathbf{y}_1, \ldots, \mathbf{y}_4$ of the four endpoints of lines coinciding with the point $\mathbf{z}_1$, the diagrams of type (a) and (b), as well as those of type (c) and (d), become indistinguishable.

The number of times with which each Feynman integral occurs among the functional derivatives in (3.2), which is equal to the number of Wick contractions leading to the same Feynman integral, is called the *multiplicity* M_G of a Feynman integral.

The multiplicity M_G is usually combined with the factor $1/4!^p p!$ accompanying the pth-order expansion term of the exponential function in (2.46). The result is the so-called *weight factor* W_G of a diagram G:

$$W_G = \frac{M_G}{4!^p p!} . \tag{3.6}$$

It is this factor which eventually accompanies each Feynman diagram in the perturbation expansion.

Initially, each vertex possesses four emerging lines and there are 4! ways of labeling them. Thus there are in general 4! diagrams of each type. In many cases, however, this number is reduced because the sum in (3.3) does not cover all possible index permutations. This can be seen analytically upon performing the integrations over $\mathbf{y}_1, \ldots, \mathbf{y}_4$. For a vertex at $\mathbf{z}$, the variables $\mathbf{y}_1, \ldots, \mathbf{y}_4$ may appear all in different propagators,

$$\begin{aligned}\int d^D y_1 \ldots d^D y_4 \delta^{(D)}(\mathbf{y}_1 - \mathbf{z})\delta^{(D)}(\mathbf{y}_2 - \mathbf{z})\delta^{(D)}(\mathbf{y}_3 - \mathbf{z})\delta^{(D)}(\mathbf{y}_4 - \mathbf{z})& \\ \times\ G_0(\mathbf{y}_1, \bar{\mathbf{y}}_1)\, G_0(\mathbf{y}_2, \bar{\mathbf{y}}_2)\, G_0(\mathbf{y}_3, \bar{\mathbf{y}}_3)\, G_0(\mathbf{y}_4, \bar{\mathbf{y}}_4)& \\ = G_0(\mathbf{z}, \bar{\mathbf{y}}_1)\, G_0(\mathbf{z}, \bar{\mathbf{y}}_2)\, G_0(\mathbf{z}, \bar{\mathbf{y}}_3)\, G_0(\mathbf{z}, \bar{\mathbf{y}}_4),&\end{aligned} \tag{3.7}$$

where $\bar{\mathbf{y}}_1, \ldots, \bar{\mathbf{y}}_4$ are an arbitrary set of variables. As there are 4! ways to arrange $\mathbf{y}_1, \ldots, \mathbf{y}_4$ in the integrand, there are 4! terms which coincide after the integration, giving an overall factor 4!. However, instead of (3.7), we could also encounter a situation in which two $\mathbf{y}_i$-variables appear in the same propagator.

$$\begin{aligned}\int d^D y_1 \ldots d^D y_4 \delta^{(D)}(\mathbf{y}_1 - \mathbf{z})\delta^{(D)}(\mathbf{y}_2 - \mathbf{z})\delta^{(D)}(\mathbf{y}_3 - \mathbf{z})\delta^{(D)}(\mathbf{y}_4 - \mathbf{z})& \\ \times\ G_0(\mathbf{y}_1, \mathbf{y}_2)\, G_0(\mathbf{y}_3, \bar{\mathbf{y}}_3)\, G_0(\mathbf{y}_4, \bar{\mathbf{y}}_4)& \\ = G_0(\mathbf{z}, \mathbf{z})\, G_0(\mathbf{z}, \bar{\mathbf{y}}_3)\, G_0(\mathbf{z}, \bar{\mathbf{y}}_4).&\end{aligned} \tag{3.8}$$

Here, the permutation of $\mathbf{y}_1$ and $\mathbf{y}_2$ amounts to an interchange of the arguments of the propagator. Such permutations are not of the relevant type and are thus excluded from the sum over the permutations in (3.3). For this reason, the $\mathbf{y}$-integration yields now only $4!/2$ identical terms, thus resulting in a factor $4!/2$. Diagrammatically, $G_0(\mathbf{z}, \mathbf{z})$ produces a self-connection of a vertex. Examples for this case are the diagrams (a) or (b) in Fig. 3.1. Instead of 4! different ways to label the lines of the vertex, we find only four possibilities to choose the line connecting the vertex to $\mathbf{x}_1$, and three possibilities to choose the line connecting the vertex to $\mathbf{x}_2$. Altogether, the diagram contributes with a factor $3 \times 4 = 12$.

