2

Definition of ϕ^4-Theory

A thermally fluctuating field theory is defined by means of a functional integral representation of the partition function, in which the fields are coupled linearly to external sources. This constitutes a generating functional, from which all thermodynamic quantities and correlation functions of the system can be obtained by functional differentiation, as we shall now see.

2.1 Partition Function and Generating Functional

All objects of study in this book are described in terms of an N-component fluctuating field $\boldsymbol{\phi} = (\phi_1(\mathbf{x}) \cdots \phi_N(\mathbf{x}))$ in D euclidean space dimensions. The field components interact with each other via a term of fourth order in the fields, $\lambda T_{\alpha\beta\gamma\delta}\phi_\alpha\phi_\beta\phi_\gamma\phi_\delta$, $(\alpha, \beta, \gamma, \delta = 1, \ldots, N)$, where the parameter $\lambda > 0$ characterizes the interaction strength and is called the *coupling constant* of the theory. The quantity $T_{\alpha\beta\gamma\delta}$ is a *coupling tensor*. Many structural properties of the theory do not depend on the number N of field components. When discussing such properties, the subscripts $\alpha, \beta, \gamma, \delta$ and the tensor $T_{\alpha\beta\gamma\delta}$ will be omitted everywhere, except in Chapters 6, 17, and 18, which derive particular consequences of the tensor structure.

The field $\phi(\mathbf{x})$ performs thermal fluctuations. We shall investigate the properties of the system mainly in the normal phase where the expectation value of the field ϕ is zero, which happens in the temperature regime in which the symmetry of the system is unbroken [see the definition following Eq. (1.1)]. Then the fluctuations take place around zero. Their size is controlled by a local energy functional consisting of two terms:

$$E\,[\phi] = E_0\,[\phi] + E_{\rm int}\,[\phi]\,. \tag{2.1}$$

The first term

$$E_0\,[\phi] = \int d^Dx\,\frac{1}{2}\Big\{\,[\partial_{\mathbf{x}}\phi(\mathbf{x})]^2\,+\,m^2\phi^2(\mathbf{x})\,\Big\} \tag{2.2}$$

is quadratic in the field, and is called the *free-field energy*. The second term

$$E_{\rm int}\,[\phi] = \int d^Dx\,\frac{\lambda}{4!}\,\phi^4(\mathbf{x}) \tag{2.3}$$

is of fourth-order in the field and is called the *interaction energy*. The parameter m is called the *mass* of the field. This name derives from the fact that if the theory were to be continued analytically in one coordinate, say x_D, to a time-like variable $t = ix_D$, this would change the squared length of a vector from $\mathbf{x}^2 = x_1^2 + \ldots + x_{D-1}^2 + x_D^2$ to $x^2 = x_1^2 + \ldots + x_{D-1}^2 - t^2$, and the free-field energy would turn into an action describing the propagation of relativistic particles of mass m in a Minkowski spacetime. Instead of thermal fluctuations, the field $\phi(\mathbf{x})$ would then perform quantum fluctuations, which can equivalently be described by a *quantum field operator* $\hat{\phi}(x)$, which is capable of creating and annihilating particles of mass m in a multiparticle Hilbert space. No such Hilbert space exists in the present euclidean formulation, but the nomenclature of calling m a mass has nevertheless become customary. Moreover, because of the intimate relationship of the theories by analytic continuation in the time variable $x_D \to -it$, the entire fluctuating field theory is often referred to as a quantum field theory, in contrast to a nonfluctuating field theory, which is similarly referred to as a classical field theory, by analogy.

