1

Introduction

Systems containing a large number of particles exhibit a great variety of phase transitions. Most common are *first-* and *second-order transitions*. A transition is said to be first-order if the internal energy changes discontinuously at a certain temperature. Such a transition is accompanied by the release or absorption of *latent heat*. Important examples are melting and evaporation processes. Second-order transitions involve no latent heat, and the internal energy changes continuously with temperature. The derivative of the internal energy with respect to the temperature diverges at the transition temperature T_c, which is also called the *critical temperature*. The most important examples for materials undergoing second-order transitions are ferromagnets, superfluids, and superconductors.

There also exist phase transitions of higher order in which the first appearance of a divergence occurs in some higher derivative of the internal energy with respect to the temperature. A famous extreme example is the *Kosterlitz-Thouless transition* [1] of a Coulomb gas in two space dimensions. The same type of transition is also found in thin films of ^{4}He at temperatures of a few degrees Kelvin where the films become superfluid. In this transition, the internal energy may be differentiated any number of times with respect to the temperature and does not show any divergence. Instead, the temperature behavior exhibits an essential singularity of the form $e^{\text{const}\times(T-T_c)^{-1/2}}$.

The present text is devoted to a field-theoretic description of second-order transitions. Transitions of the first and higher than second order will not be considered.

1.1 Second-Order Phase Transitions

An important property of second-order phase transitions is the divergence, at the critical temperature T_c, of the length scale, over which the system behaves coherently. This is accompanied by a divergence of the size of thermal fluctuations. As a consequence, many physical observables show, near T_c, a power behavior in the temperature difference $|T-T_c|$ from the critical point, i.e., they behave like $|T-T_c|^p$. The power p is called a *critical exponent* of the associated observable.

We shall focus our attention on those physical systems whose relevant thermal fluctuations near the transition temperature can be described by some N-component *order field* $\boldsymbol{\phi}(\mathbf{x}) = (\phi_1(\mathbf{x})\cdots\phi_N(\mathbf{x}))$. An order field is the space-dependent generalization of Landau's famous *order parameter*, which characterizes all second-order transitions in a molecular field approximation. The energy of a field configuration is described by some functional of the order field $E = E[\boldsymbol{\phi}(\mathbf{x})]$. To limit the number of possible interaction terms, certain symmetry properties will be assumed for the energy in the N-dimensional field space.

The thermal expectation value of the order field will be denoted by

$$\boldsymbol{\Phi}(\mathbf{x}) \equiv \langle\boldsymbol{\phi}(\mathbf{x})\rangle. \tag{1.1}$$

This expectation value is nonzero below, and zero above the transition temperature [2]. The high-temperature state with zero expectation value is referred to as the *normal phase*, the low-temperature phase with nonzero expectation value is the *ordered phase*. In most systems, the nonzero expectation value $\mathbf{\Phi}(\mathbf{x})$ is a constant, for example in ferromagnetic systems where the ordered phase shows a uniform magnetization. Only systems in the normal phase will be described in detail in this book. Thus we shall ignore a great number of interesting physical systems, for example cholesteric and smectic liquid crystals, which have a periodic $\mathbf{\Phi}(\mathbf{x})$ and are of great industrial importance.

The ferromagnetic transition is the archetype for a second-order phase transition, in which a paramagnetic normal phase goes over into a ferromagnetic ordered phase when temperature is lowered below T_c, which is here called the *Curie temperature*. The microscopic origin lies in the lattice constituents. In ferromagnetic materials like Fe, Co, and Ni, these possess permanent magnetic moments which tend to line up parallel to one another to minimize the exchange energies of the electrons. At nonzero temperatures, they are prevented from perfect alignment by thermal fluctuations which increase the entropy of the many possible directional configurations. Disorder is also favored by the fact that alignment creates magnetic field energy, which causes a breakup into Weiss domains. This mechanism will be ignored. The local magnetic moments in a continuum approximation of these systems are described by a three-component order field $\boldsymbol{\phi}(\mathbf{x}) = (\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \phi_3(\mathbf{x}))$.

At zero temperature, the alignment forces give the system a constant global order. The thermal expectation value of the field is nonzero, exhibiting a *spontaneous magnetization* $\mathbf{\Phi} \equiv \langle \boldsymbol{\phi}(\mathbf{x}) \rangle$ which serves as an order parameter, and which we shall alternatively denote by $\mathbf{M}$ when dealing with magnetic systems. The vector $\mathbf{M}$ may point in any fixed spatial direction.

If the temperature is raised, the size of $\mathbf{M}$ decreases due to the disordering effect of thermal fluctuations. When approaching the Curie temperature T_c, the spontaneous magnetization tends to zero according to a power law

$$|\mathbf{M}| \sim |T - T_c|^{\beta}. \tag{1.2}$$

Above the Curie temperature, it is identically zero, and the system is in the normal phase.

From the symmetry point of view, the existence of a spontaneous magnetization below T_c implies a *spontaneous breakdown of rotational symmetry*. The energy functional of the system is rotationally invariant in the space of the order field which happens to coincide here with the configuration space; whereas below T_c there exists a preferred direction defined by the spontaneous magnetization vector $\mathbf{M}$. The ordered state has a reduced symmetry, being invariant only under rotations around this direction.

In general one speaks of a spontaneous symmetry breakdown if the symmetry group in field space reduces to a subgroup when passing from the disordered high-temperature phase to the ordered low-temperature phase.

Another important system undergoing a second-order phase transition is liquid ^{4}He. At a temperature $T_\lambda = 2.18$K, called the λ-point, it becomes superfluid. At the transition, the phase factor $e^{i\theta(\mathbf{x})}$ of its many-body wave function which fluctuates violently in the normal state, becomes almost constant with only small long-wavelength fluctuations. Since the phase factor $e^{i\theta(\mathbf{x})}$ may be viewed as a two-dimensional vector $(\cos\theta(\mathbf{x}), \sin\theta(\mathbf{x}))$, this transition may also be described by a two-component order field $\boldsymbol{\phi}(\mathbf{x})$. For experimental studies of its phase transition, liquid ^{4}He is especially well suited since its thermal conductivity is very high in the superfluid state, which permits an extremely uniform temperature to be established in a sizable sample, with variations of less than 10^{-8} K, as we shall see below in Fig. 1.1.

1.2 Critical Exponents

It is usually possible to design scattering experiments which are sensitive to the thermal fluctuations of the order field $\boldsymbol{\phi}(\mathbf{x})$. They are capable of measuring the full correlation function

$$G_{ij}(\mathbf{x}-\mathbf{y}) = \langle \phi_i(\mathbf{x})\phi_j(\mathbf{y})\rangle \tag{1.3}$$

which in the normal phase is proportional to δ_{ij}:

$$G_{ij}(\mathbf{x}-\mathbf{y}) = \delta_{ij} G(\mathbf{x}-\mathbf{y}). \tag{1.4}$$

1.2.1 Correlation Functions

Take, for example, an order field $\boldsymbol{\phi}(\mathbf{x})$ with only a single component $\phi(\mathbf{x})$ describing the local density $\rho(\mathbf{x})$ of a liquid near its critical point. In this case, the *correlation function*

$$G(\mathbf{x}-\mathbf{y}) = \langle \phi(\mathbf{x})\phi(\mathbf{y})\rangle \tag{1.5}$$

can be measured by neutron scattering. The information on $G(\mathbf{x}-\mathbf{y})$ is contained in the so-called *structure factor* $S(\mathbf{x})$, whose experimental measurement is sketched in Appendix 1A.

In a phase with broken symmetry where the order field $\boldsymbol{\phi}(\mathbf{x})$ has a nonzero thermal expectation value $\boldsymbol{\Phi} = \langle \boldsymbol{\phi}(\mathbf{x})\rangle$, the physically interesting quantity is the *connected correlation function*, which describes the fluctuations of the deviations of the order field $\boldsymbol{\phi}(\mathbf{x})$ from its expectation value $\boldsymbol{\Phi}$, to be denoted by $\delta\boldsymbol{\phi}(\mathbf{x}) \equiv \boldsymbol{\phi}(\mathbf{x}) - \boldsymbol{\Phi}$:

$$G_{\mathrm{c}\,ij}(\mathbf{x}-\mathbf{y}) = \langle \delta\phi_i(\mathbf{x})\delta\phi_j(\mathbf{y})\rangle = \langle \phi_i(\mathbf{x})\phi_j(\mathbf{y})\rangle - \Phi_i\Phi_j. \tag{1.6}$$

In a ferromagnet, $\boldsymbol{\phi}(\mathbf{x})$ describes the local magnetization, and $\delta\boldsymbol{\phi}(\mathbf{x})$ its deviations from the spontaneous magnetization $\boldsymbol{\Phi} = \mathbf{M}$. The connected correlation function $G_{\mathrm{c}\,ij}(\mathbf{x}-\mathbf{y})$ is an anisotropic tensor, which may be decomposed into a longitudinal and a transverse part with respect to the direction of $\boldsymbol{\Phi}$:

$$G_{\mathrm{c}\,ij}(\mathbf{x}) = \frac{\Phi_i\Phi_j}{\Phi^2} G_{\mathrm{c}\,L}(\mathbf{x}) + \left(\delta_{ij} - \frac{\Phi_i\Phi_j}{\Phi^2}\right) G_{\mathrm{c}\,T}(\mathbf{x}). \tag{1.7}$$

For temperatures closely above T_c, the correlation functions have a universal scaling behavior. To characterize this behavior, we identify first a microscopic length scale a below which microscopic properties of the material begin to become relevant. The size of a is usually defined by the spacing of a lattice or the size of molecules. Only for distances much larger than a can a field-theoretic description of the system in terms of an order field be meaningful. One then observes correlation functions whose behavior can be explained by field theory. For $T \gtrsim T_c$, they have a typical scaling form

$$G(\mathbf{x}) \sim \frac{1}{r^{D-2+\eta}}\, g(\mathbf{r}/\xi), \qquad r \equiv |\mathbf{x}| \gg a\,, \tag{1.8}$$

where the function g falls off exponentially for large distances $r \to \infty$:

$$g(\mathbf{x}/\xi) \sim \exp(-r/\xi). \tag{1.9}$$

For $T \lesssim T_c$, the same type of behavior is found for $G_{\mathrm{c}\,L}(\mathbf{x})$, but with different functions $g(\mathbf{r}/\xi)$ and parameter ξ. The characteristic length scale ξ over which this falloff occurs is called the

correlation or *coherence length* ξ. For $r \ll \xi$, but still much larger than a, the function $g(\mathbf{r}/\xi)$ becomes independent of $\mathbf{r}$, so that the correlation function behaves like a pure power $1/r^{D-2+\eta}$.

For $T \lesssim T_c$, the same type of behavior is found for the longitudinal connected correlation function $G_{c\,L}(\mathbf{x})$, but with different functions $g(\mathbf{r}/\xi)$ and coherence length ξ.

Near the critical temperature, the correlation length ξ diverges according to a power law

$$\xi \sim |T - T_c|^{-\nu}, \tag{1.10}$$

which defines the critical exponent ν. In principle, the exponent could be different when T_c is approached from above or below. Experimentally, however, they seem to be equal. In the field theory of critical phenomena, this equality will emerge as a prediction [in Eqs. (1.87) and (1.90) at the mean-field level, and in Section 10.10 in general]. The proportionality constant in (1.10), however, will be different for the two approaches from above and below. The ratio of these proportionality constants is called the *amplitude ratio*.

At $T = T_c$, the correlation length is infinite and the correlation function shows a pure power behavior

$$G_c(\mathbf{x}) \sim \frac{1}{r^{D+\eta-2}}, \qquad r \equiv |\mathbf{x}| \quad \text{at} \quad T = T_c\,, \tag{1.11}$$

which defines the critical exponent η. Higher correlation functions have a similar power behavior, all characterized by the single exponent η.

As power laws do not contain any length scale, this implies that at $T = T_c$ the large-scale properties of the system are free of any length scale. Such properties are generically called *critical*. The critical properties are independent of the microscopic nature of the system. They are *universal*properties, independent of lattice structures and atomic composition. The same is true for the behavior in the immediate neighborhood of T_c, where a system approaches the critical point and the correlation function has a behavior of the form (1.8).

One therefore distinguishes systems at or very close to a phase transition by the *universality classes* of their critical exponents. Each class depends only on the symmetry properties of the energy as a functional of the order parameter, apart from the space dimension of the many-body system.

