16

Basic Resummation Theory

The power series in the coupling constant g or in ε derived so far are unsuitable for numeric calculations since they are divergent for any coupling strength. They are called *asymptotic series*. Resummation procedures are necessary to extract physical results. The crudest method to approximate a divergent function $f(g)$ whose first L expansion coefficients are known employs *Padé approximants* [1]. These are rational functions with the same power series expansions as $f(g)$. A better approximation can be found by taking into account the knowledge of the *large-order behavior* of the expansion coefficients. This is provided by a semiclassical analysis of the functional integral (2.6) of the ϕ^4-theory, from which it follows that the coefficients grow factorially with the order. Such a factorial growth can be taken into account by means of *Borel transformations*. The *Padé-Borel method* applies the Padé approximation to the Borel transform. We shall see below that the Borel transform has a left-hand cut in the complex g-plane. The Padé approximation replaces the cut by a string of poles. This approximation can be improved further by a *conformal mapping technique* in which the complex g-plane is mapped into a unit circle which contains the original left-hand cut on its circumference. More efficient resummation techniques are based on re-expansions of the asymptotic truncated series for $f(g)$ in terms of special basis functions $I_n(g)$. These can be chosen in different ways to possess precisely the analytic behavior responsible for the divergence of the original series. Thus the basis functions contain from the outset the necessary information on the large-order behavior of the expansion coefficients, thus containing information beyond the expansion coefficients of the perturbation series. All these methods will be explained in this chapter. The most efficient technique will be applied in Chapters 17 and 18 to calculate critical exponents. A completely different approach to the critical exponents and the resummation problem will be developed in Chapters 19 and 20. There we shall consider the problem of calculating the critical behavior of ϕ^4-theory as a strong-coupling problem in the *bare* coupling constant g_B.

16.1 Asymptotic Series

At large orders, the perturbation coefficients of the renormalization group functions grow like a factorial of the order, with alternating signs. This behavior has a simple origin. If we plot the potential part of the energy density in (2.1) for small negative values of the coupling constant g as shown in Fig. 16.1, we see that the fluctuations around the origin are metastable, since they can carry the field with a nonzero probability all across the barrier into the energy abyss to the right or the left. For very high barriers, the probability of such an event is given by the Boltzmann factor associated with a spherically symmetric classical solution to the field equation:

$$\frac{\delta}{\delta\phi}E[\phi] = (-\partial_{\mathbf{x}}^2 + m^2)\phi^2(\mathbf{x}) + \frac{g}{3!}\phi^3(\mathbf{x}) = 0\,. \tag{16.1}$$

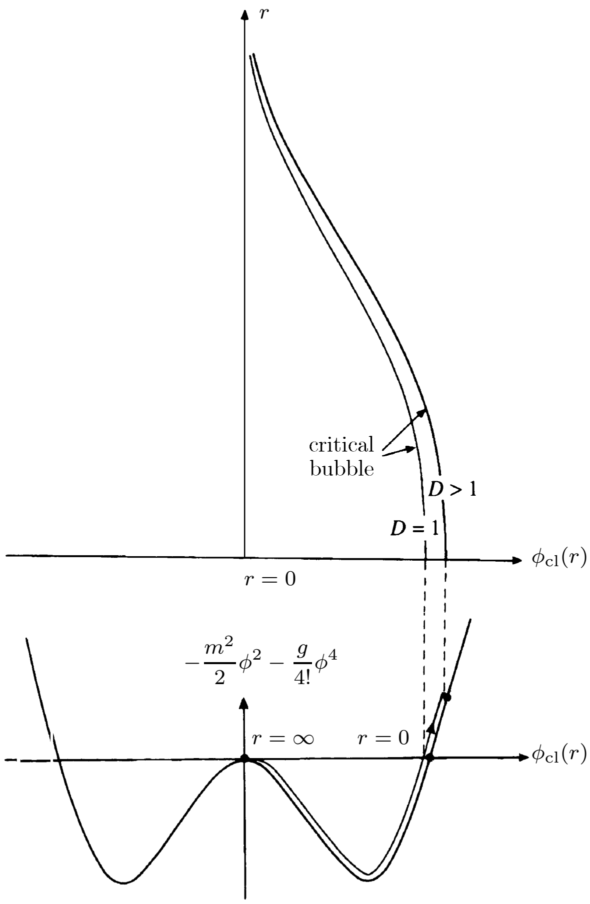

FIGURE 16.1 Fluctuations around the origin are metastable for $g < 0$ since they can reach beyond the barrier (drawn upside down) with ϕ depending on r as indicated by the curve with arrow. The solution of the field equation traversing the barrier looks like the trajectory of a mechanical point particle moving with negative friction in the potential well. It has an activation energy $1/\alpha g$ producing a left-hand cut in the complex g-plane in all correlation functions, with a discontinuity proportional to the Boltzmann factor $e^{-1/\alpha g}$ [in the natural units used in the functional integral (2.6)].

This has the form of a bubble described by a function $\phi_{\rm cl}(r)$, solving the radial equation in D dimensions

$$\left(-\frac{d^2}{dr^2} - \frac{D-1}{r}\frac{d}{dr}\right)\phi_{\rm cl}(r) + m^2\phi_{\rm cl} + \frac{g}{3!}\phi_{\rm cl}^3(r) = 0\,. \tag{16.2}$$

If we interprete $-r$ as a time t and $\phi_{\rm cl}(r)$ as a position $x(-t)$ of a point particle as a function of time, we obtain an equation of motion for this point particle in the inverse potential $V(x) = -(m^2/2)x - (g/4!)x^4$:

$$\ddot{x}(t) - \frac{D-1}{t}\dot{x}(t) - m^2 x(t) - \frac{g}{3!}x^3(t) = 0\,. \tag{16.3}$$

In point mechanics, the second term proportional to $\dot{x}(t)$ corresponds to a negative friction whose strength decreases with time like $1/t$. The trajectory is indicated in Fig. 16.1 by the curve with an arrow, from which one deduces the properties of the bubble solution. For $D = 1$, the friction is absent and a first integral of the differential equation (16.3) is given by energy conservation, fixing the maximal value of $x = \phi_{\rm cl}$ by the right-hand zero of the potential. In $D > 1$ dimensions, the negative friction makes the particle overshoot on the right-hand side. For more details see the discussion in the textbooks [2]. The Boltzmann factor is proportional to $e^{-1/\alpha g}$, where $1/\alpha g$ is the action of such a solution, playing the role of an activation energy. The instability of the fluctuations gives rise to an imaginary part in the functional integral (2.6) for the partition function and (2.10) for the correlation functions. This property will go over to all renormalization group functions such as $\beta(g)$, $\nu(g)$, $\eta(g)$. Since all these expressions are real for positive g, they have a cut from $g = 0$ to $-\infty$ with a discontinuity across it which behaves like $e^{-1/\alpha g}$ near the tip of the cut.

