Rules for Integrals over Products of Distributions from Coordinate Independence of Path Integrals

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Abstract. – In perturbative calculations of quantum-mechanical path integrals in curvilinear coordinates, one encounters Feynman diagrams involving multiple temporal integrals over products of distributions, which are mathematically undefined. In addition, there are terms proportional to powers of Dirac δ-functions at the origin coming from the measure of path integration. We derive simple rules for dealing with such singular terms from the natural requirement of coordinate independence of the path integrals.

Introduction. – While quantum mechanical path integrals in curvilinear coordinates have been defined uniquely and independently of the choice of coordinates within the time-sliced formalism [1], a perturbative definition on a continuous time axis poses severe problems which have been solved only recently [2, 3]. To exhibit the origin of the difficulties, consider the associated partition function calculated for periodic paths on the imaginary-time axis τ:

$$Z = \int \mathcal{D}q(\tau)\, \sqrt{g(q)}\, e^{-\mathcal{A}[q]}, \tag{1}$$

where $\mathcal{A}[q]$ is the euclidean action with the general form

$$\mathcal{A}[q] = \int_0^\beta d\tau \left[\frac{1}{2}\, g_{\mu\nu}(q(\tau))\dot{q}^\mu(\tau)\dot{q}^\nu(\tau) + V(q(\tau))\right]. \tag{2}$$

The dots denote τ-derivatives, $g_{\mu\nu}(q)$ is a metric, and $g = \det g$ its determinant. The path integral is defined perturbatively as follows: The metric $g_{\mu\nu}(q)$ and the potential $V(q)$ are expanded around some point q_0^μ in powers of $\delta q^\mu \equiv q^\mu - q_0^\mu$. After this, the action $\mathcal{A}[q]$ is separated into a free part $\mathcal{A}_0[q_0;\delta q] \equiv \int_0^\beta d\tau\,\left[\frac{1}{2}g_{\mu\nu}(q_0)\dot{q}^\mu\dot{q}^\nu + \frac{1}{2}\omega^2\delta q^\mu\delta q^\nu\right]$, and an interacting part $\mathcal{A}_{\rm int}[q_0;\delta q] \equiv \mathcal{A}[q] - \mathcal{A}_0[q_0;\delta q]$.

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A first problem is encountered in the measure of functional integration in (1). Taking $\sqrt{g(q)}$ into the exponent and expanding in powers of δq, we define an effective action $\mathcal{A}_{\sqrt{g}} = -\frac{1}{2}\delta(0)\int_0^\beta d\tau \log[g(q_0+\delta q)/g(q_0)]$ which contains the infinite quatity $\delta(0)$, the δ-function at the origin. It is a formal representation of the inverse infinitesimal lattice spacing on the time axis, and is equal to the linearly divergent momentum integral $\int dp/(2\pi)$.

The second problem arises in the expansion of Z in powers of the interaction. Performing all Wick contractions, Z is expressed as a sum of loop diagrams. There are interaction terms involving $\delta\dot{q}^2\delta q^n$ which lead to Feynman integrals over products of distributions. The diagrams contain three types of lines representing the correlation functions

$$\Delta(\tau-\tau') \equiv \langle \delta q(\tau)\delta q(\tau')\rangle = \quad , \tag{3}$$

$$\partial_\tau\Delta(\tau-\tau') \equiv \langle \delta \dot{q}(\tau)\delta q(\tau')\rangle = \quad , \tag{4}$$

$$\partial_\tau\partial_{\tau'}\Delta(\tau-\tau') \equiv \langle \delta \dot{q}(\tau)\delta \dot{q}(\tau')\rangle = \quad . \tag{5}$$

The right-hand sides define the line symbols to be used in Feynman diagrams to follow below.

Explictly, the first correlation function reads

$$\Delta(\tau,\tau') = \frac{1}{2\omega}e^{-\omega|\tau-\tau'|}. \tag{6}$$

The second correlation function has a discontinuity

$$\partial_\tau\Delta(\tau,\tau') = -\frac{1}{2}\epsilon(\tau-\tau')e^{-\omega|\tau-\tau'|}, \tag{7}$$

where

$$\epsilon(\tau-\tau') \equiv -1 + 2\int_{-\infty}^{\tau} d\tau''\delta(\tau''-\tau') \tag{8}$$

is a distribution which vanishes at the origin and is equal to ± 1 for positive and negative arguments, respectively. The third correlation function contains a δ-function:

$$\partial_\tau\partial_{\tau'}\Delta(\tau,\tau') = \delta(\tau-\tau') - \frac{\omega}{2}e^{-\omega|\tau-\tau'|}, \tag{9}$$

In mathematics, the temporal integrals over products of such distributions are undefined [4]. The purpose of this paper is to point out that these integrals can be defined uniquely by setting up relations between them and ordinary integrals over products of nonsingular functions, plus integrals over products of the basic distributions $\epsilon(\tau)$, $\delta(\tau)$, or their higher time derivatives. The latter will be specified uniquely by the requirement of coordinate invariance of the path integral (1).