For temperatures closely below T_c, the longitudinal correlation function $G_{c\,L}(\mathbf{x})$ has a similar behavior as (1.8) for $T \lesssim T_c$ only with a slightly different correlation length which, however, diverges with the same critical exponent ν for $T \to T_c$. At T_c it is characterized by the same critical exponent η as in (1.11).

The tranverse correlation functions have a special property: if the field energy is symmetric under O(N) symmetry transformations of the N field components, spatially constant fluctuations transverse to $\mathbf{\Phi}$ are symmetry transformations and do not expend any energy. For this reason, transverse fluctuations have an infinite correlation length and behave like (1.11) for *all* T, not just at T_c. This is a manifestation of the *Nambu-Goldstone theorem*, which states that the spontaneous breakdown of a continuous symmetry in a phase transition gives rise to long-range modes associated with each generator of the symmetry.

The exponents η and ν of the correlation function $G(\mathbf{x})$ above T_c, and of $G_{c\,L}(\mathbf{x})$, $G_{c\,T}(\mathbf{x})$ below T_c will be calculated in Chapter 10 for a great variety of physical systems.

Experimentally one can easily measure the susceptibility tensor $\chi_{ij}(\mathbf{k})$ of a system at a finite wave vector $\mathbf{k}$, denoted by $\chi_{ij}(\mathbf{k})$. It is proportional to the Fourier transforms of the correlation function $G_{c\,ij}(\mathbf{x}-\mathbf{y})$, and has the same invariant decompositions (1.4) and (1.7) as the correlation function above and below T_c, respectively. Thus we decompose, above T_c,

$$\chi_{ij}(\mathbf{x}-\mathbf{y}) = \delta_{ij}\chi(\mathbf{x}-\mathbf{y}), \tag{1.12}$$

with

$$\chi(\mathbf{x}-\mathbf{y}) \propto G(\mathbf{x}-\mathbf{y}), \tag{1.13}$$

and below T_c:

$$\chi_{\mathrm{c}\,ij}(\mathbf{x}) = \frac{\Phi_i\Phi_j}{\Phi^2}\chi_{\mathrm{c}\,L}(\mathbf{x}) + \left(\delta_{ij} - \frac{\Phi_i\Phi_j}{\Phi^2}\right)\chi_{\mathrm{c}\,T}(\mathbf{x}), \tag{1.14}$$

with longitudinal and transverse parts

$$\chi_{L,T}(\mathbf{k}) \propto G_{\mathrm{c}\,L,T}(\mathbf{k}) = \int d^D x\, e^{i\mathbf{k}\mathbf{x}}\, G_{\mathrm{c}\,L,T}(\mathbf{x}). \tag{1.15}$$

1.2.2 Other Critical Exponents

There are other critical exponents which can be measured in global thermodynamic experiments, where one observes certain thermodynamic potentials or their derivatives with respect to temperature or external fields. Important examples are the specific heat C, the susceptibility χ_{ij} at zero external field, the magnetization $\mathbf{M} = \mathbf{\Phi}$ at zero external field below the critical temperature, or the magnetic equation of state $\mathbf{M}(\mathbf{B})$ at $T = T_c$. They have the following characteristic critical power behaviors:

$$C \sim |T - T_c|^{-\alpha}, \tag{1.16}$$

$$\chi_L(\mathbf{0}) \sim |T - T_c|^{-\gamma}, \tag{1.17}$$

$$\mathbf{k}^2\chi_T(\mathbf{k}) \underset{\mathbf{k}=0}{\sim} |T - T_c|^{\eta\nu}, \tag{1.18}$$

$$|\mathbf{M}| \sim |T - T_c|^{\beta}, \quad \text{for } T < T_c, \tag{1.19}$$

$$|\mathbf{M}| \sim |\mathbf{B}|^{1/\delta}, \quad \text{for } T = T_c. \tag{1.20}$$

These will all be derived in Chapter 10. An experimental critical behavior is shown for the

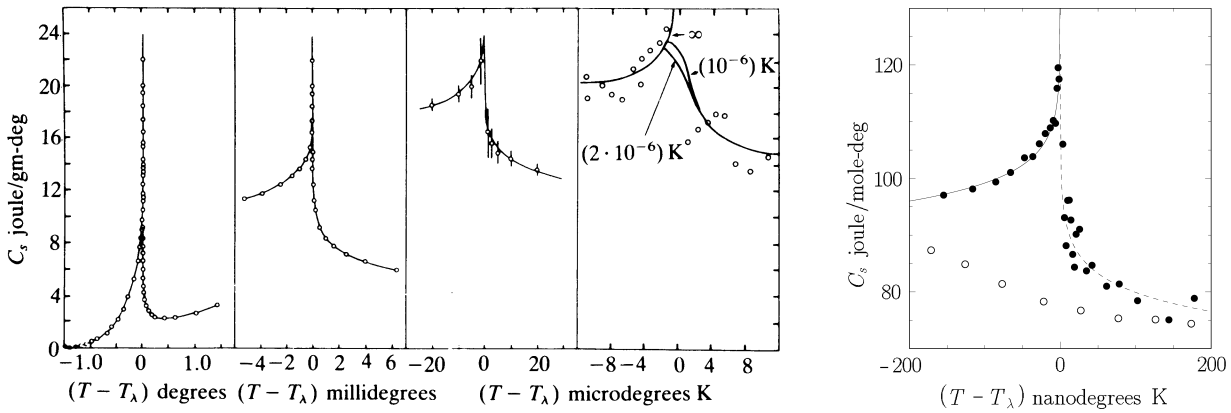

FIGURE 1.1 Specific heat near superfluid transition at $T_\lambda \approx 2.18$ K measured with increasing temperature resolutions. The curve has the typical λ-shape which is the reason for calling it λ-transition. Note that at higher resolutions, the left shoulder of the peak lies above the right shoulder. The data are from Ref. [3]. The forth plot is broadened by the pressure difference between top and bottom of the sample. This is removed by the microgravity experiment in the space shuttle yielding the last plot (open circles are irrelevant here) [4]. They show no pressure broadening even in the nK regime around the critical temperature.

specific heat of the superfluid transition of ^{4}He in Fig. 1.1. The curve for the specific heat has the form of a Greek letter λ, and for this reason one speaks here of a λ-transition, and denotes the critical temperature T_c in this case by T_λ. Just like the critical exponent ν, the

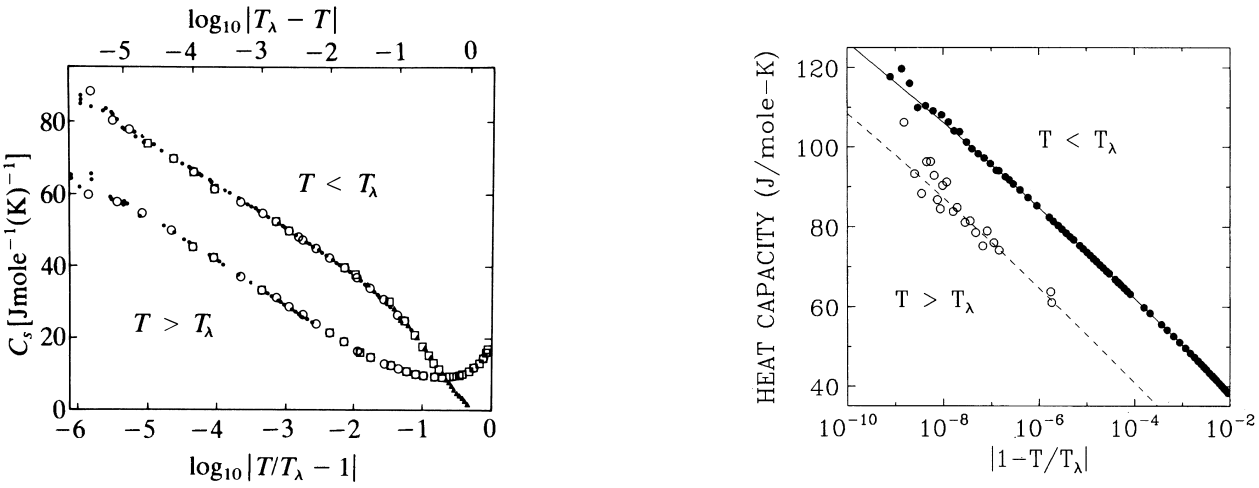

FIGURE 1.2 Specific heat of ^{4}He near superfluid transition plotted against $\log_{10}|T/T_\lambda - 1|$. The early data on the left-hand side by G. Ahlers [5] yield $\alpha \approx -0.026 \pm 0.004$ (for these an upper scale shows $\log_{10}|T_\lambda - T|$ with T measured in units of K). The right-hand side shows the space shuttle data of J.A. Lipa et al. in Refs. [4, 6], which yield $\alpha = -0.01056 \pm 0.0004$.

exponents α and γ could also in principle be different for the approaches of T_c from above and from below. Experiments, however, suggest their equality. Only the proportionality constants in (1.16)–(1.18) will be different for the two approaches, with well-defined amplitude ratios. This is illustrated by the plot of the experimental specific heat of the superfluid ^{4}He transition in Fig. 1.2 against the logarithm of $|T - T_\lambda|$. The data points lie approximately on straight lines corresponding to a logarithmic behavior

$$C \approx (A_\pm/\alpha)\,(1 - \alpha \log|T/T_\lambda - 1|) + \text{const}, \tag{1.21}$$

for the approaches from above and below, respectively. From his early data, G. Ahlers [5] determined the parameters α and $A_\pm$ by the best fits shown in Fig. 1.2, which gave $\alpha = -0.026 \pm 0.004$, $(A_- + A_+)/2 = 1.3 \pm 0.02$, and $A_+/A_- = 1.112 \pm 0.022$. The observed behavior (1.21) is compatible with a leading power $C \propto A_\pm |T - T_\lambda|^{-\alpha}$ of Eq. (1.16), owing to the smallness of the critical exponent α. For a finite α, a double-logarithmic plot would have been appropriate (as below in Fig. 1.3 determining the exponent ν).

The accuracy of Ahler's experiments was limited by the pressure difference within the sample caused by the earth's gravitational field. The small pressure dependence of the critical temperature smears out the singularity at T_λ to a narrow round peak, as seen on the right-hand plot of Fig. 1.1. The smearing was diminished by two orders of magnitude when the measurement was repeated at zero gravity in the space shuttle [4]. This permitted a much closer approach to the critical point than on earth, yielding the data on the right-hand plot of Fig. 1.2. They are fitted very well by a function

$$C = (A_\pm/\alpha)|t|^{-\alpha}\left(1 + D|t|^\Delta + E|t|\right) + B, \quad t \equiv T/T_\lambda - 1, \tag{1.22}$$

with [6]

$$\alpha = -0.01056 \pm 0.0004,\ \Delta = 0.5, \qquad A_+/A_- = 1.0442 \pm 0.001, \tag{1.23}$$

$$A_-/\alpha = -525.03, \qquad D = -0.00687, \qquad E = 0.2152, \quad B = 538.55\ (\text{J/mol K}). \tag{1.24}$$

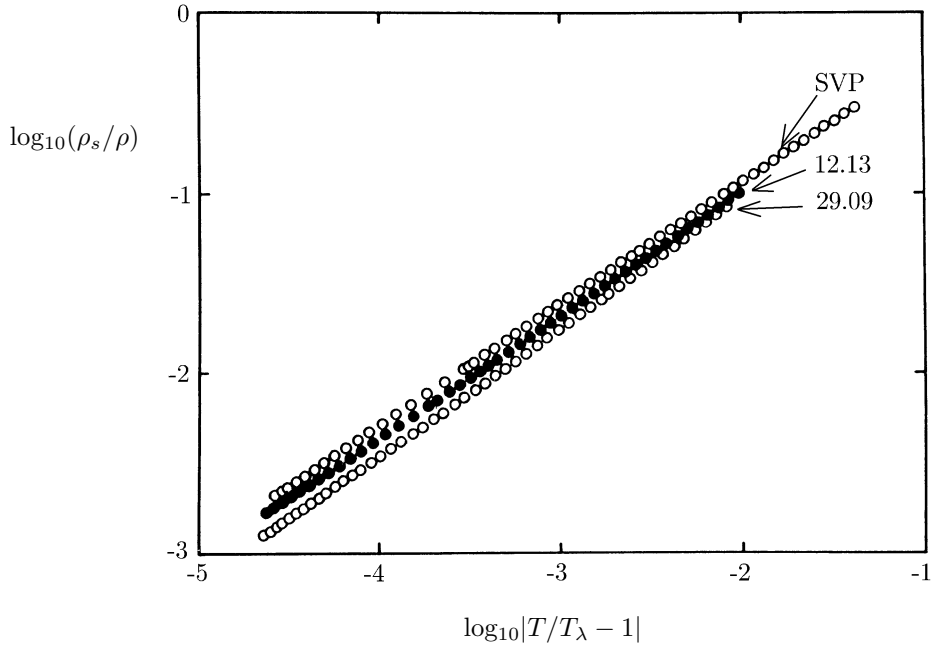

FIGURE 1.3 Doubly logarithmic plots of superfluid density ρ_s divided by the total density as a function of temperature. The slope is 2/3 [7]. The labels on the curves indicate the pressures in bar. The label SVP refers to the saturated vapor pressure.