Functions with such a cut have a vanishing radius of convergence. Their expansion coefficients grow factorially, as we can easily see as follows: let $f(g)$ be such a function with a discontinuity along the left-hand cut:

$$\operatorname{disc} f(g) = f(g + i\varepsilon) - f(g - i\varepsilon) = 2\, i \operatorname{Im} f(g + i\varepsilon) \quad \text{for } g < 0. \tag{16.4}$$

The discontinuity determines all expansion coefficients. To see this, we use a dispersion relation relating $f(g)$ on the positive real axis to the discontinuity at the cut:

$$f(g) = \frac{1}{2\pi i} \int_{-\infty}^{0} dg' \, \frac{\operatorname{disc} f(g')}{g' - g}\,. \tag{16.5}$$

Expansion in powers of g as $f(g) = \sum_k f_k g^k$ yields coefficients f_k expressed as integrals over the discontinuity of the cut:

$$f_k = \frac{1}{2\pi i} \int_{-\infty}^{0} dg' \, \frac{\operatorname{disc} f(g')}{g'^{k+1}}\,. \tag{16.6}$$

The main contribution to the integral in Eq. (16.6) for $k \to \infty$ comes from the region of small negative g', i.e. from the tip of the cut. In this limit, quasiclassical methods can be used to calculate the imaginary part of the function $f(g)$ [3]. Near the tip of the cut, the functional integral describes a system which decays through a very high barrier. The functional integral can therefore be approximated by the saddle point approximation. The leading term is produced by the classical solution. The fluctuation corrections correspond to small distortions and translations of this solution. As a result we find for $g \to 0^-$ a discontinuity of the form

$$\operatorname{disc} f(-|g|) = 2\pi i \, \frac{\gamma}{(\alpha|g|)^{\beta+1}} \, e^{-1/\alpha|g|} \left[1 + \mathcal{O}(\alpha|g|)\right]. \tag{16.7}$$

The exponential is a quantum version of the Boltzmann factor for the activation energy of the classical solution. The prefactor accounts for the fluctuation entropy. Inserting (16.7) into Eq. (16.6) yields

$$f_k \;=\; \frac{(-1)^k}{2\pi i} \int_0^\infty dg' \, \frac{\operatorname{disc} f(-g')}{g'^{k+1}} \approx \frac{\gamma\,(-1)^k}{\alpha^{\beta+1}} \int_0^\infty dg' (g')^{-(\beta+k+2)} \, e^{-1/\alpha g'}\,, \tag{16.8}$$

which leads via the integral formula

$$\int_0^\infty dg\, g^{-(\beta+k+2)} e^{-1/\alpha g} \;= \alpha^{\beta+k+1} \Gamma(\beta + k + 1) \tag{16.9}$$

to the large-order behavior of the coefficients:

$$f_k = \gamma\,(-\alpha)^k \Gamma(k + \beta + 1)\,. \tag{16.10}$$

For large k, the Gamma function $\Gamma(k + \beta + 1)$ can be approximated with the help *Stirling's formula*

$$\Gamma(pk + q) \sim \sqrt{2\pi}^{\,1-p} p^{q-1/2} k^{-1/2+q-p/2} p^{pk} (k!)^p, \tag{16.11}$$

from which we find that $\Gamma(k+\beta+1) = (k+\beta)!$ grows like $k^\beta k!$, and we may write just as well

$$f_k = \gamma(-\alpha)^k k^\beta k![1+\mathcal{O}(1/k)] \quad \text{for } k \to \infty\,. \tag{16.12}$$

In a similar way, the large-order behavior can be determined for all correlation functions of the O(N)-symmetric ϕ^4-theory [4]. This will be considered in Chapter 17.

The factorial growth of the perturbation expansion implies that we are confronted with asymptotic series which require special resummation procedures if we want to extract reliable results. Let us first recall some of their basic properties. According to Poincaré [5], a divergent series is an asymptotic expansion of a function $f(g)$ if

$$\lim_{|g|\to 0}\left[\frac{1}{g^L}\left|f(g)-\sum_{k=0}^{L} f_k g^k\right|\right] = 0\,, \qquad \text{for } L \geq 0\,. \tag{16.13}$$

This definition implies that an asymptotic series does not define a function uniquely. The expansion coefficients of a function of the type $e^{-1/\alpha g}\sum_{k=0}^{L} f_k g^k$ being identically zero, such a function can be added to $f(g)$ without changing (16.13):

$$\lim_{|g|\to 0}\left[\frac{1}{g^L}\left(e^{-1/\alpha g}\sum_{k=0}^{L} f_k g^k\right)\right] = 0 \qquad \text{for all } L\,. \tag{16.14}$$

The existence of such an ambiguity can be excluded under certain conditions by the theorem of Carleman [6], which is based on theorems of Phragmen-Lindelöf. Suppose a function $f(g)$ is analytic for $|g| < g_0$ with $|\arg\,(g)| \leq \pi\delta/2$ and has an asymptotic expansion in this region, so that

$$\Delta(L,g) = |f(g) - \sum_{k=0}^{L-1} f_k g^k| \leq M\alpha^L (L!)^\rho |g|^L\,, \quad |\arg\, g| \leq \frac{\pi\delta}{2}\,, \tag{16.15}$$

where for all k:

$$|f_k| \leq \gamma\, k!\, A^k\,, \tag{16.16}$$

with some real number A. Such a function is defined uniquely by its asymptotic expansion if $\delta > \rho \geq 1$. According to Nevanlinna [7], this is true even for $\delta \geq \rho \geq 1$. In this case, it makes sense to reconstruct a function from its asymptotic series expansion.

The error of the $(L-1)$ th partial sum of an asymptotic series is, according to Eq. (16.15) for $\rho = 1$, bounded by $\alpha^L L!|g|^L$. For small g, this error decreases for some initial orders and reaches a minimum at some $L_{\rm min}$. Indeed, from Stirling's formula (16.11) we know that

$$L! \approx L^{L+1/2}\sqrt{2\pi}e^{-L}, \tag{16.17}$$

so that we may estimate

$$\alpha^L L! g^L \approx \exp[-L + L\ln\alpha g + (L+1/2)\ln L + 1/2\ln 2\pi]. \tag{16.18}$$

This has a minimum at $L_{\rm min} \approx 1/\alpha g$, where the error becomes roughly

$$\Delta_{\rm min}(g) \equiv \Delta(L_{\rm min}, g) \approx e^{-1/\alpha g}. \tag{16.19}$$

Calculating higher and higher partial sums does not improve the approximation. The best result is reached by the partial sum of order $L_{\rm min}$, where the error is of the order $\mathcal{O}(e^{-1/\alpha g})$ such that even this result is useful provided g is very small. For larger g, more complicated methods have to be used to extract reliable information from an asymptotic series.

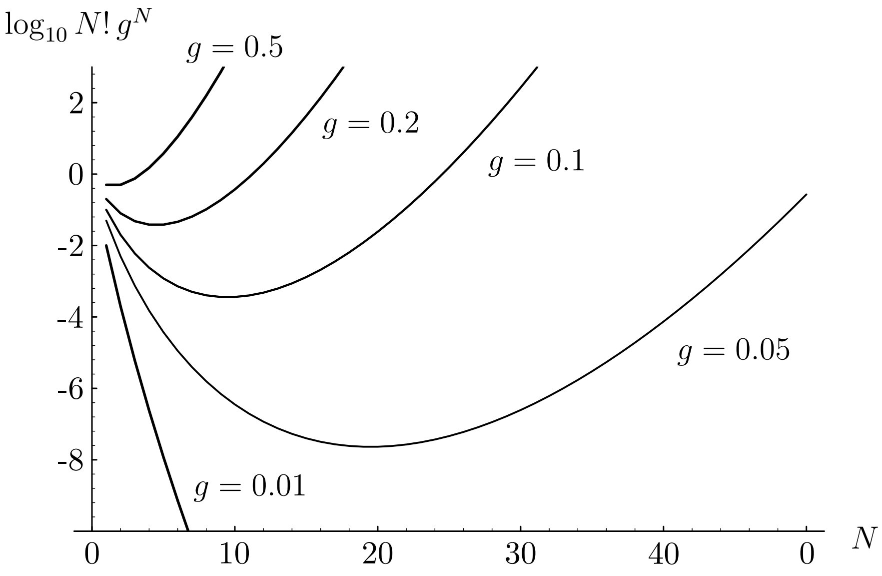

FIGURE 16.2 Minimal error $\Delta_{\min}(g)$ of an expansion with a behavior of the form $L!\, g^L$ is shifted towards larger L for smaller and smaller g.