Other irrelevant permutations occur if more than one propagator has the same combination of the arguments $\bar{\mathbf{y}}_i$

$$\begin{aligned}\int d^D y_1 \ldots d^D y_4 \delta^{(D)}(\mathbf{y}_1 - \mathbf{z})\delta^{(D)}(\mathbf{y}_2 - \mathbf{z})\delta^{(D)}(\mathbf{y}_3 - \mathbf{z})\delta^{(D)}(\mathbf{y}_4 - \mathbf{z})& \\ \times\ G_0(\mathbf{y}_1, \bar{\mathbf{y}}_1)\, G_0(\mathbf{y}_2, \bar{\mathbf{y}}_1)\, G_0(\mathbf{y}_3, \bar{\mathbf{y}}_1)\, G_0(\mathbf{y}_4, \bar{\mathbf{y}}_4)& \\ =\ G_0(\mathbf{z}, \bar{\mathbf{y}}_1)\, G_0(\mathbf{z}, \bar{\mathbf{y}}_1)\, G_0(\mathbf{z}, \bar{\mathbf{y}}_1)\, G_0(\mathbf{z}, \bar{\mathbf{y}}_4).&\end{aligned} \tag{3.9}$$

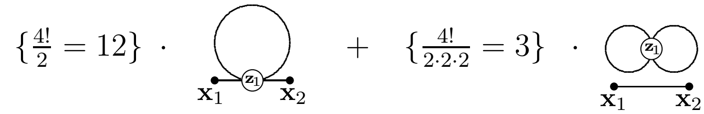
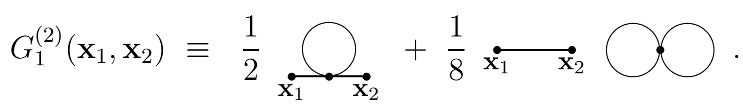


FIGURE 3.2 Diagrams in expansion of $G_1^{(2)}$. The multiplicities add up to $(4p+n-1)!!$ with $n=2$ and $p=1$: $12+3=(4\times 1+2-1)!!$.

In this case, permutations of the indices $1 \leftrightarrow 2 \leftrightarrow 3$ are irrelevant and the sum contains only $4!/3! = 4$ identical terms. In the diagrammatic illustration, this case corresponds to a triple connection of the vertices $\bar{\mathbf{y}}_1$ and $\mathbf{z}$. For double and fourfold connections we find a factor $4!/2! = 12$ and $4!/4! = 1$, respectively.

It will be convenient to use the one-to-one relation between Feynman integrals and diagrams to write *diagrammatic equations* for any correlation function to be calculated. The first order two-point function whose diagrams we just calculated has then the diagrammatic equation:



(3.10)

The diagrammatic contributions to the correlation function $G_1^{(2)}$ are summarized in Fig. 3.2. The first diagram has a self-connection giving rise to a factor 1/2. The second diagram contains two self-connections and one double connection leading to the factor $1/(2\cdot 2\cdot 2)$. To check the resulting multiplicative factor 3 we note that there are obviously only three different ways of choosing two out of four points if the order is irrelevant. So only the following combinations appear in the sum: $G_0(\mathbf{y}_1,\mathbf{y}_2)G_0(\mathbf{y}_3,\mathbf{y}_4)$, $G_0(\mathbf{y}_1,\mathbf{y}_3)G_0(\mathbf{y}_2,\mathbf{y}_4)$, $G_0(\mathbf{y}_1,\mathbf{y}_4)G_0(\mathbf{y}_2,\mathbf{y}_3)$. The second-order contributions $G_2^{(2)}$ to the two-point function are shown in Fig. 3.3.