The exponent Δ of the subleading singularity in t governs the approach to scaling and defines an associated critical exponent ω by the relation $\Delta = \omega\nu$, as will be derived in Eq. (10.152).

The equality of the critical exponents will also be explained by the Ginzburg-Landau theory in Eqs. (1.43) and (1.50), and in general in Section 10.10.

For superfluid ^{4}He below T_c, an important experimental quantity is the *superfluid density* ρ_s which determines the stiffness of the fluctuations of the phase angle $\theta(\mathbf{x})$ of the ground state wave function (recall the discussion at the end of Section 1.1). It will be defined below in Eq. (1.122), being proportional to the $\mathbf{k} \to 0$ limit of $\mathbf{\Phi}^2/\mathbf{k}^2\chi_T(\mathbf{k})$. From Eqs. (1.18) and (1.19) we see that its critical behavior is $2\beta - \eta\nu = (D-2)\nu$. Its measurement therefore supplies us directly with information on the exponent ν which determines the rate of divergence of the coherence length in Eq. (1.10):

$$\rho_s \propto |T_c - T|^{(D-2)\nu}. \tag{1.25}$$

This will be proved in Subsection 10.10.5. The experimental data in Fig. 1.3 show precisely this behavior with $\nu \approx 2/3$.

1.2.3 Scaling Relations

Several relations between the critical exponents were suggested on general theoretical grounds and found to hold experimentally. Different exponents in thermodynamic potentials were related to each other by Widom [8], who assumed these potentials near T_c to be generalized homogeneous functions of their variables [9]. For the free energy of a system with a magnetization M, he conjectured the general functional form

$$F(t, M) = |t|^{2-\alpha}\, \psi(|t|/M^{1/\beta}), \tag{1.26}$$

where t is the reduced temperature characterizing the relative temperature distance from the critical point,

$$t \equiv T/T_c - 1, \tag{1.27}$$

and ψ some smooth function of its arguments. The scaling property (1.26) was first proposed by Kadanoff [10], who argued semiphenomenologically that not only thermodynamic potentials but also the correlation functions should be generalized homogeneous functions of two variables. For these he deduced the general form for $t \gtrsim 0$

$$G(\mathbf{x}) = \frac{f(\mathbf{x}/|t|^{-\nu})}{r^{D-2+\eta}}, \qquad r \equiv |\mathbf{x}|, \tag{1.28}$$

thus combining the properties (1.8), (1.10), and (1.11) in a single expression.

Such considerations showed that only two of the thermodynamic critical exponents (1.16)–(1.20) are independent of each other. There are two *scaling relations* between them:

$$\alpha + \beta(1+\delta) = 2, \qquad \text{(Griffith)} \tag{1.29}$$

$$\gamma + \beta(1-\delta) = 0. \qquad \text{(Widom)} \tag{1.30}$$

A combination of these is

$$\alpha + 2\beta + \gamma = 2. \qquad \text{(Rushbrook)} \tag{1.31}$$

Moreover, the two independent thermodynamic critical exponents are directly related to the two critical exponents η and ν which parametrized the power behavior of the correlation function in (1.28). The relations are:

$$\alpha = 2 - \nu D, \tag{1.32}$$

$$\beta = \frac{\nu}{2}(D - 2 + \eta), \tag{1.33}$$

$$\gamma = \nu(2 - \eta), \tag{1.34}$$

$$\delta = \frac{D + 2 - \eta}{D - 2 + \eta}. \tag{1.35}$$

The scaling relations (1.32), (1.33), and (1.35) involving the dimension D are referred to as *hyperscaling relations*.

These relations are satisfied in all exactly solvable models (for instance, the two-dimensional Ising model, and the spherical model for $D = 1, 2, 3$, both to be described in more detail in the next section) and by computer simulations of nonsolvable models. They are also confirmed by experimental observations of critical properties in many physical systems described by these models.

It is instructive to derive one of the scaling relations (1.32)–(1.35) between thermodynamic critical exponents and those of the correlation functions, for example (1.34). The susceptibility tensor at zero wave vector and external field in the normal phase is obtained from the correlation function (1.4) by the spatial integral

$$\chi_{ij} \propto \int d^D x \, \langle \phi_i(\mathbf{x}) \phi_j(\mathbf{y}) \rangle = \delta_{ij} \chi = \delta_{ij} \int d^D x \, G_c(\mathbf{x} - \mathbf{y}). \tag{1.36}$$

Inserting the general scaling form (1.8) with (1.9), we obtain

$$\chi \propto \int d^D x \frac{1}{r^{D-2+\eta}} e^{-r/\xi} \propto \xi^{2-\eta} \propto |T - T_c|^{-\nu(2-\eta)}. \tag{1.37}$$

Comparison with (1.17) yields the scaling relation (1.34). Similar derivation holds for $T < T_c$ and the longitudinal susceptibility χ_L defined in (1.15).

1.3 Models for Critical Behavior

In order to describe correctly a second-order phase transition, a model must have the following properties. At T_c, fluctuations must occur on arbitrarily large length scales. In the neighborhood of T_c, a correlation length which diverges for $T \to T_c$ should be the only relevant length scale of the system, making the correlation functions of the model at the critical point scale independent. The experimentally observed scaling forms (1.26) and (1.28) of the thermodynamic functions and the correlation functions should be reproduced.

1.3.1 Landau Theory

The construction of field-theoretic models is guided by the existence of successful self-consistent *molecular-field theories* for a great variety of many-body systems. In these theories, the interacting system is approximated by a noninteracting system subject to the influence of a self-consistent external field, whose temperature dependence is determined by an averaging procedure over molecular properties of the system. The prototype for a mean-field approximation is the Weiss theory for ferromagnetism. Another well-known example is the Van der Waals theory of the fluid-gas transitions.

Existing mean-field theories of second-order phase transitions are unified in *Landau's theory of phase transitions* [11]. The Gibbs free energy $\Gamma(T, \mu, V, \mathbf{M})$ is assumed to depend on the temperature T, the chemical potential μ, the volume V, and some order parameter $\mathbf{M}$, which has in general several components. The energy is invariant under some group of linear transformations among these components, depending on the symmetry of the system, which for simplicity will be assumed to be isotropic. The symmetry restricts the form of the different powers of the multi-component order parameter. For a vector $\mathbf{M}$, for instance, rotational symmetry does not permit a cubic term $\mathbf{M}^3$. The power series expansion is truncated after the fourth power in $\mathbf{M}$, whose coefficient is assumed to be positive, to guarantee stability.

It is then easy to show that if symmetry forbids the existence of a cubic term, the transition can only be of second order. A cubic term would change the transition to first order. In a magnetic system with rotational symmetry, where the order parameter is the magnetization vector $\mathbf{M}$, the Landau expansion of the Gibbs free energy has the general form

$$\Gamma(T, \mu, V, \mathbf{M}) = V \left[\frac{A_2}{2} \mathbf{M}^2 + \frac{A_4}{4!} \left(\mathbf{M}^2 \right)^2 \right], \tag{1.38}$$

where V is the volume of the system. The coefficients A_2, A_4 depend only on T and μ. Minimizing $\Gamma(T, \mu, V, \mathbf{M})$ with respect to $\mathbf{M}$ gives two possible minima, in which the vector $\mathbf{M}$ has an arbitrary direction and a size $M = |\mathbf{M}|$ given by

$$M = 0, \qquad \text{for} \quad A_2 \geq 0, \quad A_4 > 0, \tag{1.39}$$

$$M = \sqrt{-6A_2/A_4}, \qquad \text{for} \quad A_2 < 0, \quad A_4 > 0. \tag{1.40}$$

A phase transition occurs if A_2 changes its sign. We identify this place with the critical point by setting near the zero

$$A_2 \equiv a_2 \left(\frac{T}{T_c^{\mathrm{MF}}} - 1 \right) \equiv a_2 \, \tau, \tag{1.41}$$

and approximating a_2 and A_4 by a constant near T_c^{MF}. The relative temperature distance from the mean-field critical temperature T_c^{MF} is denoted by τ, in contrast to the similar quantity

with the true critical temperature T_c defined in Eq. (1.27). From Eqs. (1.40) and (1.41) we obtain the temperature behavior

$$M \propto \tau^{1/2} \quad \text{for} \quad T < T_c^{\rm MF}, \tag{1.42}$$

corresponding to a critical exponent $\beta = 1/2$ in Eq. (1.19) within the Landau theory.

At the minimum, the Gibbs free energy (1.38) is

$$\Gamma_{\rm min} = \begin{cases} 0 & \text{for} \quad T \geq T_c^{\rm MF}, \\ -V\dfrac{3a_2^2}{2A_4}\tau^2 & \text{for} \quad T < T_c^{\rm MF}. \end{cases} \tag{1.43}$$

The energy below $T_c^{\rm MF}$ is the so-called *condensation energy* F_s. From this we deduce that the specific heat C looks like a step function $\Theta(t)$ in the immediate neighborhood of $T_c^{\rm MF}$. This determines the critical exponent α in Eq. (1.16) as being zero in this *Landau approximation*, for the approach to $T_c^{\rm MF}$ from above as well as from below. It also shows that the proportionality constants in Eq. (1.16) are different in these two approaches, with a well-defined amplitude ratio.

The magnetic susceptibility tensor χ_{ij} of the system with zero wave vector $\mathbf{k}$ is found from the Gibbs free energy as follows. First we add to the energy (1.38) an interaction term with an external magnetic field $\mathbf{B}$:

$$\Gamma_{\rm ext} = -V\mathbf{M}\cdot\mathbf{B}. \tag{1.44}$$

The total Gibbs free energy

$$\Gamma + \Gamma_{\rm ext} = V\left[\frac{A_2}{2}\mathbf{M}^2 + \frac{A_4}{4!}\left(\mathbf{M}^2\right)^2 - \mathbf{M}\cdot\mathbf{B}\right] \tag{1.45}$$

must now be minimized in the presence of $\mathbf{B}$ to obtain the magnetization $\mathbf{M}$ produced by the magnetic field. Its direction is parallel to $\mathbf{B}$ and its magnitude $M(B)$ is given by the solution of the cubic equation

$$A_2 M + \frac{A_4}{6}M^3 - B = 0, \tag{1.46}$$

where $B = |\mathbf{B}|$. At the critical point where $A_2 = 0$, we have $M^3 \propto B$, so that the critical exponent of the relation (1.20) is determined to be $\delta = 3$.

The susceptibility tensor $\chi_{ij}(\mathbf{M})$ at zero wave vector and finite magnetization is given by the derivative matrix $\partial M_i/\partial B_j$, which is the inverse of the second-derivative matrix of the Gibbs free energy

$$\chi_{ij}^{-1}(\mathbf{M}) = \frac{\delta B_i}{\delta M_j} = \frac{\delta^2\Gamma[\mathbf{M}]}{\delta M_i \delta M_j}. \tag{1.47}$$

In the ordered phase, the susceptibility is anisotropic, and has the same decomposition into longitudinal and transverse parts with respect to the direction of $\mathbf{M}$ as $G_{{\rm c}\,ij}$ in Eq. (1.7):

$$\chi_{ij}(\mathbf{M}) = \chi_L(M)\frac{M_i M_j}{M^2} + \chi_T(M)\left(\delta_{ij} - \frac{M_i M_j}{M^2}\right). \tag{1.48}$$

The longitudinal part is found by differentiating (1.46) with respect to B yielding

$$\left(A_2 + \frac{A_4}{2}M^2\right)\chi_L(B) = 1. \tag{1.49}$$

Setting $B = 0$ and inserting (1.39), (1.40), and (1.41), we obtain the longitudinal susceptibility at zero field

$$\chi_L(0) = \begin{cases} \dfrac{1}{a_2}\tau^{-1} & \text{for } T \geq T_c^{\rm MF}, \\ \dfrac{1}{2a_2}|\tau|^{-1} & \text{for } T < T_c^{\rm MF}. \end{cases} \tag{1.50}$$

The critical exponent γ defined in (1.17) is thus predicted by Landau's theory to be equal to unity. This holds for the approach to $T_c^{\rm MF}$ from above as well as from below. Moreover, the proportionality constants in Eq. (1.17) are different in these two approaches with a well-defined amplitude ratio.