All thermodynamic properties of the system are described by the partition function of the fluctuating field which is given by the functional integral

$$Z^{\rm phys} = \int \mathcal{D}\phi(\mathbf{x})\; e^{-E[\phi]/k_{\rm B}T}, \tag{2.4}$$

where $k_{\rm B}$ is Boltzmann's constant and T the temperature. The measure of functional integration is defined by the product of integrals at each space point $\mathbf{x}$, multiplied by some irrelevant normalization factor $\mathcal{N}$:

$$\int \mathcal{D}\phi(\mathbf{x}) \equiv \mathcal{N} \prod_{\mathbf{x}} \int d\phi(\mathbf{x}). \tag{2.5}$$

The space points are initially assumed to lie on a narrow spatial lattice, whose spacing is assumed to approach zero.

We shall abbreviate the notation by renormalizing the field, the mass, and the coupling constant λ in such a way that $k_{\rm B}T = 1$. Thus we shall write the partition function as

$$Z^{\rm phys} = \int \mathcal{D}\phi(\mathbf{x})\; e^{-E[\phi]}. \tag{2.6}$$

For zero coupling constant λ, this reduces to the free-field partition function

$$Z_0^{\rm phys} = \int \mathcal{D}\phi(\mathbf{x})\; e^{-E_0[\phi]}. \tag{2.7}$$

The functional integral is now Gaussian, and can be evaluated exactly.

For the upcoming derivation of perturbation expansions it will be convenient to renormalize the partition function by this value, and define the reduced partition function

$$Z \equiv \frac{Z^{\rm phys}}{Z_0^{\rm phys}}, \tag{2.8}$$

whose free part Z_0 is equal to unity:

$$Z_0 = 1. \tag{2.9}$$

While the partition function (2.4) describes all thermodynamic properties of the system, it does not give any information on the local properties of the system which are observed in scattering experiments. This information is carried by the *correlation functions* of the field $\phi(\mathbf{x})$ (see Appendix 1A for details):

$$\begin{aligned} G^{(n)}(\mathbf{x}_1,\dots,\mathbf{x}_n) &\equiv \langle \phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_n)\rangle \\ &= \frac{1}{Z^{\rm phys}} \int \mathcal{D}\phi(\mathbf{x})\; \phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_n)\, e^{-E[\phi]}. \end{aligned} \tag{2.10}$$

They are also called *n-point functions* or *Green functions*. There is a compact way of describing all correlation functions of the system with the help of one functional object. One introduces an auxiliary external field $j(\mathbf{x})$ called a *current*, and adds to the energy functional (2.1) a linear interaction energy of this current with the field $\phi(\mathbf{x})$,

$$E_{\rm source}[\phi, j] = -\int d^Dx\; \phi(\mathbf{x}) j(\mathbf{x}), \tag{2.11}$$

thus extending it to a total energy

$$E[\phi, j] = E[\phi] + E_{\rm source}[\phi, j]. \tag{2.12}$$

The partition function formed with this energy

$$Z[j] = (Z_0^{\rm phys})^{-1} \int \mathcal{D}\phi(\mathbf{x})\, e^{-E[\phi,j]} \tag{2.13}$$

is a functional of $j(\mathbf{x})$ whose value at $j(\mathbf{x}) \equiv 0$ is equal to the normalized partition function (2.8).

The functional derivatives of $Z[j]$ with respect to $j(\mathbf{x})$ evaluated at $j \equiv 0$ yield obviously the correlation functions of the system:

$$G^{(n)}(\mathbf{x}_1, \dots, \mathbf{x}_n) \;=\; Z^{-1} \left[\frac{\delta}{\delta j(\mathbf{x}_1)} \cdots \frac{\delta}{\delta j(\mathbf{x}_n)}\, Z[j] \right]_{j\equiv 0}. \tag{2.14}$$

For this reason, $Z[j]$ is referred to as the *generating functional* of the theory.

