Nature always creates the best of all options.
ARISTOTELES (384 BC–322 BC)

8

Continuous Symmetries and Conservation Laws. Noether's Theorem

In many physical systems, the action is invariant under some continuous set of transformations. In such systems, there exist local and global *conservation laws* analogous to current and charge conservation in electrodynamics. The analogs of the charges can be used to generate the symmetry transformation, from which they were derived, with the help of Poisson brackets, or after quantization, with the help of commutators.

8.1 Point Mechanics

Consider a simple mechanical system with a generic action

$$\mathcal{A} = \int_{t_a}^{t_b} dt\, L(q(t), \dot{q}(t), t). \tag{8.1}$$

8.1.1 Continuous Symmetries and Conservation Law

Suppose $\mathcal{A}$ is invariant under a continuous set of transformations of the dynamical variables:

$$q(t) \to q'(t) = f(q(t), \dot{q}(t)), \tag{8.2}$$

where $f(q(t), \dot{q}(t))$ is some functional of $q(t)$. Such transformations are called symmetry transformations. Thereby it is important that the equations of motion are not used when establishing the invariance of the action under (8.2).

If the action is subjected successively to two *symmetry transformations*, the result is again a symmetry transformation. Thus, symmetry transformations form a group called the *symmetry group* of the system. For infinitesimal symmetry transformations (8.2), the difference

$$\delta_s q(t) \equiv q'(t) - q(t) \tag{8.3}$$

will be called a *symmetry variation*. It has the general form

$$\delta_s q(t) = \epsilon \Delta(q(t), \dot{q}(t), t). \tag{8.4}$$

Symmetry variations must not be confused with ordinary variations $\delta q(t)$ used in Section 1.1 to derive the Euler-Lagrange equations (1.8). While the ordinary variations $\delta q(t)$ vanish at initial and final times, $\delta q(t_b) = \delta q(t_a) = 0$ [recall (1.4)], the symmetry variations $\delta_{\rm s} q(t)$ are usually nonzero at the ends.

Let us calculate the change of the action under a symmetry variation (8.4). Using the chain rule of differentiation and an integration by parts, we obtain

$$\delta_{\rm s}\mathcal{A} = \int_{t_a}^{t_b} dt \left[\frac{\partial L}{\partial q(t)} - \partial_t \frac{\partial L}{\partial \dot{q}(t)}\right] \delta_{\rm s} q(t) + \left.\frac{\partial L}{\partial \dot{q}(t)} \delta_{\rm s} q(t)\right|_{t_a}^{t_b}. \tag{8.5}$$

For orbits $q(t)$ that satisfy the Euler-Lagrange equations (1.8), only boundary terms survive, and we are left with

$$\delta_{\rm s}\mathcal{A} = \epsilon \left.\frac{\partial L}{\partial \dot{q}} \Delta(q, \dot{q}, t)\right|_{t_b}^{t_a}. \tag{8.6}$$

Under the symmetry assumption, $\delta_{\rm s}\mathcal{A}$ vanishes for *any* orbit $q(t)$, implying that the quantity

$$Q(t) \equiv \frac{\partial L}{\partial \dot{q}} \Delta(q, \dot{q}, t) \tag{8.7}$$

is the same at times $t = t_a$ and $t = t_b$. Since t_b is arbitrary, $Q(t)$ is *independent* of the time t, i.e., it satisfies

$$Q(t) \equiv Q. \tag{8.8}$$

It is a *conserved quantity*, a *constant of motion*. The expression on the right-hand side of (8.7) is called *Noether charge*.

The statement can be generalized to transformations $\delta_{\rm s} q(t)$ for which the action is not directly invariant but its symmetry variation is equal to an arbitrary boundary term:

$$\delta_{\rm s}\mathcal{A} = \epsilon \left.\Lambda(q, \dot{q}, t)\right|_{t_a}^{t_b}. \tag{8.9}$$

In this case,

$$Q(t) = \frac{\partial L}{\partial \dot{q}} \Delta(q, \dot{q}, t) - \Lambda(q, \dot{q}, t) \tag{8.10}$$

is a conserved Noether charge.