The transverse susceptibility $\chi_T(0)$ is for $T \geq T_c^{\rm MF}$ the same as the longitudinal one, since the system is isotropic at zero magnetic field. Below the critical temperature, however, the nonzero spontaneous magnetization (1.40) breaks the rotational symmetry of the Gibbs free energy (1.38) making $\chi_T(0)$ different from $\chi_L(0)$. In fact, to the lowest order, the magnetization changes by a large amount upon a small transverse variation of the magnetic field, making the transverse susceptibility infinite. This is a manifestation of the general *Nambu-Goldstone theorem* which states that the spontaneous breakdown of a continuous symmetry in a phase transition causes a divergent susceptibility for variations associated with the remaining symmetry.

The critical exponents of the Landau theory

$$\alpha = 0, \quad \beta = 1/2, \quad \gamma = 1, \quad \delta = 3 \tag{1.51}$$

can be shown to satisfy the scaling relations (1.29)–(1.31).

Landau's theory makes no predictions for the exponents η and ν of the correlation functions since the order parameter is assumed to take the same value everywhere. In order to calculate these, the order parameter has to be generalized to a space-dependent order field, as was first done by Ginzburg and Landau. This extension is discussed in Section 1.4.

The Landau values for the critical exponents give only a rough estimate of the size of the critical exponents. They differ qualitatively and quantitatively from experimental values. In particular, they do not distinguish between systems of different symmetry and different spontaneous symmetry breakdown, whereas the experimental critical exponents depend sensitively on these. As we shall see below, this is caused by the neglect of local fluctuations, which become extremely important near the transition temperature.

1.3.2 Classical Heisenberg Model

A famous model which serves to demonstrate the importance of fluctuations is the *classical Heisenberg model* of ferromagnetic systems. The energy is specified on a lattice as a sum over nearest-neighbor interactions between local spin vectors $\mathbf{S}_i$, which are conventionally normalized to unit length,

$$E = -\frac{J}{2}\sum_{\{i,j\}} \mathbf{S}_i \cdot \mathbf{S}_j \quad \text{with } J > 0. \tag{1.52}$$

The index pairs i, j run over all nearest neighbor pairs. The spin vectors $\mathbf{S}_i$ fluctuate and represent the order field. The expectation $\mathbf{M} \equiv \langle \mathbf{S}_i \rangle$ is the magnetization of the system. It is nonzero only at low temperatures. Above a critical temperature T_c, the system is normal with $\mathbf{M} = 0$.

A special case of this model is the famous *Ising model* where the direction of the vector $\mathbf{S}_i$ is restricted to a single axis, pointing parallel or antiparallel to it. The phase transition of magnetic systems with strong anisotropy can be described by this model. Then $\mathbf{S}$ can be replaced by a scalar with positive and negative signs. The order field of this model $\phi(\mathbf{x})$ has only one component $S(\mathbf{x})$. The symmetry which is spontaneously broken in the low-temperature phase is the reflection symmetry $S(\mathbf{x}) \to -S(\mathbf{x})$.

The energy (1.52) can be generalized from three-vectors $\mathbf{S}_i$ to N-dimensional vectors to describe a spontaneous breakdown of O(N) symmetry.

The partition function associated with the generalized Heisenberg model is

$$Z = \sum_{\text{spin configurations}} e^{-E/k_B T}, \tag{1.53}$$

where the sum runs over all possible spin configurations. In order to specify this sum mathematically, let us rescale the spin vectors to unit length by replacing $\mathbf{S} \to |\mathbf{S}|\mathbf{n}$, and denote each lattice point by a vector $\mathbf{x}$ on a simple cubic lattice, i.e., we let $\mathbf{x}$ take any value

$$\mathbf{x}_{(m_1,m_1,\ldots,m_D)} \equiv \Sigma_{i=1}^{D} m_i \, \mathbf{i}, \tag{1.54}$$

where $\mathbf{i}$ are the basis vectors of a D-dimensional hypercubic lattice, and m_i are integer numbers. The sum over products $\sum_{\{i,j\}} \mathbf{n}_i \mathbf{n}_j$ can then be rewritten as

$$\begin{aligned} \sum_{\{i,j\}} \mathbf{n}_i \mathbf{n}_j &= \sum_{\{i,j\}} [\mathbf{n}_i(\mathbf{n}_j - \mathbf{n}_i) + 1] = 2 \sum_{\mathbf{x}} \sum_{\mathbf{i}} \{\mathbf{n}(\mathbf{x})[\mathbf{n}(\mathbf{x}+\mathbf{i}) - \mathbf{n}(\mathbf{x})] + 1\} \\ &= \sum_{\mathbf{x}} \left\{ - \sum_{\mathbf{i}} [\mathbf{n}(\mathbf{x}+\mathbf{i}) - \mathbf{n}(\mathbf{x})]^2 + 2D \right\}. \end{aligned} \tag{1.55}$$

We then introduce the lattice gradients

$$\nabla_{\mathbf{i}} \, \mathbf{n}(\mathbf{x}) \equiv \frac{1}{a} \left[\mathbf{n}(\mathbf{x}+\mathbf{i}) - \mathbf{n}(\mathbf{x}) \right], \tag{1.56}$$

$$\overline{\nabla}_{\mathbf{i}} \, \mathbf{n}(\mathbf{x}) \equiv \frac{1}{a} \left[\mathbf{n}(\mathbf{x}) - \mathbf{n}(\mathbf{x}-\mathbf{i}) \right], \tag{1.57}$$

where a is the lattice spacing, and rewrite (1.55) as

$$\sum_{\{i,j\}} \mathbf{n}_i \mathbf{n}_j = \sum_{\mathbf{x}} \left\{ -a^2 \sum_{\mathbf{i}} [\nabla_{\mathbf{i}} \, \mathbf{n}(\mathbf{x})]^2 + 2D \right\}. \tag{1.58}$$

We can now use the lattice version of partial integration [12]

$$\sum_{\mathbf{x}} \sum_{\mathbf{i}} \nabla_{\mathbf{i}} \, f(\mathbf{x}) g(\mathbf{x}) = - \sum_{\mathbf{x}} \sum_{\mathbf{i}} f(\mathbf{x}) \overline{\nabla}_{\mathbf{i}} \, g(\mathbf{x}), \tag{1.59}$$

which is valid for all lattice functions with periodic boundary conditions, to express (1.58) in terms of a lattice version of the Laplace operator $\overline{\nabla}_{\mathbf{i}} \nabla_{\mathbf{i}}$, where repeated lattice unit vectors are summed. Then

$$\sum_{\{i,j\}} \mathbf{n}_i \mathbf{n}_j = \sum_{\mathbf{x}} \left\{ a^2 \, \mathbf{n}(\mathbf{x}) \overline{\nabla}_{\mathbf{i}} \nabla_{\mathbf{i}} \, \mathbf{n}(\mathbf{x}) + 2D \right\}. \tag{1.60}$$

Ignoring the irrelevant constant term $\Sigma_{\mathbf{x}} 2D$, we can write the sum over all spin configurations in the partition function (1.53) as a product of integrals over a unit sphere at each lattice point

$$Z = \prod_{\mathbf{x}} \left[\int \mathbf{n}(\mathbf{x}) \right] \exp \left\{ \frac{1}{2} \sigma a^2 \sum_{\mathbf{x}} \mathbf{n}(\mathbf{x}) \overline{\nabla}_{\mathbf{i}} \nabla_{\mathbf{i}} \mathbf{n}(\mathbf{x}) \right\} \tag{1.61}$$

where we have introduced the dimensionless parameter

$$\sigma \equiv \frac{J}{k_B T}. \tag{1.62}$$

This parameter characterizes how easily the direction of the unit vectors can be turned in the opposite direction at a certain temperature. It will be called the *stiffness* of the directional fluctuations.

It is easy to estimate the temperature at which this partition function has a phase transition. For this we liberate the vectors $\mathbf{n}(\mathbf{x})$ from lying on a unit sphere, allowing them to run through the entire N-dimensional space, calling them $\boldsymbol{\phi}(\mathbf{x})$. To keep the system unchanged, we enforce the unit length with the help of Lagrange multipliers $\lambda(\mathbf{x})$. Thus we rewrite the partition function (1.61) as

$$Z = \prod_{\mathbf{x}} \left[\int \boldsymbol{\phi}(\mathbf{x}) \int \lambda(\mathbf{x}) \right] \exp \left(-\frac{1}{2} \sigma a^2 \sum_{\mathbf{x}} \left\{ -\boldsymbol{\phi}(\mathbf{x}) \overline{\nabla}_{\mathbf{i}} \nabla_{\mathbf{i}} \boldsymbol{\phi}(\mathbf{x}) + \lambda(\mathbf{x}) [\boldsymbol{\phi}^2(\mathbf{x}) - 1] \right\} \right). \tag{1.63}$$

A continuous field formulation of this generalized Heisenberg model with the same critical properties has the field energy

$$E[\boldsymbol{\phi}, \lambda] = \frac{1}{2} \int d^D x \left\{ \partial_i \boldsymbol{\phi}(\mathbf{x}) \partial_i \boldsymbol{\phi}(\mathbf{x}) + a^2 \lambda(\mathbf{x}) \left[\boldsymbol{\phi}^2(\mathbf{x}) - 1 \right] \right\}. \tag{1.64}$$

The integrals in the lattice partition function (1.63) are now Gaussian and can be performed using the generalization of the integral formula

$$\int d\xi e^{-a\xi^2/2} = \sqrt{\frac{2\pi}{a}} \tag{1.65}$$

for positive c-numbers a to symmetric positive $L \times L$ matrices $A_{\alpha\beta}$:

$$\prod_{\alpha} \left[\int d\xi_\alpha \right] e^{-\xi_\alpha A_{\alpha\beta} \xi_\beta / 2} = \frac{\sqrt{2\pi}^L}{\sqrt{\det A}}, \tag{1.66}$$

where $\det A$ is the determinant of the matrix A. With the help of the well-known matrix formula

$$\det A = e^{\mathrm{tr} \log A}, \tag{1.67}$$

where tr denotes the trace of the matrix A, formula (1.66) can be rewritten as

$$\prod_{\alpha} \left[\int d\xi_\alpha \right] e^{-\xi_\alpha A_{\alpha\beta} \xi_\beta / 2} = \sqrt{2\pi}^L e^{-\mathrm{tr} \log A/2}. \tag{1.68}$$

Since the lattice Laplacian $\overline{\nabla}_{\mathbf{i}} \nabla_{\mathbf{i}}$ is a symmetric positive matrix, we can express (1.63) in the form

$$Z = \mathrm{const} \times \prod_{\mathbf{x}} \left[\int \lambda(\mathbf{x}) \right] \exp \left[-\frac{N}{2} \mathrm{tr} \log \left[-\overline{\nabla}_{\mathbf{i}} \nabla_{\mathbf{i}} + \lambda(\mathbf{x}) \right] + \frac{\sigma a^2}{2} \sum_{\mathbf{x}} \lambda(\mathbf{x}) \right]. \tag{1.69}$$

In the limit of large N, the integrals over $\lambda(\mathbf{x})$ in this partition function can be performed exactly. The generalized Heisenberg model is then referred to as the *spherical model*.