2.2 Free-Field Theory

The properties of the theory defined by the generating functional (2.13) will be investigated with the help of perturbative expansions. This is an expansion of $Z[j]$ around the generating functional $Z_0[j]$ of the free-field theory in powers of the coupling constant λ. For this we rewrite the free-field energy functional (2.2) after a partial integration of the gradient term as

$$\begin{aligned} E_0[\phi] &= \frac{1}{2} \int d^D x\, \phi(\mathbf{x}) \left(-\partial_{\mathbf{x}}^2 + m^2 \right) \phi(\mathbf{x}) \\ &= \frac{1}{2} \int d^D x_1 d^D x_2\, \phi(\mathbf{x}_1)\, D(\mathbf{x}_1, \mathbf{x}_2)\, \phi(\mathbf{x}_2). \end{aligned} \tag{2.15}$$

In the second line we have expressed the differential operator $-\partial_{\mathbf{x}}^2 + m^2$ as a symmetric *functional matrix* $D(\mathbf{x}_1, \mathbf{x}_2) = D(\mathbf{x}_2, \mathbf{x}_1)$. This is a matrix with continuous indices $\mathbf{x}_1$ and $\mathbf{x}_2$. Its explicit form is

$$D(\mathbf{x}_1, \mathbf{x}_2) = \delta^{(D)}(\mathbf{x}_1 - \mathbf{x}_2)\, (-\partial_{\mathbf{x}_2}^2 + m^2). \tag{2.16}$$

The difference between (2.15) and (2.2) is a surface term which vanishes if the ϕ-field satisfies periodic boundary conditions. This will always be assumed.

Given such a quadratic energy, we can immediately calculate the free-field functional integral (2.7). Remembering the definition (2.5), we see that

$$Z_0^{\rm phys} = \int \mathcal{D}\phi(\mathbf{x})\, e^{-E_0[\phi]} = \mathcal{N} \left[\prod_{\mathbf{x}} \int d\phi(\mathbf{x}) \right] e^{-E_0[\phi]} = \frac{1}{\sqrt{{\rm Det} D}}, \tag{2.17}$$

where ${\rm Det}\, D$ is the functional determinant of $D(\mathbf{x}_1, \mathbf{x}_2)$. This is defined by a straightforward generalization of a well-known identity for matrices:

$${\rm Det}\, D = e^{{\rm Tr}\, \log D} = e^{{\rm Tr}\, \log[1 + (D-1)]}. \tag{2.18}$$

Hence

$$Z_0^{\rm phys} = e^{-\frac{1}{2}{\rm Tr}\, \log D}. \tag{2.19}$$

The proof of this formula is simple. A single Gaussian integral yields

$$\int \frac{d\phi}{\sqrt{2\pi}} e^{-a\phi^2/2} = \frac{1}{\sqrt{a}}. \tag{2.20}$$

For a diagonal matrix M with matrix elements $M_{ii} = a_i$, we obtain similarly

$$\prod_i \int \frac{d\phi_i}{\sqrt{2\pi}} e^{-a_i\phi_i^2/2} = \frac{1}{\sqrt{\Pi_i a_i}}. \tag{2.21}$$

An arbitrary symmetric matrix M can be diagonalized by a rotation of the integration variables ϕ_i which does not change the volume of integration, so that we find

$$\prod_i \int \frac{d\phi_i}{\sqrt{2\pi}} e^{-\phi_i M_{ij}\phi_j/2} = \frac{1}{\sqrt{\Pi_i a_i}} = \frac{1}{\sqrt{\det M}}. \tag{2.22}$$

Since the measure of functional integration in (2.17) is defined by the product of integrals at each space point $\mathbf{x}$, this formula applies with the indices i replaced by $\mathbf{x}$, and the determinant on the right-hand side of (2.22) replaced by the functional determinant on the right-hand side of (2.17). The normalization factor $\mathcal{N}$ picks up all infinite factors arising in the limit of zero lattice spacing.