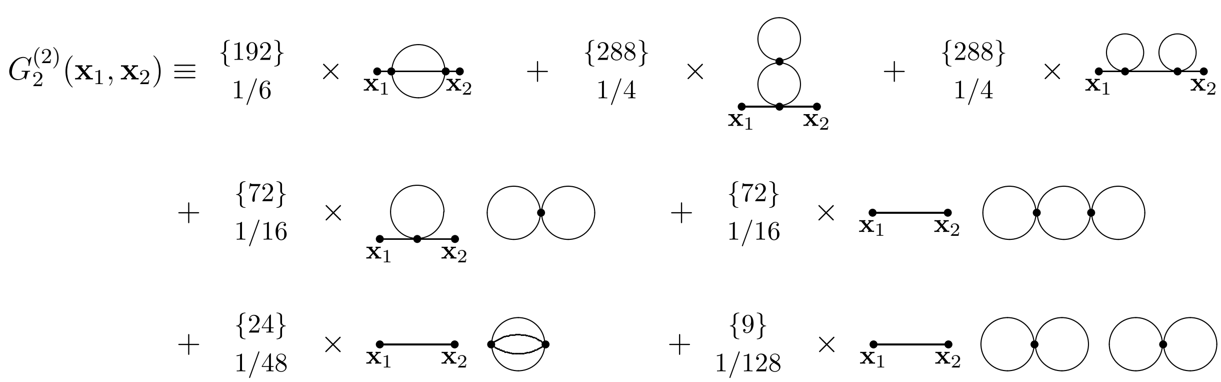


FIGURE 3.3 Diagrams in expansion of $G_2^{(2)}$. The curly brackets indicate the multiplicities, the numbers underneath them show the weight factors. The vertices are indicated with the short notation (3.5).

After performing all the $\mathbf{y}$-integration in (3.3), the function $G_p^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n)$ is a sum of terms which differ in the arrangement of their $\mathbf{x}_i$ and $\mathbf{z}_i$ arguments in the propagators. The terms with permuted $\mathbf{z}_i$-arguments differ only in the labeling of the vertices, and this is irrelevant to the $\mathbf{z}_i$-integrations. Thus the $\mathbf{z}_i$-integration produces only an integer factor. In

pth order perturbation theory, we have p vertices allowing initially for $p!$ permutations. This number is reduced to the relevant permutations in the sum (3.3). The irrelevant one are the $\mathbf{z}_i$-permutations, which are equivalent to a mere rearrangement of some propagators $G_0(\mathbf{z}_i, \mathbf{z}_j)$. They are called *identical vertex permutations* (IVP), since in the diagrammatic representation the reordering of the arguments of the propagators is equivalent to interchanging the corresponding vertices, performed with the vertices attached to the same lines. The number of identical vertex permutations will be denoted by $N_{\rm IVP}$.

FIGURE 3.4 (a) Fourth-order diagram in $G_4^{(2)}$. (b) Same diagram with assignment of integration variables to internal points. The vertices $\mathbf{z}_3$ and $\mathbf{z}_4$ can be interchanged without cutting any line. This is an identical vertex permutation.

An example is shown in Fig. 3.4. The analytic expression for the right-hand diagram (b) is

$$\int d^D z_1 d^D z_2 d^D z_3 d^D z_4 G_0(\mathbf{x}_1, \mathbf{z}_1) G_0(\mathbf{z}_1, \mathbf{z}_2) G_0(\mathbf{z}_1, \mathbf{z}_4) G_0(\mathbf{z}_1, \mathbf{z}_3)$$
$$\times [G_0(\mathbf{z}_3, \mathbf{z}_4)]^2 G_0(\mathbf{z}_3, \mathbf{z}_2) G_0(\mathbf{z}_4, \mathbf{z}_2) G_0(\mathbf{z}_2, \mathbf{x}_2). \tag{3.11}$$

The interchange $\mathbf{z}_3 \leftrightarrow \mathbf{z}_4$ is irrelevant. The number of identical vertex permutations is therefore $N_{\rm IVP} = 2$. In contrast, the permutations involving $\mathbf{z}_1$ or $\mathbf{z}_2$ are relevant. The corresponding vertices cannot be interchanged in the diagram without cutting the lines to the external vertices.

