The only place outside of Heaven where you can be perfectly safe
from all the dangers and perturbations of love is Hell.
C. S. Lewis (1898–1963)

10

Quantum Field Theoretic Perturbation Theory

In this chapter we would like to develop a method for calculating the physical consequences of a small interaction in a nearly free quantum field theory. All results will be expressed as power series in the coupling strength. These powers series will have many unpleasant mathematical properties to be discussed in later chapters. In this chapter, we shall ignore such problems and show only how the power series can be calculated in principle. More details can be found in standard textbooks [2, 3, 4].

10.1 The Interacting n-Point Function

We consider an interacting quantum field theory with a time-independent Hamiltonian. All physical information of the theory is carried by the n-point functions

$$G^{(n)}(x_1, \ldots, x_n) \quad = \quad {}_H\langle 0|T\phi_H(x_1)\phi_H(x_2)\cdots\phi_H(x_n)|0\rangle_H. \tag{10.1}$$

Here $|0\rangle_H$ is the Heisenberg ground state of the interaction system, i.e., the lowest steady eigenstate of the full Schrödinger Hamiltonian:

$$H|0\rangle_H = E|0\rangle_H. \tag{10.2}$$

The fields $\phi_H(x)$ are the fully interacting time-dependent Heisenberg fields, i.e., they satisfy

$$\phi_H(\mathbf{x}, t) = e^{-iHt}\phi_S(\mathbf{x})e^{iHt}, \tag{10.3}$$

where

$$H = H_0 + V \tag{10.4}$$

is the full Hamiltonian of the scalar field. Note that the field on the right-hand side of Eq. (10.3) has the time argument $t = 0$ and is therefore the same in any picture $\phi_I(\mathbf{x}, 0) = \phi_H(\mathbf{x}, 0) = \phi(\mathbf{x}, 0)$ so that we shall also write

$$\phi_H(\mathbf{x}, t) = e^{-iHt}\phi(\mathbf{x}, 0)e^{iHt}. \tag{10.5}$$

We now express $\phi_H(x)$ in terms of the field $\phi_I(x)$ of the interaction picture and rewrite

$$\begin{aligned} G^{(n)}(x_1,\ldots,x_n) &= {}_H\langle 0|T\,[U_I(0,t_1)\phi_I(x_1)U_I(t_1,t_2)\phi_I(x_2)\cdots \\ &\qquad \cdots U_I(t_{n-1},t_n)\phi_I(x_n)U_I\,(t_n,0)]\,|0\rangle_H, \end{aligned} \tag{10.6}$$

where we have used the properties of the time displacement operator

$$\begin{aligned} U_I^{-1}(t,0) &= U_I^\dagger(t,0) \;=\; U_I(0,t), \\ U_I(t_1,t_2) &= U_I(t_1,0)U_I^{-1}(0,t_2). \end{aligned} \tag{10.7}$$

We shall now assume that the state $|0\rangle_H$ is a non-degenerate eigenstate of the full Hamiltonian. Then we can make use of the switching-on procedure of the interaction. Then, in the limit $t \to -\infty$, the vacuum state will develop towards the vacuum of the free field Hamiltonian H_0. According to the Gell-Mann–Low formula we may write [5, 6]

$$\begin{aligned} |0\rangle_H &= \frac{U_I\,(0,-\infty)\,|0\rangle}{\langle 0|U_I\,(0,\infty)\,|0\rangle}, \\ {}_H\langle 0| &= \frac{\langle 0|U_I\,(\infty,0)}{\langle 0|U_I\,(\infty,0)\,|0\rangle}, \end{aligned} \tag{10.8}$$

where $|0\rangle$ is the free-particle vacuum. The presence of a switching parameter η and its limit $\eta \to 0$ at the end are tacitly assumed. After this, formula (10.6) becomes

$$\begin{aligned} G^{(n)}\,(x_1,\ldots,x_n) &= \langle 0|U_I\,(\infty,0)\,T\,\Big(U_I\,(0,t_1)\,\phi_I(x_1)U_I\,(t_1,t_2)\,\phi_I(x_2)\cdots \\ &\qquad \cdots U_I(t_{n-1},t_n)\phi_I(x_n)U_I\,(t_n,0)\big)\,U_I\,(0,-\infty)\,|0\rangle \\ &\times\;\; 1/\langle 0|U_I\,(\infty,0)\,|0\rangle\langle 0|U_I\,(0,-\infty)\,|0\rangle. \end{aligned} \tag{10.9}$$

The product in the denominator can be combined to a single expression using the relation (9.95):

$$\langle 0|U_I\,(\infty,0)\,|0\rangle\langle 0|U_I\,(0,-\infty)\,|0\rangle = \langle 0|U_I\,(\infty,-\infty)\,|0\rangle = \langle 0|S|0\rangle. \tag{10.10}$$

The numerator consists of the S-matrix operator $U_I\,(\infty,-\infty)$, time-sliced into $n+1$ pieces at $t_1,\ldots,t_n$, with n fields $\phi(x_i)$, $i = 1,\ldots,n$, inserted successively. It is gratifying to observe that due to the definition of the time-ordering operator, the expression can be written in the much more compact fashion

$$T\Big(S\,\phi_I(x_1)\phi_I(x_2)\cdots\phi_I(x_n)\Big), \tag{10.11}$$

so that we arrive at the simple formula

$$\begin{aligned} G^{(n)}\,(x_1,\ldots,x_n) &= \langle 0|T\Big(S\,\phi_I(x_1)\cdots\phi_I(x_n)\Big)|0\rangle/\langle 0|S|0\rangle \qquad (10.12) \\ &= \frac{\langle 0|Te^{-i\int_{-\infty}^{\infty}dt\,V_I(t)}\phi_I(x_1)\cdots\phi_I(x_n)|0\rangle}{\langle 0|e^{-i\int_{-\infty}^{\infty}dtV_I(t)}|0\rangle}. \end{aligned}$$

The fields $\phi_I(x)$ are now expressed as

$$\begin{aligned} \phi_I(\mathbf{x},t) &= e^{iH_0t}\phi_S(\mathbf{x})e^{-iH_0t} \\ &= e^{iH_0t}\phi(\mathbf{x},0)\,e^{-iH_0t}. \end{aligned} \tag{10.13}$$

This implies that the field $\phi_I(\mathbf{x},t)$ changes in time in the same way as the Heisenberg field $\phi_H(\mathbf{x},t)$ would do if the Hamiltonian H were without interaction. This observation is the key to the upcoming evaluation of the n-point functions.

What is the interaction picture of the interaction V_I itself? We assumed V to be an arbitrary time-independent functional of $\phi_S(\mathbf{x})$,

$$V = V[\phi_S(\mathbf{x})]. \tag{10.14}$$

But then we may use (10.13) to calculate

$$V_I(t) = V[\phi_I(\mathbf{x},t)]. \tag{10.15}$$

Thus the potential $V_I(t)$ in the interaction picture is simply the Schrödinger interaction V with the fields $\phi_S(\mathbf{x})$ replaced by $\phi_I(\mathbf{x},t)$, which develop from the initial configuration $\phi(\mathbf{x},0)$ according to the free-field equations of motion.

The state $|0\rangle$ is the ground state of the free Hamiltonian H_0, i.e., the vacuum state arising in the free-field quantization of Chapters 2 and 4. If we drop the indices I, we can state the interacting n-particle Green function as

$$G^{(n)}(x_1,\dots,x_n) = \frac{\langle 0|Te^{-i\int_{-\infty}^{\infty}dtV[\phi(\mathbf{x},t)]}\phi(x_1)\cdots\phi(x_n)|0\rangle}{\langle 0|Te^{-i\int_{-\infty}^{\infty}dtV[\phi(\mathbf{x},t)]}|0\rangle}, \tag{10.16}$$

where $\phi(\mathbf{x},t)$ is the free field and $|0\rangle$ the vacuum associated with it.

Note that the functional brackets only hold for the spatial variable $\mathbf{x}$. All fields in $V[\phi(\mathbf{x},t)]$ have the same time argument. In a local quantum field theory, the functional is a spatial integral over a density

$$e^{-i\int_{-\infty}^{\infty}dt\,V[\phi(\mathbf{x},t)]}e^{-i\int_{-\infty}^{\infty}dt\int d^3x\,v(\phi(\mathbf{x},t))}. \tag{10.17}$$

10.2 Perturbation Expansion for Green Functions

In general, it is very hard to evaluate expressions like (10.16). If the interaction term V_I is very small, however, it is suggestive to perform a power series expansion and write

$$\begin{aligned} e^{-i\int_{-\infty}^{\infty}dtV[\phi(\mathbf{x},t)]} &= 1 - i\int_{-\infty}^{\infty}dtV[\phi(\mathbf{x},t)] \\ &+ \frac{(-i)^2}{2!}\int_{-\infty}^{\infty}dt_1dt_2T\left(V[\phi(\mathbf{x}_1,t_1)]\,V[\phi(\mathbf{x}_2,t_2)]\right) + \dots\,. \end{aligned} \tag{10.18}$$

In this way, we are confronted in (10.16) with the vacuum expectation value of many free fields $\phi(x)$, of which n are from the original product $\phi(x_1)\cdots\phi(x_n)$, the others from the interaction terms. From Wick's theorem we know that we may reduce the expression to a sum over products of free-particle Green functions $G_0(x-x')$, with all possible pair contractions. The simplest formulation of this theorem was given in terms of the generating functional of all free-particle Green functions [recall (7.840)],

$$Z_0[j] = \langle 0|Te^{i\int d^4x\,\phi(x)j(x)}|0\rangle, \tag{10.19}$$

where the subscript 0 emphasizes now the absence of interactions. This functional can also be used to compactly specify the perturbation expansion. Let us also introduce the generating functional for the interacting case, where the Green functions are expectation values of products of the Heisenberg fields $\phi_H(x)$ in the full Heisenberg vacuum state,

$$Z_H[j] \equiv {}_H\langle 0|Te^{i\int d^4x\,\phi_H(x)j(x)}|0\rangle_H. \tag{10.20}$$

The functional derivatives of this yield the full n-point functions (10.1). The perturbation expansion derived above can now be phrased compactly in the formula

$$Z_H[j] \equiv Z_D[j]/Z[0], \tag{10.21}$$

where $Z[j]$ is the generating functional in the interaction or Dirac picture:

$$Z[j] \equiv \langle 0|Te^{-i\int_{-\infty}^{\infty} dtV[\phi(x)]+i\int d^4x\,\phi(x)j(x)}|0\rangle. \tag{10.22}$$

The fields and the vacuum state in $Z[j]$ are those of the free field theory. Note that $Z_H[j]$ and $Z[j]$ differ only by an irrelevant constant $Z[0]$ which appears in the denominators of all Green functions (10.16), and which has an important physical meaning to be understood in Section 10.3.1. The main difference between them is the prescription of how they have to be evaluated.

As functionals of the sources j, the generating functional yields the perturbation expansion for all n-point functions. This can be verified by functional differentiations with respect to j and comparison of the results with (10.16) and (10.1).

Note that while the generating functional $Z_H[j]$ is normalized to unity for $j \equiv 0$, this is not the case for the auxiliary functional $Z[j]$. However, for generating n-point functions, $Z[j]$ is just as useful as the properly normalized $Z_H[j]$ if one only modifies the differentiation rule by the overall factor $Z[0]^{-1}$:

$$G^{(n)}(x_1,\dots,x_n) = \left[\frac{1}{Z[j]}\frac{\delta}{i\delta j(x_1)}\cdots\frac{\delta}{i\delta j(x_n)}Z[j]\right]_{j=0}. \tag{10.23}$$

This is what we shall do from now on so that we can refer to $Z[j]$ as a generating functional. This will also be of advantage when enumerating the different perturbative contributions to each Green function. Indeed, formula (10.23) enables us to write down an immediate, although formal and implicit, solution for the interacting generating functional: Since differentiation $\delta/\delta j(x)$ produces a field $\phi(x)$ we may rewrite the interaction $V[\phi(x)]$ as $V[-i\delta/\delta j(x)]$. Then it is no longer a field operator. It may be removed from the vacuum expectation value by rewriting $Z[j]$ as

$$Z[j] = e^{-i\int_{-\infty}^{\infty} dt V[-i\delta/\delta j(x)]} Z_0[j]. \tag{10.24}$$

Recall that $Z_0[j]$ was calculated explicitly in (7.843) from Wick's theorem

$$Z_0[j] = \exp\left\{-\frac{1}{2}\int d^4y_1 d^4y_2 j(y_1)G_0(y_1, y_2)j(y_2)\right\}. \tag{10.25}$$

The perturbation series of all n-point functions are now found by expanding the exponential in (10.24) in powers of V and performing the derivatives with respect to $\delta/\delta j(x)$. These produce precisely all Wick contractions involving the fields in the interaction.

The explicit evaluation is quite difficult for an arbitrary interaction. It is therefore advisable to learn dealing with such expressions by considering simple examples.