The integration is possible since for large N, the integrals can be treated by the saddle point approximation. For any function $f(\xi)$ with a single smooth extremum at ξ_m, this approximation is

$$\int d\xi\, e^{-Nf(\xi)} \approx \sqrt{\frac{2\pi}{Nf''(\xi_m)}} e^{-Nf(\xi_m)}. \tag{1.70}$$

For the functional in the exponent of (1.69), the extremum is given by a constant $\lambda(\mathbf{x}) \equiv \lambda$ satisfying the so-called gap equation [13]

$$N v^D_{a^2\lambda}(\mathbf{0}) = \sigma, \tag{1.71}$$

where

$$v^D_{a^2\lambda}(\mathbf{x}) \equiv a^{-2}\Big[(-\overline{\nabla}_{\mathbf{i}}\nabla_{\mathbf{i}}) + a^2\lambda(\mathbf{x})\Big]^{-1}(\mathbf{x}) \tag{1.72}$$

is the dimensionless lattice version of the Yukawa potential in D dimensions

$$V^D_{m^2}(\mathbf{x}) = \int \frac{d^Dk}{(2\pi)^D} \frac{e^{i\mathbf{kx}}}{\mathbf{k}^2 + m^2}. \tag{1.73}$$

The lattice potential has the Fourier representation

$$v^D_{a^2\lambda}(\mathbf{x}) = \prod_i \left[\int_{-\pi/a}^{\pi/a} \frac{d(ak_i)}{2\pi}\right] \frac{e^{i\mathbf{kx}}}{\sum_{i=1}^D (2 - 2\cos ak_i) + a^2\lambda}. \tag{1.74}$$

The denominator can be rewritten as $\int_0^\infty ds\, e^{-s\left[\sum_{i=1}^D (2-2\cos ak_i) + a^2\lambda\right]}$, leading to the multiple integral

$$v^D_{a^2\lambda}(\mathbf{0}) = \int_0^\infty ds e^{-s(a^2\lambda+2D)} \prod_{i=1}^D \left[\int_{-\pi}^{\pi} \frac{d\kappa_i}{2\pi} e^{2s\cos\kappa_i}\right]. \tag{1.75}$$

The integrations over k_i can now easily be performed, and we obtain the integral representation

$$v^D_{a^2\lambda}(\mathbf{0}) = \int_0^\infty ds e^{-s(a^2\lambda+2D)} [I_0(2s)]^D, \tag{1.76}$$

where $I_0(2s)$ is the modified Bessel function. Integrating this numerically, we find for $D = 3, 4, \ldots, \ldots$ the values shown in Table 1.1 [14]. A power series expansion of the Dth power of

TABLE 1.1 Values of lattice Yukawa potential $v^D_{l^2}(\mathbf{0})$ of reduced mass l^2 at origin for different dimensions and l^2. The lower entries show the approximate values from the hopping expansion (1.78).

D	$v^D_0(\mathbf{0})$	$v^D_1(\mathbf{0})$	$v^D_2(\mathbf{0})$	$v^D_3(\mathbf{0})$	$v^D_4(\mathbf{0})$
3	0.2527	0.1710	0.1410	0.1214	0.1071
	0.2171	0.1691	0.1407	0.1214	0.1071
4	0.1549	0.1271	0.1105	0.0983	0.0888
	0.1496	0.1265	0.1104	0.0983	0.0888

the modified Bessel function in (1.76),

$$[I_0(2s)]^D = 1 + Ds^2 + D(2D-1)\frac{s^4}{4} + D(6D^2 - 9D + 4)\frac{s^3}{36} + \ldots\,, \tag{1.77}$$

leads to the so-called *hopping expansion* for $v^D_{a^2\lambda}(\mathbf{0})$:

$$v^D_{a^2\lambda}(\mathbf{0}) = \frac{1}{2D+a^2\lambda} + \frac{2D}{(2D+a^2\lambda)^3} + \frac{6D(2D-1)}{(2D+a^2\lambda)^5} + \frac{20D(6D^2-9\ D+4)}{(2D+a^2\lambda)^7} + \mathcal{O}(D^{-9}), \quad (1.78)$$

which converges rapidly for large D, and yields for $D = 3, 4$ to the approximate values shown in the lower entries of Table 1.1 . They lie quite close to the exact values in the upper part.

The lattice potential at the origin $v^D_{a^2\lambda}(\mathbf{0})$ in the gap equation (1.71) is always smaller than the massless potential $v^D_0(\mathbf{0})$. A nonzero value for λ can therefore only be found for sufficiently small values of the stiffness σ, i.e., for sufficiently high temperatures T [recall (1.62)]. The temperature T_c at which the gap equation (1.71) has the solution $\lambda = 0$ determines the Curie point. Thus we have

$$T_c = \frac{J}{\sigma_c k_B}, \quad (1.79)$$

where σ_c is the critical stiffness

$$\sigma_c = N v^D_0(\mathbf{0}). \quad (1.80)$$

This result, derived for large N, turns out to be amazingly accurate even for rather small N. As an important example, take $N = 2$ where the model consists of planar spins and is referred to as XY-model. For $D = 3$, it describes accurately the critical behavior of the superfluid transition in helium near the λ-transition. From the approximation (1.80) and the value $v^D_0(\mathbf{0}) \approx 0.2527$ in Table 1.1 we estimate

$$\sigma_c \approx 0.5054. \quad (1.81)$$

In Monte-Carlo simulation of this model one obtains, on the other hand [15],

$$\sigma_c \approx 0.45. \quad (1.82)$$

Thus, in three dimensions, we can use the large-N result (1.80) practically for all $N \geq 2$.

Apart from the spherical model in all dimensions, there also exist exact results also for the two-dimensional Ising model. In general, only approximate results are available. They are obtained from analyzing series expansions in powers of the temperature or the inverse temperature (low- and high-temperature expansions, respectively). Another tool to obtain approximate results are computer simulations using so-called Monte Carlo techniques.

Critical exponents of the different models are listed in Table 1.2 . The differences show that the critical exponents depend sensitively on the internal symmetry of the order parameter, apart from the space dimension. By studying these models on different types of lattices, one finds that the critical exponents are *not* influenced by the symmetry group of the lattice nor by any microscopic property of the system.

1.4 Fluctuating Fields

The long-range character of critical fluctuations and the universality of the critical properties suggest that it is possible to calculate these exponents by a phenomenological field theory rather than a microscopic model. One may neglect the lattice completely and describe the partition function of the system near T_c^{MF} by a functional integral over a continuous local order field $\boldsymbol{\phi}(\mathbf{x})$. The order field may be considered as an average of the localized magnetic moments of some lattice model over a few lattice spacings. The energy functional is assumed to have a Taylor expansion in the order field $\boldsymbol{\phi}(\mathbf{x})$ and in its gradients. The lowest gradient energy accounts for the nearest-neighbor interactions. It has the effect of suppressing short-wavelength fluctuations.

Exponent	Landau	Ising $D=2$	Ising $D=3$	Helium II $D=3$	Heisenberg $D=3$	Spherical $D=3$
α	0 (disc)	0 (log)	0.1097 (.0012)	-0.011 (.004)	-0.122 (.009)	-1
β	1/2	1/8	0.3258(.0014)	0.3470(.0014)	0.3662(.0025)	1/2
γ	1	7/4	1.2378(.0006)	1.3178(.0010)	1.3926(.0010)	2
δ	3	15	4.8055(.0140)	4.7950(.0140)	4.7943(.0140)	5
η	0	1/4	0.0355(.0009)	0.0377(.0006)	0.0374(.0004)	0
ν	1/2	1	0.6301(.0005)	0.6715(.0007)	0.7096(.0008)	1

TABLE 1.2 Critical exponents. The numbers in the Landau column of the last two rows are derived in Section 1.4 after introducing gradient terms into the Gibbs free energy as proposed by Ginzburg and Landau. For the remaining columns see Chapters 17, 19, 20, and **??**, in particular Table 20.2 and Eq. (20.97). The numbers in parentheses are error estimates.

1.4.1 Ginzburg-Landau Energy Functional

The symmetry group of the high-temperature phase limits the possible interaction terms in the energy functional $E[\boldsymbol{\phi}]$. In a ferromagnet, the order parameter is a real scalar three-component field. Reflection symmetry $\boldsymbol{\phi} \to -\boldsymbol{\phi}$ eliminates all terms with odd powers. The restriction to the vicinity of the critical point eliminates all but three terms:

$$E[\boldsymbol{\phi}] = \int d^D x \Big\{ \frac{A_1}{2}\, \partial_i \boldsymbol{\phi}(\mathbf{x}) \partial_i \boldsymbol{\phi}(\mathbf{x}) + \frac{A_2}{2} \boldsymbol{\phi}^2(\mathbf{x}) + \frac{A_4}{4!} [\boldsymbol{\phi}^2(\mathbf{x})]^2 \Big\}. \tag{1.83}$$

This is the famous *Ginzburg-Landau energy functional*. It differs from Landau's expansion of the Gibbs free energy (1.38) by the first gradient term (apart from the more general notation). If the system is symmetric only under a subgroup of the group $\mathrm{O}(N)$, there may be more terms. An important case where the symmetry is only hypercubic (cubic for $N = 3$) will be considered in Chapter 18.

The parameters A_1 and A_4 in the expansion (1.83) depend weakly on the temperature, whereas A_2 vanishes at the critical temperature T_c^{MF} according to (1.41), with a weakly temperature-dependent coefficient a_2, as in the expansion (1.38). An approximate energy functional of the form (1.83) can be derived by a few simple manipulations from the functional integral representation of the partition function of the classical Heisenberg model (1.52) [16].

The Ginzburg-Landau energy functional (1.83) allows us to derive immediately an approximate critical exponent η governing the power behavior of the correlation function at the critical point [see (1.11)]. There A_2 vanishes according to (1.41). Neglecting the interaction, the connected correlation function (1.6) is given by the inverse of the differential operator between the ϕ-fields, as we shall see later in Eq. (2.34):

$$G_{ij}(\mathbf{x}) = \langle \phi_i(\mathbf{x}) \phi_j(\mathbf{0}) \rangle = \delta_{ij} G(\mathbf{x}) = \delta_{ij} \frac{k_B T}{A_1} \int \frac{d^D k}{(2\pi)^D} \frac{e^{i\mathbf{k}\mathbf{x}}}{\mathbf{k}^2} = \delta_{ij} \frac{k_B T}{A_1} \frac{1}{r^{D-2}}. \tag{1.84}$$

Comparison with (1.11) yields the approximation $\eta = 0$ listed in Table 1.2 .

The exponent ν governing the temperature behavior of the correlation length [see (1.10)] is obtained by neglecting the fourth-order terms in the fields in (1.83), considering only the quadratic part

$$E[\boldsymbol{\phi}] = \int d^D x \Big[\frac{A_1}{2}\, \partial_i \boldsymbol{\phi}(\mathbf{x}) \partial_i \boldsymbol{\phi}(\mathbf{x}) + \frac{A_2}{2} \boldsymbol{\phi}^2(\mathbf{x}) \Big]. \tag{1.85}$$

For $T \geq T_c^{\rm MF}$, this can be rewritten as

$$E[\phi] = \frac{A_2}{2} \int d^D x \Big[\xi^2 \, \partial_i \phi(\mathbf{x}) \partial_i \phi(\mathbf{x}) + \phi^2(\mathbf{x}) \Big], \tag{1.86}$$

where

$$\xi^2 \equiv \frac{A_1}{A_2} = \xi_0^2 \, \tau^{-1} \quad \text{with} \quad \xi_0^2 \equiv \frac{A_1}{a_2} \tag{1.87}$$

defines the characteristic length scale of the fluctuations. The correlation function is now given by

$$G_{ij}(\mathbf{x}) = \langle \phi_i(\mathbf{x}) \phi_j(\mathbf{0}) \rangle = \delta_{ij} \frac{k_B T}{A_1} \int \frac{d^D k}{(2\pi)^D} \frac{e^{i\mathbf{kr}}}{\mathbf{k}^2 + 1/\xi^2} = \text{const} \times \delta_{ij} \frac{e^{-r/\xi}}{r^{D-2}}. \tag{1.88}$$

Thus ξ measures directly the coherence length defined in Eq. (1.9).

From Eq. (1.87) we derive the mean-field approximation for the critical exponent ν as T approaches $T_c^{\rm MF}$ from above. Comparison with (1.10) shows that $\nu = 1/2$, as listed in Table 1.2 .