Let us count the multiplicity M_G of the diagram. Each vertex can be connected to four lines in 4! ways, and since there are 4 vertices, there are initially $4! \cdot 4! = 7\,962\,624$ ways of drawing the diagram. There is one double-connection between vertices $\mathbf{z}_3$ and $\mathbf{z}_4$ so that this number has to be divided by 2. These two vertices can, moreover, be interchanged by an identical vertex permutation, such that we arrive at a multiplicity

$$M_G = \frac{4!^4 \cdot 4!}{2 \cdot 2} = 1\,990\,656. \tag{3.12}$$

This number will be found in the complete list of all multiplicities of two-point diagrams in Table 14.3 on page 265, where it appears with the label 4-3 or No. 3 in the first and last columns, respectively. The number 3 refers to the running number of this diagram in Appendix A.2 on page 436.

The associated weight of the diagram is, according to the definition (3.6),

$$W_G = \frac{1\,990\,656}{7\,962\,624} = \frac{1}{4}. \tag{3.13}$$

This number will be found in Appendix B.2 on page 449 (in the third column of the third four-loop entry).

The reader may proceed in this way to check the weight factors of the diagrammatic expansion of the second-order correction to the two-point function $G_2^{(2)}(\mathbf{x}_1, \mathbf{x}_2)$ in Fig. 3.3. There are altogether $(2 + 4 \times 2 - 1)!! = 945$ expansion terms.

Collecting the factors we find the general formula for the multiplicity M_G:

$$M_G = \frac{4!^p\, p!}{2!^{S+D} 3!^T 4!^F N_{\rm IVP}}\,, \qquad \begin{aligned} S &= \text{number of self-connections,}\\ D &= \text{number of double connections,}\\ T &= \text{number of triple connections,}\\ F &= \text{number of fourfold connections,}\\ N_{\rm IVP} &= \text{number of identical vertex permutations,}\end{aligned} \tag{3.14}$$

and for the weight W_G

$$W_G = \frac{1}{2!^{S+D} 3!^T 4!^F N_{\rm IVP}}\,, \tag{3.15}$$

Formula (3.15) allows us to determine the multiplicity of simple diagrams by inspection, although the application to higher-order diagrams has the difficulty mentioned above of finding the number of identical vertex permutations. One way to solve this problem will be indicated in Chapter 14.

The above formula (3.15) determines the weight of a diagram with a fixed configuration of positions $\mathbf{x}_1, \ldots, \mathbf{x}_n$ at the ends of the external legs. These appear, however, in various permutations. To find the number of these permutations, we connect the endpoints of the external lines to an extra fictitious vertex labeled by $\mathbf{x}_0$. There are initially $n!$ ways of doing this. But because of the identity of the lines, the number of different connections is smaller by a factor $2!^{D'} 3!^{T'} 4!^{F'}$, where D', T', F' count the number of double, triple, and fourfold connections from the vertices of G to the extra vertex, respectively. Actually, fourfold connections do not appear. Also, the vertices where external lines enter can show additional symmetries if the labeling of the external vertices is neglected. Such a symmetry is accounted for by a factor $N_{\rm IVP}^{\rm ext}$ in the denominator. Thus there are

$$N_{\rm perm} = \frac{n!}{2!^{D'} 3!^{T'} 4!^{F'} N_{\rm IVP}^{\rm ext}} \tag{3.16}$$

different configurations of position labels of the external lines $\mathbf{x}_1, \ldots, \mathbf{x}_n$ for each diagram.

The weight of a diagram G irrespective of the external positions is then given by:

$$W_G^{\rm unlabeled} = W_G \times N_{\rm perm} = \frac{1}{2!^{S+D} 3!^T 4!^F N_{\rm IVP}} \times \frac{n!}{2!^{D'} 3!^{T'} 4!^{F'} N_{\rm IVP}^{\rm ext}}. \tag{3.17}$$

The value for $W_G^{\rm unlabeled}$ equals the value W_G of the corresponding amputated diagram which will be introduced in Chapter 14, in Eq. (14.1).

In the example of the diagram in Fig. 3.4, there are two external lines. These give rise to a factor $n! = 2$ in the numerator of formula (3.17), and thus of (3.13). However, when neglecting the labeling of the vertices $\mathbf{x}_1$ and $\mathbf{x}_2$ the vertices at $\mathbf{z}_1$ and $\mathbf{z}_2$ can also be permuted without changing the diagram giving rise to a factor $N_{\rm IVP}^{\rm ext} = 2$. Hence the number of irreducible vertex permutations in the denominator in (3.13) is increased by a factor 2 as well, thus yielding the same multiplicity as before.