10.3 Feynman Rules for ϕ^4-Theory

In order to understand the systematics of the perturbation expansion let us focus our attention on a very simple scalar field theory with the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4. \tag{10.26}$$

This is usually referred to as ϕ^4-theory. Here m is the mass of the free particles, and g the interaction strength. We shall assume g to be small enough to be able to expand all interacting Green functions in a power series in g. It is well-known that the resulting series will be divergent since the coefficients of g^k at large order k will grow like $k!$. Fortunately, however, the limiting behavior of the coefficients is exactly known. This has made it possible to develop powerful resummation techniques for extracting reliable results from this series.

The interaction in the Schrödinger picture is

$$V[\phi_S(\mathbf{x})] = \frac{g}{4!}\int d^3x \phi_S^4(\mathbf{x}). \tag{10.27}$$

In the interaction picture, after substituting $\phi_S(x)$ by the free field $\phi(x)$, the exponents in the formulas (10.16), (10.22) become

$$e^{-i\int_{-\infty}^{\infty} dt V[\phi_S(\mathbf{x},t)]} = e^{-i\frac{g}{4!}\int d^4x\, \phi^4(x)}. \tag{10.28}$$

In the functional formulation of the perturbation expansion, we have to calculate the series

$$Z[j] = e^{-i\frac{g}{4!}\int d^4x(-i\delta/\delta j(x))^4} Z_0[j]$$

$$
\begin{aligned}
&= \left[1 - i\frac{g}{4!}\int d^4x \left(-i\frac{\delta}{\delta j(x)}\right)^4\right.\\
&\qquad \left. + \frac{(-i)^2}{2!}\left(\frac{g}{4!}\right)^2 \int d^4x_1 d^4x_2 \left(-i\frac{\delta}{\delta j(x_1)}\right)^2 \left(\frac{-i\delta}{\delta j(x_2)}\right)^2 + \dots\right]\\
&\times \; e^{-\frac{1}{2}\int d^4y_1 d^4y_2 j(y_1) G_0(y_1,y_2) j(y_2)}.
\end{aligned}
\tag{10.29}
$$

The n-point functions are obtained according to (10.23) by expanding the exponential on the right-hand side in a power series, forming the nth functional derivatives with respect to j, and setting j to zero. The result has to be divided by $Z[0]$ which is also a power series in g. Certainly, n has to be even, otherwise the result vanishes. If we want to calculate $G^{(n)}$ up to a given power in g, say g^k, there are many different contributions. The denominator $Z[0]$ has to be expanded in powers of g, and its kth-order contributions come from the $\left(-\frac{1}{2}\int jG_0 j\right)^{2k}/(2k)!$ term in the expansion of the exponential. Here and in most of the following structural formulas we shall omit the integration variables, for brevity. The kth-order term has the form

$$
Z_k[0] = \left(-i\frac{g}{4!}\right)^k \int \left(-i\frac{\delta}{\delta j_1}\right)^4 \cdots \left(-i\frac{\delta}{\delta j_k}\right)^4 \frac{1}{(2k)!}\left(-\frac{1}{2}\int jG_0 j\right)^{2k}. \tag{10.30}
$$

In the numerator of (10.23), there are contributions of zeroth order in g to $G^{(n)}$ from the $(n/2)$th terms which have the form $\left(-\frac{1}{2}\int jG_0 j\right)^{n/2}$. Then there are those of first order in g from the $(n/2+2)$nd terms $\left(-\frac{1}{2}\int jG_0 j\right)^{n/2+2}$, of second order in g from the $(n/2+4)$th term $\left(-\frac{1}{2}\int jG_0 j\right)^{n/2+4}$, etc. Forming the product of four derivatives $(\delta/\delta j)^4$ associated with every order in g, as well as the n derivatives for the Green function $G^{(n)}$, the expressions of kth order have the structure

$$
\begin{aligned}
&\left(-i\frac{g}{4!}\right)^k \left(-i\frac{\delta}{\delta j_1}\right) \cdots \left(-i\frac{\delta}{\delta j_n}\right)\\
&\qquad \times \int \left(-i\frac{\delta}{\delta j}\right)^4 \cdots \left(-i\frac{\delta}{\delta j}\right)^4 \frac{1}{(n/2+2k)!}\left(-\frac{1}{2} jG_0 j\right)^{\frac{n}{2}+2k}.
\end{aligned}
\tag{10.31}
$$

The Green functions accurate to order g^k are then obtained by dividing the two power series (10.31) and (10.30) through each other and expanding the result again up to order g^k.

This all seems to be a horrendous task. It is, however, possible to devise a diagrammatic procedure for keeping track of the different contributions which will cause many simplifications. In particular, the division process is really quite trivial due to the fact that $Z[0]$ appears automatically as a factor in the calculation of the numerator of each n-point function.

Actually, formula (10.29), although it gives the most explicit answer to the problem, is quite cumbersome when it comes to actual calculations. The derivatives are an efficient analytic way of accounting for the set of all Wick contractions of pairs of field operations. In the calculation of a specific n-point function, however, it is much more advantageous to insert the expansion (10.18) into formula (10.16), and to separate numerator and denominator by writing

$$G^{(n)}(x_1,\dots,x_n) \equiv \frac{1}{Z[0]}\bar{G}^{(n)}(x_1,\dots,x_n). \tag{10.32}$$

Here the unnormalized Green function $\bar{G}^{(n)}(x_1,\dots,x_n)$ is the unnormalized Green function. This has the expansion

$$\begin{aligned}\bar{G}^{(n)}(x_1,\dots,x_n) &\equiv \\ &= \langle 0|T\Big\{\Big(1-\frac{ig}{4!}\int d^4z\phi(z)+\frac{1}{2!}\left(\frac{-ig}{4!}\right)^2\int d^4z_1d^4z_2\phi^4(z_1)\phi^4(z_2)+\dots\Big) \\ &\qquad\qquad \times\,\phi(x_1)\cdots\phi(x_n)\Big\}|0\rangle,\end{aligned} \tag{10.33}$$

whereas the denominator $Z[0]$ in (10.32) has the series

$$Z[0]=\langle 0|T\left(1-\frac{ig}{4!}\int d^4z\phi^4(z)+\frac{1}{2!}\left(\frac{-ig}{4!}\right)^2\int d^4z_1d^4z_2\phi^4(z_1)\phi^4(z_2)+\dots\right)|0\rangle. \tag{10.34}$$

By performing the Wick contractions in the two expansions explicitly we obtain $\bar{G}^{(n)}_p$ and $Z_p[0]$, respectively, to be divided by one another.

10.3.1 The Vacuum Graphs

Because of its formal simplicity let us start a more explicit perturbation expansion with the calculation of $Z[0]$. To first-order in the coupling constant g we have to evaluate

$$Z_1[0] = -i\frac{g}{4!}\int d^4z\langle 0|T\left(\phi(z)\phi(z)\phi(z)\phi(z)\right)|0\rangle, \tag{10.35}$$

where we have written down the four powers of $\phi(z)$ separately in order to see better how to perform all pair contractions. The first field can be contracted with the three others. After this the second field has only one choice. Thus there are $3\cdot 1$ contractions, all of the form

$$-i\frac{g}{4!}\int d^4zG_0(z,z)G_0(z,z), \tag{10.36}$$

so that

$$Z_1[0] = -i3\frac{g}{4!}\int d^4zG_0(z,z)G_0(z,z). \tag{10.37}$$

To order g^2 we have to evaluate

$$\frac{1}{2!}\left(-i\frac{g}{4!}\right)^2\int d^4z_1d^4z_2\,\langle 0|T\left(\phi(z_1)\phi(z_1)\phi(z_1)\phi(z_1)\phi(z_2)\phi(z_2)\phi(z_2)\phi(z_2)\right)|0\rangle. \tag{10.38}$$

Expanding the expectation value of the product of eight fields into a sum over pair contractions, we obtain $7 \cdot 5 \cdot 3 \cdot 1 = 105$ contractions, 3^2 of them with $\phi(z_1)$'s and $\phi(z_2)$'s contracting among each other, for example,

$$\phi(z_1)\phi(z_1)\phi(z_1)\phi(z_1)|\phi(z_2)\phi(z_2)\phi(z_2)\phi(z_2), \tag{10.39}$$

where we have explicitly separated the two interactions by a vertical line. There are further $4 \cdot 3 \cdot 2 = 24$ contractions, where each $\phi(z_1)$ connects with a $\phi(z_2)$, for example,

$$\phi(z_1)\phi(z_1)\phi(z_1)\phi(z_1)|\phi(z_2)\phi(z_2)\phi(z_2)\phi(z_2), \tag{10.40}$$

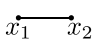
and $6 \cdot 6 \cdot 2 = 72$ of the mixed type, for example,

$$\phi(z_1)\phi(z_1)\phi(z_1)\phi(z_1)|\phi(z_2)\phi(z_2)\phi(z_2)\phi(z_2)\,. \tag{10.41}$$

The factors six counts the six choices of one contraction within each factor ϕ^4 after which there are only two possible interconnections.
The 105 terms obtained in this way correspond to the following integrals

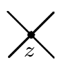
$$\frac{1}{2!}\left(-i\frac{g}{4!}\right)^2\left[9\left(\int d^4z_1 G_0(z_1,z_1)^2\right)^2 + 24\int d^4z_1 d^4z_1 G_0\left(z_1,z_2\right)^4 \right.$$
$$\left. + 72\int d^4z d^4z G_0(z_1,z_1)G_0(z_1,z_2)^2 G_0\left(z_1,z_2\right)^2 G_0(z_2,z_2)\right]. \tag{10.42}$$

It is useful to picture the different contributions by means of so-called *Feynman diagrams*: A line with x_1, x_2 at the ends



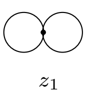
$$= G_0(x_1, x_2) \tag{10.43}$$

represents a free-particle propagator. A vertex with four emerging lines



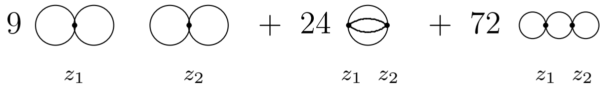
$$= -i\frac{g}{4!} \tag{10.44}$$

stands for the $\phi^4(z)$ interaction at the point z with the convention to carry a coupling constant $-ig/4!$. The spacetime variables of each vertex have to be integrated over. Then the only diagram to first order is

3 

. (10.45)

To second order there are three diagrams



. (10.46)

To third order we find $11 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1 = 10\,395$ terms. The total number rapidly proliferates. Diagrams of this type consisting only of lines which close back into themselves are called *vacuum diagrams*.

When we discussed Eq. (7.140) we noticed an important statistical interpretation of the relativistic euclidean propagator $G(x, x')$. It describes the probability for a random walk of any length τ to go from x to x', provided that its lengths are distributed with an exponential Boltzmann-like factor $e^{-\mu\tau}$. The loop expansion of the partition function in terms of vacuum diagrams may therefore be interpreted as a direct picture of the various topologies of random walks in a grand-canonical ensemble of walks of any length. For this reason, relativistic quantum field theories may be used to study random walk ensembles, in which case they are called *disorder field theories*. As mentioned in Chapter 7, such random walks appear in many physical systems in the form of vortex lines and defect lines.

This line interpretation of quantum fields has led to an entire quantum field theory of physical systems in which the statistical mechanics of line like excitations play an important role for understanding the observed behavior. Consider, for example, the phase transitions in liquid helium. Conventionally, they are understood by describing the liquid as an ensemble of a large number of atoms interacting by a van der Waals type of potential. At low temperatures, below the so-called λ-point $T_\lambda \approx 2.17\,\mathrm{K}$, the atoms enter the superfluid phase in which all atoms behave in a coherent fashion [1]. At zero temperature the entire system lies in a ground state. As temperature rises, thermal fluctuations create small loops of vortex lines. Their average length grows, and at T_λ it diverges. The vortex lines proliferate and fill the entire sample. Since the inside of each vortex line contains a normal liquid, the superfluid becomes normal. This picture gives rise to a completely alternative quantum field theoretic description of superfluid He. At zero temperature, the superfluid is a vacuum for vortex lines, i.e., the disorder field describing them has a zero expectation value. As the temperature rises, more and more disorder excitations are generated, and the field acquires a finite expectation value.

The reader who wants to understand this interesting development is referred to the original literature[1]. Some details will also be discussed in Chapter 19.2.

At this place it is also worth mentioning that the opposite direction of research has been pursued by a number of people, who are trying to understand as field theory as a system of an ensemble of lines. The formalism arising in this way is refereed to as *string-inspired approach* to quantum field theory. It abandons the marvellous power of the field theoretic description of ensembles of lines in favor of some calculational advantages [10]. By construction, this approach conserves the

[1]See Refs. [7, 8, 9].

number of lines, i.e., the number of particles. Thus it will not be efficient when particle condensation processes are important, since then vortex line numbers are certainly not conserved.