16.2 Padé Approximants

The *Padé approximant* for a series expansion $f(z) = \sum_{i=0}^{k} f_i z^i + \ldots$ up to the order k is given by

$$[M/N] = \frac{a_0 + a_1 z + \ldots + a_M z^M}{1 + b_1 z + \ldots + b_N z^N}\,, \quad M + N = k, \tag{16.20}$$

where a_i and b_i are chosen such that the series expansion of $[M/N]$ up to the order k equals the original series

$$\sum_{i=0}^{k} f_i z^i = [M/N] + O[z^{M+N+1}]. \tag{16.21}$$

The $M + N + 1$ unknown coefficients are given uniquely by the $k + 1$ coefficients f_i. The calculation of the b_i proceeds in principle by multiplication of Eq. (16.21) with the denominator of $[M/N]$ and comparison of the coefficients of the equations for z^n with $M < n < M + N + 1$:

$$\begin{aligned} b_N f_{M-N+1} + b_{N-1} f_{M-N+2} + \ldots + b0 f_{M+1} &= 0 \\ b_N f_{M-N+2} + b_{N-1} f_{M-N+3} + \ldots + b0 f_{M+2} &= 0 \\ &\vdots \\ b_N f_M + b_{N-1} f_{M+1} + \ldots + b0 f_{M+N} &= 0\,, \end{aligned} \tag{16.22}$$

where $f_j = 0$ for $j < 0$. The coefficients of the numerator follow after inserting the results for the b_i into Eq. (16.21). For actual calculations, there are many algorithms which can be found in Ref. [1]. The symmetric $[N/N]$ Padé approximant usually gives the fastest approximation for increasing N although this depends on the function.

Being quotients of polynomials, the Padé approximants can describe functions with poles even in lowest order, in contrast to power series which are good approximations only inside some circle of convergence $|z| < R$. Therefore, the method can also be used to approximate functions outside of the region of convergence of their power series. A sequence of Padé approximants which converge outside the region of $|z| \le R$ can be used to define an analytical continuation of the function for $|z| > R$. An example is shown in Fig. 16.3. The function has a power series with a convergence radius $R = 1/2$:

$$f(z) = \sqrt{\frac{1+z/2}{1+2z}} = 1 - \frac{3}{4}z + \frac{39}{32}z^2 - \dots . \tag{16.23}$$

Its [1/1] Padé approximant is

$$[1/1] = \frac{1+7z/8}{1+13z/8}, \tag{16.24}$$

which deviates by less than 8% from the exact function for $z \to \infty$, where $f(\infty) = 0.5$, whereas $[1/1](\infty) \approx 0.54$.

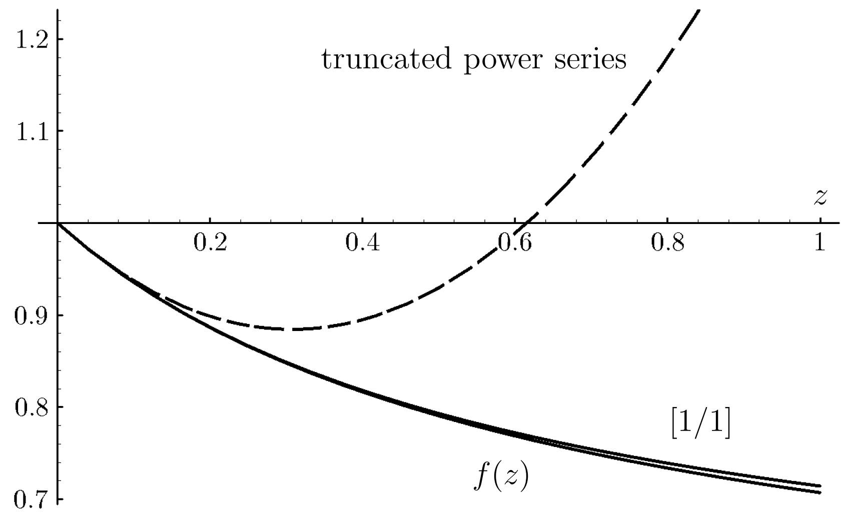

FIGURE 16.3 Plot of the function $f(z) = \sqrt{(1+z/2)/(1+2z)}$, its truncated power series expansion $1-3z/4+39z^2/32 - \dots$, and its Padé approximant $[1/1] = (1+7z/8)/(1+13z/8)$.

For $R = 0$, an asymptotic power series has only a formal character. With the help of a converging sequence of Padé approximants, however, it is possible to resum such a series.

The method has been extended [8] to two-variable approximants $[M, M]/[N, N]$, which could not be used in our applications since they lead to poles in the positive coupling constant plane. For the purpose of extracting reliable results from five-loop calculations of quantum field theory, the Padé approximants converge too slowly. It is necessary to include the knowledge of the large-order behavior which, as argued before, is possible using a Borel transformation of the asymptotic series.

16.3 Borel Transformation

Let the asymptotic series which describes a function $f(g)$ have a zero radius of convergence. Let this divergence be caused by a factorial growth of the coefficients f_k. If we divide each term in the expansion by a factor $k!$, we can obtain a series with a nonzero radius of convergence. This series is called a *Borel sum* [9]. The function $f(g)$ is called *Borel summable*, if the Borel sum can be summed and analytically continued over the whole positive axis. This analytical continuation can be used to recover the original function via a Borel transformation. The Borel transformation restores the factorial in each term of the expansion. For details and citations see Ref. [10].

For a function $f(g) = \sum_k f_k g^k$ with $f_k \propto (-\alpha)^k k!$ for large k, the Borel sum is defined by

$$B(g) \equiv \sum_k^\infty B_k g^k, \qquad \text{with} \quad B_k \equiv \frac{f_k}{k!}. \tag{16.25}$$

The denominator in B_k removes the factorial growth of the expansion coefficients f_k and leads to a function $B(g)$ which is analytic in some neighborhood of the origin in the complex g-plane. In fact, for large k, the expansion coefficients B_k behave like

$$B_k \propto (-\alpha)^k, \quad \text{for large } k\,, \tag{16.26}$$

so that the Borel sum converges for $|g| < 1/\alpha$, and has an expansion

$$\sum_k^\infty B_k g^k = \frac{\text{const}}{1+\alpha g}\left[1 + \mathcal{O}(g)\right], \quad \text{for } |g| < \frac{1}{\alpha}\,. \tag{16.27}$$

From this convergent expansion, an analytic continuation $B(g)$ can be defined for $g > 1/\alpha$.

Another way of defining the Borel transform (16.25) is via a contour integral:

$$B(t) \equiv \frac{1}{2\pi i}\oint_C \frac{dz}{z} e^z\, f(t/z), \tag{16.28}$$

where the contour C encircles the origin. Inserting the expansion $f(g) = \sum_n f_n g^n$, the integral produces in each term a factor $1/n!$, thus converting $f(g)$ into $B(t) = \sum_n f_n t^n/n!$.