For $T < T_c^{\rm MF}$, the quadratic term is found by expanding the expectation value $\mathbf{\Phi} = \langle \phi(\mathbf{x}) \rangle$, yielding

$$E[\phi] = |A_2| \int d^D x \Big\{ \xi^2 \, \partial_i \delta\phi(\mathbf{x}) \partial_i \delta\phi(\mathbf{x}) + [\delta\phi(\mathbf{x})]^2 \Big\}, \tag{1.89}$$

where

$$\xi^2 = -\frac{A_1}{2A_2} = \frac{A_1}{2a_2} \, |\tau|^{-1} = \frac{1}{2} \, \xi_0^2 \, |\tau|^{-1}, \quad \text{with} \quad \xi_0^2 \equiv \frac{A_1}{a_2}. \tag{1.90}$$

The *zero-temperature coherence length* in the Ginzburg-Landau theory is

$$\xi(0) = \xi_0 / \sqrt{2}. \tag{1.91}$$

Comparing the temperature behaviors of (1.90) and (1.87) we see that the critical exponents ν are equal for the approach to $T_c^{\rm MF}$ from below and from above. We also see that the proportionality constants in Eq. (1.10) are different in these two approaches.

Below $T_c^{\rm MF}$, the connected correlation function $G_{{\rm c}\,ij}(\mathbf{x}) = \langle \delta\phi_i(\mathbf{x}) \delta\phi_j(\mathbf{0}) \rangle$ has the decomposition (1.7) into a longitudinal part and a transverse part with respect to $\mathbf{\Phi}$. These have the spatial behavior

$$G_{{\rm c}\,L}(\mathbf{x}) = \frac{k_B T}{A_1} \int \frac{d^D k}{(2\pi)^D} \frac{e^{i\mathbf{kr}}}{\mathbf{k}^2 + 1/\xi^2} = \text{const} \times \frac{e^{-r/\xi}}{r^{D-2}}. \tag{1.92}$$

$$G_{{\rm c}\,T}(\mathbf{x}) = \frac{k_B T}{A_1} \int \frac{d^D k}{(2\pi)^D} \frac{e^{i\mathbf{kr}}}{\mathbf{k}^2} = \text{const} \times \frac{1}{r^{D-2}}. \tag{1.93}$$

While the longitudinal part of the connected correlation function below $T_c^{\rm MF}$ has the same r-dependence as the full correlation function above $T_c^{\rm MF}$ in (1.88), except for the different correlation length (1.90) by a factor $\sqrt{2}$, the transverse part has an infinite range. At zero momentum, the longitudinal susceptibility $\chi_L(\mathbf{k})$ defined in (1.15) is proportional to $\xi^2 \propto |t|^{-1}$, so that we find once more the Landau approximation to the critical exponent γ in Eq. (1.17) to have the value $\gamma = 1$, as in (1.51).

Note that the hyperscaling relations (1.32), (1.33), and (1.35) involving the dimension D are fulfilled by the critical exponents of the Ginzburg-Landau theory only for $D = 4$. The special role of this dimension in the ϕ^4-theory will be fully appreciated in Chapter 7.

1.4.2 Ginzburg Criterion

A rough understanding of this special role is already possible at the level of the Ginzburg-Landau theory, as first pointed out by Ginzburg in 1960 [17]. Let us estimate the fluctuations of the order field. For this we separate the fields $\phi_i(x)$ and their expectation values Φ_i into size and direction parts by writing

$$\phi_i(x) = \phi(x)\, n_i(x), \qquad \langle \phi_i \rangle \equiv \Phi_i \equiv \Phi n_i. \tag{1.94}$$

The unit direction vector $\mathbf{n}$ breaks spontaneously the O(N) symmetry, and the decomposition (1.7) of the connected correlation function (1.6) becomes

$$G_{\mathrm{c}\,ij}(\mathbf{x}-\mathbf{y}) = \langle \delta\phi_i(\mathbf{x})\delta\phi_j(\mathbf{y})\rangle = \langle \phi_i(\mathbf{x})\phi_j(\mathbf{y})\rangle - \Phi_i\Phi_j = \frac{n_i n_j}{n^2} G_{\mathrm{c}\,L}(\mathbf{x}) + \left(\delta_{ij} - \frac{n_i n_j}{n^2}\right) G_{\mathrm{c}\,T}(\mathbf{x}). \tag{1.95}$$

Using Eq. (1.92) we now see that the mean square deviation of the field from its expectation value is

$$\langle [\phi(\mathbf{x}) - \Phi]^2 \rangle = \frac{k_B T}{A_1} \int \frac{d^D k}{(2\pi)^D} \frac{1}{\mathbf{k}^2 + 1/\xi^2} = \frac{k_B T}{A_1} V^D_{1/\xi^2}(\mathbf{0}) \tag{1.96}$$

where $V^D_{m^2}(\mathbf{0})$ is the momentum integral

$$V^D_{m^2}(\mathbf{0}) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{\mathbf{k}^2 + m^2} \tag{1.97}$$

which diverges at large k. The integral is equal to the Yukawa potential (1.73) for a particle of mass m, evaluated at the origin.

In order to derive the desired fluctuation information we imagine decomposing the field system into a lattice of cubic patches with an edge length $\xi_l = l\xi$ of the order of the coherence length ξ. One usually assumes size parameters l of the order of unity which ensures an approximate independence of the patches. The fluctuation width (1.96) within such a patch is calculated with the momenta in the integral (1.97) limited by $k_{\max} \approx \pi/\xi_l$. Denoting the surface of a unit sphere in D dimensions by

$$S_D \equiv \frac{2\pi^{D/2}}{\Gamma(D/2)}, \tag{1.98}$$

which will be derived later in Appendix 8A, we obtain the patch version of (1.97):

$$V^D_{1/\xi^2}(\mathbf{0}) \approx \xi_l^{2-D} w^D_{l^2}(\mathbf{0}), \tag{1.99}$$

with the reduced patch version of the Yukawa potential at the origin

$$w^D_{l^2}(\mathbf{0}) \approx \frac{S_D}{(2\pi)^D} \int_0^\pi dq\, q^{D-1} \frac{1}{q^2 + l^2}. \tag{1.100}$$

For $D = 3, 4$, this has the values shown in Table 1.3 . Ginzburg decomposed in three dimensions $1/(k^2+l^2) = 1/k^2 - l^2/k^2(k^2+l^2)$, and rewrote

$$w^3_{l^2}(\mathbf{0}) \approx \frac{S_3}{(2\pi)^3} \frac{1}{l} \left[\pi - \int_0^\pi dq\, \frac{l^2}{q^2 + l^2}\right]. \tag{1.101}$$

TABLE 1.3 Values of reduced patch version $w_{l^2}^D(\mathbf{0})$ of Yukawa potential of mass l^2 at origin for different dimensions and l^2.

D	$w_0^D(\mathbf{0})$	$w_1^D(\mathbf{0})$	$w_2^D(\mathbf{0})$	$w_3^D(\mathbf{0})$	$w_4^D(\mathbf{0})$
3	0.1591	0.0952	0.0863	0.0809	0.0769
4	0.0625	0.0474	0.0439	0.0416	0.0400

The second integral is now convergent and, assuming the lattice spacing parameter l to be equal to unity, Ginzburg approximated it as $\int_0^\pi dk\ 1/(k^2+1) \approx \pi/2$, thus estimating $w_1^3(\mathbf{0}) \approx 1/(2\pi^2) \times \pi/2 \approx 0.0795$, which lies reasonably close to the value 0.0952 in Table 1.3 .

Alternatively, we can approximate the patch version of the Yukawa potential (1.97) by a Yukawa potential on a simple cubic lattice of spacing ξ_l, or equivalently, replace the reduced version $w_{l^2}^D(\mathbf{0})$ of Eq. (1.100) by the reduced lattice potential [compare (1.74)]

$$v_{l^2}^D(\mathbf{0}) = \int_{-\pi}^{\pi} \frac{d^D\kappa}{(2\pi)^D} \frac{1}{\sum_{i=1}^D (2-2\cos\kappa_i) + l^2}. \tag{1.102}$$

In terms of $v_{l^2}^D(\mathbf{0})$, the fluctuation width (1.96) can be written as

$$\langle [\phi(x) - \Phi]^2 \rangle = k_B T\, l^{2-D} (2|\tau|)^{D/2-1} \frac{1}{a_2} \left(\frac{a_2}{A_1}\right)^{D/2} v_{l^2}^D(\mathbf{0}). \tag{1.103}$$

Fluctuations will become important if this width is of the order of $\Phi^2 = 6|\tau|a_2/A_4$. This gives the condition

$$|\tau| \le \tau_G \equiv [l^{2-D} K v_{l^2}^D(\mathbf{0})]^{2/(4-D)} \tag{1.104}$$

where K is the parameter

$$K \equiv 2^{D/2-1} k_B T_c^{\rm MF} \frac{A_4}{6a_2^2} \left(\frac{a_2}{A_1}\right)^{D/2}. \tag{1.105}$$

The right-hand side of Eq. (1.104) defines the *Ginzburg temperature*

$$T_G = T_c^{\rm MF}\,(1-\tau_G)\,, \tag{1.106}$$

above which the mean-field approximation becomes unreliable. The size fluctuations shift the critical temperature below the mean-field critical temperature (*Ginzburg's criterion*).

It is interesting to compare the energy δE_G in a fluctuation pocket of coherence volume ξ^D near the Ginzburg temperature with the thermal energy $k_B T_c^{\rm MF}$. From the condensation energy density (1.43) and the coherence length (1.90), we find

$$\delta E_G = \xi^D \frac{3a_2^2}{2A_4} \tau_G^2 = \frac{1}{2^{2+D/2}} \frac{6a_2^2}{A_4} \left(\frac{A_1}{a_2}\right)^{D/2} \tau_G^{2-D/2}. \tag{1.107}$$

Using (1.105) and (1.104), this yields an energy ratio

$$\frac{\delta E_G}{k_B T_c^{\rm MF}} = \frac{1}{8K} \tau_G^{2-D/2} = \frac{1}{8} v_{l^2}^D(\mathbf{0}) l^{2-D}. \tag{1.108}$$

This ratio is considerably smaller than unity. For instance in three dimensions with $l=1$, the right-hand side is roughly equal to 0.02. Thus, at the Ginzburg temperature, the Boltzmann factor for the fluctuation pockets is close to unity $e^{-\delta E_G/k_B T} \approx e^{-0.02}$. This implies that an

alternative estimate of the Ginzburg temperature from the condition $\delta E_G \approx k_B T_c^{\rm MF}$, which is sometimes found in the literature [18], grossly overestimates the size of τ_G.

We are now in a position to understand the underlying reason for the special role of $D = 4$ dimensions. From Eq. (1.103), and also (1.107), we see that as T approaches $T_c^{\rm MF}$ from below in less than four dimensions, the fluctuations *increase*. In more than four dimensions, on the other hand, they decrease and become irrelevant. For this reason one always observes mean-field behavior in more than four dimensions. This will be seen in more detail in Chapter 10. The dimension $D_u = 4$ is called the *upper critical dimension*.

The length parameter l is a free parameter in this criterium. The above and Landau's estimates are both based on the assumption that $l \approx 1$ which corresponds to the physical situation that the order field $\Phi(\mathbf{x})$ is properly defined only up to a length scale of the order of the coherence length. In an ordinary superconductor, this is indeed the case. The order field describes Cooper pairs of electrons whose wave functions extend over a coherence length, requiring a cutoff related to this size. In modern superconductors, where the phase transition occurs at higher critical temperature of the order of 100 K, however, there is the theoretical possibility that Cooper pairs could have a much smaller diameter than the coherence length [19]. In this case, l would be much smaller than unity. If the Cooper pairs are bound extremely strongly, another effect becomes important: quantum fluctuations begin driving the phase transition. In this limit, the Cooper pairs form an almost free gas of almost point-like bosons which undergo Bose-Einstein condensation [20]. The relevant length scale is then the De Broglie wavelength of thermal motion

$$\lambda = \frac{2\pi\hbar}{\sqrt{2Mk_BT}}, \tag{1.109}$$

where M is the mass of the Cooper pairs.