10.4 The Two-Point Function

Let us now turn to the numerator in the perturbation expansion for the n-point function (10.29). We shall first study the two-point function. Clearly, to zeroth order there is only the free-particle expression

$$\bar{G}^{(2)}(x_1, x_2) = \langle 0|T\phi(x_1)\phi(x_2)|0\rangle = G_0\,(x_1, x_2) \tag{10.47}$$

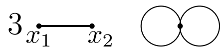
corresponding to the Feynman diagram (10.43). To first order we have to find all contractions of the expression

$$\frac{-ig}{4!}\int d^4z\langle 0|T\phi(z)\phi(z)\phi(z)\phi(z)\phi(z_1)\phi(z_2)|0\rangle. \tag{10.48}$$

There are $5{\cdot}3{\cdot}1 = 15$ of them. They fall into two classes: 3 diagrams contain contractions only among the four fields $\phi(z)$ with the same z, multiplied by a contraction of $\phi(x_1)$ with $\phi(x_2)$. Analytically, they correspond to

$$3\left(\frac{-ig}{4!}\right)\int d^4zG_0^2(z,z)G_0\,(x_1, x_2)\,, \tag{10.49}$$

i.e., this expression carries the same factor 3 that was found in the calculation (10.36) of the vacuum diagrams by themselves. The diagrammatic representation consists of a line and the vacuum diagram side by side



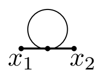
. (10.50)

Such a diagram is called *disconnected*. In general, the analytic expression represented by the disconnected diagram is the product of the expressions corresponding to the individual pieces.

The second class of first-order diagrams collects the contractions between the four $\phi(z)$ and $\phi(x_1)$ or $\phi(x_2)$. There are 12 of them with the analytic expression

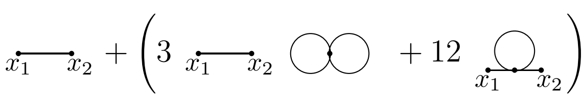
$$12\left(\frac{-ig}{4!}\right)\int d^4zG_0\,(x_1, z)\,G_0\,(z, z)\,G_0\,(z, x_2)\,. \tag{10.51}$$

They are pictured by the connected Feynman diagram



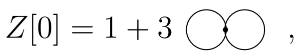
. (10.52)

Thus the expansion to first order has the diagrammatic expansion

$\bar{G}^{(2)}(x_1, x_2) =$


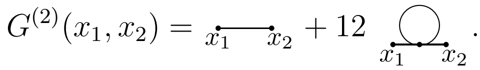
. (10.53)

Remembering the expansion of the denominator to this order



(10.54)

we see that the two-particle Green function is, to order g, given by the free diagram plus the diagram in Fig. 10.52:



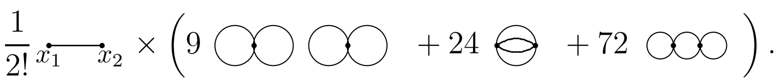
(10.55)

The disconnected pieces involving the vacuum diagram have disappeared.

Consider now the second-order contributions. To obtain $\bar{G}$ we have to form all contractions of

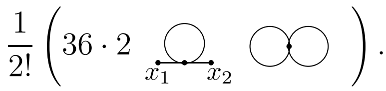
$$\frac{1}{2!}\left(\frac{-ig}{4!}\right)^2 \int d^4z_1 d^4z_2 \langle 0|T\phi(z_1)\phi(z_1)\phi(z_1)\phi(z_1)\phi(z_2)\phi(z_2)\phi(z_2)\phi(z_2) \times \phi(x_1)\phi(x_2)|0\rangle. \tag{10.56}$$

There are $9\cdot 7\cdot 5\cdot 3\cdot 1 = 945$ of them. These decompose into three classes. The first is disconnected and contains the 105 vacuum diagrams multiplied by $G_0(x_1, x_2)$,



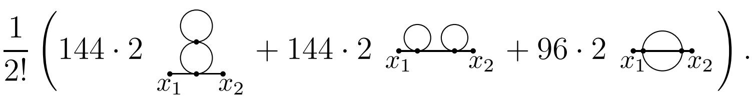
(10.57)

The second consists of mixed contributions in which the first order correction to $\bar{G}$ is combined with a first-order vacuum diagram to



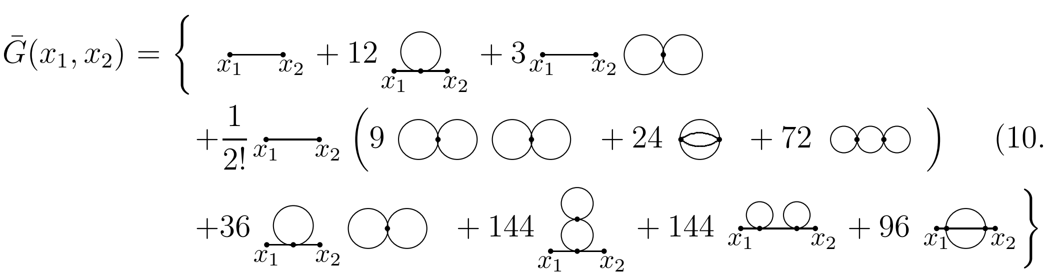
(10.58)

The third contains only the connected diagrams



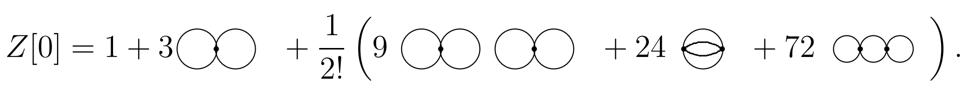
(10.59)

In order to calculate the two-point function up to order g^2 we consider the expansion



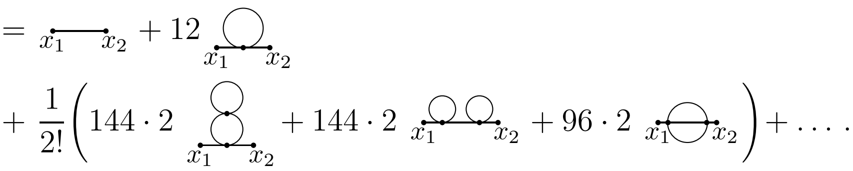
(10.60)

and divide it by the expansion of $Z[0]$ calculated up to the same order. This consists of the diagrams



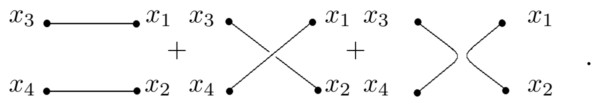
(10.61)

Dividing $\bar{G}(x_1, x_2)$ by $Z[0]$ gives the two-point function

$G^{(2)}(x_1, x_2) =$  (10.62)

10.5 The Four-Point Function

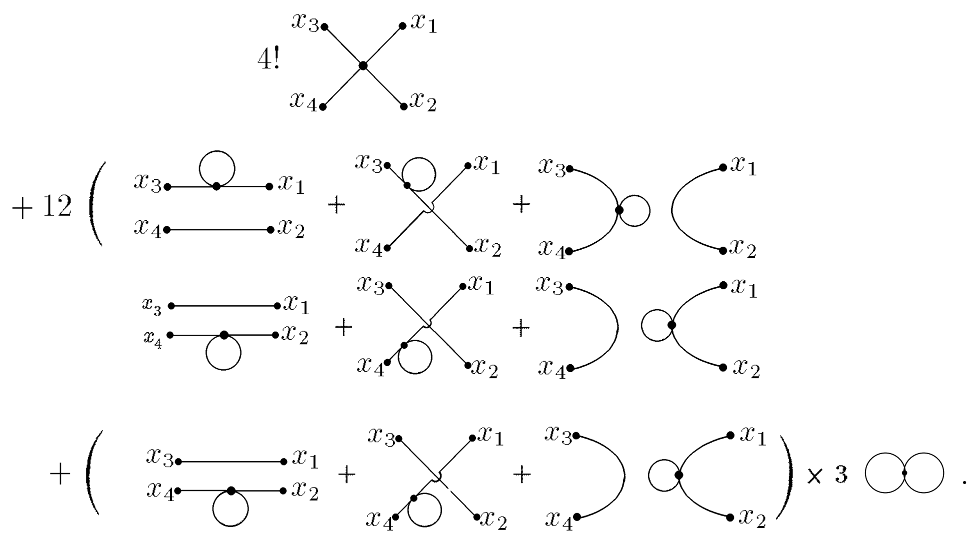
Let us now study the four-point function. To zeroth order, the numerator has the following trivial contributions in which all particles propagate freely

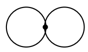
$\bar{G}^{(4)}(x_1, x_2, x_3, x_4) =$  (10.63)

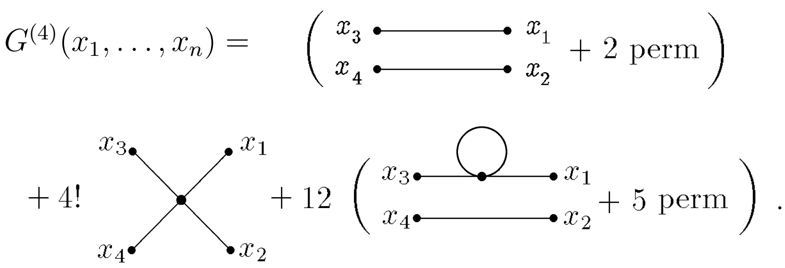
To first order we must form all contractions in

$$\frac{-ig}{4!} \int d^4z_1 d^4z_2 \langle 0|T\phi\,(z_1)\,\phi(z_1)\phi(z_1)\phi(z_1) \times \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle, \quad (10.64)$$

which yield the diagrams

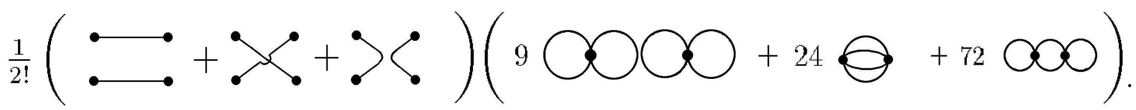
 (10.65)

We observe again the appearance of a factor $1 + 3$  containing the first-order vacuum diagram, which is canceled when forming the quotient (10.32). Thus $G^{(4)}$ can be written to this order as



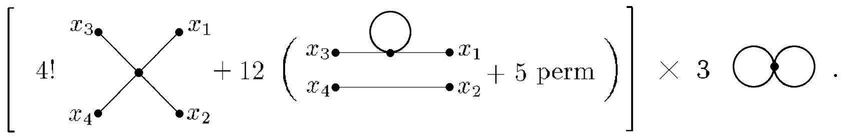
(10.66)

We now turn to the second-order diagrams for which we must form all contractions in (10.56) after exchanging its second line by $\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle$. Their total number is $11 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1 = 10395$ diagrams in $\bar{G}^{(4)}(x_1, x_2, x_3, x_4)$. These can be grouped into 105 vacuum diagrams of second order multiplied with the previously calculated zeroth-order diagram in $\bar{G}^{(4)}(x_1, x_2, x_3, x_4)$:



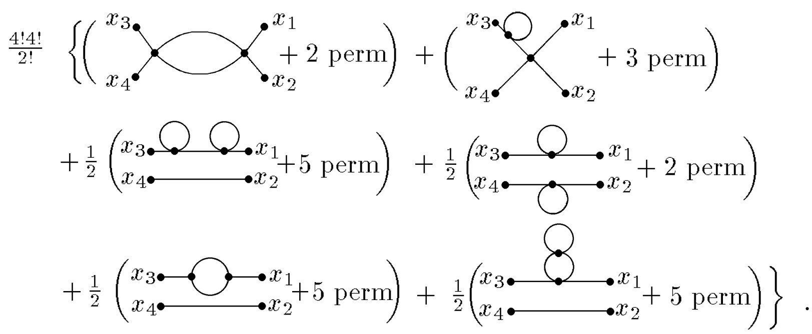
(10.67)