It is also possible to derive the constant of motion (8.10) without invoking the action, but starting from the Lagrangian. For it we evaluate the symmetry variation as follows:

$$\delta_{\rm s} L \equiv L\left(q + \delta_{\rm s} q, \dot{q} + \delta_{\rm s}\dot{q}\right) - L(q, \dot{q}) = \left[\frac{\partial L}{\partial q(t)} - \partial_t \frac{\partial L}{\partial \dot{q}(t)}\right] \delta_{\rm s} q(t) + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}(t)} \delta_{\rm s} q(t)\right]. \tag{8.11}$$

On account of the Euler-Lagrange equations (1.8), the first term on the right-hand side vanishes as before, and only the last term survives. The assumption of invariance of the action up to a possible surface term in Eq. (8.9) is equivalent to assuming that the symmetry variation of the Lagrangian is a *total time derivative* of some function $\Lambda(q, \dot{q}, t)$:

$$\delta_{\rm s} L(q, \dot{q}, t) = \epsilon \frac{d}{dt} \Lambda(q, \dot{q}, t). \tag{8.12}$$

Inserting this into the left-hand side of (8.11), we find

$$\epsilon \frac{d}{dt} \left[\frac{\partial L}{\partial \dot{q}} \Delta(q, \dot{q}, t) - \Lambda(q, \dot{q}, t) \right] = 0, \tag{8.13}$$

thus recovering again the conserved Noether charge (8.8).

The existence of a conserved quantity for every continuous symmetry is the content of *Noether's theorem* [1].

8.1.2 Alternative Derivation

Let us do the substantial variation in Eq. (8.5) explicitly, and change a classical orbit $q_c(t)$, that extremizes the action, by an *arbitrary variation* $\delta_{\rm a} q(t)$. If this does *not* vanish at the boundaries, the action changes by a pure boundary term that follows directly from (8.5):

$$\delta_{\rm a} \mathcal{A} = \left. \frac{\partial L}{\partial \dot{q}} \delta_{\rm a} q \right|_{t_a}^{t_b}. \tag{8.14}$$

From this equation we can derive Noether's theorem in yet another way. Suppose we subject a classical orbit to a new type of symmetry variation, to be called *local symmetry transformations*, which generalizes the previous symmetry variations (8.4) by making the parameter ϵ time-dependent:

$$\delta_{\rm s}^t q(t) = \epsilon(t) \Delta(q(t), \dot{q}(t), t). \tag{8.15}$$

The superscript t of $\delta_{\rm s}^t q(t)$ indicates the new time dependence in the parameter $\epsilon(t)$. These variations may be considered as a special set of the general variations $\delta_{\rm a} q(t)$ introduced above. Thus also $\delta_{\rm s}^t \mathcal{A}$ must be a pure boundary term of the type (8.14). For the subsequent discussion it is useful to introduce the infinitesimally transformed orbit

$$q^\epsilon(t) \equiv q(t) + \delta_{\rm s}^t q(t) = q(t) + \epsilon(t) \Delta(q(t), \dot{q}(t), t), \tag{8.16}$$

and the associated Lagrangian:

$$L^\epsilon \equiv L(q^\epsilon(t), \dot{q}^\epsilon(t)). \tag{8.17}$$

Using the time-dependent parameter $\epsilon(t)$, the local symmetry variation of the action can be written as

$$\delta_{\rm s}^t \mathcal{A} = \int_{t_a}^{t_b} dt \left[\frac{\partial L^\epsilon}{\partial \epsilon(t)} - \frac{d}{dt} \frac{\partial L^\epsilon}{\partial \dot{\epsilon}(t)} \right] \epsilon(t) + \frac{d}{dt} \left. \left[\frac{\partial L^\epsilon}{\partial \dot{\epsilon}} \right] \epsilon(t) \right|_{t_a}^{t_b}. \tag{8.18}$$

Along the classical orbits, the action is extremal and satisfies the equation

$$\frac{\delta \mathcal{A}}{\delta \epsilon(t)} = 0, \tag{8.19}$$

which translates for a local action to an Euler-Lagrange type of equation:

$$\frac{\partial L^\epsilon}{\partial \epsilon(t)} - \frac{d}{dt}\frac{\partial L^\epsilon}{\partial \dot{\epsilon}(t)} = 0. \tag{8.20}$$