Another example with a nontrivial configuration of external points is discussed at the end of Section 3.4.

3.2 Diagrammatic Expansion of the Partition Function

Let us also find a diagrammatic representation for the perturbation series of the partition function in Eq. (2.44). The expansion terms (2.45) are quite similar to those for $G_p^{(n)}$ in (2.48), except that all spatial variables are integrated out. Expanding the free correlation functions in (2.45) à la Wick, we obtain an expression analogous to (3.3):

$$
\begin{aligned}
Z_p \equiv \frac{1}{p!}\left(\frac{-\lambda}{4!}\right)^p \int d^D z_1 \cdots d^D z_p \int d^D y_1 \cdots d^D y_{4p} \\
\times \prod_{k=1}^{p}\left[\delta^{(D)}(\mathbf{y}_{4k-3}-\mathbf{z}_k)\delta^{(D)}(\mathbf{y}_{4k-2}-\mathbf{z}_k)\delta^{(D)}(\mathbf{y}_{4k-1}-\mathbf{z}_k)\delta^{(D)}(\mathbf{y}_{4k}-\mathbf{z}_k)\right] \\
\times \sum_{i=1}^{(4p-1)!!}\prod_{j=1}^{2p} G_0(\mathbf{y}_{\pi_i^{(4p)}(2j-1)}, \mathbf{y}_{\pi_i^{(4p)}(2j)}) \, .
\end{aligned}
\tag{3.18}
$$

The right-hand side contains a sum over products of free propagators whose lines all end at internal points. There are no external vertices or legs. The corresponding diagrams are called *vacuum diagrams*. The diagrammatic expansion of the partition function consists of the sum of all vacuum diagrams, as shown in Fig. 3.5 up to order λ^2, including their weight factors. These are the same vacuum diagrams which appeared before in disconnected pieces of the diagrams contributing to the expansion terms $G_p^{(2)}$ shown in Fig. 3.3. The simultaneous appearance of vacuum diagrams in these expansions has the pleasant consequence that, when dividing the sum of all diagrams in $G^{(2)}$ by those in Z and re-expanding the ratio in powers of λ, the contributions from the vacuum diagrams will cancel each other out. This will be seen in more detail below.

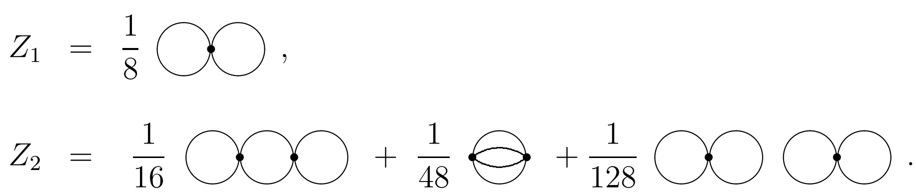


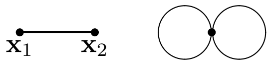
FIGURE 3.5 Vacuum diagrams in Z_1 and Z_2 with their respective weight factors.

3.3 Connected and Disconnected Diagrams

It is useful to distinguish *connected* and *disconnected diagrams*. For a disconnected diagram, the associated Feynman integral in $G_p^{(n)}$ or in Z_p factorizes. Examples are shown in Fig. 3.6. The connected diagrams contained in $G_p^{(n)}$ are denoted by $G_{p\,\mathrm{c}}^{(n)}$.

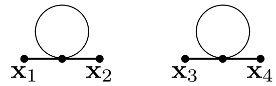
3.3.1 Multiplicities of Disconnected Diagrams

The multiplicity M_G of a diagram G consisting of n disconnected parts is calculated in the usual way. If a number n_{id} of the n disconnected parts are identical, the number of identical vertex permutations includes a factor $n_{\mathrm{id}}!$, since these parts can be permuted in $n_{\mathrm{id}}!$ ways. In general, there may be m sets of $n_{\mathrm{id}}^{(i)}$ $(i = 1, \ldots, m)$ identical diagrams, with a number of permutations of identical diagrams $(n_{\mathrm{id}}^{(1)})! \cdots (n_{\mathrm{id}}^{(m)})!$. The weight-factor of the disconnected diagram is given by the product of the weight factors of the connected diagrams.