With the explicit result (2.17) we can write the reduced partition function (2.8) as

$$Z = e^{\frac{1}{2}\mathrm{Tr}\,\log D} \int \mathcal{D}\phi(\mathbf{x})\, e^{-E[\phi]}. \tag{2.23}$$

It will be useful to absorb the prefactor into the measure of integration and define

$$\int \mathcal{D}'\phi(\mathbf{x}) \equiv e^{\frac{1}{2}\mathrm{Tr}\,\log D} \int \mathcal{D}\phi(\mathbf{x}), \tag{2.24}$$

so that we write the reduced partition function as

$$Z = \int \mathcal{D}'\phi(\mathbf{x})\, e^{-E[\phi]}, \tag{2.25}$$

and the generating functional (2.13) as

$$Z[j] = \int \mathcal{D}'\phi(\mathbf{x})\, e^{-E[\phi,j]}. \tag{2.26}$$

Setting here $\lambda = 0$, we obtain the generating functional of the free-field theory:

$$Z_0[j] = \int \mathcal{D}'\phi(\mathbf{x})\, e^{-\{E_0[\phi]+E_{\mathrm{source}}[\phi,j]\}}. \tag{2.27}$$

This can be integrated as easily as for zero sources. The field still occurs at most quadratically in the exponent, and this permits us to absorb the current term into the field $\phi(\mathbf{x})$ by a shift

$$\phi(\mathbf{x}) \to \phi'(\mathbf{x}) = \phi(\mathbf{x}) - \int d^D x' D^{-1}(\mathbf{x},\mathbf{x}')j(\mathbf{x}'), \tag{2.28}$$

where the symbol $D^{-1}(\mathbf{x},\mathbf{x}')$ denotes the inverse functional matrix of $D(\mathbf{x},\mathbf{x}')$, defined by

$$\int d^D x'\, D(\mathbf{x},\mathbf{x}')D^{-1}(\mathbf{x}',\mathbf{x}'') = \delta^{(D)}(\mathbf{x}-\mathbf{x}''). \tag{2.29}$$

With $D(\mathbf{x}, \mathbf{x}')$, the inverse functional matrix $D^{-1}(\mathbf{x}, \mathbf{x}')$ is also symmetric in its arguments. The shift (2.28) transforms the generating functional (2.27) into

$$Z_0[j] = \int \mathcal{D}'\phi'(\mathbf{x})\, e^{-E_0[\phi']} e^{\frac{1}{2}\int d^D x d^D x'\, j(\mathbf{x}) D^{-1}(\mathbf{x},\mathbf{x}') j(\mathbf{x}')}. \tag{2.30}$$

The functional integral over the shifted field $\phi'(\mathbf{x})$ yields the same constant as in (2.19), leading to

$$Z_0[j] = e^{\frac{1}{2}\int d^D x d^D x'\, j(\mathbf{x}) D^{-1}(\mathbf{x},\mathbf{x}') j(\mathbf{x}')}. \tag{2.31}$$

For $j \equiv 0$, this coincides with the reduced free-field partition function (2.7) with unit normalization.

The correlation functions of the free-field theory are now obtained from the functional derivatives of $Z_0[j]$ at $j(\mathbf{x}) \equiv 0$:

$$\begin{aligned} G_0^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n) &= \left[\frac{\delta}{\delta j(\mathbf{x}_1)} \cdots \frac{\delta}{\delta j(\mathbf{x}_n)}\, Z_0[j]\right]_{j\equiv 0} \\ &= \left[\frac{\delta}{\delta j(\mathbf{x}_1)} \cdots \frac{\delta}{\delta j(\mathbf{x}_n)}\, e^{\frac{1}{2}\int d^D x d^D x'\, j(\mathbf{x}) D^{-1}(\mathbf{x},\mathbf{x}') j(\mathbf{x}')}\right]_{j\equiv 0}. \end{aligned} \tag{2.32}$$

Only correlation functions with an even number of fields are nonzero. They are given by the functional derivatives

$$G_0^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \frac{1}{2^{n/2}\,(n/2)!} \left\{\frac{\delta}{\delta j(\mathbf{x}_1)} \cdots \frac{\delta}{\delta j(\mathbf{x}_n)} \left[\int d^D x d^D x'\, j(\mathbf{x}) D^{-1}(\mathbf{x},\mathbf{x}') j(\mathbf{x}')\right]^{n/2}\right\}_{j\equiv 0}. \tag{2.33}$$