Then there are those in which the vacuum diagrams appear in first order



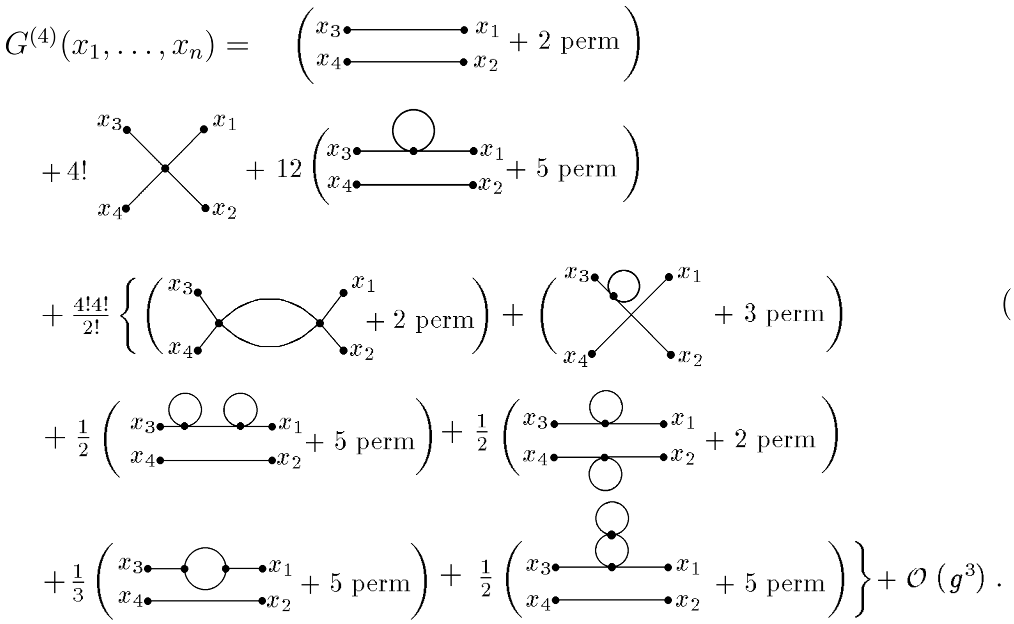
(10.68)

Finally, there are 9504 terms without vacuum contributions



(10.69)

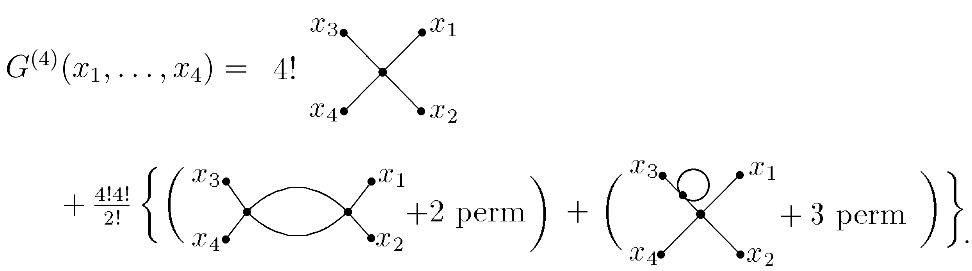
If the vacuum diagrams are divided out, we remain with



(10.70)

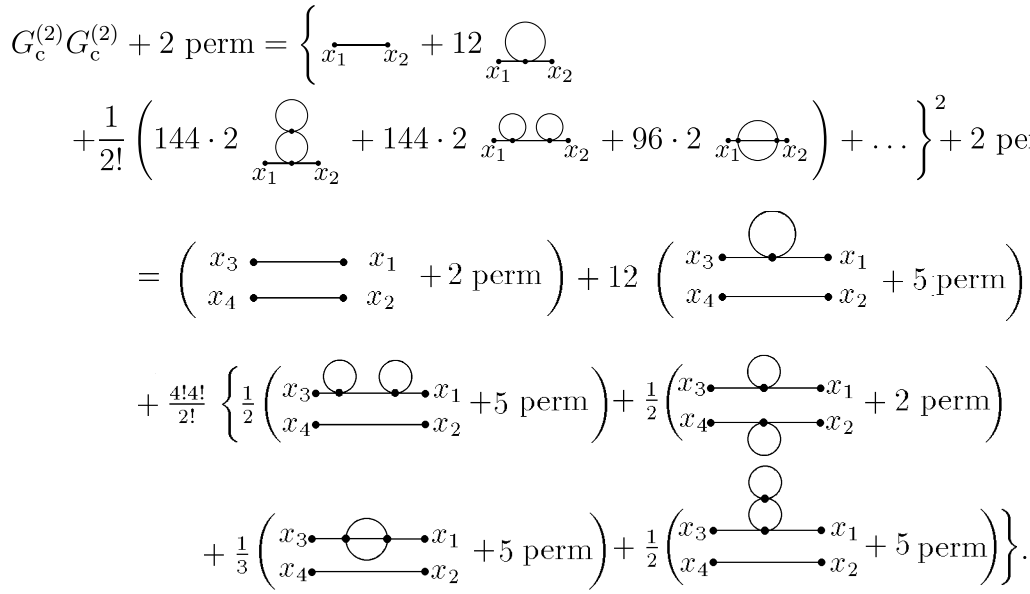
10.6 Connected Green Functions

Faced with the rapid proliferation of diagrams for increasing order in the coupling constant, there is need to economize the calculation procedure. The cancellation of all disconnected pieces involving vacuum diagrams was a great simplification. But the remaining diagrams are still many, even at low order in perturbation theory. Fortunately, not all of these diagrams really require a separate calculation. First of all, there are many diagrams which consist of disconnected pieces, each of which already occurs in the expansion (10.62) of the two-point function. The total amplitude factorizes into the product of expressions, of which each is known from the calculation of $G^{(2)}$. Thus, we can save a great deal of labor if we separate the connected diagrams in $G^{(4)}$ and consider them separately. They are called the connected four-point functions with the notation $G_c^{(4)}(x_1, x_2, x_3, x_4)$. Their low-order expansion is simply



(10.71)

We only have to learn how to recover the full Green function from the connected one and the omitted diagrams which are all known from $G^{(2)}$. In our example it can easily be verified that these omitted parts are simply the product of two propagators $G^{(2)}$ together with the three perturbative corrections on the external legs



(10.72)

We shall see later that this is a completely general law if the field theory is in the so-called *normal phase*. In that phase, the general connectedness structure is

$$G^{(4)}\left(x_1,\ldots,x_n\right)=G_{\rm c}^{(4)}\left(x_1,\ldots,x_n\right)+\left[G_{\rm c}^{(2)}\left(x_1,x_2\right)G_{\rm c}^{(2)}\left(x_3,x_n\right)+2\text{ perm}\right]. \tag{10.73}$$

Note that the expansion (10.62) of $G^{(2)}$ is connected. This is a general feature for a system that is in the normal phase, which will be contrasted with the *condensed phase* in a separated study in Chapters 16, 17, and 18.

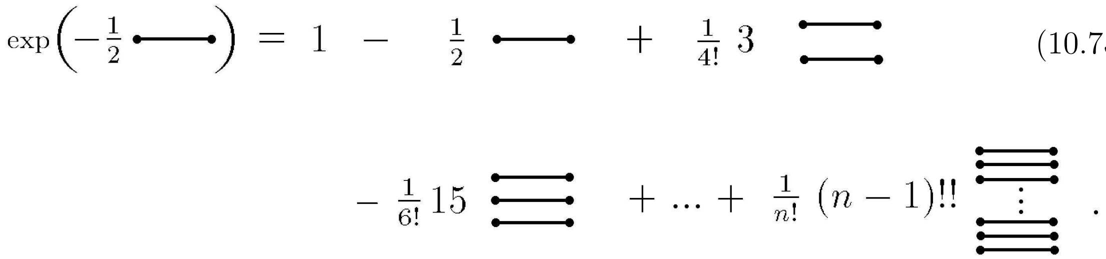
For the higher Green functions we expect more elaborate connectedness relations than (10.73) and an even more drastic reduction of labor using these relations when calculating all diagrams. The question arises as to the general composition law of n-point functions from connected subunits.

To gain a first idea what this law could be, consider the free theory. Its generating functional is [see (5.438)]

$$Z_0[j]=\exp\left\{-\frac{1}{2}\int d^4x\,d^4y\,j(x)G_0(x,y)j(y)\right\}. \tag{10.74}$$

If expanded in powers of j, it gives the sum of all n-point functions of the free theory. In accordance with Wick's theorem all these free n-point functions are disconnected and consist of sums of products of free Green functions G_0, which themselves are the only connected two-point diagrams of the theory. The important point is that the exponential tells us in which way the connected diagrams G_0 can be combined such as to form all diagrams.

This may best be visualized diagrammatically by expanding the exponential in (10.74) in a power series



(10.75)

The numbers behind the factors $1/n!$ in the second line show how many combinations of $n/2$ powers of G_0 occur in Wick's expansion. To obtain these numbers we have rewritten the $(n/2)$th coefficient $\frac{1}{(n/2)!}(1/2)^{n/2}$ as $\frac{1}{n!}(n-1)!!$. This establishes contact with the previous counting rules: The denominator $n!$ is factorized out since it is canceled when going to the n-point function (which involves n differentiations $\delta/\delta j$). This leaves $(n-1)!!$ diagrams in agreement with the result found earlier when counting the diagrams directly. Thus we have verified, in the free-field case, the simple rule for the reconstruction and proper counting of *all* n-point functions, given only the connected ones (of which in this case there is only one). By expanding the exponential of the connected diagrams, which is here $\exp\{-\frac{1}{2}$ •——• $\}$, we can read off *all* connected plus disconnected diagrams behind the factors $1/n!$. In this way, the exponential of the connected diagram yields all diagrams.

Does this simple statement also hold in the interacting case? Here the generating functional is given by

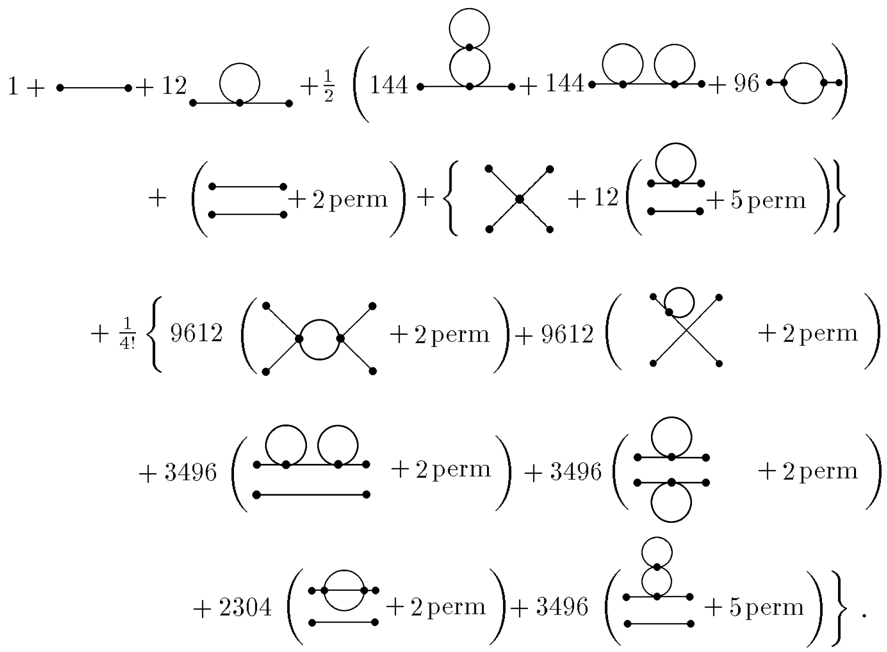
$$Z_0[j] = \exp\left\{i\int d^4x\,\mathcal{L}_{\rm int}\left(\frac{\delta}{i\delta j}\right)\right\}\exp\left\{-\frac{1}{2}\int d^4x\,d^4y\,j(x)G_0(x,y)j(y)\right\}. \quad (10.76)$$

The interactions also enter exponentially. It is then suggestive that also here the sum of all Green functions can be obtained by exponentiating all connected ones. The proof will be given later after having developed more powerful formal techniques. Let us here state only the result which may be written as a relation

$$1+\sum_{n=0,k=0}^{\infty}\frac{1}{n!k!}G_k^{(n)} = \exp\left\{\sum_{n=0,k=1}^{\infty}\frac{1}{n!k!}G_{c\,k}^{(n)}\right\}, \quad (10.77)$$

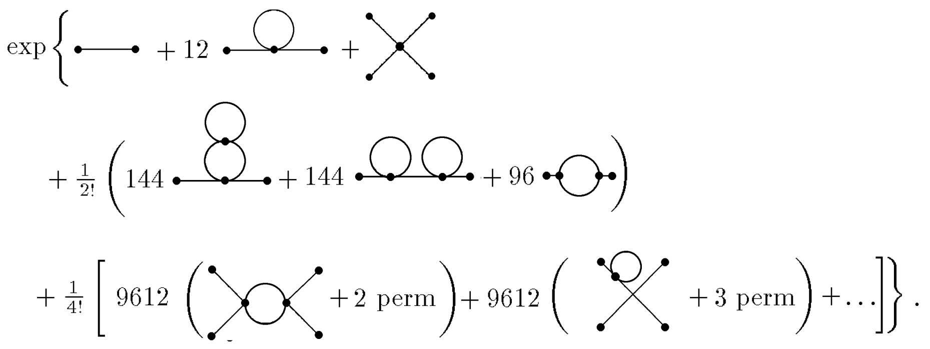
where $G_k^{(n)}$ are all diagrams and $G_{c\,k}^{(n)}$ all connected diagrams in kth-order perturbation theory. A similar relation holds separately for each number of external lines. This will be of great help when it comes to calculating physical scattering amplitudes and cross sections.