The factorials can be restored using an integral representation of the Gamma function:

$$k! = \Gamma(k+1) = \int_0^\infty dt\, e^{-t} t^k\,. \tag{16.29}$$

This shows that the original power series expansion for $f(g)$ is recovered from the Borel sum by the Borel transformation:

$$f(g) = \sum_k^\infty f_k g^k = \sum_k^\infty B_k g^k k! = \sum_k^\infty B_k \int_0^\infty dt\, e^{-t}(gt)^k. \tag{16.30}$$

Interchanging the order of integration and summation, the right-hand side becomes

$$\int_0^\infty dt\, e^{-t} \sum_k^\infty B_k (gt)^k\,, \tag{16.31}$$

and leads to an integral transformation

$$f(g) = \int_0^\infty dt\, e^{-t} B(gt)\,. \tag{16.32}$$

This is the Borel transformation of the Borel sum $B(g)$. If only a finite number of terms in the Borel sum is known, the integral in Eq. (16.31) just reinserts the factors $k!$ and leads back to the initial diverging series. This can be avoided by trying to find a nonpolynomial function whose initial Taylor coefficients coincide with the known terms in the Borel sum. Such a function could be a Padé approximant. But we can also try to construct an analytical continuation using the knowledge of the large-order behavior.

For finding an analytical continuation it is crucial to know the radius of convergence from the position of the singularity closest to the origin. But this singularity can be deduced from the large-order behavior as is stated by the *theorem of Darboux* [5]. Usually, the singularity will be a cut starting at $g = -1/\alpha$. Indeed, suppose the presence of such a cut in $B(g)$ in the form

$$B(g) = \frac{\gamma}{(1+\alpha g)^{\beta+1}}. \tag{16.33}$$

Expanding this in powers of g yields

$$\frac{\gamma}{(1+\alpha g)^{\beta+1}} = \gamma \sum_{k=0}^{\infty} B_k g^k = \gamma \sum_{k=0}^{\infty} \frac{\Gamma(\beta+k+1)}{\Gamma(\beta+1)\Gamma(k+1)} (-\alpha)^k g^k \,, \tag{16.34}$$

with coefficients behaving for large k like

$$B_k \propto (-\alpha)^k k^\beta. \tag{16.35}$$

The coefficients of the original series $f(g) = \sum_k f_k g^k$ grow therefore like

$$f_k \propto (-\alpha)^k k^\beta k!. \tag{16.36}$$

If $B(g)$ contains a superposition of cuts like (16.34), the cut lying closest to the origin determines the convergence radius of the Borel sum, and dominates the large-k behavior of f_k.

In order to carry out the integration over the Borel sum $B(g)$, it not only has to be analytic in the circle of radius $1/\alpha$, it also has to be regular on the positive real axis. This is assured by a theorem of Watson [11] under conditions which are somewhat stricter than those required for the uniqueness of the asymptotic expansion. The theorem states that a function analytic in $|\arg g| \leq \pi/2 + \delta$ for $\delta > 0$ and $|g| < g_0$ with an asymptotic expansion in this region fulfilling

$$\Delta(L, g) = |f(g) - \sum_{k=0}^{L-1} f_k g^k| = \mathcal{O}(\alpha^L L! |g|^L) \,, \quad |\arg g| \leq \pi/2 + \delta, \tag{16.37}$$

with f_k growing like $k! C^k$ for large k, has a Borel sum $B(t) = \sum_k f_k t^k / k!$ convergent in the circle $|t| < 1/\alpha$ and regular in the angle $|\arg t| < \delta$. The Borel sum can be integrated to give back $f(g)$:

$$f(g) = \int_0^\infty e^{-t} B(gt) \,, \quad \text{for } |g| < g_0, \ |\arg g| < \delta \,. \tag{16.38}$$

This justifies the interchange of integration and summation under these conditions.

It is not easy to find out whether these conditions are satisfied for various field theories. It has been possible to prove Borel summability for ϕ^4-theory in $D = 0, 1, 2, 3$ dimensions [12], but not yet in $D = 4$ dimensions. We expect it to hold for all $D = 4 - \varepsilon < 4$.

Instead of a Borel transform with a leading cut at $-1/\alpha$, we may also define a generalized Borel transform in which the leading cut becomes a leading pole, called *Borel-Leroy transform* $B^\beta(g)$. It arises from the expansion of $f(g)$ by dividing each coefficient f_k by $\Gamma(k+\beta+1)$ rather than by $k!$ [13]. The basic inverse Borel-Leroy transform is

$$\Gamma(k+\beta+1) = \int_0^\infty dt \, e^{-t} t^{k+\beta}, \tag{16.39}$$

which serves to reinsert the growth factor $\Gamma(k+\beta+1)$ to recover $f(g)$:

$$f(g) = \int_0^\infty dt\, e^{-t} t^\beta B^\beta(gt). \tag{16.40}$$

A pole term $1/(1+\alpha tg)$ in $B^\beta(gt)$ corresponds obviously to expansion coefficients of $f(g)$ growing like

$$f_k \propto (-\alpha)^k \Gamma(k+\beta+1) \approx (-\alpha)^k k! k^\beta. \tag{16.41}$$

The simplest way to apply the technique of Borel transformation to the resummation of perturbation expansions for critical exponents proceeds by dividing the expansion coefficients f_k by the leading growth $\Gamma(k+\beta+1)$, using the Padé approximation to sum the resulting Borel series [14], and recovering $f(g)$ by doing the integral (16.40) numerically. If no high accuracy is required, this method can be applied without knowledge of the large-order behavior. We may try out a few different values of β and optimize the speed of convergence for increasing orders of the expansion.

The Padé approximants are useful to approximate the singularities on the negative axis by a string of poles. This makes them particularly useful for approximating Borel and Borel-Leroy transforms. Unfortunately, they often introduce unphysical poles on the positive axis, in which case the integral (16.40) no longer exists, making it impossible to recover the original function $f(g)$.

For the critical exponents to be evaluated in this book, the radius of convergence of the Borel sum is known from the large-order behavior of the expansions. This knowledge can be exploited to set up conformal mappings to a new complex variable, by which the Borel sum can be continued analytically to a regime outside the circle of convergence, as will now be explained.

16.4 Conformal Mappings

The method of conformal mappings [15, 16] exploits the knowledge of the location of the closest singularity in $B^\beta(g)$ from the large-order behavior of the expansion coefficients of $f(g)$. This knowledge is used to map the entire cut g-plane into the unit circle such that the singularities on the negative axis are moved onto the circumference of the circle. The result is an analytic continuation of the Borel sum to the entire g-plane. For a function $f(g) = \sum_k f_k g^k$ with the large-order behavior

$$f_k = \gamma k^\beta (-\alpha)^k k! [1 + \mathcal{O}(1/k)], \quad k \to \infty, \quad \alpha > 0\,, \tag{16.42}$$

the Borel-Leroy sum is

$$B^{\beta_0}(g) \;=\; \sum_{k=0}^{\infty} \frac{f_k}{\Gamma(k+\beta_0+1)}\, g^k = \sum_{k=0}^{\infty} B_k^{\beta_0} g^k. \tag{16.43}$$

Its expansion coefficients grow for large k like

$$B_k^{\beta_0} = \gamma(-\alpha)^k k^{\beta-\beta_0}\,, \text{ for } k \to \infty, \tag{16.44}$$

as a consequence of the singularity in $B^{\beta_0}(g)$ at $g = -1/\alpha$ which is closest to the origin. It is assumed that all other singularities of $B^{\beta_0}(g)$ lie on the negative axis further away from the origin. In ϕ^4-theory, this was proven for $D = 0, 1$ but not for $D = 2, 3$. Such singularities come from subleading instanton contributions.