For $n = 2$, this equation yields the *free two-point function* or the *free propagator* of the field

$$\begin{aligned} G_0^{(2)}(\mathbf{x}_1, \mathbf{x}_2) &= \frac{1}{2}\left[\frac{\delta}{\delta j(\mathbf{x}_1)} \frac{\delta}{\delta j(\mathbf{x}_2)} \int d^D x d^D x'\, j(\mathbf{x}) D^{-1}(\mathbf{x},\mathbf{x}') j(\mathbf{x}')\right]_{j\equiv 0} \\ &= \frac{1}{2}\left[D^{-1}(\mathbf{x}_1, \mathbf{x}_2) + D^{-1}(\mathbf{x}_2, \mathbf{x}_1)\right] \\ &= D^{-1}(\mathbf{x}_1, \mathbf{x}_2). \end{aligned} \tag{2.34}$$

Thus we may replace $D^{-1}(\mathbf{x}, \mathbf{x}')$ in Eqs. (2.31)–(2.33) by the free two-point function $G_0^{(2)}(\mathbf{x}, \mathbf{x}')$. This function will occur so often in this text that it is convenient to drop the superscript and write $G_0(\mathbf{x}, \mathbf{x}')$ for $G_0^{(2)}(\mathbf{x}, \mathbf{x}')$.

Equation (2.33) implies that all n-point functions of the free-field theory can be expressed as sums of products of two-point functions. The product rule of differentiation leads to a sum of $n!$ terms. Since $G_0^{(2)}(\mathbf{x}, \mathbf{x}')$ is symmetric in its arguments, and multiplication is commutative, the $n!$ terms decompose into groups of $2^{n/2} \times (n/2)!$ identical terms. This number is canceled out by the denominator in front of the right-hand side of (2.33) coming from the Taylor expansion of the exponential in (2.32). Hence we remain with a sum of $n!/[2^{n/2}(n/2)!] = (n-1)!!$ different products of $n/2$ propagators $G_0^{(2)}$ with no numerical prefactor:

$$\begin{aligned} G_0^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n) &= \sum_{i=1}^{(n-1)!!} G_0^{(2)}(\mathbf{x}_{\pi_i(1)}, \mathbf{x}_{\pi_i(2)}) \cdots G_0^{(2)}(\mathbf{x}_{\pi_i(n-1)}, \mathbf{x}_{\pi_i(n)}) \\ &= \sum_{i=1}^{(n-1)!!} \prod_{j=1}^{n/2} G_0^{(2)}(\mathbf{x}_{\pi_i(2j-1)}, \mathbf{x}_{\pi_i(2j)}). \end{aligned} \tag{2.35}$$

The indices $\pi_i(j)$ $(1 \le i \le (n-1)!!,\ 1 \le j \le n)$ label the position arguments for all independent set of pairs: $\{\ [\pi_i(1), \pi_i(2)], \ldots, [\pi_i(n-1), \pi_i(n)]\ \}$. The right-hand side of (2.35) is known as *Wick's expansion.*

The easiest way to enumerate the $(n-1)!!$ different pair combinations in Wick's expansion is by lining up the n indices

$$1\ 2\ 3\ 4\ \ldots\ n-2\ n-1\ n, \tag{2.36}$$

and introducing a *pair contraction* marked by two common dots, for instance:

$$\dot{1}\ \dot{2}\ 3\ 4\ \ldots\ n-2\ n-1\ n. \tag{2.37}$$

One then forms the $n-1$ single contractions starting out from the index 1:

$$\begin{aligned} &\dot{1}\ \dot{2}\ 3\ 4\ \ldots\ n-2\ n-1\ n \quad + \quad \dot{1}\ 2\ \dot{3}\ 4\ \ldots\ n-2\ n-1\ n \\ + \ &\dot{1}\ 2\ 3\ \dot{4}\ \ldots\ n-2\ n-1\ n \quad + \quad \ldots \quad + \quad \dot{1}\ 2\ 3\ 4\ \ldots\ \dot{n-2}\ n-1\ n \\ + \ &\dot{1}\ 2\ 3\ 4\ \ldots\ n-2\ \dot{n-1}\ n \quad + \quad \dot{1}\ 2\ 3\ 4\ \ldots\ n-2\ n-1\ \dot{n}. \end{aligned} \tag{2.38}$$