$G_0(\mathbf{x}_1, \mathbf{x}_2) \int d^D z\, [G_0(\mathbf{z}, \mathbf{z})]^2$

(a)



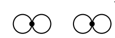
$\int d^D z\, G_0(\mathbf{x}_1, \mathbf{z}) G_0(\mathbf{x}_2, \mathbf{z}) G_0(\mathbf{z}, \mathbf{z})$
$\times \int d^D z'\, G_0(\mathbf{x}_3, \mathbf{z}') G_0(\mathbf{x}_4, \mathbf{z}') G_0(\mathbf{z}', \mathbf{z}')$

(b)

FIGURE 3.6 Two disconnected diagrams and the associated integral expressions.

The composition rules of disconnected diagrams from their connected components are expressed most compactly in terms of the generating functional of all correlation functions. If we write

$$Z[j] \;=\; e^{W[j]}, \tag{3.19}$$

then $W[j]$ is the generating functional for the connected correlation functions. We shall prove this in Chapter 5. Here we only note that an exponential function of any diagram yields, with the correct weight factors, the sum of all disconnected diagrams composed of any number of copies of this diagram. For a disconnected diagram with two or more identical components, the inverse factorials in the expansion coefficients of the exponential produce reduction factors to the products of individual weights of the components, yielding a combined weight factor specified by Eq. (3.17). In the example  , they produce a factor 1/2 correcting the product of the individual weights 1/64 to the correct total weight 1/128.

For $j \equiv 0$, Eq. (3.19) relates the disconnected and connected diagrams for the vacuum to each other.

There exist simple relations between the numbers of diagrams of two- and four-point functions and those of the vacuum diagrams. These will be found in Chapters 5 and used further in Chapter 14.

3.3.2 Cancellation of Vacuum Diagrams

The diagrammatic expression for $G_p^{(n)}$ consists of all possible Feynman diagrams that can be drawn for n external points and p vertices. It contains connected and disconnected diagrams. The disconnected diagrams are composed of *external components* involving the external lines, and *vacuum components* containing only vacuum diagrams. If the number of vertices of the vacuum component is called p_V, the number of vertices of the external component is $p - p_V$. For $n = 2$, the external part contains only one connected lower-order diagram. For $n = 4$, the external part may be composed of two two-point diagrams, and so on.

As all possible contractions are contained in $G_p^{(n)}$, all possible compositions of external and vacuum diagrams occur. The weight factors for disconnected diagrams are the product of the weight factors of the constituents times the factor for the permutations of the identical diagrams. Thus $G_p^{(n)}$ can be written as:

$$G_p^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \sum_{k=0}^{p} G_{p-k\,\mathrm{ext}}^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n) Z_k\,, \tag{3.20}$$

where G_{ext} denotes the diagrams of the corresponding order which have no vacuum component and contain only external parts. If Eq. (3.20) is summed over all orders, the partition sum can be factored out:

$$\begin{aligned}\sum_{p=0}^{\infty} G_p^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n) &= \sum_{p=0}^{\infty}\sum_{k=0}^{p} G_{p-k\,\text{ext}}^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n)\, Z_k \\ &= \sum_{p'=0}^{\infty} G_{p'\,\text{ext}}^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n) \sum_{k=0}^{\infty} Z_k \\ &= \sum_{p'=0}^{\infty} G_{p'\,\text{ext}}^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n)\, Z. \qquad (3.21)\end{aligned}$$

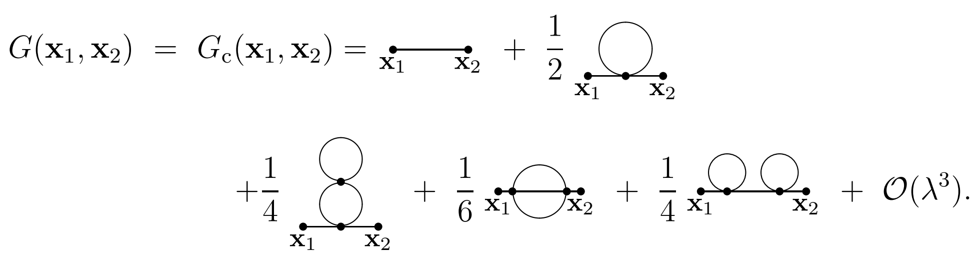
Insertion of this equation into Eq. (3.1) yields

$$G^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \sum_{p=0}^{\infty} G_{p\,\text{ext}}^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n), \qquad (3.22)$$

where the two-point function satisfies $G_{p\,\text{ext}} = G_{p\,\text{c}}$.