The internal consistency of these definitions is ensured by previous work of the present authors. In Ref. [2], we have shown that Feynman integrals in *momentum space* can be uniquely defined as $\epsilon \to 0$ -limits of $1-\epsilon$ -dimensional integrals via an analytic continuation à la 't Hooft and Veltman [5]. This definition makes path integrals coordinate independent. In Ref. [3] we have given rules for calculating the same results directly from Feynman integrals in a $1-\epsilon$ -dimensional space.

The calculation procedure developed in this paper avoids the cumbersome evaluation of Feynman integrals in $1-\epsilon$ dimensions. In fact, it does not require specifying any regularization scheme. As a fundamental byproduct, it lays the foundation for a new extension of the theory of distributions, in which also integrals over products are defined, not only linear combinations.

Perturbation Expansion. – The envisaged identities will be derived from the requirement of coordinate independence of the exactly solvable path integral of a point particle of unit mass in a harmonic potential $\omega^2 x^2/2$, whose action is

$$\mathcal{A}_\omega = \frac{1}{2}\int_0^\beta d\tau \left[\dot{x}^2(\tau) + \omega^2 x^2(\tau)\right]. \tag{10}$$

For a large imaginary-time interval β, the partition function is given by the path integral

$$Z_\omega = \int \mathcal{D}x(\tau)\, e^{-\mathcal{A}_\omega[x]} = e^{-(1/2)\mathrm{Tr}\log(-\partial^2+\omega^2)} = e^{-\beta\omega/2}. \tag{11}$$

A coordinate transformation turns this into a path integral of the type (1) with a singular perturbation expansion. From our work in Refs. [2, 3] we know that all terms in this expansion vanish in dimensional regularization, thus ensuring the coordinate invariance of the perturbatively defined path integral. In this paper, we proceed in the opposite direction: we *require* the vanishing of all expansion terms to find the desired identities for integrals over products of distributions.

For simplicity we assume the coordinate transformation to preserve the symmetry $x \leftrightarrow -x$ of the initial oscillator, such that its power series expansion starts out like $x(\tau) = f(q(\tau)) = q - gq^3/3 + g^2 a q^5/5 - \cdots$, where g is a smallness parameter, and a some extra parameter. We shall see that the identities are independent of a, such that a will merely serve to check the calculations. The transformation changes the partition function (11) into

$$Z = \int \mathcal{D}q(\tau)\, e^{-\mathcal{A}_J[q]} e^{-\mathcal{A}[q]}, \tag{12}$$

where is $\mathcal{A}[q]$ is the transformed action, whereas $\mathcal{A}_J[q]$ an effective action coming from the Jacobian of the coordinate transformation:

$$\mathcal{A}_J[q] = -\delta(0)\int_0^\beta d\tau\, \log\frac{\delta f(q(\tau))}{\delta q(\tau)}. \tag{13}$$

The transformed action is decomposed into a free part

$$\mathcal{A}_\omega[q] = \frac{1}{2}\int_0^\beta d\tau[\dot{q}^2(\tau) + \omega^2 q^2(\tau)], \tag{14}$$

and an interacting part, which reads to second order in g:

$$\begin{aligned}\mathcal{A}_{\rm int}[q] &= \frac{1}{2}\int_0^\beta d\tau \Bigg\{ -g\left[2\dot{q}^2(\tau)q^2(\tau) + \frac{2\omega^2}{3}q^4(\tau)\right] \\ &+ g^2\left[(1+2a)\,\dot{q}^2(\tau)q^4(\tau) + \omega^2\left(\frac{1}{9}+\frac{2a}{5}\right)q^6(\tau)\right]\Bigg\}.\end{aligned} \tag{15}$$

To the same order in g, the Jacobian action (13) is

$$\mathcal{A}_J[q] = -\delta(0)\int_0^\beta d\tau\left[-gq^2(\tau) + g^2\left(a - \frac{1}{2}\right)q^4(\tau)\right]. \tag{16}$$