We may illustrate the relation (10.77) for the previously calculated diagrams with $n=2$ and $n=4$. The left-hand side of relation (10.77) looks as follows



(10.78)

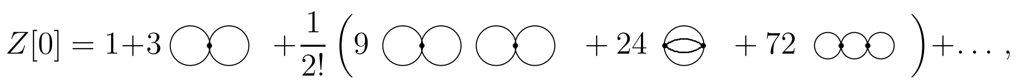
The right-hand side has the form



(10.79)

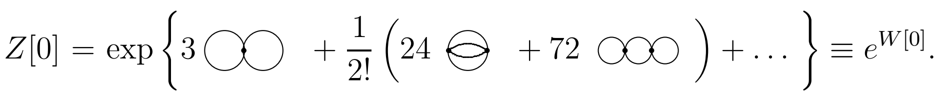
Indeed, by multiplying out the square in the last line we recover the correct sum of disconnected diagrams of the four-point function.

Also the vacuum diagrams satisfy the law of exponentiation: Up to the second order we have for all disconnected pieces



(10.80)

and see that this can be obtained as an exponential of the connected vacuum diagrams



(10.81)

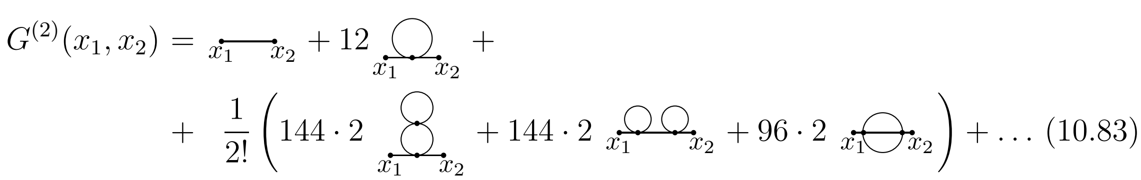
This corresponds to equation (10.77) for $n = 0$:

$$1 + \sum_{k=0}^{\infty} \frac{1}{k!} G_k^{(0)} = \exp\left[\sum_{k=1}^{\infty} \frac{1}{k!} G_{c\,k}^{(0)}\right], \tag{10.82}$$

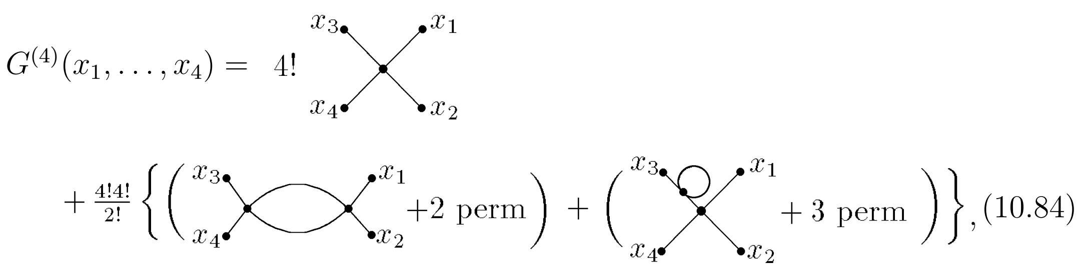
where $G_k^{(0)}$ collect all vacuum diagrams and $G_{c\,k}^{(0)}$ all connected ones in kth-order perturbation theory.

10.6.1 One-Particle Irreducible Graphs

The decomposition into connected diagrams does not yet exhaust the possibilities of reducing calculational labor. If we inspect the connected diagrams for two and four-point functions

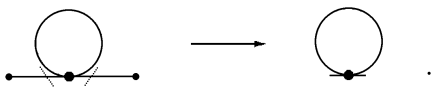


and



we discover that some of the diagrams contain a portion of others calculated at a lower order of perturbation theory in the same connected two-point functions $G_{\rm c}^{(2)}$ or $G_{\rm c}^{(4)}$. An example is the fourth diagram in $G_{\rm c}^{(2)}$, which is a simple repetition of the second one. Similarly, the second diagram in $G_{\rm c}^{(4)}$ is the composition of the first in $G_{\rm c}^{(4)}$ with the second in $G_{\rm c}^{(2)}$. It would be useful to find the rule according to which lower subdiagrams of $G_{\rm c}^{(2)}, G_{\rm c}^{(4)}$ reappear in higher ones of $G_{\rm c}^{(2)}$ and $G_{\rm c}^{(4)}$.

As far as $G_{\rm c}^{(2)}$ is concerned, this rule turns out to be really simple: Let us characterize the repetition of a former subdiagram of $G_{\rm c}^{(2)}$ topologically by noting that the diagram falls into two pieces by cutting one internal line. Such diagrams are called *one-particle reducible* (OPR); otherwise irreducible (OPI). Then the full connected two-point function $G_{\rm c}^{(2)}$ may be composed from all OPI subdiagrams as follows: Consider the set of all OPI diagrams to the two-point function. They all carry a free Green function $G_0^{(2)}$ at the end of each leg which describes propagation of the particle up to the first interaction vertex. Cutting off these last Green functions amounts diagramically to amputating the two legs of the diagram. The lowest order correction to the two-point function is amputated as follows:



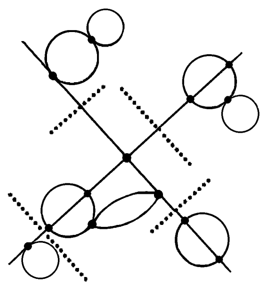
The two short little trunks indicate the places of amputation. Let $-i\Sigma$ be the sum of all these *amputated* OPI two-point functions. Then the geometric series

$$G_{\rm c}^{(2)} = \frac{1}{G_0^{-1} + \Sigma} = G_0 + G_0(-i\Sigma)G_0 + G_0(-i\Sigma)G_0 + \dots \tag{10.85}$$

gives precisely the connected two-point function $G_{\rm c}^{(2)}$. Thus the one-particle reducibility in the two-point function exhausts itself in a simple geometric series type of repetition of the irreducible pieces, each term in the string having the same factor. Also this result will be proved later in Chapter 13 when studying the general formal properties of perturbation theory.

The sum of all OPI connected two-point functions $-i\Sigma$ is usually referred to as self-energy.

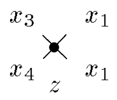
Consider now the four-point function $G_{\rm c}^{(4)}$. Here we recognize that any ornamentation of external legs can be taken care of by replacing the legs by the interacting two-point function. Thus we decide to introduce the concept of an arbitrary one-particle irreducible amputated Green function, shortly called the *vertex function* $\Gamma^{(n)}(x_1, \dots, x_n)$. For any connected n-point function, cut all simple lines such that the diagrams decompose. What remains are parts with two, four, or more trunks sticking out. The first set consists of the OPI self-energy diagrams discussed before. The others are called three-, four-, n-point vertex parts $\Gamma^{(n)}$, $n = 3, 4, \dots$. For example,

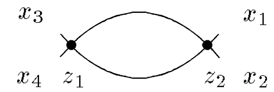


can be cut into four proper self-energy diagrams and one four-point vertex part. The sum of all composite diagrams obtained in this way composes the n-point vertex function denoted by a fat dot. The important reconstruction principle for all diagrams can now be states as follows: The set of all connected diagrams in a four-point function is obtained by connecting all vertex functions in the four-point vertex function $G_{\rm c}^{(4)}$ with the full connected Green function $G_{\rm c}^{(2)}$ at each truncated leg. Analytically, this amounts to the formula (valid in normal systems)

$$G_c^{(4)}(x_1,\dots,x_4)=\int d^4x_1'\,d^4x_2'\,d^4x_3'd^4x_4'\;G_c^{(2)}(x_1,x_1')\cdots G_c^{(2)}(x_4,x_4')\,\Gamma^{(4)}(x_1',\dots,x_4'). \tag{10.86}$$

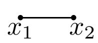
For the Green function $G_c^{(4)}$ we see that, as far as it has been calculated in (10.84), it can indeed be decomposed into a sum of a direct term plus a vertex function to order g^2:

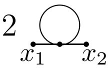
$i\Gamma_c^{(4)}(x_1,\dots,x_n) =$ 4! 

$+\ \frac{1}{2!}4!^2$ (

$+$ 2 perm), (10.87)

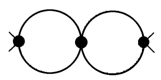
with each pair of vertices being connected to each other by a two-point function

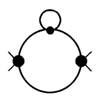
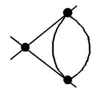
$G^{(2)}(x_1,x_2) =$ 

$+\ 12$ 

. (10.88)

This decomposition of Green functions in terms of vertex functions shows its particular strength when going to higher orders in perturbation theory. Then the number of diagrams to be calculated is greatly reduced. For example, the third-order contributions to the vertex function $i\Gamma_c^{(4)}(x_1,x_2,x_3,x_4)$ are

$i\Gamma_c^{(4)}(x_1,x_2,x_3,x_4) = 144$ ( $+2$ perm)

$+\ 288$ ( $+\ 2$ perm) $+\ 144$ ( $+\ 5$ perm). (10.89)

We leave it up to the reader to compare this with the diagrams in the connected four-point function $G_c^{(4)}$ up to g^3.

For theories with more general interactions than ϕ^4, the composition law is more involved. It will be discussed in Chapter 15.

10.6.2 Momentum Space Version of Diagrams

The spacetime formulation of Feynman rules is inconvenient when it comes to an explicit evaluation of diagrams. It will be of great advantage to exploit the translational invariance of the theory by going to momentum space. The free Green function $G_0(q_1,q_2)$, i.e. the propagator in momentum space, has the very simple Fourier representation

$$\begin{aligned} G_0(q_1,q_2) &\equiv \int d^4x_1 d^4x_2 e^{i(q_1x_1+q_2x_2)} G_0(x_1,x_2) \\ &= \int d^4x_1 d^4x_2 e^{i(q_1x_1+q_2x_2)} \int \frac{d^4q}{(2\pi)^4} e^{-iq(x_1-x_2)} \frac{i}{q^2-m^2} \\ &= (2\pi)^4 \delta^{(4)}\left(q_1+q_2\right) G_0(q_1). \end{aligned} \tag{10.90}$$

There is an overall $(2\pi)^4\,\delta^{(4)}$-function which ensures the conservation of four-momenta. This is a consequence of the translational invariance of $G_0(x_1, x_2) = G_0(x_1 - x_2)$. The same factor appears in the Fourier transform of all interacting n-point functions since $G^{(n)}(x_1, \ldots, x_1)$ depends only on the differences between the coordinates

$$G^{(n)}(x_1, \ldots, x_n) = G^{(n)}(x_1 - x_n, x_2 - x_n, \ldots, x_{n-1} - x_n, 0) \tag{10.91}$$

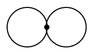
such that we can write

$$\begin{aligned}
&\int d^4x_1 \ldots d^4x_n e^{i\Sigma_{i=1}^n q_i x_i} G^{(n)}(x_1, \ldots x_n) \\
&= \int d^4(x_1 - x_n) \cdots d^4(x_{n-1} - x_n) e^{i\sum_{i=1}^{n-1} q_i(x_i - x_n)} \left(\int d^4x_n e^{i\Sigma_{i=1}^{\infty} q_i x_n}\right) \\
&\qquad \times G^{(n)}(x_1 - x_n, x_2 - x_n, \ldots, x_{n-1} - x_n, 0) \,.
\end{aligned} \tag{10.92}$$

Thus we may define the Fourier transform of an n-point function directly without the factor of momentum conservation as

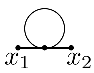
$$(2\pi)^4\delta^{(4)}(q_1 + \ldots + q_n)\, G^{(n)}(q_1, \ldots, q_n) \equiv \int d^4x_1 \cdots d^4x_n\ e^{i\Sigma_{i=1}^n q_i x_i} G^{(n)}(x_1, \ldots, x_n) \,. \tag{10.93}$$

Consider now the vacuum diagrams evaluated via the Fourier transforms. To first order we have

3 
$$= -3i\frac{g}{4!}\int d^4z\ \ G^2(z,z) = -3i\frac{g}{4!}\int d^4z \left[\int \frac{d^4q}{(2\pi)^4}\frac{i}{q^2 - m^2}\right]^2 . \tag{10.94}$$

The integral over z can be defined meaningfully only if the system is enclosed in a finite box of volume V and studied in a finite time interval T. Then the integral $\int d^4z$ gives a factor VT. This would become infinite for large VT which is called the *thermodynamic limit.*

Even if VT is finite, there is still a divergence coming from the integral over the momenta p at large p. This is called an *ultraviolet divergence*. It reflects the singularity of $G_0(x_1, x_2)$ for $x_1 \to x_2$ (a so-called *short-distance singularity*). It will be the subject of the next chapter to show how to deal with this type of divergence. For $\bar{G}^{(2)}(x_1, x_2)$, the diagram of first order in g is

12 

$$= -12i\frac{g}{4!}\int dz G(x_1, z)G(z, z)G(z, x_2) \,. \tag{10.95}$$

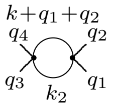
Going to the Fourier transform this gives

$$\int \frac{d^4q}{(2\pi)^4} e^{-iq(x_1 - x_2)} \frac{i}{q^2 - m^2}\left(\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m^2}\right)\frac{i}{q^2 - m^2}, \tag{10.96}$$

which amounts to a contribution to the Fourier-transformed Green function:

$$\bar{G}^{(2)}(q) = -i\frac{g}{4!}12\frac{i}{q^2 - m^2}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m^2}\frac{i}{q^2 - m^2} \,. \tag{10.97}$$

As another example take



$$= \left(-\frac{ig}{4!}\right)^2 \int d\!\!{}^-{}^4k \frac{i}{k^2 - m^2 + i\eta} \frac{i}{(k + q_1 + q_2)^2 - m^2 + i\eta} \, . \qquad (10.98)$$

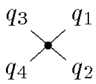
It is easy to see that the following rules hold for the translation of the spacetime diagrams to the analytic expression for the Fourier transformed Green function:

1. With each line associate a momentum label q and specify its direction of flow. Such a line represents a Fourier-transformed two-point function $G_0(q)$ pictured by

$$\text{———} \quad = \quad G_0(q) = \frac{i}{q^2 - m^2} \, . \qquad (10.99)$$

   The arrow has been omitted since its direction is irrelevant on account of the symmetry of $G_0(q)$ under $q \to -q$.