The uncontracted $n-2$ indices in each term are now treated once more in the same way, etc., until all indices are contracted. The $(n-1)!!$ terms obtained in this way are precisely the indices of the position arguments of the free propagators on the right-hand side of Eq. (2.35).

Note that the functional matrix (2.16) is invariant under arbitrary translations $\mathbf{a}$ of the spatial arguments of the field $\phi(\mathbf{x})$

$$\mathbf{x} \to \mathbf{x} + \mathbf{a}. \tag{2.39}$$

This property goes over to its inverse, the propagator, which therefore depends only on the difference between its arguments. It is then useful to introduce a propagator function with only a single argument:

$$G_0^{(2)}(\mathbf{x}, \mathbf{x}') = G_0(\mathbf{x}, \mathbf{x}') \equiv G_0(\mathbf{x} - \mathbf{x}'). \tag{2.40}$$

For the free n-point functions $G_0^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$, translational invariance has the consequence

$$G_0^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = G_0^{(n)}(\mathbf{x}_1 - \mathbf{x}_n, \ldots, \mathbf{x}_{n-1} - \mathbf{x}_n, 0), \tag{2.41}$$

which can be directly verified using the Wick expansion (2.35) and the translational invariance (2.40) of the two-point function.

2.3 Perturbation Expansion

An expansion of the generating functional $Z[j]$ in (2.13) in a power series of the coupling constant λ yields the so-called *perturbation expansion*

$$Z[j] \ = \ Z_0[j] \ + \ \sum_{p=1}^{\infty} Z_p[j], \tag{2.42}$$

which reads more explicitly

$$\begin{aligned} Z[j] \ &= \ Z_0[j] + \sum_{p=1}^{\infty} \frac{1}{p!}\, \lambda^p \left[\frac{d^p}{d\lambda^p} Z[j] \right]_{\lambda=0} \\ &= \ Z_0[j] + \sum_{p=1}^{\infty} \frac{1}{p!} \left(\frac{-\lambda}{4!} \right)^p \int \mathcal{D}'\phi(\mathbf{x}) \int d^D z_1 \cdots d^D z_p \\ &\qquad \times\ \phi^4(\mathbf{z}_1) \cdots \phi^4(\mathbf{z}_p)\, e^{-\{E_0[\phi] + E_{\text{source}}[\phi, j]\}}. \end{aligned} \tag{2.43}$$

For zero external current, the terms on the right-hand side contain integrals over correlation functions of $4p$ fields. Thus we may write the perturbation series of the partition function as

$$Z \;\equiv\; Z[j\equiv 0] \;= 1+\sum_{p=1}^{\infty} Z_p, \tag{2.44}$$

where the pth-order contribution is given by

$$Z_p \;=\; \frac{1}{p!}\left(\frac{-\lambda}{4!}\right)^p \int d^Dz_1\cdots d^Dz_p\, G_0^{(4p)}(\mathbf{z}_1,\mathbf{z}_1,\mathbf{z}_1,\mathbf{z}_1,\,\dots\,,\mathbf{z}_p,\mathbf{z}_p,\mathbf{z}_p,\mathbf{z}_p), \tag{2.45}$$

and where $G_0^{(4p)}$ are free $4p$-point functions with the Wick expansion (2.35).