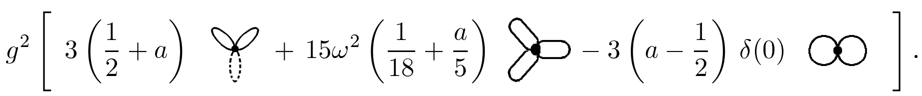
For $g = 0$, the transformed partition function (12) coincides, of course, with (11). When expanding Z of Eq. (12) in powers of g, we obtain Feynman integrals to each order in g, whose sum must vanish to ensure coordinate invariance. By considering only connected Feynman diagrams, we study directly the energy of the ground state energy.

Ground State Energy. – The graphical expansion for the ground state energy will be calculated here only up to three loops. To order g^n, there exist Feynman diagrams with $L = n+1, n$, and $n-1$ number of loops coming from the interaction terms (15) and (16), respectively. The diagrams are composed of the three line types in (3)–(5), and new interaction vertices arising for each power of g. The diagrams coming from the Jacobian action (16) are easily recognized by accompanying factors $\delta^n(0)$.

To first order in g, there exists only three diagrams, two originated from the interaction (15), and one from the Jacobian action (16):

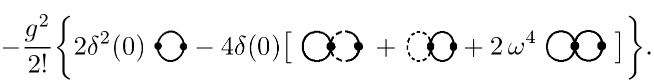
(17)

To order g^2, we distinguish several contributions. First there are two three-loop local diagrams coming from the interaction (15), and one two-loop local diagram from the Jacobian action (16):



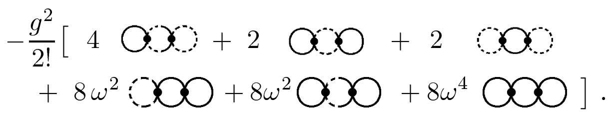
(18)

We call a diagram *local* if it involves no temporal integral. The Jacobian action (16) contributes further the nonlocal diagrams:



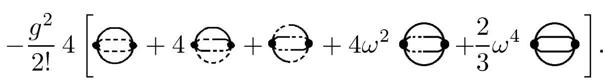
(19)

The remaining diagrams come from the interaction (15) only. They are either of the three-bubble type, or of the watermelon type, each with all possible combinations of the three line types (3)–(5): The sum of all three-bubbles diagrams is



(20)

The watermelon-like diagrams contribute



(21)

Since the equal-time expectation value $\langle \dot{q}(\tau)\, q(\tau)\rangle$ vanishes according to Eq. (7) there are, in addition, a number of trivially vanishing diagrams, which have been omitted.

In our previous work [2,3], all integrals were calculated in $D = 1-\varepsilon$ dimensions, taking the limit $\varepsilon \to 0$ at the end. In this way we confirmed that the sums of all Feynman diagrams contributing to each order in g vanishes. Here we proceed in the opposite direction, deriving from the vanishing of the sums the rules for integrating prodicts of distributions.

Rules for Integrals over Distributions. – As a first step to derive these rules we express integrals containing singular time derivatives $\dot{\Delta}(\tau)$, $\ddot{\Delta}(\tau)$ in terms of integrals over regular correlation functions $\Delta(\tau)$, plus integrals containing fundamental distributions ϵ- and δ-functions.

Most simply, we find by explicit integration for integrals over products of two singular correlation functions the relation

$$\int d\tau\, \left[\dot{\Delta}^2(\tau) + \omega^2 \Delta^2(\tau)\right] = \Delta(0). \tag{22}$$

Well-defined integrals over products of four correlation functions are

$$\int d\tau\, \Delta^4(\tau) = \frac{1}{4\omega^2}\Delta^3(0), \quad \int d\tau\, \dot{\Delta}^2(\tau)\Delta^2(\tau) = \frac{1}{4}\Delta^3(0), \quad \int d\tau\, \dot{\Delta}^4(\tau) = \frac{1}{4}\omega^2\Delta^3(0). \quad (23)$$

We now turn to integrals involving the singular function $\ddot{\Delta}^2(\tau)$. These can be expressed in terms of integrals over regular correlation functions $\Delta(\tau)$, plus integrals containing pure products of ϵ- and δ-functions, using the inhomogeneous field equation satisfied by the correlation function (6):

$$\ddot{\Delta}(\tau) = -\int \bar{d}k\, \frac{k^2}{k^2+\omega^2} e^{ik\tau} = -\,\delta(\tau) + \omega^2\Delta(\tau)\,. \quad (24)$$