The following mapping preserves the origin and transforms $g = \infty$ to $w = 1$ such that the whole cut plane will be in a circle of unit radius, and the singularities of the negative axis from $-1/\alpha$ to $-\infty$ will be on the border of the unit circle:

$$w(g) = \frac{\sqrt{1+\alpha g} - 1}{\sqrt{1+\alpha g} + 1}\,, \qquad g = \frac{4}{\alpha}\frac{w}{(1-w)^2}. \tag{16.45}$$

This implies that the function $W(w) = B^{\beta_0}(g(w))$ has a convergent Taylor expansion for $|w| < 1$ and so will the Borel sum which is re-expanded in the new variable $w(g)$ in the whole cut plane of g:

$$B^{\beta_0}(g) = \sum_{k=0}^{\infty} W_k \, [w(g)]^k\,. \tag{16.46}$$

A Borel sum that is known up to the L-th order can be re-expanded in the new variable:

$$B^{\beta_0[L]}(g) = \sum_{k=0}^{L} B_k^{\beta_0} g^k = \sum_{k=0}^{L} W_k \, [w(g)]^k + \mathcal{O}(g^{L+1})\,, \tag{16.47}$$

where the coefficients W_k are given by

$$W_k = \sum_{n=0}^{k} B_n^{\beta_0} \left(\frac{4}{\alpha}\right)^n \frac{(n+k-1)!}{(k-n)!(2n-1)!}\,. \tag{16.48}$$

The re-expanded series on the right-hand side of Eq. (16.47) will give a much better approximation to the Borel sum. The reason is that the analytic behavior which causes the divergent large-order behavior is already incorporated in each of the expansion terms $[w(g)]^k$. This can be seen by expanding $[w(g)]^k$ near $g = -1/\alpha$, which is the starting point of a square root branch cut, as

$$[w(g)]^k = (-1)^k \left[1 - 2k(1+\alpha g)^{1/2} + \mathcal{O}(1+\alpha g)\right], \quad \text{for} \quad g \to -1/\alpha\,. \tag{16.49}$$

According to the remarks made in the context of Eq. (16.33), each term leads to a growth of f_k of the form (16.42). Inserting (16.46) into the integral transform (16.40) gives an expansion for $f^{[L]}(g)$:

$$f^{[L]}(g) = \sum_{k=0}^{L} W_k \int_0^\infty e^{-t} t^{\beta_0} [w(gt)]^k\,. \tag{16.50}$$

In the case of ϕ^4-theory in 3, 4 dimensions it is not known whether the new expansion of $f^{[L]}$ will have a nonzero radius of convergence or is just a new kind of an asymptotic expansion.

The convergence can be improved by an appropriate choice of the parameter β_0. A functions with expansion coefficients of the form

$$B_k^{\beta_0} = \frac{\gamma(-\alpha)^k k^\beta k!}{\Gamma(k+\beta_0+1)}[1+\mathcal{O}(1/k)] \;\to\; \gamma(-\alpha)^k k^{\beta-\beta_0}[1+\mathcal{O}(1/k)],\; k \to \infty, \tag{16.51}$$

behaves close to its leading singularity at $-1/\alpha$ as:

$$B^{\beta_0}(g) = \gamma\Gamma(1+\beta-\beta_0)(1+\alpha g)^{\beta_0-\beta-1}[1+\mathcal{O}(1+\alpha g)], \quad g \to -1/\alpha\,. \tag{16.52}$$

The parameter β_0 can be chosen such that the singularity in Eq. (16.52) is of the same form as in Eq. (16.49):

$$\beta_0 = \beta + 3/2\,. \tag{16.53}$$

Furthermore, we can adapt a possibly known *strong-coupling power behavior* of $f(g)$:

$$f(g) \overset{g\to\infty}{\sim} g^s. \tag{16.54}$$

This is done by re-expanding the Borel sum in a modified manner:

$$B^{\beta_0[L]}(g) \equiv \sum_{k=0}^{L} B_k^{\beta_0} g^k = \frac{1}{[1-w(g)]^{2s}} \sum_{k=0}^{L} W_k(s)[w(g)]^k\,, \tag{16.55}$$

where the W_k are now depending on s:

$$W_k(s) = \sum_{n=0}^{k} \frac{B_n^{\beta_0}}{\Gamma(n+\beta_0+1)} \left(\frac{4}{\alpha}\right)^n \frac{(k+n-2s-1)!}{(k-n)!(2n-2s-1)!}\,. \tag{16.56}$$

Noting that for $g\to\infty$, $w(g)\to 1$ for $g\to\infty$ and

$$\frac{1}{[1-w(g)]^{2s}} = \left(\frac{\sqrt{1+\alpha g}+1}{2}\right)^{2s} \overset{g\to\infty}{\to} g^s \left(\frac{\alpha}{4}\right)^s\,, \tag{16.57}$$

the strong-coupling behavior is obviously reproduced order by order and is unchanged by the integral transform of Eq. (16.50).

16.5 Janke-Kleinert Resummation Algorithm

The conformal mapping results in a re-expansion of the Borel sum into powers of the functions $w(gt)$. The large-order behavior of the Borel sum is included in the functions $[w(gt)]^k$ which have the proper branch cut on the negative axis to generate the correct divergences after the integral transformation. It was realized by Janke and Kleinert [10] that these properties can be implemented in a simple systematic way by re-expanding the asymptotic expansion in a complete set of basis functions $I_n(g)$:

$$f_L(g) = \sum_{k=0}^{L} f_k g^k = \sum_{n=0}^{L} h_n I_n(g) \overset{L\to\infty}{\to} f(g)\,. \tag{16.58}$$

The basis functions $I_n(g)$ possess the following properties:

1. They have a simple *Borel-Leroy representation*

$$I_n(g) = \int_0^\infty dt e^{-t} t^{\beta_0} B_{I_n}^{\beta_0}(gt). \tag{16.59}$$

2. They possess the *large-order behavior* of the expansion coefficients f_k:

$$f_k \to \gamma(-\alpha)^k k^\beta k!, \quad \text{for} \quad k\to\infty. \tag{16.60}$$

3. They exhibit a power-like *strong-coupling behavior*

$$f(g) \to g^s, \quad \text{for} \quad g\to\infty. \tag{16.61}$$

4. The expansion coefficients h_n in (16.58) are determined by a triangular system of equations from f_k, so that the first known coefficients $f_1,\ f_2,\ldots,f_L$ are correctly reproduced.