1.4.3 Kleinert Criterion

The Ginzburg temperature does not tell us the full story about the onset of fluctuations. In an O(N)-symmetric theory, the order field performs not only size fluctuations, but also directional fluctuations. These are of long-range and therefore will be more violent than the size fluctuations. They destroy the order at a much lower temperature than Ginzburg's T_G, as shown by Kleinert [21]. To find the relevant temperature where these become important we ignore size fluctuations and introduce a normalized order field $\mathbf{n} \equiv \boldsymbol{\phi}/\sqrt{\phi^2}$. From Eq. (1.83) we see that this field has a pure gradient energy

$$E_{\mathbf{n}} = \frac{A_1}{2}\boldsymbol{\Phi}^2 \int d^D x\, [\partial_i \mathbf{n}(\mathbf{x})]^2. \tag{1.110}$$

We now proceed in analogy with Ginzburg's analysis and imagine that the field system consists of a simple-cubic lattice of patches of size $\xi_l = l\xi$. Then we can associate with (1.110) the energy of a classical O(N)-symmetric Heisenberg model on a simple cubic lattice with unit spacing

$$E_{\mathbf{n}} = \frac{A_1}{2}\boldsymbol{\Phi}^2 \xi_l^{D-2} \sum_{\{i,j\}} \mathbf{n}_i(\mathbf{n}_j - \mathbf{n}_i) = \frac{A_1}{2}\boldsymbol{\Phi}^2 \xi_l^{D-2} \sum_{\{i,j\}} \mathbf{n}_i\mathbf{n}_j - DA_1\boldsymbol{\Phi}^2\xi_l^{D-2}, \tag{1.111}$$

which is completely equivalent to the energy (1.52) with a parameter J given by

$$J \equiv A_1 \boldsymbol{\Phi}^2\, \xi_l^{D-2}. \tag{1.112}$$

The ratio J/k_BT is the stiffness σ of the directional fluctuations introduced in Eq. (1.62). Inserting here the mean-field temperature behavior of $|\mathbf{\Phi}| = |\mathbf{M}|$ from Eq. (1.40), and the coherence length $\xi = \xi_0/\sqrt{2|\tau|} = \sqrt{A_1/2a_2|\tau|}$ from (1.90), we see that J depends on the temperature as follows:

$$J = \frac{l^{D-2}}{2^{D/2-1}} \frac{6a_2^2}{A_4} \left(\frac{A_1}{a_2}\right)^{D/2} |\tau|^{2-D/2} = k_B T_c^{\mathrm{MF}} \frac{l^{D-2}}{K} |\tau|^{2-D/2} . \tag{1.113}$$

Now, as discussed above in Subsection 1.3.2, the classical Heisenberg model has a phase transition where directional fluctuations disorder the ordered state if J is of the order of k_BT. More specifically, the transition for an O(N)-symmetric energy occurs for $D > 3$ at a critical value J_c which is roughly given by the critical stiffness σ_c in Eq. (1.80).

$$\sigma_c \approx \frac{J_c}{k_B T_c^{MF}} \approx N v_0^D(\mathbf{0}). \tag{1.114}$$

In $D = 3$ dimensions, the exact value of $v_0^D(0)$ gives an estimate for the critical stiffness $\sigma_c \approx 0.2527N$. For the XY-model with $N = 2$, this is equal to 0.5054. Monte-Carlo simulations for $N = 2$, on the other hand, yield $\sigma_c \approx 0.45$ [22].

Thus we see that the directional fluctuations destroy the order in a reduced temperature interval

$$\tau_K \equiv [l^{2-D} K N v_0^D(\mathbf{0})]^{2/(4-D)}. \tag{1.115}$$

This is the *Kleinert criterion*, the directional analog of the Ginzburg criterion in Eq. (1.104). Due to directional fluctuations, the transition occurs roughly at a temperature

$$T_K = T_c^{MF} (1 - \tau_K) \tag{1.116}$$

smaller than T_G (and, of course, much smaller than T_c^{MF}).

In the limit of large N, directional fluctuations are always responsible for the destruction of the ordered state. This explains why the critical exponents of the ϕ^4-theory to be derived in this text and those of the Heisenberg model have the same series expansions in powers of $1/N$ in any dimension $D > 2$.

For finite N, a small coherence length $\xi(0)$ requires a very small coupling strength in the mean-field energy to make directional fluctuations important. Note that in the symmetry-broken phase, a small $A_4 > 0$ implies a deep degenerate minimum with O(N) symmetry. The coupling must be larger than zero to have a potential minimum at $\tau < 0$.

What is a simple experimental signal for the dominance of directional fluctuations? In magnetic systems one measures, in the neighborhood of the critical point, the spontaneous magnetization Φ, the coherence length ξ, and further the longitudinal and transverse parts of the susceptibility defined in Eq. (1.15), all as a function of temperature. From the longitudinal and transverse parts of the susceptibility one finds the correlation functions $G_L(\mathbf{k})$ and $G_T(\mathbf{k})$. From these one may determine the mean-field transition temperature T_c^{MF} by plotting ξ^{-2} or $1/G_L(\mathbf{0})$ versus temperature. In the mean-field regime these are straight lines which intercept the temperature axis at $T = T_c^{\mathrm{MF}}$. We may extract the mean-field parameter (1.105) by plotting any of the combinations

$$K_L(t) \equiv |t|^{2-D/2} \frac{1}{\xi^D} \frac{G_L(\mathbf{0})}{k_B T_c \Phi^2}, \tag{1.117}$$

$$K_T(t) \equiv |t|^{2-D/2} \frac{\mathbf{q}^2}{\xi^{D-2}} \frac{G_T(\mathbf{q})}{k_B T_c \Phi^2}\bigg|_{\mathbf{q}\to 0}, \tag{1.118}$$

$$\bar{K}_L(t) \equiv |t|^{2-D/2} \frac{1}{\xi^{D-2}} \frac{[dG_L^{-1}(\mathbf{q})/d\mathbf{q}^2]^{-1}}{k_B T_c \Phi^2}\bigg|_{\mathbf{q}\to 0}, \tag{1.119}$$

versus $t = T/T_c - 1$. In the mean-field regime, all three combinations determine experimentally the constant K in (1.105), as can easily be verified using Eqs. (1.93), (1.92), (1.90), and (1.40). From these K-values we calculate the reduced temperatures τ_G and τ_K using (1.104) and (1.115). The ratio of the two is

$$\frac{\tau_K}{\tau_G} = \left(\frac{N v_0^D(\mathbf{0})}{v_{l^2}^D(\mathbf{0})}\right)^{2/(4-D)}, \tag{1.120}$$

which in three dimensions with $l = 1$ is roughly $2.26\, N^2$. Thus the temperature T_K of Eq. (1.116), where directional fluctuations destroy the order, always lies *far below* the Ginzburg temperature T_K of Eq. (1.106).

In superfluid helium, we may plot, by analogy with (1.118),

$$K_T(t) \equiv |t|^{2-D/2} \frac{M^2 k_B T_c}{\xi^{D-2} \hbar^2 \rho_s}, \tag{1.121}$$

where M is the atomic mass and ρ_s is the superfluid mass density, which at the mean-field level is defined by writing the gradient energy (1.110) as

$$E_{\mathbf{n}} = \frac{\rho_s}{2k_B T} \frac{\hbar^2}{M^2} \int d^3x\, [\partial \mathbf{n}(\mathbf{x})]^2. \tag{1.122}$$

In the critical regime, the three quantities in (1.118) and (1.121) go universally to zero like $|t|^{2-D/2}$, since $\xi \propto |t|^{-\nu}$, $\chi_L(\mathbf{0}) \approx |t|^{(\eta-2)\nu}$, $q^2 \chi_T(\mathbf{q})|_{\mathbf{q}\to 0} \approx |t|^{\eta\nu}$, $\Phi^2 \approx |t|^{\nu(D-2+\eta)}$, $\rho_s \approx |t|^{(D-2)\nu}$, where $t \equiv T/T_c - 1$, and ν and η are the critical exponents defined above. This dependence follows directly from (1.10), (1.17), (1.18), (1.19), (1.33), and (1.25), and will be derived in Chapter 10.

Experimentally, the superfluid density of bulk helium cannot be fitted by a mean-field approximation [23]. If we nevertheless try to fit roughly a mean-field curve to the superfluid density, we obtain $\rho_s/\rho \approx 2(1 - T/T_c^{\rm MF})$, where $\rho = M/a^3$ is the total mass density, with $a \approx 3.59\,\mathbf{r}$A [24], and $T_c^{\rm MF}$ differs from T_c by only about 5%, such that it may be neglected for the present estimate. The factor $k_B T_c$ at $T_c = 2.18\,$K can be expressed as $k_B T_c \approx 2.35 \hbar^2/Ma^3$. With $\xi(0) \approx 2\,\mathbf{r}$A, Eq. (1.121) yields an estimate for K of about 2. Inserting this into Eqs. (1.104) and (1.115), we find

$$\tau_K \approx 1, \qquad \tau_G \approx 0.12. \tag{1.123}$$

The large size of $\tau_K \approx 1$ reflects the bad quality of a mean-field approximation. The numbers in the estimates (1.123) depend quite sensitively on the choice of the lattice spacing parameter. An increase of l by a factor 2 decreases τ_K by a factor 4, which is an appropriate estimate for τ_K. In any case, we may conclude that the superfluid transition in helium is initiated by size fluctuations, not by directional fluctuations. Still, the immediate neighborhood of the critical point contains both types of fluctuations in a universal way. The above statement concerns only the onset of the critical behavior and the type of fluctuations which drive the system into the critical behavior.

We may define an experimental quantity $X_K(T)$ which exhibits directly the onset of directional fluctuations:

$$X_K(T) = k_B T_c |t|^{D/2-1} \xi^{D-2} \Phi^2 \left.\frac{G_T^{-1}(\mathbf{q})}{\mathbf{q}^2}\right|_{\mathbf{q}\to 0} = k_B T_c |t|^{D/2-1} \xi^{D-2} \Phi^2 \left.\frac{dG_L^{-1}(\mathbf{q})}{d\mathbf{q}^2}\right|_{\mathbf{q}\to 0}, \tag{1.124}$$

or in superfluid helium

$$X_K(T) \;=\; |t|^{D/2-1}\xi^{D-2}\frac{\hbar^2\rho_s}{M^2k_BT_c}. \tag{1.125}$$

In the mean-field regime, this falls off linearly towards the mean-field critical temperature $T_c^{\rm MF}$. Near the critical value σ_c, however, the linear falloff changes into the critical power behavior $|t|^{D/2-1}$.

As in the case of the Ginzburg criterion, the parameter l may differ considerably from unity in extreme cases, such as in some models of high-T_c superconductors. The discussion at the end of the previous subsection applies here as well. The directional fluctuations play a crucial role in pion physics [25].

1.5 General Remarks

In order to properly define a field theory, it is necessary to regularize the short-distance behavior of the system. This may be done with the help of the original lattice of the system or by an auxiliary mathematical one. Another possibility is to work in a continuous spacetime, but assuming all momenta to be confined to a sphere of a large radius Λ, called a *momentum space cutoff*. Such a cutoff supplies a smallest length scale over which the order parameter can vary. Since very short distances become visible only under ultraviolet light, Λ is also called an *ultraviolet cutoff* or *UV cutoff*. A continuous field theory with an ultraviolet cutoff Λ has no ultraviolet divergences.

In quantum field theory, a system described by the above Ginzburg-Landau functional (1.83) is called a ϕ^4-theory. Equipped with different symmetries, these ϕ^4-theories have become the most appealing theoretical tool to study the critical phenomena in a great variety of statistical systems. Their relevance for understanding these phenomena was emphasized by Wilson, and Fisher [26]. Using field-theoretic techniques, it has been possible to understand quite satisfactorily all second-order phase transitions of magnetic systems in Table 1.2 and their generalizations. These explain other important phase transitions. An O(2)-symmetric ϕ^4-theory, for example, has the same critical properties as superfluid ^{4}He at the transition to the normal phase. If the critical exponents of O(N) models are continued analytically to $N = 0$, one obtains the values observed in diluted solutions of *polymers* [27].

The calculation of observable consequences of such a quantum field theory proceeds via perturbation theory. Certain correlation functions are obtained as power series in the coupling strength A_4, which is usually denoted by g. The expansion coefficients receive contributions from a rapidly growing number of *Feynman integrals* which are multiple integrals in energy-momentum space.

Initially, quantum field theory is formulated in a Minkowski space with one time and $D-1$ space dimensions. The associated Feynman integrals have singularities in the energy subintegrals, which can be removed by a rotation of the integration contour in the complex energy plane by 90 degrees. This operation, called *Wick rotation*, makes the integrations run along the imaginary energy axis. In quantum field theory, the singularities in the integral correspond to states in a Hilbert space representing particle-like excitations. The Wick rotation removing these singularities changes the quantum field theory into a theory of statistically fluctuating fields. After the Wick rotation, the Planck constant $\hbar$, which in the quantum field theory controls the size of quantum fluctuations, plays the role of the temperature T governing the size of thermal fluctuations. In applications to statistical physics, $\hbar$ may directly be replaced

by k_BT/E_0, where E_0 is some energy scale. The squared mass m^2 of the particle-like excitations in the original quantum field is proportional to the parameter A_2 in the Ginzburg-Landau functional (1.83). The critical point where thermal fluctuations become violent is characterized by the vanishing of the mass m.