With this and (22), we obtain the relation

$$\int d\tau \Big[\ddot{\Delta}^2(\tau) + 2\omega^2\dot{\Delta}^2(\tau) + \omega^4\Delta^2(\tau)\Big] = \int d\tau\, \delta^2(\tau)\,. \quad (25)$$

The last integral is undefined. Its value will be specified in the next section. Before coming to this, however, let us derive relations for integrals over $\ddot{\Delta}(\tau)\dot{\Delta}^2(\tau)\Delta(\tau)$ and $\ddot{\Delta}^2(\tau)\Delta^2(\tau)$. Applying again the field equation (24), we find the relations

$$\int d\tau\, \ddot{\Delta}(\tau)\dot{\Delta}^2(\tau)\Delta(\tau) = -\int d\tau\, \dot{\Delta}^2(\tau)\Delta(\tau)\delta(\tau) + \omega^2 \int d\tau\, \dot{\Delta}^2(\tau)\Delta^2(\tau) \quad (26)$$

and

$$\int d\tau\, \ddot{\Delta}^2(\tau)\Delta^2(\tau) \;=\; \int d\tau\, \Delta^2(\tau)\delta^2(\tau) - 2\omega^2\Delta^3(0) + \omega^4 \int d\tau\, \Delta^4(\tau). \quad (27)$$

Combining these with (23), we obtain

$$\int d\tau\, \ddot{\Delta}(\tau)\dot{\Delta}^2(\tau)\Delta(\tau) = -\frac{1}{8\omega}\int d\tau\, \epsilon^2(\tau)\delta(\tau) + \frac{1}{4}\omega^2\Delta^3(0), \quad (28)$$

$$\int d\tau\, \ddot{\Delta}^2(\tau)\Delta^2(\tau) = \int d\tau\, \Delta^2(\tau)\delta^2(\tau) - \frac{7}{4}\omega^2\Delta^3(0). \quad (29)$$

Relations (25), (28), and (29) reduce all integrals over singular products of correlation functions to regular integrals, plus undefined integrals containing $\delta^2(\tau)$ and $\epsilon^2(\tau)\delta(\tau)$. We are now going to show, that the physically necessary coordinate independence of path integrals yields the following rules for integrals over products of two δ-functions occuring in Eqs. (25) and (29):

$$\int d\tau\, f(\tau)\delta^2(\tau) = f(0)\delta(0), \quad (30)$$

and for integral over product of two ϵ-functions with one δ-function encountered in Eq. (28):

$$\int d\tau\, f(\tau)\epsilon^2(\tau)\delta(\tau) = \frac{1}{4}f(0), \quad (31)$$

for any smooth test function $f(\tau)$.

Imposing Coordinate Independence . – To first order in g, the sum of Feynman diagrams (17) must vanish:

 $+\ \omega^2$  $-\ \delta(0)$  $= 0. \qquad (32)$

The analytic form of this relation is

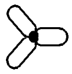
$$\left[-\ddot{\Delta}(0) + \omega^2\Delta(0) - \delta(0)\right]\, \Delta(0) = 0, \tag{33}$$

the zero on the right-hand side being a direct consequence of the equation of motion (24) for the correlation function at origin.

To order g^2, the same equation reduces the sum of all local diagrams in (18) to a finite result plus a term proportional to $\delta(0)$:

$$\left[-3\left(\frac{1}{2}+a\right)\ddot{\Delta}(0)+15\left(\frac{1}{18}+\frac{a}{5}\right)\omega^2\Delta(0) - 3\left(a-\frac{1}{2}\right)\delta(0)\right]\Delta^2(0) = \left[3\delta(0)-\frac{2}{3}\,\omega^2\Delta(0)\right]\Delta^2(0).$$

Representing right-hand side diagrammatically, we obtain the identity

$\Sigma\,(18) = 3\delta(0)$  $-\ \frac{2}{3}\,\omega^2$ $, \qquad (34)$

where $\Sigma\,(18)$ denotes the sum of all diagrams in Eq. (18). Using the identity (22) together with the field equation (24), we reduce the sum (19) of all one and two-loop bubbles diagrams to terms involving $\delta(0)$ and $\delta^2(0)$:

$$-\frac{1}{2!}\left\{2\delta^2(0)\int d\tau\,\Delta^2(\tau) - 4\delta(0)\int d\tau\,\left[\Delta(0)\dot{\Delta}^2(\tau) - \ddot{\Delta}(0)\Delta^2(\tau) + 2\omega^2\Delta(0)\Delta^2(\tau)\right]\right\}$$
$$= 2\Delta^2(0)\,\delta(0) + \delta^2(0)\int d\tau\Delta^2(\tau). \tag{35}$$