Applying the derivatives in (2.14) to (2.43), we find the perturbation series for n-point functions

$$\begin{aligned} G^{(n)}(\mathbf{x}_1,\dots,\mathbf{x}_n) \;&=\; Z^{-1}\left[\frac{\delta}{\delta j(\mathbf{x}_1)}\cdots\frac{\delta}{\delta j(\mathbf{x}_n)}\,Z[j]\right]_{j\equiv 0} \\ &= Z^{-1}\left[G_0^{(n)}(\mathbf{x}_1,\dots,\mathbf{x}_n)+\sum_{p=1}^{\infty}\frac{1}{p!}\left(\frac{-\lambda}{4!}\right)^p\!\int d^Dz_1\cdots d^Dz_p \right. \\ &\quad\left.\times\int\mathcal{D}'\phi(\mathbf{x})\,\phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_n)\phi^4(\mathbf{z}_1)\cdots\phi^4(\mathbf{z}_p)\,e^{-\{E_0[\phi]+E_{\rm source}[\phi,j]\}}\right]_{j=0}. \end{aligned} \tag{2.46}$$

The functional integral on the right-hand side may be expressed in terms of free n-point functions $G_0^{(n)}(\mathbf{x}_1,\dots,\mathbf{x}_n)$ as

$$G^{(n)}(\mathbf{x}_1,\dots,\mathbf{x}_n) \;=\; Z^{-1}\left[G_0^{(n)}(\mathbf{x}_1,\dots,\mathbf{x}_n)+\sum_{p=1}^{\infty}G_p^{(n)}(\mathbf{x}_1,\mathbf{x}_2,\,\dots\,,\mathbf{x}_n),\right] \tag{2.47}$$

with the contributions of pth order with $p\geq 1$ given by the integrals

$$\begin{aligned} G_p^{(n)}(\mathbf{x}_1,\dots,\mathbf{x}_n) = \frac{1}{p!}\left(\frac{-\lambda}{4!}\right)^p\int d^Dz_1\cdots d^Dz_p \\ \times\;\; G_0^{(n+4p)}(\mathbf{z}_1,\mathbf{z}_1,\mathbf{z}_1,\mathbf{z}_1,\,\dots\,,\mathbf{z}_p,\mathbf{z}_p,\mathbf{z}_p,\mathbf{z}_p,\mathbf{x}_1,\dots,\mathbf{x}_n). \end{aligned} \tag{2.48}$$

The free-field correlation functions $G_0^{(n+4p)}$ in the sum may now be Wick-expanded as in Eq. (2.35) into sums over products of propagators G_0. Due to the proliferating number of terms with increasing order p, the evaluation of these series rapidly becomes complicated. Even their organization is involved, although it may be economized with the help of Feynman diagrams to be introduced in the next chapter.

An important property should be observed at this place. By (2.41), the free n-point functions $G_0^{(n)}(\mathbf{x}_1,\dots,\mathbf{x}_n)$ are invariant under the spatial translations (2.39):

$$G_0^{(n)}(\mathbf{x}_1,\dots,\mathbf{x}_n)=G_0^{(n)}(\mathbf{x}_1+\mathbf{a},\dots,\mathbf{x}_n+\mathbf{a}). \tag{2.49}$$

The perturbation expansion (2.48) shows that this property goes over to the interacting n-point functions:

$$G^{(n)}(\mathbf{x}_1,\dots,\mathbf{x}_n)=G^{(n)}(\mathbf{x}_1+\mathbf{a},\dots,\mathbf{x}_n+\mathbf{a}). \tag{2.50}$$

This is, of course, a general consequence of the fact that the full interacting field energy (2.1) is invariant under arbitrary translations of the spatial argument of the field:

$$\phi(\mathbf{x}) \to \phi(\mathbf{x}+\mathbf{a}). \tag{2.51}$$

For the interacting two-point function, translational invariance implies that it depends only on the difference of its arguments:

$$G^{(2)}(\mathbf{x},\mathbf{x}') \equiv G(\mathbf{x},\mathbf{x}') \equiv G(\mathbf{x}-\mathbf{x}'), \tag{2.52}$$

where we have introduced a bilocal short notation $G(\mathbf{x},\mathbf{x}')$ for the interacting propagator, just as for the free propagator in (2.40).