3.4 Connected Diagrams for Two- and Four-Point Functions

After dropping the vacuum components, the diagrams for two- and four-point functions can easily be written down. They are always connected for the two-point function (this would not be true for a ϕ^3-interaction). The lowest contributions are

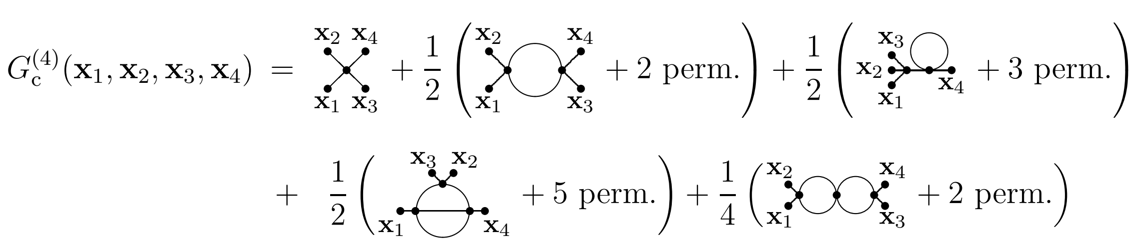


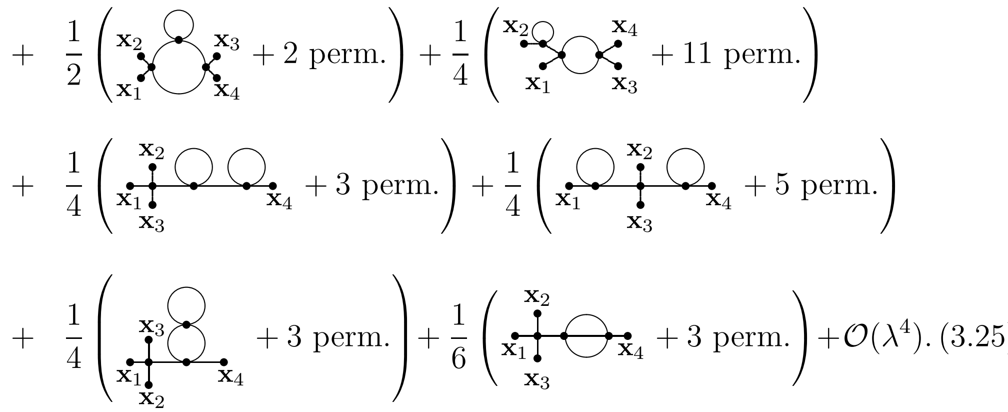
(3.23)

The diagrams for the 4-point function without vacuum components are either connected, or they decompose into a product of two connected diagrams which occur in the expansion of the two-point function. With the help of the product rule in the last section we can always separate out the disconnected contributions, writing

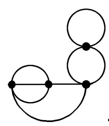
$$\begin{aligned}G^{(4)}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) &= G_{\text{c}}^{(4)}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) + G_{\text{c}}(\mathbf{x}_1,\mathbf{x}_2)G_{\text{c}}(\mathbf{x}_3,\mathbf{x}_4) \qquad (3.24)\\ &+ G_{\text{c}}(\mathbf{x}_1,\mathbf{x}_3)G_{\text{c}}(\mathbf{x}_2,\mathbf{x}_4) + G_{\text{c}}(\mathbf{x}_1,\mathbf{x}_4)G_{\text{c}}(\mathbf{x}_2,\mathbf{x}_3).\end{aligned}$$

The connected component has the expansion





It is useful to illustrate the calculation of the weights for one of the diagrams with a nontrivial configuration of external points $\mathbf{x}_1, \ldots, \mathbf{x}_4$, for instance any one the diagrams in the second-last parentheses. Each has one self- and one double connection, and the number of identical vertex permutations is $N_{\mathrm{IVP}} = 1$. From Eq. (3.15) we obtain its weight $W_G = 1/4$. If the ends are connected to an external vertex, we obtain the diagram


.