Hence we find the diagrammatic identity

$-\frac{1}{2!}\ \Sigma\,(19) = 2\delta(0)$  $+\ \delta^2(0)$  $. \qquad (36)$

Now, the terms accompanying $\delta^2(0)$ turn out to cancel similar terms coming from the sum of all three-loop bubbles diagrams in (20). In fact, the identities (22) and (25) lead to

$$-\frac{1}{2!}\int d\tau\,\left[-4\Delta(0)\ddot{\Delta}(0)\dot{\Delta}^2(\tau) + 2\Delta^2(0)\ddot{\Delta}^2(\tau) + 2\ddot{\Delta}^2(0)\Delta^2(\tau) + 8\omega^2\Delta^2(0)\dot{\Delta}^2(\tau)\right.$$
$$\left.-8\omega^2\Delta(0)\ddot{\Delta}(0)\Delta^2(\tau) + 8\omega^4\Delta^2(0)\Delta^2(\tau)\right] = -\left[\int d\tau\,\delta^2(\tau) + 2\delta(0)\right]\Delta^2(0) - \delta^2(0)\int d\tau\,\Delta^2(\tau).$$

Thus we find the diagrammatic identity for all bubbles diagrams

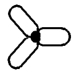
$-\frac{1}{2!}\ \Sigma\,(19) - \frac{1}{2!}\ \Sigma(20) = -\int d\tau\,\delta^2(\tau)$  $. \qquad (37)$

Finally, the relations (23), (23), (23), (28), and (29) reduce the sum (21) of all watermelon-like diagrams to a finite contribution, plus integrals involving $\delta^2(\tau)$ and $\epsilon^2(\tau)\delta(\tau)$:

$$-\frac{4}{2!}\int d\tau\,\left[\Delta^2(\tau)\ddot{\Delta}^2(\tau) + 4\Delta(\tau)\dot{\Delta}^2(\tau)\ddot{\Delta}(\tau) + \dot{\Delta}^4(\tau) + 4\omega^2\Delta^2(\tau)\dot{\Delta}^2(\tau) + \frac{2}{3}\omega^4\Delta^4(\tau)\right]$$
$$= \ -\ 2\int d\tau\,\Delta^2(\tau)\delta^2(\tau) + \frac{1}{\omega}\int d\tau\,\epsilon^2(\tau)\delta(\tau) - \frac{4}{3}\,\omega^2\Delta^3(0)\,. \tag{38}$$

Combining these with all local diagrams (34), we obtain the diagrammatic identity

$$\Sigma\,(18) - \frac{4}{2!}\Sigma(21)$$

$$= \left[3\delta(0) - 2\Delta^{-2}(0)\int d\tau\, \Delta^2(\tau)\delta^2(\tau)\right]$$  $$+\ 2\omega^2\left[4\int d\tau\, \epsilon^2(\tau)\delta(\tau) - 1\right]$$  . (39)

If all terms in Eqs. (37) and (39) are to sum up to zero, as required by coordinate independence, we must have the integration rules for the distributional products (30) and (48), and these determine completely the right-hand sides of relations (25), (29), and (28).

The procedure can be continued to higher-loop diagrams yielding calculation rules for integrals over higher products of ϵ- and δ-function, and their time derivatives.

It is important to realize that at no place do we have to specify the value of $\delta(0)$, or a regularization scheme for the singular integrals. There is perfect cancellation of all powers of $\delta(0)$ arising from the expansion of the Jacobian action, which is the fundamental reason why the heuristic *Veltman rule* of setting $\delta(0) = 0$ can be used everywhere without problems.

Note that in this purely one-dimensional approach we are *not*, in general, allowed to use partial integration for integrals containing ϵ- and δ-function, or their time derivatives. This is possible only for the simplest integrals of this kind. An example is the first integral in the relation (22) which is compatible with partial integration:

$$\int d\tau\, \dot{\Delta}^2(\tau) = -\int d\tau\, \Delta(\tau)\ddot{\Delta}(\tau), \tag{40}$$

as can be verified by inserting the equation of motion (24). The same is true for Eq. (23).