2. Each vertex



   is associated with an amplitude

$$-i\frac{g}{4!}(2\pi)^4\delta^{(4)}(q_1 + q_2 + q_3 + q_4)\frac{i}{q_4^2 - m^2}\frac{i}{q_3^2 - m^2}\frac{i}{q_2^2 - m^2}\frac{i}{q_1^2 - m^2}$$

   containing a four-momentum conserving $(2\pi)^4\,\delta^{(4)}$-function for the incoming momenta. This is a consequence of the integration over z in the interaction.

3. We now distinguish external and internal lines, the latter connecting pairs of vertices with each other. We also call the corresponding momenta q_i external and internal, respectively. Then for every internal line, we integrate over all internal four momenta with the invariant measure $\int d^4q_i/(2\pi)^4$. At the end we remove the associated overall δ-function of energy-momentum conservation.

Obviously, each $(2\pi)^4\delta^{(4)}$-function on the vertices cancels one internal momentum integral $\int d^4p/(2\pi)^4$. Thus in a diagram of nth order with I internal lines, only $I - n$ internal integrations remain. These are referred to as *loop integrals.*

In general, loop integrations diverge at large loop momenta, the so-called *ultraviolet regime.* We shall learn in the next chapter how to deal with such divergent momentum space integrals.

10.7 Green Functions and Scattering Amplitudes

The Green functions carry all informations contained in the theory. In particular they can be used to extract scattering amplitudes. For definiteness, let us discuss here the simplest and most important case of the elastic scattering among two particles. The free initial state that exists long before the interaction takes place is

$$|\psi_{\rm in}\rangle = a^\dagger_{\mathbf{q}_2} a^\dagger_{\mathbf{q}_1}|0\rangle. \tag{10.100}$$

Long after the interaction, the state is given by

$$U_I^\eta(\infty,-\infty)\, a^\dagger_{\mathbf{q}_2} a^\dagger_{\mathbf{q}_1}|0\rangle. \tag{10.101}$$

If we analyze this state with respect to its free-particle content we find the amplitude

$$\langle 0|a_{\mathbf{q}_4} a_{\mathbf{q}_3} U_I^\eta(\infty,-\infty)\, a^\dagger_{\mathbf{q}_2} a^\dagger_{\mathbf{q}_1}|0\rangle = \langle 0|a_{\mathbf{q}_4} a_{\mathbf{q}_3} S^\eta a_{\mathbf{q}_2} a_{\mathbf{q}_1}|0\rangle. \tag{10.102}$$

We shall soon observe that this amplitude has a divergent phase arising in the limit of the switching parameter η tending to zero. It is caused by the same vacuum diagrams as before in the corresponding Green function. In order to obtain a well-defined $\eta \to 0$ -limit we define the 2×2 scattering amplitude as the ratio

$$S(\mathbf{q}_4,\mathbf{q}_3|\mathbf{q}_1,\mathbf{q}_2) \;\equiv\; \frac{S_N(\mathbf{q}_4,\mathbf{q}_3|\mathbf{q}_1,\mathbf{q}_2)}{Z[0]}, \tag{10.103}$$

with the numerator

$$S_N(\mathbf{q}_4,\mathbf{q}_3|\mathbf{q}_1,\mathbf{q}_2) \;\equiv\; \langle 0|a_{\mathbf{q}_4} a_{\mathbf{q}_3} T e^{-i\int_{-\infty}^{\infty} dt\, V_I(t)} a^\dagger_{\mathbf{q}_2} a^\dagger_{\mathbf{q}_1}|0\rangle, \tag{10.104}$$

and the denominator

$$Z[0] \;\equiv\; \langle 0|e^{-i\int_{-\infty}^{\infty} dt\, V_I(t)}|0\rangle. \tag{10.105}$$

We shall often use the four-momentum notation $S(q_4,q_3|q_1,q_2)$ for $S(\mathbf{q}_4,\mathbf{q}_3|\mathbf{q}_1,\mathbf{q}_2)$ with the tacit understanding that, in the S-matrix, the energies are always on the mass shells $q^0 = \sqrt{\mathbf{q}^2+m^2}$.

It is now easy to see how these amplitudes can be extracted from the Green functions calculated in the last section. There exists a mathematical framework to do this known as the *Lehmann-Symanzik-Zimmermann formalism* (LSZ-reduction formulas) [11]. Rather than presenting this we sketch here a simple pedestrian approach to obtain the same results.

We begin with the observation that if the energies q^0 on the mass shells of the particles, i.e., if $q^0 = \omega_{\mathbf{q}} = \sqrt{\mathbf{p}^2+M^2}$, the particle operators $a_{\mathbf{q}}, a^\dagger_{\mathbf{q}}$ can be written as the large-time limits

$$a_{\mathbf{q}} \;=\; \lim_{x^0\to-\infty}\sqrt{\frac{2q^0}{V}}\int d^3x\, e^{i(q^0x^0-\mathbf{q}\mathbf{x})}\phi(x), \tag{10.106}$$

$$a^\dagger_{\mathbf{q}} \;=\; \lim_{x^0\to\infty}\sqrt{\frac{2q^0}{V}}\int d^3x\, e^{-i(q^0x^0-\mathbf{q}\mathbf{x})}\phi(x). \tag{10.107}$$

The limits have the important effect of eliminating undesired frequency contents in $\phi(x)$. Indeed, if we expand the field into creation and annihilation operators, we see that the right-hand side of Eq. (10.106) becomes

$$\lim_{x^0\to-\infty}\sqrt{\frac{2q_0}{V}}\int d^3x e^{i(q^0x^0-\mathbf{qx})}\sum_{\mathbf{q}}\frac{1}{\sqrt{2p^0V}}\left(e^{-ipx}a_{\mathbf{p}}+\text{c.c.}\right)$$
$$=\lim_{x^0\to-\infty}\sum_{\mathbf{p}}\delta_{\mathbf{p},\mathbf{q}}\left[e^{i(q^0-p^0)x^0}a_{\mathbf{p}}+e^{i(q^0+p^0)x^0}a^\dagger_{\mathbf{p}}\right]. \tag{10.108}$$

The spatial δ-function enforces $\mathbf{q}=\mathbf{p}$ and thus $q^0=p^0$, so that the right-hand side becomes

$$a_{\mathbf{q}}+\lim_{x^0\to-\infty}e^{i2q^0x^0}a^\dagger_{\mathbf{q}}. \tag{10.109}$$

In the limit $x^0\to-\infty$, the second exponential function oscillates rapidly with diverging frequency. Such an oscillating expression can be set equal to zero. The reason why this makes sense uses the fact that no physical state is completely sharp in momentum space but contains some, possibly very narrow, distribution function $f(\mathbf{q}-\mathbf{q}')$ in the momenta. Thus, instead of $a_{\mathbf{q}}$, we really deal with a packet state

$$\int\frac{d^3q'}{(2\pi)^3}f(\mathbf{q}-\mathbf{q}')a_{\mathbf{q}'},$$

with $f(\mathbf{q}-\mathbf{q}')$ sharply peaked around $\mathbf{q}$. Then Eq. (10.114) has to be smeared out with such a would-be δ-function, and the second term in (10.109) becomes

$$\lim_{x^0\to-\infty}\int\frac{d^3q'}{(2\pi)^3}e^{i2q'^0x^0}f(\mathbf{q}-\mathbf{q}')a^\dagger_{\mathbf{q}'}\to 0. \tag{10.110}$$

The vanishing of this in the limit $x^0\to-\infty$ is a well-known consequence of the Riemann-Lebesgue Lemma (recall the remarks on p. 262). The other equation (10.107) is proved similarly.

We can now make use of formula (10.107), replace the operators $a_{\mathbf{q}}, a^\dagger_{\mathbf{q}}$ by time-ordered fields $\phi(x)$ and obtain, for the numerator part of the S matrix elements in Eq. (10.104), the following expression:

$$S_N(\mathbf{q}_4\mathbf{q}_3|\mathbf{q}_2\mathbf{q}_1)=\frac{\sqrt{2^4q_1^0q_2^0q_3^0q_4^0}}{V^2}\lim_{\substack{x_1^0>x_2^0\to\infty\\ x_4^0<x_3^0\to-\infty}}e^{i\left[q_4^0x_4^0-\mathbf{q}_4\mathbf{x}_4+q_3^0x_3^0-\mathbf{q}_3\mathbf{x}_3-q_2^0x_2^0+\mathbf{q}_2\mathbf{x}_2-q_1^0x_1^0+\mathbf{q}_1\mathbf{x}_1\right]}$$
$$\times\langle 0|T\phi(x_4)\phi(x_3)S\phi(x_2)\phi(x_1)|0\rangle. \tag{10.111}$$

The last factor is precisely the four-point function $G^{(4)}(x_4,x_3,x_2,x_1)$.

This formula looks somewhat cumbersome to implement in an actual calculation of the scattering amplitudes and it is useful to simplify it by evaluating the infinite-time limits more explicitly. We observe that the perturbation series for

$G^{(n)}(x_n, \dots, x_1)$ consists of sums of products of free two-point functions G_0 which contain, for each spacetime argument $x_1, x_2, \dots, x_n$ in $G^{(n)}$, a two-point function G_0 whose line ends at that point. Consider, for example, the point x_1, and an associated Green function $G_0(z_1, x_1)$. The operation (10.108) at this point corresponds to taking the limit

$$\sqrt{\frac{2q_1^0}{V}} \lim_{x_1^0 \to -\infty} \int d^3x_1 e^{-i(q_1^0 x_1^0 - \mathbf{q}_1 \mathbf{x}_1)} \int \frac{d^4q}{(2\pi)^4} e^{-iq(z-x_1)} \frac{i}{q^2 - m^2 + i\eta}. \tag{10.112}$$

The spatial integral over $\mathbf{x}_1$ enforces $\mathbf{q} = \mathbf{q}_1$. The remaining integral over dq^0 can be done via Cauchy's residue theorem. Since $x_1^0 \to -\infty$, the contour of integration may be closed in the lower half of the complex q^0-plane, where it contains only a pole at $q^0 = \omega_{\mathbf{q}_1} = \sqrt{\mathbf{q}_1^2 + m^2} - i\eta$. Thus the integral over q^0 can be done trivially, and we find, that the limit (10.111) has the effect of replacing, in the Green function $G(z, x_1)$ of the external leg, the amplitude by

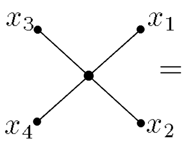
$$G(z, x_1) \longrightarrow \frac{1}{\sqrt{2q_1^0 V}} e^{-iq_1 z}, \tag{10.113}$$

with q_1^0 on the mass shell $q_1^0 = \sqrt{\mathbf{q}_1^2 + m^2}$. The right-hand side is simply the wave function $e^{-iq_1 z}/\sqrt{2q_1^0 V}$ of the incoming particle with the argument z of the nearest vertex in the Feynman diagram.

Similarly, we obtain for an outgoing particle of momentum q_3 the replacement

$$G(x_3, z) \longrightarrow \frac{1}{\sqrt{2q_3^0 V}} e^{iq_3 z}. \tag{10.114}$$

As a specific example, consider the simple vertex diagram in (10.84):



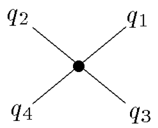
$$= -ig \int d^4z \;\; G(x_4, z)G(x_3, z)G(z, x_2)G(z, x_1). \tag{10.115}$$

Taking the limits in (10.111) this becomes

$$\prod_{i=1}^{4} \left(\frac{1}{\sqrt{2Vq_i^0}} \right) (-ig) \int d^4z \;\; e^{i(q_4 z + q_3 z - q_1 z - q_2 z)}$$
$$= -ig \prod_{i=1}^{4} \left(\frac{1}{\sqrt{2Vq_i^0}} \right) (2\pi)^4 \delta^{(4)} (q_4 + q_3 - q_2 - q_1). \tag{10.116}$$

Thus, up to a factor $1/\sqrt{2Vq_i^0}$ for each particle, we remain precisely with the vertex contribution in momentum space, including the total four-momentum conservation factor.