There is a triple connection to the extra fictitious vertex on the left, such that the number N_{perm} of configurations of Eq. (3.16) is $N_{\mathrm{perm}} = 4!/3! = 4$, which explains the number of terms in parentheses. Multiplying this additional factor to the weight factor $W_G = 1/4$ of the labeled diagram according to formula (3.17), we find $W_G^{\mathrm{unlabeled}} = 1$ which is the total weight of all diagrams in the second-last parentheses, irrespective of the configurations of the external points.

3.5 Diagrams for Composite Fields

In Eq. (2.53), we introduced correlation functions containing composite operators $\phi^2(\mathbf{x})$. For these we showed in Eqs. (2.57) and (2.58) that their spatial integrals within an expectation value of ordinary fields can be generated by forming derivatives of the n-point functions $G^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ with respect to the mass. In this book, we shall deal in detail only with a single insertion of $\phi^2(\mathbf{x})$ into G (see Chapter 12):

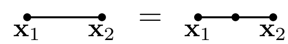
$$G^{(1,2)}(\mathbf{y}, \mathbf{x}_1, \mathbf{x}_2) \equiv \frac{1}{2}\langle \phi^2(\mathbf{y})\phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\rangle. \tag{3.26}$$

Its spatial integral is generated by the derivative with respect to the squared mass:

$$\frac{\partial}{\partial m^2} G(\mathbf{x}_1, \mathbf{x}_2) = -\int d^D y\, G^{(1,2)}(\mathbf{y}, \mathbf{x}_1, \mathbf{x}_2). \tag{3.27}$$

The perturbative calculation of the quantity on the right-hand side follows Wick's expansion rule. The resulting Feynman diagrams are quite simply related to those of G, whose differentiation with respect to the squared mass generates successively in every line a $-\phi^2/2$ -insertion. The latter is represented diagrammatically by a $-\phi^2$-vertex, which is a vertex with two legs carrying a factor -1 (since there are two ways of connecting two lines to it). In a Feynman diagram, it will be denoted by a dot on the line. The negative sign is implied by this point. For the free correlation function G we obtain the diagrammatic equation

$$-\int d^D y\, G^{(1,2)}(\mathbf{y}, \mathbf{x}_1, \mathbf{x}_2) = \frac{\partial}{\partial m^2}$$



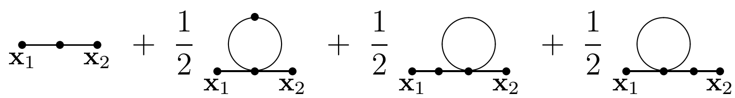
(3.28)

Since two legs can be connected with the $-\phi^2$-vertex in two ways without generating a different diagram, every diagram appears twice, thereby canceling the factor 1/2 of the $-\phi^2/2$ -insertion in (3.26).

Lines in a diagram are called *topologically equivalent* if they are part of a double, triple, or quadruple connection, or if they are transformed into one another by an identical vertex permutation. The differentiation of topologically equivalent lines leads to identical diagrams with a $-\phi^2$-vertex, causing a multiplicity factor. For example, the $-\phi^2/2$-insertion on a threefold connection leads to three equivalent diagrams.

The first terms of the diagrammatic expansion of $G^{(1,2)}$ are

$$\int d^D y\; G^{(1,2)}(\mathbf{y}, \mathbf{x}_1, \mathbf{x}_2) \;=$$



$$+\; \mathcal{O}(\lambda^2).$$

The number of terms grows rapidly with increasing order, so that we restrict the display here to the first order. The second-order diagrams will be shown explicitly in Eq. (4.43).

The correlation function $G^{(1,2)}$ is always connected, i.e., $G^{(1,2)} = G_c^{(1,2)}$, because of the restriction of our study to the normal phase of the system.

Notes and References

For more details see the textbooks on quantum field theory listed at the end of Chapter 1. We have followed the textbook

H. Kleinert, *Gauge Fields in Condensed Matter*, Vol. I, *Superflow and Vortex Lines, Disorder Fields and Phase Transitions*, World Scientific, Singapore, 1989 (www.physik.fu-berlin.de/˜kleinert/re.html#b1).