16.5.1 Reexpansion Functions

The second property implies the existence of the same tip of the branch cut on the negative real axis in the functions $I_n(g)$ as in $f(g)$. Equivalently, the associated basis functions $B^{\beta_0}_{I_n}(gt)$ of the Borel-Leroy transforms have their smallest singularity in absolute size at $t = -1/\alpha|g|$. Janke and Kleinert chose for these a set of *hypergeometric functions* of the argument αgt, which are indeed analytic functions in gt with a cut on the negative axis starting at $-1/\alpha$:

$$ {}_2F_1(a,b;c;-\alpha gt) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k} \frac{(-\alpha gt)^k}{k!}\,, \tag{16.62}$$

where $(a)_n$ are the *Pochhammer symbols*:

$$ (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)(a+2)\dots(a+k-1)\,, \qquad (a)_0 = 1\,. \tag{16.63}$$

The hypergeometric functions are standard special functions whose properties are well-known. The Borel integral can immediately be calculated:

$$ \int_0^\infty dt\, \mathrm{e}^{-t}\, t^{\beta_0}\, {}_2F_1(a,b,c;-\alpha gt) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\, E(a,b,\beta_0+1;c;1/\alpha g)\,. \tag{16.64}$$

The resulting functions are the *MacRobert functions*. Their asymptotic expansions are [17]

$$ \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\, E(a,b,\beta_0+1;c;1/\alpha g) \equiv \sum_{k=0}^{\infty} e_k g^k\,, \tag{16.65}$$

with coefficients of the form:

$$ e_k \longrightarrow \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} (-1)^k k! k^{a+b-c+\beta_0-1} \alpha^k, \qquad \text{for } k \to \infty, \tag{16.66}$$

thus reproducing the large-order behavior of (16.60). This behavior is unchanged if the hypergeometric function is multiplied by $(\alpha gt)^n$. A set of functions is therefore

$$ B^{\beta_0}_{I_n}(gt) = (\alpha gt)^n\, {}_2F_1(a,b;c;-\alpha gt) = (\alpha gt)^n \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k} \frac{(-\alpha gt)^k}{k!}\,. \tag{16.67}$$

The parameters a, b, and c are not completely fixed by the large-order behavior. The parameter β in (16.60) imposes the following relation upon a, b, c, and β_0 of (16.66):

$$ a + b + \beta_0 - c - 1 = \beta. \tag{16.68}$$

This leaves three parameters undetermined. They may be adjusted to reduce the hypergeometric function to simple algebraic expressions. Two such choices are

$$ b = a + \frac{1}{2},\ c = 2a+1:\ F^{(1)}(a, a+\tfrac{1}{2}; 2a+1; -\alpha gt) = 4^a \left(1 + \sqrt{1+\alpha gt}\right)^{-2a}, \tag{16.69}$$

$$ b = a + \frac{1}{2},\ c = 2a:\ F^{(2)}(a, a+\tfrac{1}{2}; 2a; -\alpha gt) = 4^a \left(1 + \sqrt{1+\alpha gt}\right)^{-2a} \frac{1+\sqrt{1+\alpha gt}}{2\sqrt{1+\alpha gt}}\,, \tag{16.70}$$

leaving only one free parameter. From Eq. (16.68) we see that the parameter β_0 is in these two cases equal to $\beta_0 = \beta + 3/2$ and $\beta_0 = \beta + 1/2$, respectively. The two hypergeometric functions possess precisely the cut of square root type in the complex g-plane that was previously generated by the analytic mapping procedure. The hypergeometric functions ${}_2F_1(a,b;c;-\alpha gt)$ are real for real arguments smaller than one, i.e., for $-\alpha gt < 1$. The imaginary part of the function for negative gt is found by rewriting the hypergeometric function as

$$\begin{aligned} {}_2F_1(a,b;c;-\alpha gt) &= C_1\, {}_2F_1(a,b;a+b-c+1;1+\alpha gt) \\ &+ (1+\alpha gt)^{c-a-b} C_2\, {}_2F_1(c-a,c-b;c-a-b+1;1+\alpha gt)\,, \end{aligned} \tag{16.71}$$

$$\text{with} \quad C_1 = \frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \text{and} \quad C_2 = \frac{\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}. \tag{16.72}$$

The hypergeometric functions on the right-hand side are real for real negative gt as $1+\alpha gt < 1$. Since the parameters a, b, c satisfy the relation $c-a-b = \pm 1/2$ for the two sets of functions, the term $(1+\alpha gt)^{\pm 1/2}$ possesses a branch cut of the square root type. Its explicit form reads, for $g = -|g| + i\varepsilon$,

$$(1+\alpha gt)^{\pm\frac{1}{2}} = \begin{cases} (1-\alpha|g|t)^{\pm\frac{1}{2}} & \text{for } t < 1/\alpha|g|, \\ \pm i\,(\alpha|g|t-1)^{\pm\frac{1}{2}} & \text{for } t > 1/\alpha|g|. \end{cases} \tag{16.73}$$

Inserting the special values of a, b, c, β_0 and the two functions (16.69) and (16.70) into (16.59), and carrying out the t-integration we obtain for $I_n(g)$ an imaginary part of the correct form $e^{-1/\alpha|g|}(1/\alpha|g|)^{\beta_0}$ responsible for the large-order behavior of $f(g)$. This ensures that the re-expansion (16.58) has an improved convergence.

The functions Eq. (16.69) and (16.70) contain a free parameter a which determines the strong-coupling exponent or strong-coupling parameter s in (16.61). For large g, the functions $F^{(1)}$ and $F^{(2)}$, multiplied by the factor $(\alpha g)^n$ to get $B_{I_n}^{\beta_0}(gt)$, grow like g^{n-a}, so that a is related to s by

$$a = n - s. \tag{16.74}$$

Including for later convenience a factor $1/4^n\Gamma(\beta_0+1)$ and $2/4^n\Gamma(\beta_0+1)$, we arrive at the basis functions for the Borel-Leroy transforms:

$$\begin{aligned} B_{I_n}^{\beta_0(1)}(gt) &= \frac{(\alpha gt)^n}{\Gamma(\beta_0+1)4^n}\, {}_2F_1(n-s, n-s+\tfrac{1}{2}, 2(n-s)+1, -\alpha gt) \\ &= \frac{1}{\Gamma(\beta_0+1)} \left(\tfrac{1}{2}+\tfrac{1}{2}\sqrt{1+\alpha gt}\right)^{2s} \frac{(\alpha gt)^n}{(1+\sqrt{1+\alpha gt})^{2n}}\,, \qquad \text{with } \beta_0 = \beta + 3/2\,, \end{aligned} \tag{16.75}$$

$$\begin{aligned} B_{I_n}^{\beta_0(2)}(gt) &= \frac{2(\alpha gt)^n}{\Gamma(\beta_0+1)4^n}\, {}_2F_1(n-s, n-s+\tfrac{1}{2}, 2(n-s), -\alpha gt) \\ &= \frac{1}{\Gamma(\beta_0+1)} \left(\tfrac{1}{2}+\tfrac{1}{2}\sqrt{1+\alpha gt}\right)^{2s} \frac{1+\sqrt{1+\alpha gt}}{\sqrt{1+\alpha gt}} \frac{(\alpha gt)^n}{(1+\sqrt{1+\alpha gt})^{2n}}\,, \quad \text{with } \beta_0 = \beta + 1/2\,; \end{aligned} \tag{16.76}$$

with the associated basis functions

$$I_n^{(1)}(g) = \int_0^\infty dt\, e^{-t} t^{\beta_0} B_{I_n}^{\beta_0(1)}(gt), \tag{16.77}$$

$$I_n^{(2)}(g) = \int_0^\infty dt\, e^{-t} t^{\beta_0} B_{I_n}^{\beta_0(2)}(gt). \tag{16.78}$$

