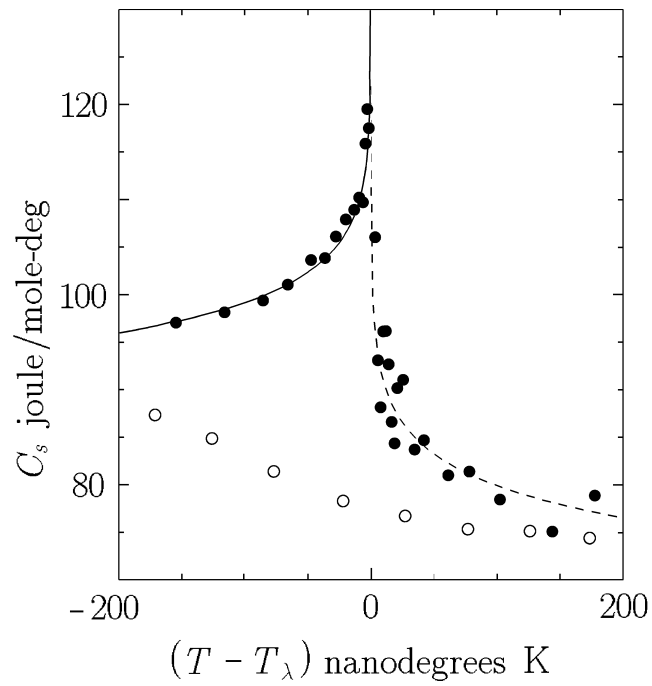


Critical Properties of ϕ^4 -Theories

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Preface

During the past 25 years, field theory has given us much understanding of critical phenomena. Development in this area was extremely rapid and has reached a certain degree of maturity. Perturbative calculations of the critical exponents in $4 - \varepsilon$ dimensions have been carried out to the level of five loops, in 3 dimensions to seven loops, with great effort. The resulting power series diverge and their evaluation requires resummation methods, which have been developed at the same time to a considerable degree of accuracy.

The present monograph started life as lecture notes for a course on quantum field theory delivered regularly by the first author (H.K.) at the Freie Universität Berlin since the early seventies. In 1989, two of his students, J. Neu and the second author (V. S.-F.) attended this course while working on their Master's thesis. Ambitiously, they undertook the arduous task of recalculating the five-loop renormalization constants of $O(N)$ -symmetric ϕ^4 -theories published by K.G. Chetyrkin, S.G. Gorishny, S.A. Larin, and S.V. Tkachov, and discovered several errors.¹ They traveled to Moscow to discuss their findings with the Russian authors who confirmed them after repeated checks. The correct results were subsequently published in a joint paper.² In her Ph.D. thesis, V. S.-F. extended the five-loop calculations to a mixture of interactions with $O(N)$ and cubic symmetry.³ The complete five-loop results are contained in this book.

At present it would be extremely difficult to increase the number of loops in the exact calculations any further without injection of new ideas. We therefore believe it is time to put together the available field-theoretic techniques in this monograph, so that future workers on this subject may profit from it.

We are grateful to J. Neu for many discussions at an early stage in the preparation of this book until 1991. He wrote the computer program for enumerating the Feynman diagrams with the associated weight factors in Chapter 14. More recently, Dr. E. Babaev, Dr. A. Pelster, Dr. Pai-Yi Hsiao, Dr. C. Bevilier contributed with comments.

We would further like to thank Dr. J.A. Gracey for several useful communications on the large- n limits of the critical exponents, and to Drs. Butera and Comi for permission to use their high-temperature expansions in Chapter 20 which we made available as files on the internet pages of this book (<http://www.physik.fu-berlin.de/~kleinert/re.html#b8>).

Most importantly, we are indebted to Dr. B. Kastening for his intensive reading of the book. His corrections and useful suggestions greatly helped improve the final draft. Many printing and stylistic errors were pointed out by Dr. Annemarie Kleinert and by Jeremiah Kwok, our editor of World Scientific Publishing Company.

None of the above persons can, of course, be blamed for the errors introduced in the subsequent correction process.

¹J. Neu, FU-Berlin MS Thesis, 1991, V. Schulte-Frohlinde, FU-Berlin MS Thesis, 1991.

²H. Kleinert, J. Neu, V. Schulte-Frohlinde, K.G. Chetyrkin, S.A. Larin, Phys. Lett. B *272*, 39 (1991) (hep-th/9503230); Erratum ibid. *319*, 545 (1993).

³V. Schulte-Frohlinde, FU-Berlin PhD Thesis, 1996. Results are published in H. Kleinert and V. Schulte-Frohlinde, Phys. Lett. B *342*, 284 (1995) (cond-mat/9503038).

All quoted papers published by our research group in Berlin can be downloaded from the internet, the more recent ones from the Los Alamos server (<http://xxx.lanl.gov/find>), the older ones from our local server (<http://www.physik.fu-berlin.de/~kleinert/re0.html>).

Finally, H.K. thanks his secretary Ms. S. Endrias for her invaluable help in finishing the book.

Hagen Kleinert and Verena Schulte-Frohlinde
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