2.4 Composite Fields

Later, in Section 7.3.1, we shall see a special role played by a certain class of correlation functions with coinciding spatial arguments of some of its fields. For these we introduce the notation

$$G^{(l,n)}(\mathbf{y}_1,\ldots,\mathbf{y}_l,\mathbf{x}_1,\ldots,\mathbf{x}_n) \equiv \frac{1}{2^l}\langle\phi^2(\mathbf{y}_1)\cdots\phi^2(\mathbf{y}_l)\phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_n)\rangle. \tag{2.53}$$

They can, of course, all be generated by appropriate functional derivatives of $Z[j]$. More conveniently, however, we introduce a new generating functional extended by a source term coupled to the square of the field $\phi(\mathbf{x})$:

$$Z[j,K] = \int \mathcal{D}'\phi(\mathbf{x})\, e^{-E[\phi]+\int d^Dx[j(\mathbf{x})\phi(\mathbf{x})+\frac{1}{2}K(\mathbf{x})\phi^2(\mathbf{x})]}. \tag{2.54}$$

From this, the correlation functions (2.53) are obtained by the functional derivatives

$$G^{(l,n)}(\mathbf{y}_1,\ldots,\mathbf{y}_l,\mathbf{x}_1,\ldots,\mathbf{x}_n) \tag{2.55}$$
$$= Z^{-1}\left[\frac{\delta}{\delta K(\mathbf{y}_1)}\cdots\frac{\delta}{\delta K(\mathbf{y}_l)}\frac{\delta}{\delta j(\mathbf{x}_1)}\cdots\frac{\delta}{\delta j(\mathbf{x}_n)}Z[j,K]\right]_{j\equiv K\equiv 0}.$$

For $K(\mathbf{x}) = \text{constant} = k$, the source term can be combined with the mass term in the energy functional (2.2). As a consequence, the correlation functions for a mass m^2-k can be written as a power series in k:

$$G^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n)\Big|_{m^2->m^2-k} = \sum_{l=0}^{\infty}\frac{k^l}{l!}\int d^Dy_1\cdots d^Dy_l\, G^{(l,n)}(\mathbf{y}_1,\ldots,\mathbf{y}_l,\mathbf{x}_1,\ldots,\mathbf{x}_n), \tag{2.56}$$

where the replacement of the square mass is done only in the energy functional of the defining equation (2.10), not in the normalization factor which stays fixed. In particular the first expansion term, the integral $\int d^Dy\, G^{(1,n)}(\mathbf{y},\mathbf{x}_1,\ldots,\mathbf{x}_n)$, can be obtained by differentiating $G^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n)$ with respect to the mass. Let us emphasize the mass dependence of $G^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n)$ coming from the energy functional in (2.2) by an extra argument m, we have

$$-\frac{\partial}{\partial m^2}G^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n,m) =, \tag{2.57}$$

or, expressed in terms of field expectation values:

$$-\frac{\partial}{\partial m^2}\langle\phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_n)\rangle = \frac{1}{2}\int d^Dy\,\langle\phi^2(\mathbf{y})\phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_n)\rangle. \tag{2.58}$$

The differentiation $-m^2\partial_{m^2}$ is called a *mass insertion*, since it inserts a mass term $m^2\phi^2(\mathbf{x})/2$ into a correlation function.

Notes and References

Textbooks on quantum field theory are listed at the end of Chapter 1. The notation in the present book follows

H. Kleinert, *Gauge Fields in Condensed Matter*, Vol. I, *Superflow and Vortex Lines, Disorder Fields and Phase Transitions*, World Scientific, Singapore, 1989 (www.physik.fu-berlin.de/~kleinert/re.html#b1).