In momentum space, the Feynman diagram (10.115) reads



$$= \quad -ig\frac{i}{q_4^2-m^2}\frac{i}{q_3^2-m^2}\frac{i}{q_2^2-m^2}\frac{i}{q_1^2-m^2} \quad .$$

Thus, the replacement of the four external lines by external physical states of momenta q_i corresponds to dropping the factors

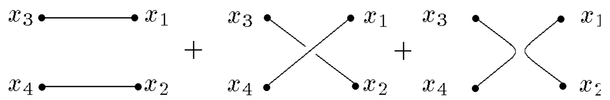
$$\frac{i}{q_i^2-m^2}, \tag{10.117}$$

and replacing them by

$$\frac{1}{\sqrt{2Vq_i^0}}. \tag{10.118}$$

This corresponds to the amputation of the Feynman diagram for the four-point function introduced earlier when defining the vertex functions.

How about the disconnected diagrams in (10.70)? Consider first the three diagrams containing two disconnected lines



In each of the diagrams we have to do the operations of the type (10.100). Take the first diagram and consider $G_0(x_3, x_1)$. It contributes a factor

$$\sqrt{\frac{2q_3^0}{V}}\sqrt{\frac{2q_1^0}{V}}\lim_{\substack{x_3^0\to\infty \\ x_1^0\to-\infty}}\int d^3x_3 d^3x_1 \quad e^{i(q_3^0x_3^0-\mathbf{q}_3\mathbf{x}_3-q_1^0x_1^0+\mathbf{q}_1\mathbf{x}_1)}G(x_3,x_1). \tag{10.119}$$

Performing the first limit $x_1 \to -\infty$ we get

$$\sqrt{\frac{2q_3^0}{V}}\lim_{x_3^0\to\infty}\int d^3x_3 \quad e^{iq_3x_3}e^{-iq_1x_3}\frac{1}{\sqrt{2q_1^0V}} = \lim_{x_3^0\to\infty} e^{i(q_3^0-q_1^0)x_3^0}\delta_{\mathbf{q}_3\mathbf{q}_1} = \delta_{\mathbf{q}_3\mathbf{q}_1}. \tag{10.120}$$

This is just the amplitude for the particle 1 running to the final state 2 without interaction:

$$\langle\mathbf{q}_3|\mathbf{q}_1\rangle = \langle 0|a_{\mathbf{q}_3}a^\dagger_{\mathbf{q}_1}|0\rangle = \delta_{\mathbf{q}_3,\mathbf{q}_1} \tag{10.121}$$

The same factor appears for the lower line of the diagram

$$\langle\mathbf{q}_4|\mathbf{q}_2\rangle. \tag{10.122}$$

Thus the first of the above three diagrams corresponds to the S-matrix element

$$\delta_{\mathbf{q}_3\mathbf{q}_1}\delta_{\mathbf{q}_2\mathbf{q}_4}. \tag{10.123}$$

The second diagram results in the same product of δ-functions, except that $\mathbf{q}_3$ and $\mathbf{q}_4$ are interchanged. The third diagram is different. Here the Fourier limit to be done is

$$\sqrt{\frac{2q_1^0}{V}}\sqrt{\frac{2q_2^0}{V}}\lim_{t_1<t_2\to-\infty}\int d^3x_1 d^4x_2^3 e^{-i(q_2^0x_2^0-\mathbf{q}_2\mathbf{x}_2-q_1^0x_1^0+\mathbf{q}_1\mathbf{x}_1)}G(x_2,x_1). \tag{10.124}$$

The first limit on x_1 leads to

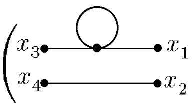
$$\sqrt{\frac{2q_2^0}{V}}\lim_{t_2\to-\infty}\int d^3x_2 e^{-iq_2x_2-iq_1x_2}\frac{1}{\sqrt{2q_1^0V}}\,, \tag{10.125}$$

and the integration gives

$$\lim_{x_2\to-\infty} e^{-i(q_2^0+q_1^0)x_2^0}\delta_{\mathbf{q}_2,-\mathbf{q}_1}\,. \tag{10.126}$$

The limit $x_0 \to -\infty$ of the exponential oscillates infinitely rapid, so that the result vanishes for the same reasons as before.

Let us now look at the diagrams

12 ( + 5 perm) (10.127)

contained in the set (10.70). From the lowest line, this obviously contains a factor

$$\langle \mathbf{q}_4|\mathbf{q}_2\rangle = \delta_{\mathbf{q}_4\mathbf{q}_2}, \tag{10.128}$$

just as in (10.120). The upper line corresponds to the integral

$$12\left(-\frac{ig}{4!}\right)\int d^4z \;\; G(x_3,z)G(z,z)G(z,x_1)\,. \tag{10.129}$$

Adding this to the line without the loop correction leads to the one-loop corrected Green function

$$G_R(x_3,x_1) = 12\left(-\frac{ig}{4!}\right)\int d^4z \;\; G(x_3,z)G(z,z)G(z,x_1)\,. \tag{10.130}$$

The Fourier representation of the free Green function $i/(q^2-m^2)$ is replaced by

$$\frac{i}{q^2-m^2} \to \frac{i}{q^2-m^2} - i\,\frac{g}{2}\int\frac{d^4q'}{(2\pi)^4}\frac{i}{q'^2-m^2}\left[\frac{i}{q^2-m^2}\right]^2. \tag{10.131}$$

This may be viewed as the first-order corrected Fourier transform of the renormalized propagator

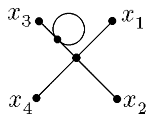
$$G_R(x_2,x_1) = \int\frac{d^4q}{(2\pi)^4}e^{-iq(x_2-x_1)}\frac{i}{q^2-m^2-\delta m^2} \tag{10.132}$$

with a mass shift

$$\delta m^2 = \frac{g}{2} \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2}. \tag{10.133}$$

The scattering amplitude is extracted from the amplitude with the renormalized Green function as before, the only difference being that the factor (10.118) contains now q_i^0 with the renormalized masses.

A similar mass shift occurs in the diagrams

$\frac{4!\,4!}{2!}$ ( $+3$ perm) (10.134)

contained in the set (10.70). The remaining diagrams of (10.70) account for the second-order mass shifts.

To study such mass shifts in general we make use of the Gell-Mann–Low formula for the energy shift [5, 6]. For the vacuum, the energy shifts by ΔE_0 can be taken from the matrix element

$$e^{-i\Delta E_0(t_2-t_1)} = \langle 0|U_I^\eta\,(t_2,t_1)\,|0\rangle \tag{10.135}$$

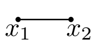
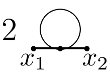
by going to the limit $t_2 \to \infty$, $t_1 \to -\infty$. Consider now the single-particle state $a_{\mathbf{q}}^\dagger|0\rangle$. If the interaction is applied for $t_1 \to -\infty$ and $t_2 \to \infty$, the state

$$U_I^\eta\,(t_2,t_1)\,a_{\mathbf{q}}^\dagger|0\rangle \tag{10.136}$$

will be again a solution of the free Hamiltonian. Because of energy and momentum conservation, it must be equal to $a_{\mathbf{q}}^\dagger|0\rangle$ up to a phase which, due to (10.135), contains the information on the energy shift of this state. It consists of ΔE_0 for the vacuum plus $\Delta E_{\mathbf{q}}$ for the particle. Collecting both together, we may write

$$e^{-\Delta E(t_1-t_0)} \underset{\substack{t_1\to\infty \\ t_2\to-\infty}}{=} \frac{\langle 0|a_{\mathbf{q}}U_I^\eta\,(t_2t_1)\,a_{\mathbf{q}}^\dagger|0\rangle}{\langle 0|U_I^\eta\,(t_2,t_1)\,|0\rangle}. \tag{10.137}$$

Expanding U_I^η in powers of g, the lowest diagrams on the right-hand side are precisely the diagrams that appeared before:

$G^{(2)}(x_1,x_2) =$  $+\,12$  $+\ \ldots\ .$

Indeed, we find from (10.137) with $t_1 - t_0 = T$:

$$\begin{aligned} 1 - i\Delta E\,T &= 1 + 12\left(\frac{-ig}{4!}\right)\frac{1}{2q^0V}\int d^4z \;\; G(z,z) \\ &= 1 - \frac{i}{2}g\frac{T}{2q^0}\int \frac{d^4q}{(2\pi)^4}\frac{i}{q^2-m^2}\,. \end{aligned} \tag{10.138}$$

Similarly, we may use formulas (10.106), (10.107), and the energy shifts $q^0 \to q^0 + \Delta E = q_R^0$, to write

$$a_{\mathbf{q}}^\dagger = \lim_{x^0\to-\infty} \sqrt{\frac{2q_R^0}{V}} \int d^3x\, e^{i(q_R^0 x^0 - \mathbf{qx})}\phi(x), \tag{10.139}$$

$$a_{\mathbf{q}} = \lim_{x^0\to\infty} \sqrt{\frac{2q_R^0}{V}} \int d^3x\, e^{-i(q_R^0 x^0 - \mathbf{qx})}\phi(x). \tag{10.140}$$

Now every particle line automatically receives a factor $e^{i\Delta E_q T}$, which removes precisely the phase factor (10.135).

To higher orders in g, it is somewhat hard to proceed in this fashion. The problem will be solved in the next section with more elegance.

Note that formula (10.137) allows us to evaluate the shift in the particle mass in another way. If $q^0 = \sqrt{\mathbf{q}^2 + m^2}$ is the energy before turning on the interaction, then

$$q^0 + \Delta E = \sqrt{\mathbf{q}^2 + m^2 + \delta m^2} = q^0 + \frac{\delta m^2}{2q^0} \tag{10.141}$$

is the energy afterwards, and we find once more the mass shift (10.133).

10.8 Wick Rules for Scattering Amplitudes

The possibilityX of obtaining external particle states from fields with the help of temporal limiting procedures of the type (10.107) allows us to incorporate the creation and annihilation operators of these particles into the general framework of Wick's contraction rules. Confronted with the numerator of the perturbative scattering amplitude in (10.103),

$$S_N\left(\mathbf{q}_4\mathbf{q}_3|\mathbf{q}_1\mathbf{q}_2\right) \equiv \langle 0|a_{\mathbf{q}_4}a_{\mathbf{q}_3}Te^{-i\int_{-\infty}^{\infty}dtV_I(t)}a_{\mathbf{q}_2}^\dagger a_{\mathbf{q}_1}^\dagger|0\rangle, \tag{10.142}$$

we may imagine the incoming particle operators on the one side to carry negative infinite time arguments, and the outgoing ones on the other side to carry positive infinite times. Then they can be brought inside the parentheses of the time-ordering operator. This, in turn, can be evaluated as usual via Wick contractions.

The results of the last section show that the Wick contractions leading to an external particles are

$$\overbrace{\phi(x)a_{\mathbf{p}}^\dagger} = [\overbrace{\phi(x),a_{\mathbf{p}}^\dagger}] = \frac{1}{\sqrt{2V\omega_{\mathbf{p}}}}e^{-iqx}, \tag{10.143}$$

$$\overbrace{a_{\mathbf{p}}\phi}(x) = [\overbrace{a_{\mathbf{p}},\phi}(x)] = \frac{1}{\sqrt{2V\omega_{\mathbf{p}}}}e^{-iqx}, \tag{10.144}$$

where $\omega_{\mathbf{p}} = \sqrt{\mathbf{p} + M^2}$.

For Dirac particles, there are the corresponding contraction rules

$$\overline{\psi(x) a^\dagger_{\mathbf{p},s_3}} = \{\overline{\phi(x), a^\dagger_{\mathbf{p}}}\} = \frac{1}{\sqrt{VE_{\mathbf{p}}}} u(\mathbf{p}, \mathbf{s}_3) e^{-ipx}, \tag{10.145}$$

$$\overline{a_{\mathbf{p},s_3} \psi}(x) = \{\overline{a_{\mathbf{p},s_3}, \psi}(x)\} = \frac{1}{\sqrt{VE_{\mathbf{p}}}} e^{ipx} \bar{u}(\mathbf{p}, \mathbf{s}_3), \tag{10.146}$$

and for antiparticles:

$$\overline{\bar{\psi}(x) b^\dagger_{\mathbf{p},s_3}} = \{\overline{\bar{\phi}(x), b^\dagger_{\mathbf{p}}}\} = \frac{1}{\sqrt{VE_{\mathbf{p}}}} e^{-ipx} \bar{v}(\mathbf{p}, \mathbf{s_3}), \tag{10.147}$$

$$\overline{b_{\mathbf{p},s_3} \psi}(x) = \{\overline{a_{\mathbf{p},s_3}, \psi}(x)\} = \frac{1}{\sqrt{VE_{\mathbf{p}}}} v(\mathbf{p}, \mathbf{s}_3) e^{ipx}, \tag{10.148}$$

where $E_{\mathbf{p}} \equiv \sqrt{\mathbf{p} + M^2}$.