For integrals containing more time derivatives, however, partial integration would lead to inconsistencies. To illustrate this problem consider the integral

$$\int d\tau\, \epsilon^2(\tau)\delta(\tau)e^{-\lambda|\tau|}, \tag{41}$$

with an arbitrary parameter λ. Applying partial integration reduces this to the completly regular form

$$\int d\tau\, \epsilon^2(\tau)\delta(\tau)e^{-\lambda|\tau|} = \frac{\lambda}{6}\int d\tau\, \epsilon^4(\tau)e^{-\lambda|\tau|} = \frac{1}{3}, \tag{42}$$

for any value of λ. For $\lambda = 3\omega$ this yields the relation

$$\int d\tau\, \ddot{\Delta}(\tau)\dot{\Delta}^2(\tau)\Delta(\tau) = -\frac{1}{3}\int d\tau\, \dot{\Delta}^4(\tau). \tag{43}$$

On the other hand, we may use Eqs. (28) and (23) to transform (42) by partial integration to

$$\begin{aligned}\int d\tau\, \epsilon^2(\tau)\delta(\tau)e^{-\lambda|\tau|} &= \int d\tau\, \epsilon(\tau)\delta(\tau)\frac{d}{d\tau}\left[-\frac{1}{\lambda}e^{-\lambda|\tau|}\right] = \\ &= \frac{1}{\lambda}\left[2\int d\tau\, \delta^2(\tau)e^{-\lambda|\tau|} + \int d\tau\, \epsilon(\tau)\dot{\delta}(\tau)e^{-\lambda|\tau|}\right].\end{aligned} \tag{44}$$

This implies the relation

$$\int d\tau\, \ddot{\Delta}(\tau)\dot{\Delta}^2(\tau)\Delta(\tau) = -\frac{1}{2}\int d\tau\, \ddot{\Delta}^2(\tau)\Delta^2(\tau) - \frac{1}{2}\int d\tau\, \frac{d}{d\tau}\left[\ddot{\Delta}(\tau)\right]\dot{\Delta}(\tau)\Delta^2(\tau). \tag{45}$$

Inserting here the equation of motion (24), and using the explicit representations (6) and (7), a comparison with (44) yields

$$\int d\tau\, \epsilon^2(\tau)\delta(\tau)e^{-\lambda|\tau|} = 1. \tag{46}$$

instead of (42). Thus, partial integration is inconsistent in integrals over products of distributions.

From the perspective of our previous papers [2,3] where all integrals were defined in $d = 1-\epsilon$ dimensions and continued to $d \to 1$ at the end, this failure is obvious: Partial integration is forbidden whenever several dots can correspond to different contractions of partial derivatives ∂_α, $\partial_\beta, \ldots$, from which they arise in the $d \to 1$ -limit. The different contractions may lead to different integrals. The simplest example is

$$\int d^d\tau\, \Delta^2(\tau)\Delta^2_{\alpha\alpha}(\tau) - \int d^d\tau\, \Delta^2(\tau)\Delta^2_{\alpha\beta}(\tau) \underset{d\to 1}{\longrightarrow} -\frac{1}{8\omega}. \tag{47}$$

The integrals on the left-hand side are indistinguishable in $d = 1$ dimension. In the present purely one-dimensional approach, this ambiguity is accounted for by abandoning partial integration for integrals containing ϵ- and δ-function, or their time derivatives. The requirement of coordinate invariance, on the other hand, fixes such integrals uniquely. For example, the rule (48) implies the integral

$$\int d\tau\, \epsilon^2(\tau)\delta(\tau) = \frac{1}{4}, \tag{48}$$

in contrast to the inconistent values 1/3 and 1 in the false equation (43) and (46).

Summary. – In this note we have set up simple rules for calculating products of distributions which allow us to evaluate singular Feynman integrals in one dimension, thus avoiding the tedious treatment of dimensionally regularized integrals in $1-\epsilon$ dimensions. The rules follow directly from the physically necessary invariance of perturbatively defined path integral under coordinate transformations. Our procedure works without specifying any regularization scheme. It uses only the equations of motion. In contrast to the $1-\epsilon$ -dimensional calculations in Refs. [2,3], partial integration is not applicable to integrals containing ϵ- and δ-function, or their time derivatives.

Just as in the time-sliced definition of path integrals in curved spcae in Ref. [1], there is absolutely no need for extra compensating potential terms found necessary in the treatments in Refs. [6–8].

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