We now turn to the determination of the re-expansion coefficients h_n in (16.58). They are found by comparing the coefficients of g^k in $f(g)$ and $\sum_{n=0}^{L} h_n I_n(g)$. For this we need the coefficients of the expansion of $I_n(g)$. Inserting the expansion of the hypergeometric function in (16.59) and performing the Laplace transforms in (16.77) and (16.78), we obtain

$$\begin{aligned}
I_n^{(1)} &= \int_0^\infty dt\, \frac{e^{-t} t^{\beta_0} (\alpha g t)^n}{\Gamma(\beta_0+1) 4^n}\, {}_2F_1(n-s, n-s+\tfrac{1}{2}; 2(n-s)+1; -\alpha g t) \\
&= \sum_{k=0}^{\infty} \frac{\Gamma(\beta_0+n+1)}{\Gamma(\beta_0+1)4^n} \frac{(n-s)_k (n-s+\frac{1}{2})_k (\beta_0+n+1)_k}{(2n-2s+1)_k} \frac{(-1)^n (-\alpha g)^{k+n}}{k!} \\
&\equiv \sum_{k'=n}^{\infty} I_{k',n}^{(1)}\, g^{k'} .
\end{aligned} \tag{16.79}$$

Expressing the Pochhammer symbols in terms of Gamma functions, and using the identity where

$$\Gamma(2a) = \frac{\Gamma(a)\Gamma(a+1/2)}{\Gamma(1/2)} 2^{2a-1}, \tag{16.80}$$

the expansion coefficients $I_{k,n}^{(1)}$ become

$$I_{k,n}^{(1)} = \frac{(-1)^n (-\alpha)^k (n-s)}{4^s \sqrt{\pi}} \frac{\Gamma(k-s)\Gamma(k-s+\frac{1}{2})\Gamma(\beta_0+k+1)}{\Gamma(\beta_0+1)\Gamma(n+k-2s+1)\Gamma(k-n+1)}. \tag{16.81}$$

They have the important property that $I_{k,n}^{(1)} = 0$ for $n > k$, i.e., the matrix $I_{k,n}^{(1)}$ has a convenient triangular form. This is due to the fact that the basis functions $B_{I_n}^{\beta_0 (1)}(gt)$ in Eq. (16.75) start out with powers $(gt)^n$. The triangular form will make the inversion of these matrices trivial.

For the second set of functions we find similarly

$$I_{k,n}^{(2)} = \frac{(-1)^n (-\alpha)^k}{4^s \sqrt{\pi}} \frac{\Gamma(k-s)\Gamma(k-s+1/2)\Gamma(\beta_0+k+1)}{\Gamma(\beta_0+1)\Gamma(n+k-2s)\Gamma(k-n+1)}, \tag{16.82}$$

where again $I_{k,n}^{(2)} = 0$ for $n > k$, i.e., the matrix $I_{k,n}^{(2)}$ has once again the convenient triangular form.

The behavior of $I_{k,n}$ for large k follows from $\Gamma(k+1+a) \to k! k^a (1 + \mathcal{O}(1/k))$ as:

$$I_{k,n}^{(1)} \to c_1 (-1)^n (-\alpha)^k k! k^{\beta_0 - 3/2} = c_1 (-1)^n (-\alpha)^k k! k^{\beta} \tag{16.83}$$

$$I_{k,n}^{(2)} \to c_2 (-1)^n (-\alpha)^k k! k^{\beta_0 - 1/2} = c_2 (-1)^n (-\alpha)^k k! k^{\beta}, \tag{16.84}$$

in accordance with (16.60). We now compare the expansion coefficients in the re-expansion

$$\sum_{k=0}^{L} f_k g^k = \sum_{n=0}^{L} h_n I_n = \sum_{k=0}^{L} g^k \sum_{n=0}^{k} h_n I_{k,n}. \tag{16.85}$$

Inverting the relation

$$f_k = \sum_{n=0}^{k} h_n I_{k,n}\,, \tag{16.86}$$

the triangular form of the matrix $I_{k,n}$ leads to a recursion relation for the expansion coefficients h_n:

$$h_k = \frac{1}{I_{k,k}} \left(f_k - \sum_{n=0}^{k-1} h_n I_{k,n} \right). \tag{16.87}$$

16.5.2 Convergent Strong-Coupling Expansion

Having re-expanded the function $f(g)$ as $h_n I_n(g)$, we must find an efficient way of evaluating the functions $I_n(g)$. The power series for $I_n(g)$ is useless since it has the same vanishing radius of convergence as the original expansion of $f(g)$. However, there exists a convergent strong-coupling expansion in powers of $1/g$. Since we shall ultimately need the values of $f(g)$ with g of the order of unity, this expansion is expected to converge fast enough for practical calculations. In order to find the strong-coupling expansion, we rewrite once more the Laplace transform of the hypergeometric function ${}_2F_1$ in terms of MacRobert's function:

$$\int_0^\infty dt\, \mathrm{e}^{-t}\, t^{\beta_0}\, {}_2F_1(a,b,c;-\alpha g t) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\, E(a,b,\beta_0+1;c;1/\alpha g)\,. \tag{16.88}$$

For MacRobert's function, a convergent $1/g$-expansion is well-known. Thus we write

$$\begin{aligned} I_n^{(1)} &= \int_0^\infty dt \frac{e^{-t} t^{\beta_0} (\alpha g t)^n}{\Gamma(\beta_0+1)4^n}\, {}_2F_1(n-s, n-s+\tfrac{1}{2}; 2(n-s)+1; -\alpha g t) \\ &= \frac{\Gamma(2(n-s)+1)(\alpha g)^n}{\Gamma(n-s)\Gamma(n-s+\frac{1}{2})\Gamma(\beta_0+1)4^n} \mathrm{E}(n-s, n-s+\tfrac{1}{2}, \beta_0+n+1; 2(n-s)+1; 1/\alpha g). \end{aligned} \tag{16.89}$$

Expressing MacRobert's function in terms of generalized hypergeometric functions ${}_2F_2(a,b;c,d;1/\alpha g)$, we obtain

$$\begin{aligned} I_n^{(1)} = \frac{(n-s)(\alpha g)^n}{\sqrt{\pi}\Gamma(\beta_0+1)4^s} \times \Bigg[&\frac{\Gamma(1/2)\Gamma(\beta_0+s+1)\Gamma(n-s)}{\Gamma(n-s+1)(\alpha g)^{n-s}}\, {}_2F_2(n-s, s-n, \tfrac{1}{2}, -s-\beta_0; 1/\alpha g) \\ + &\frac{\Gamma(-1/2)\Gamma(\beta_0+s+\frac{1}{2})\Gamma(n-s+\frac{1}{2})}{\Gamma(n-s+1/2)(\alpha g)^{n-s+1/2}}\, {}_2F_2(n-s+\tfrac{1}{2}, s-n+\tfrac{1}{2}; \tfrac{3}{2}, \tfrac{1}{2}-s-\beta_0; 1/\alpha g) \\ + &\frac{\Gamma(-s-\beta_0-1)\Gamma(-\beta_0-s-\frac{1}{2})\Gamma(\beta_0+n+1)}{\Gamma(n-2s-\beta_0)(\alpha g)^{\beta_0+n+1}}\, {}_2F_2(\beta_0+n+1, \beta_0+2s-n-1; s+\beta_0+2, s+\beta_0+\tfrac{3}{4}; 1/\alpha g) \Bigg]. \end{aligned}$$

The expansion of ${}_2F_2(a,b;c,d;1/\alpha g)$ in powers of $1/\alpha g$,

$${}_2F_2(a,b;c,d;1/\alpha g) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k (d)_k} \frac{1}{k!} \left(\frac{1}{\alpha g}\right)^k, \tag{16.90}$$

is convergent for sufficiently large g.