The different types of Feynman integrals are organized most efficiently by means of *Feynman diagrams*. The coefficients of the different powers g^p are associated with Feynman diagrams containing, for a ϕ^4 interaction, a number of closed lines called *loops*. The number of diagrams grows exponentially fast with the power p, like $(p-1)!! \equiv 1 \cdot 3 \cdots (p-1)$.

At the critical point, i.e. for zero mass, a system of fluctuating fields with a quartic self-interaction has the important scaling properties described above. In $D > 4$ dimensions, all exponents have their mean-field values.. The dimension $D = 4$ separating the two types of behavior is called the *critical dimension* of the theory, more precisely the above-defined upper critical dimension $D_u = 4$. In this text, we shall only be concerned with the upper critical dimension. There exists also a *lower critical dimension* $D_l = 2$ which appears in field-theoretic descriptions of the same systems by means of vector fields of unit length with an energy functional (1.69). Expansions around the lower critical dimension are not the subject of this book, although they will be related to the expansions around four dimensions in Section 19.8. We shall use the generic term critical dimension mainly for D_u.

The critical exponents governing the power laws of correlation functions at the critical dimension $D_u = 4$ are different from those observed in nature. It was an important discovery of Wilson and Fisher [26] that the differences could be explained by the difference between the physical space dimension D and the upper critical dimension $D_u = 4$. They found a way to continue the correlation functions analytically from their four-dimensional form to the physical three-dimensional one. During this continuation, they maintained their pure power form of the correlation functions and changed only the numerical values of the critical exponents. The important mathematical tool for this continuation was the calculation of all Feynman integrals in an arbitrary continuous number of dimensions $D = 4 - \varepsilon$, via a power series expansion in ε. The physical exponents are obtained for $\varepsilon = 1$.

Critical phenomena are determined by the long-wavelength fluctuations of a system. As such, they are independent of the short-distance properties of the system. They are indistinguishable for a wide variety of microscopically quite different physical systems. This is the universality property discussed before in Section 1.2.

The initial idea to study critical phenomena of a field theory by Kadanoff [28], employed a repeated application of a so-called *blocking transformation*. It is based on integrating out the fluctuations with short wavelengths while rescaling in a specific way the parameters mass, coupling constant, and field normalization which govern the remaining long-range fluctuations. After a few iterations, the changes stabilize in a fixed point. At this point, the correlation length becomes infinite, corresponding to the limit $m \to 0$, and all correlation functions show a power behavior typical for critical phenomena. Ultimately, this method turned out to be completely equivalent to an application of the so-called *renormalization group* in quantum field theory.

Field-theoretic calculations of critical phenomena in a spacetime continuum have two obstacles. First, all perturbative terms consist of divergent integrals and require a regularization to control the divergences. This obstacle is overcome by the fact that if we limit the interaction to a quartic term in the fields, the theory is renormalizable. This implies that these divergences can be absorbed into a few physical parameters characterizing the theory. They differ from the initial parameters by factors called *renormalization constants*. This removes the first obstacle.

Note that the renormalizability is not of physical relevance for any real many-body system which always possess an intrinsic short-distance scale such as a lattice spacing. The renormalizability is merely of technical advantage enabling us to apply field-theoretic techniques developed for theories in continuous spacetime to the critical phenomena in many-body systems.

The effort in isolating the infinities is useful since the renormalization constants turn out to contain all necessary information on the critical exponents. This comes about as follows. The critical theory near $D = 4$ dimensions can be renormalized only after introducing some arbitrary fixed mass parameter μ. This appears in all correlation functions in conjunction with the physical coupling constant g. The renormalizability of the theory has the consequence that theories with different μ and different g are not completely independent from one another. There exist families of indistinguishable correlation functions characterized by a set of parameters $\mu, g(\mu), m(\mu)$. These families are found by solving the so-called *renormalization group equations*, which are first-order differential equations in the mass parameter μ. These differential equations contain so-called *renormalization group functions* whose power series expansions in ε govern the critical exponents. They are completely determined by the divergences of the Feynman integrals.

The second obstacle in calculating the critical exponents is more serious: Due to the exponentially-fast growing number of Feynman diagrams, the expansion coefficients of the powers g^p increase in size like $(p-1)!!$. This implies that the power series diverge even for very small values of the coupling strength g. Their radius of convergence is zero, and the same thing is true for the series expansion in powers of $\varepsilon = 4 - D$ for the critical exponents.

In order to extract useful information from such expansions, elaborate *resummation methods* have been developed. The progress in the field theory of critical phenomena was possible only by simultaneous progress in resummation theory.

The purpose of the present book is to give an introduction to the techniques of calculating the power series for the critical exponents up to the order ε^5 and resumming them. These series will first be obtained for single quartic self-interaction term with $\mathrm{O}(N)$ symmetry, then for a combination of such terms with $\mathrm{O}(N)$ and cubic symmetry.

As explained above, perturbation theory requires the generation of a great number of diagrams and their weight factors. For higher orders, this can be done reliably only with the help of computer-algebraic calculations. The counting displayed in this text relies on programs developed first in collaboration with J. Neu [29]. These revealed a counting error of the five-loop diagrams in the standard literature. Helpful for the counting process was a unique representation of the diagrams by a matrix with integer-valued elements, and a simple method for extracting the multiplicity of the diagrams from this matrix (see Chapter 14).

The calculation of Feynman integrals corresponding to the various diagrams is performed in dimensional regularization according to the rules of 't Hooft and Veltman [30]. The divergences of the theory are removed by counterterms defined in a so-called *minimal subtraction scheme*. (MS-scheme). The renormalization group functions obtained in this scheme are of maximal simplicity.

The calculation of the renormalization constants requires a recursive subtraction of the divergences of subdiagrams. This is done with a so-called *R-operation* invented by Bogoliubov [31] which works diagram-wise.

In the minimal subtraction scheme, the renormalization constants are independent of mass and external momenta. This allows their calculation via massless integrals with only one external momentum. The external vertices where this momentum enters a diagram may be chosen arbitrarily as long as the infrared behavior of the integral remains unchanged (*IR-rearrangement)*. A Russian group, Chetyrkin, Kataev, and Tkachov [32] developed algorithms

for the reduction of such massless integrals to nested one-loop integrals, which turn out to be expressible in terms of Gamma functions. These algorithms are applicable to only a few generic two- and three-loop diagrams, since they are based on a subdiagram of triangle form. They have so far not been generalized to other forms, such as square diagrams.

A further class of diagrams becomes calculable when transforming the massless, dimensionally regularized integrals into a dual form by Fourier transformation. These dual integrals are solvable or may be reduced to solvable integrals by applying certain reduction algorithms developed in **x**-space by Kazakov [33]. He called this algorithm *method of uniqueness*. We prefer instead the name *method of ideal index constellation*.

Most of the diagrams can be brought to one of the calculable forms by IR-rearrangement. For many diagrams, only one arrangement of the external vertices enables us to calculate the corresponding integral. In some cases, IR-rearrangement is successful only if IR-divergences are taken into account. In dimensional regularization, the IR-divergences manifest themselves in the same way as ultraviolet ones as poles in ε. They can, therefore, be subtracted by a procedure analogous to the R-operation, called the R^*-*operation* [34, 35].

In this way, all integrals up to five loops can be calculated algebraically, with only six exceptions. These require special individual methods combining partial integrations, clever differentiations and applications of the R-operation. For the calculation of all integrals up to five loops, a computer-algebraic program was developed. The methods were available in the literature, but required several corrections.

At the six-loop level, new generic types of diagrams are encountered requiring new methods for their calculation. Subdiagrams of the square type cannot be avoided, and IR-rearrangement is of no help. This is the reason why calculations have not yet been extended to six loops [36].

In the case of fields with several indices and tensorial interactions, each Feynman diagram describes not only a Feynman integral in momentum space, but also a corresponding index sum. This sum leads to certain symmetry factors for each diagram. In this work, a combination of O(N) and cubic symmetry was considered as an interaction, thus allowing the description of many universality classes of critical exponents. For $N=2,3$, the O(N) and the mixed O(N)-cubic symmetry cover all possible symmetries, assuming only one length scale. For $N=4$, there are, in principle, many more symmetries. However, two-loop calculations by Toledano, Michel, and Brézin [37] found only two other universality classes of critical behaviors, besides the isotropic and the cubic one. The five-loop calculation of the integrals presented in this book permits extension of these results to the five-loop level.

The results of the integrals together with the symmetry factors yield the critical exponents of the systems as an expansion in ε up to ε^5. For the combination of O(N) and cubic symmetry, these expansions extend former calculations in $D=4-\varepsilon$ by two orders, and former calculations by Mayer, Sokolov, and Shalayev [38] in fixed dimension $D=3$ by one order.

For a comparison with experiments, the series for the critical exponents have to be evaluated at $\varepsilon=1$. As pointed out before, the series are divergent requiring special techniques for their resummation. The most simple technique, the *Padé approximation*, is the easiest to apply, since it does not require additional information on the series. A more powerful method, however, uses the knowledge of the behavior of all power series at high orders g^k. The series are re-expanded into functions with the same behavior at higher orders. This method can be applied to both, the series in g and the series in ε. It has recently been employed by the present authors [39] for the resummation of series which contain an additional interaction in cubic symmetry, thereby confirming the result from Padé approximations.

The most efficient method for evaluating the perturbation expansions for the critical exponents, however, is variational perturbation theory which has been developed only recently by

one of the authors [40]. This method has led to theoretical values for the critical exponent α whose accuracy matches that of the satellite experiments described above. This method will be described in detail and applied in Chapters 19 and 20.

Amplitude ratios will not be discussed in this book. There are two reasons for this: First, they are much harder to calculate than critical exponents, such that their perturbation expansions have only been carried only up to the third order. Second, experiments do not yet provide us with reliable data which can be compared with the theoretical results. We therefore refer the reader to the literature on this subject [41].

Appendix 1A Correlations and Structure Factor

In Born approximation, the differential cross section of neutron scattering by a liquid whose molecules form an ensemble of scattering potentials $V(\mathbf{x}_j)$, reads

$$\frac{d\sigma}{d\Omega d\omega} = \frac{m^2}{4\pi^2\hbar^4 p}\frac{p'}{p}V(\mathbf{q})S(\mathbf{q},\omega), \qquad \omega = (\mathbf{p}^2 - \mathbf{p}'^2)/2M,$$

where

$$S(\mathbf{q},\omega) \equiv \sum_n e^{-E_n/k_BT}\sum_{n'}\left|\left[\sum_{j=1}^N e^{i\mathbf{q}\mathbf{x}_j/\hbar}\right]_{n'n}\right|^2 \delta\left(\omega + E_{n'} - E_n\right), \qquad p \equiv |\mathbf{p}|,\ p' \equiv |\mathbf{p}'|$$

is the *dynamic structure factor* of the liquid, and $\mathbf{p}, \mathbf{p}'$ are the initial and final neutron momenta. The labels n, n' refer to initial and final wave functions of the liquid, $d\Omega$ the solid angle, $d\omega$ the energy interval of the outgoing neutrons, and $V(\mathbf{q}) \equiv \int d^3x e^{i\mathbf{q}\mathbf{x}/\hbar}V(\mathbf{x})$ is the Fourier transformed interaction potential. In $\mathbf{x}$-space, the structure factor is

$$S(\mathbf{x},t) = \int \frac{d^3q}{(2\pi\hbar)^3}\int \frac{d\omega}{2\pi\hbar}e^{i(\mathbf{q}\mathbf{x}-\omega t)/\hbar}S(\mathbf{q},\omega) = \langle\rho(\mathbf{x},t)\rho(\mathbf{0},0)\rangle.$$

Integrating this over all times yields the (static) structure factor

$$S(\mathbf{x}) \equiv \int \frac{d^3q}{(2\pi\hbar)^3}e^{i\mathbf{q}\mathbf{x}/\hbar}S(\mathbf{q},0). \tag{1A.1}$$

It can be extracted from elastic scattering data and yields direct information on the correlation function $G(\mathbf{x}-\mathbf{y})$.

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