10.9 Thermal Perturbation Theory

Since Wick's theorem was valid for thermal Green functions, we expect all perturbation expansions to have a simple generalization to the thermal case. Let us define a thermal Heisenberg picture for operators by

$$O_H(\tau) = e^{H\tau/\hbar} O_S e^{-H\tau/\hbar}, \tag{10.149}$$

and an interaction picture which moves according to the free equations of motion,

$$O_I(\tau) = e^{H_{0G}\tau/\hbar} O_S e^{-H_{0G}\tau/\hbar}. \tag{10.150}$$

Thus, a Heisenberg operator can be transformed to the free operator via

$$O_H(\tau) = U_I(0, \tau) O_I(\tau) U_I(\tau, 0) \tag{10.151}$$

where

$$U_I(\tau_2, \tau_1) = e^{H_{0G}\tau_2/\hbar} e^{-H_G(\tau_2 - \tau_1)} e^{-H_{0G}\tau_1/\hbar} \tag{10.152}$$

is the time displacement operator along the euclidean time axis τ. Therefore it has the same factorization property as the quantum mechanical operator $U_I(t_2, t_1)$ in Eq. (9.17):

$$U_I(\tau_3, \tau_2) U_I(\tau_2, \tau_1) = U_I(\tau_3, \tau_1). \tag{10.153}$$

Certainly

$$U_I(\tau_1, \tau_1) = 1. \tag{10.154}$$

However, in contrast to $U_I(t_2, t_1)$, this operator does *not* satisfy the unitarity relation (9.26), but instead:

$$U_I(\tau_2, \tau_1)^\dagger = U_I(-\tau_1, -\tau_2). \tag{10.155}$$

It obeys the following equation of motion

$$\begin{aligned} \hbar\partial_\tau U_I(\tau, \tau') &= e^{H_0\tau/\hbar}\,(H_{0G} - H_G)\, e^{-H_G(\tau-\tau)/\hbar} e^{-H_{0G}\tau/\hbar} \\ &= -V_I(\tau)\, U_I(\tau, \tau'), \end{aligned} \tag{10.156}$$

where $V_I(\tau)$ is the time-dependent thermal interaction picture of the time independent Schrödinger perturbation $V = V_S$

$$V_I(\tau) = e^{H_{0G}\tau/\hbar} V e^{-H_{0G}\tau/\hbar}. \tag{10.157}$$

Therefore we can solve (10.156) for $U_I(\tau, \tau')$ using the standard exponential Neumann-Liouville expansion (9.19), now time-ordered with respect to the euclidean time τ:

$$\begin{aligned} U_I(\tau, \tau') &= \sum_{n=0}^{\infty} \left(-\frac{1}{\hbar}\right)^n \frac{1}{n!} \int_{\tau'}^{\tau} d\tau_1 \int_{\tau'}^{\tau} d\tau_2 \ldots \int_{\tau'}^{\tau} d\tau_n T_T \left[V_I(\tau_1) \cdots V_I(\tau_n)\right] \\ &= T_\tau e^{-\frac{1}{\hbar}\int_{\tau'}^{\tau} d\tau'' V_I(\tau'')}. \end{aligned} \tag{10.158}$$

By construction, $U_I(\tau, 0)$ satisfies the relation

$$e^{-H_G\tau/\hbar} = e^{-H_0\tau/\hbar} U_I(\tau, 0). \tag{10.159}$$

Thus, using the thermal value for the imaginary time $\tau/\hbar = 1/k_BT \equiv \beta$, we obtain directly a perturbation expansion of the partition function

$$\begin{aligned} Z &= \mathrm{Tr}\left(e^{-\beta H_G}\right) = \mathrm{Tr}\left[e^{-\beta H_{0G}} U_I\left(\beta, 0\right)\right] \qquad (10.160) \\ &= \sum_{n=0}^{\infty} \left(-\frac{1}{\hbar}\right)^n \frac{1}{n!} \int_0^{\beta} d\tau_1 \cdots \int_0^{\beta} d\tau_n \mathrm{Tr}\left[e^{-\beta H_{0G}} T_\tau \left(V_I(\tau_1) \cdots V_I(\tau_n)\right)\right]. \end{aligned}$$

Consider now a thermal two-body Green function in the presence of interaction, defined for $\tau > \tau'$ as follows:

$$\begin{aligned} G(\mathbf{x}, \tau; \mathbf{x}', \tau') &\equiv \mathrm{Tr}\left[e^{-\beta H_G} \psi(\mathbf{x}, \tau)\psi^\dagger(\mathbf{x}', \tau')\right] / \mathrm{Tr}\left(e^{-\beta H_G}\right) \\ &= \mathrm{Tr}\left[e^{-\beta H_{0G}} U_I\left(\beta, 0\right) U_I(0, \tau)\psi_I(\mathbf{x}, \tau) U_I(\tau, 0)\right. \\ &\quad \left.\times\, U_I(0, \tau')\psi_I^\dagger(\mathbf{x}', \tau') U_I(\tau', 0)\right] / \mathrm{Tr}\left[e^{-\beta H_{0G}} U_I\left(\beta, 0\right)\right]. \end{aligned} \tag{10.161}$$

For $\tau < \tau'$, on the other hand, we have

$$\begin{aligned} G(\mathbf{x},\tau;\mathbf{x}',\tau') &\equiv \pm\mathrm{Tr}\left[e^{-\beta H_G}\psi^\dagger(\mathbf{x}',\tau')\psi(\mathbf{x},\tau)\right]/\mathrm{Tr}\left(e^{-\beta H_G}\right) \\ &= \mathrm{Tr}\left[e^{-\beta H_{0G}}U_I\left(0,\tau'\right)\psi_I^\dagger(\mathbf{x}',\tau')U_I(\tau',0)\right. \\ &\qquad\left.\times\, U_I(0,\tau)\psi_I(\mathbf{x},\tau)U_I(\tau,0)\right]/\mathrm{Tr}\left[e^{-\beta H_{0G}}U_I\left(\beta,0\right)\right]. \end{aligned} \qquad (10.162)$$

Both equations may be combined in the single formula

$$G(\mathbf{x},\tau;\mathbf{x}',\tau') = \mathrm{Tr}\left[e^{-\beta H_{0G}}T_\tau U_I\left(\beta,0\right)\psi(\mathbf{x},\tau)\psi^\dagger\left(\mathbf{x}',\tau'\right)\right]/\mathrm{Tr}\left[e^{-\beta H_{0G}}U_I\left(\beta,0\right)\right]. \qquad (10.163)$$

In comparison with the field theoretic formulas (10.9) and (10.12), the vacuum expectation values are replaced by the Boltzmann-weighted thermal traces, and the vacuum expectation value of the S matrix operator $U_I\left(\infty,-\infty\right)$ in the denominators is replaced by the Boltzmann-weighted trace of the interaction operator $U_I\left(\hbar/k_BT,0\right)$ along the euclidean time axis τ.

Note that the denominator ensures the existence of the zero temperature limit in just the same way as the phase factor did in the switching-on limit $\eta\to 0$. In fact, the grand canonical partition function in the denominator may be written as

$$Z_G = e^{\beta\Omega} = \sum_n e^{-(\beta E_n-\mu N_n)}, \qquad (10.164)$$

such that the limit $T\to 0$ renders a pure exponential with the ground state energy

$$Z_G \underset{T\to 0}{\to} e^{-\beta(E_0-\mu N_0)} \to e^{\pm\infty}. \qquad (10.165)$$

This is the analogue of the infinite phase factor for the field theoretic denominator in Eq. (10.9).

Let us now expand the interaction operator $U_I\left(\hbar/k_BT,0\right)$ in powers of the interaction, just as before in real time. This leads to a series of thermally averaged products of many fields $\psi_I(\mathbf{x},\tau)$ which move according to the free field equations. Therefore Wick's theorem can be applied and we obtain an expansion of $G(\mathbf{x},\tau;\mathbf{x}',\tau')$ completely analogous to the field theoretic one. The only difference is the finite-time interaction. When going to Fourier transformed space, the finite euclidean time interval is fully taken into account by the sum over the discrete imaginary Matsubara frequencies.

We have seen before that the evaluation of field-theoretic Green functions proceeds best via a Wick rotation of the energy in the perturbative digrams to an imaginary axis. This is precisely the axis along which the Matsubara frequencies are situated. Thus, as far as perturbation theory in Fourier transformed space is concerned, the diagrammatic rules are exactly the same, except that the Wick rotated integrals over the imaginary energy have to be replaced by the Matsubara frequency sums [recall Eq. (2.415)]:

$$\int\frac{dp^0}{2\pi}\to i\int_{-\infty}^{\infty}\frac{dp_E}{2\pi}\to\frac{k_BT}{\hbar}\sum_{\omega_m}=\frac{1}{\beta}\sum_{\omega_m}, \qquad \omega_m=\frac{2\pi}{\hbar\beta}\begin{cases} m & \text{for bosons,}\\ m+\frac{1}{2} & \text{for fermions.}\end{cases} \qquad (10.166)$$

In the limit of small temperatures, the Matsubara frequencies move closer and closer to each other, and sums over them tend to frequency integrals.

The two descriptions coincide, apart from the trivial presence of the chemical potential in the grand canonical energy. We conclude that the Wick-rotated calculation of field theoretic Green functions really amounts to thermal equilibrium physics in the limit of zero temperature.

Let us point out that the result (10.163) implies the same periodicity in $\hbar/k_BT = \beta$ of the full Green function $G(\mathbf{x},\tau;\mathbf{x}',\tau')$ as in the free case (2.413). First, we observe that due to the time independence of the Hamiltonian, there is now translational invariance in euclidean time T such that $G(\mathbf{x},\tau;\mathbf{x}',\tau')$ depends only on the difference $\tau-\tau'$:

$$G(\mathbf{x},\tau;\mathbf{x}',\tau') = \mathrm{Tr}\left[e^{-H_G\left(\beta-\frac{\tau}{\hbar}\right)}\psi(\mathbf{x},0)e^{-H_G\frac{\tau-\tau'}{\hbar}}\psi^\dagger(\mathbf{x}',0)e^{-H_G\tau'/\hbar}\right] / \,\mathrm{Tr}\,(e^{-\beta H_G})$$
$$= \mathrm{Tr}\left[e^{-H_G\left(\beta-\frac{\tau}{\hbar}\right)}\psi(\mathbf{x},0)e^{-H_G\tau-\tau'/\hbar}\psi^\dagger(\mathbf{x}',0)\right] / \,\mathrm{Tr}\,(e^{-\beta H_G}). \qquad (10.167)$$

Similarly, we conclude the dependence only on $\mathbf{x}-\mathbf{x}'$ by translational invariance in space, using $\psi\,(\mathbf{x},\tau) = e^{i\mathbf{P}\mathbf{x}/\hbar}\psi(\mathbf{0},\tau)e^{-i\mathbf{P}\mathbf{x}/\hbar}$ with the momentum operator $\mathbf{P}$. Thus we may write

$$G(\mathbf{x},\tau;\mathbf{x}',\tau') = G(\mathbf{x}-\mathbf{x}',\tau-\tau'), \qquad (10.168)$$

just as for the field theoretic Green functions in the vacuum. Now it is easy to see the periodicity. For the interval $\tau-\tau' \in (-\hbar/k_BT, 0)$ we calculate

$$\begin{aligned} G\,(\mathbf{x}-\mathbf{x}',\tau-\tau') &= \pm\mathrm{Tr}\left[e^{-\beta H_G}\psi^\dagger(x',\tau')\psi(x,\tau)\right] / \,\mathrm{Tr}\,(e^{-\beta H_G}) \\ &= \pm\mathrm{Tr}\left[\psi(\mathbf{x},\tau)e^{-\beta H_G}\psi^\dagger(\mathbf{x}',\tau')\right] / \,\mathrm{Tr}\,(e^{-\beta H_G}) \\ &= \pm\mathrm{Tr}\left[e^{-\beta H_G}\psi\,(\mathbf{x},\tau+\beta)\,\psi^\dagger(\mathbf{x}',\tau')\right] / \,\mathrm{Tr}\,(e^{-\beta H_G}) \\ &= \pm G\,(\mathbf{x}-\mathbf{x}',\tau-\tau'+\beta)\,, \end{aligned} \qquad (10.169)$$

thus showing that interacting Boson and Fermion thermal Green functions are periodic and antiperiodic under the replacement $\tau \to \tau+\beta$, just as in the free case in Eq. (2.413).

Notes and References

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