16.5.3 Relation with Conformal Mapping Technique

The present algorithm is mathematically equivalent to the conformal mapping technique. Its main advantage lies in its systematic formal nature, which allows us to write a simple resummation program. The above re-expansion corresponds to choosing the analytic mapping in Eq. (16.45) as

$$w(\alpha g t) = \frac{\alpha g t}{(1+\sqrt{1+\alpha g t})^2} = \frac{\sqrt{1+\alpha g t}-1}{\sqrt{1+\alpha g t}+1}, \tag{16.91}$$

which implies that

$$\frac{1}{(1-w)^{2s}} = \left(\frac{\sqrt{1+\alpha g t}+1}{2}\right)^{2s}, \tag{16.92}$$

and

$$\alpha g t = \frac{4w}{(1-w)^2}. \tag{16.93}$$

We now take the re-expansion equation $\sum_n h_n I_n(g) = \sum_k f_k g^k$ and rewrite both sides as Borel integrals using (16.59) and (16.40). Extracting $B_{I_n}^{\beta_0}$ from (16.75), the integrands satisfy the equation

$$\sum_{n=0}^{\infty} \frac{h_n}{\Gamma(\beta_0+1)} \frac{(\frac{1}{2}+\frac{1}{2}\sqrt{1+\alpha g t})^{2s}\,(\alpha g t)^n}{(1+\sqrt{1+\alpha g t})^{2n}} = \sum_{k=0}^{\infty} \frac{f_k (gt)^k}{\Gamma(k+\beta_0+1)}\,. \tag{16.94}$$

Expressing g in terms of w via (16.93), this becomes

$$\sum_{n=0}^{\infty} h_n \frac{w^n}{(1-w)^{2s}} \quad = \quad \sum_{k=0}^{\infty} \frac{f_k}{(\beta_0+1)_k} \left(\frac{4}{\alpha}\right)^k \left[\frac{w}{(1-w)^2}\right]^k, \tag{16.95}$$

or

$$\sum_{n=0}^{\infty} h_n w^n \quad = \quad \sum_{k=0}^{\infty} \frac{f_k}{(\beta_0+1)_k} \left(\frac{4}{\alpha}\right)^k \frac{w^k}{(1-w)^{2k-2s}}. \tag{16.96}$$

If we expand $1/(1-w)^{2k-2s}$ in powers of w as

$$\frac{w^k}{(1-w)^{2k-2s}} = \sum_{l=0}^{\infty} \frac{\Gamma(l+2k-2s)}{\Gamma(l+1)\Gamma(2k-2s)}\; w^{l+k}\,, \tag{16.97}$$

and replace $l+k \to n$, Eq. (16.96) becomes

$$\sum_{n=0}^{\infty} h_n w^n = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \frac{f_k}{(\beta_0+1)_k} \left(\frac{4}{\alpha}\right)^k \frac{\Gamma(n+k-2s)}{\Gamma(n-k+1)\Gamma(2k-2s)}\right] w^n. \tag{16.98}$$

This is precisely the expansion (16.56) obtained via the conformal mapping technique up to an additional factor $\Gamma(\beta_0+1)$. Comparison of equal powers of w gives

$$h_n = \sum_{k=0}^{n} \frac{f_k}{(\beta_0+1)_k} \left(\frac{4}{\alpha}\right)^k \frac{(n+k-2s-1)!}{(n-k)!(2k-2s-1)!}\,, \tag{16.99}$$

which coincides with the present re-expansion coefficients in (16.87).

16.6 Modified Reexpansions

There exists a simple modification of the algorithm based on the observation in Eqs. (16.87) or (16.99), that changing only one coefficient f_k in the expansion of $f(g)$ changes the coefficients h_n for all $n \geq k$. Suppose, for example, that we subtract the constant term from the expansion of $f(g)$. Then, we obtain a new expansion $\hat{f}(g) \equiv f(g) - f_0 = \sum_k \hat{f}_k g^k$ whose coefficients are related to f_k as follows:

$$\hat{f}_0 = 0, \qquad \hat{f}_k = f_k, \quad \text{for } k \neq 0\,. \tag{16.100}$$

Alternatively, we may write $f(g) = f_0 + g\tilde{f}(g)$ with expansion coefficients

$$\tilde{f}_k = f_{k+1}\,. \tag{16.101}$$

The function $f(g)$ is recovered from

$$f(g) = \sum_{k=0}^{\infty} f_k g^k = f_0 + \sum_{k=0}^{\infty} \hat{f}_k g^k = f_0 + g \sum_{k=0}^{\infty} \tilde{f}_k g^k. \tag{16.102}$$

The parameters $\hat{s}$ and $\hat{\beta}$ in the re-expansion functions of $\hat{f}(g)$ have obviously the same values as those of $f(g)$. In contrast, the parameters $\tilde{s}$ and $\tilde{\beta}$ in the re-expansion functions of $\tilde{f}(g)$ are related to those of $f(g)$ by $\tilde{s} = s - 1$ and $\tilde{\beta} = \beta + 1$. Let us denote the associated expansion functions by $\tilde{I}_n(g)$. They possess the modified large-order behavior $\tilde{f}_k \sim -\alpha(-\alpha)^k \Gamma(k + 1 + \beta + 1)$. If $f(g)$ behaves for large g as g^s, then $\tilde{f}$ behaves as g^{s-1}, and so do the re-expansion functions $\tilde{I}_n(g)$. With $\tilde{s} - 1$ and $\tilde{b}_0 = \tilde{\beta} + 3/2 = \beta + 5/2$, Eq. (16.99) for $\tilde{f}_n$ takes the same form as for $\hat{f}_{n+1}$, the relation between the two expansion coefficients being

$$\tilde{f}_n = (\beta_0 + 1)\frac{\alpha}{4} \sum_{l=1}^{n+1} \frac{f_l}{(\beta_0 + 1)_l} \left(\frac{4}{\alpha}\right)^l \binom{n + 1 + l - 1 - 2s}{n + 1 - l} = (\beta_0 + 1)\frac{\alpha}{4} \hat{f}_{n+1}\,. \tag{16.103}$$

An analogous relation holds for the expansion functions:

$$\tilde{I}_n(g) = \left(\frac{4}{\alpha g}\right) \frac{1}{\beta_0 + 1} I_{n+1}(g)\,, \tag{16.104}$$

so that the two re-expansions satisfy

$$\sum_{n=0}^{L} \hat{f}_n I_n(g) = g \sum_{n=0}^{L-1} \tilde{f}_n \tilde{I}_n(g). \tag{16.105}$$

16.6.1 Choosing the Strong-Coupling Growth Parameter s

In the ϕ^4-theories at hand, the leading strong-coupling power s is not known. It will be chosen to ensure the best convergence of the resummation procedure. We may start with $s = 0$, say, and plot the resummed partial sums against the order in g. This usually yields a smooth curve approaching the final value. When raising s, the convergence is improved. After a certain value of s, the points begin to jump around the smooth curve. The optimal value of s is selected by the condition that the difference between the last two points is minimal. The smallest difference serves as an estimate for the systematic error, and the last value is taken as the result.

Another possibility of selecting an optimal value of s is to raise s until the last and the third-last points have the same value. In this case the average of the last two points is taken as the final result, the error being estimated by half the difference between the last two points. Both procedures give about the same result, as we shall see in Section 17.4.

Notes and References

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