This can also be checked explicitly by differentiating (8.17) according to the chain rule of differentiation:

$$\frac{\partial L^\epsilon}{\partial \epsilon(t)} = \frac{\partial L^\epsilon}{\partial q(t)}\Delta(q,\dot{q},t) + \frac{\partial L^\epsilon}{\partial \dot{q}(t)}\dot{\Delta}(q,\dot{q},t); \tag{8.21}$$

$$\frac{\partial L^\epsilon}{\partial \dot{\epsilon}(t)} = \frac{\partial L^\epsilon}{\partial \dot{q}(t)}\Delta(q,\dot{q},t), \tag{8.22}$$

and inserting on the right-hand side the ordinary Euler-Lagrange equations (1.8).

We now invoke the symmetry assumption that the action is a pure surface term under the time-independent transformations (8.15). This implies that

$$\frac{\partial L^\epsilon}{\partial \epsilon} = \frac{d}{dt}\Lambda. \tag{8.23}$$

Combining this with (8.20), we derive a conservation law for the charge:

$$Q = \frac{\partial L^\epsilon}{\partial \dot{\epsilon}} - \Lambda. \tag{8.24}$$

Inserting here Eq. (8.22), we find that this is the same charge as that derived by the previous method.

8.2 Displacement and Energy Conservation

As a simple but physically important example consider the case that the Lagrangian does not depend explicitly on time, i.e., that $L(q,\dot{q},t) \equiv L(q,\dot{q})$. Let us perform a time translation on the coordinate frame:

$$t' = t - \epsilon. \tag{8.25}$$

In the new coordinate frame, the *same* orbit has the new description

$$\dot{q}(t') = q(t), \tag{8.26}$$

i.e., the orbit $\dot{q}(t)$ at the translated time t' is precisely the same as the orbit $q(t)$ at the original time t. If we replace the argument of $\dot{q}(t)$ in (8.26) by t', we describe a time-translated orbit in terms of the original coordinates. This implies the symmetry variation of the form (8.4):

$$\begin{aligned} \delta_s q(t) &= q'(t) - q(t) = q(t' + \epsilon) - q(t) \\ &= q(t') + \epsilon \dot{q}(t') - q(t) = \epsilon \dot{q}(t). \end{aligned} \tag{8.27}$$

The symmetry variation of the Lagrangian is in general

$$\delta_s L = L(q'(t), \dot{q}'(t)) - L(q(t), \dot{q}(t)) = \frac{\partial L}{\partial q} \delta_s q(t) + \frac{\partial L}{\partial \dot{q}} \delta_s \dot{q}(t). \tag{8.28}$$

Inserting $\delta_s q(t)$ from (8.27) we find, without using the Euler-Lagrange equation,

$$\delta_s L = \epsilon \left(\frac{\partial L}{\partial \dot{q}} \dot{q} + \frac{\partial L}{\partial \dot{q}} \ddot{q} \right) = \epsilon \frac{d}{dt} L. \tag{8.29}$$

This has precisely the form of Eq. (8.12), with $\Lambda = L$ as expected, since time translations are symmetry transformations. Here the function Λ in (8.12) happens to coincide with the Lagrangian.

According to Eq. (8.10), we find the Noether charge

$$Q = \frac{\partial L}{\partial \dot{q}} \dot{q} - L(q, \dot{q}) \tag{8.30}$$

to be a constant of motion. This is recognized as the *Legendre transform* of the Lagrangian which is, of course, the *Hamiltonian* of the system. s

Let us briefly check how this Noether charge is obtained from the alternative formula (8.10). The time-dependent symmetry variation is here

$$\delta_s^t q(t) = \epsilon(t) \dot{q}(t), \tag{8.31}$$

under which the Lagrangian is changed by

$$\delta_s^t L = \frac{\partial L}{\partial q} \epsilon \dot{q} + \frac{\partial L}{\partial \dot{q}} (\dot{\epsilon} \dot{q} + \epsilon \ddot{q}) = \frac{\partial L^\epsilon}{\partial \dot{\epsilon}} \epsilon + \frac{\partial L^\epsilon}{\partial \dot{\epsilon}} \dot{\epsilon}, \tag{8.32}$$

with

$$\frac{\partial L^\epsilon}{\partial \dot{\epsilon}} = \frac{\partial L}{\partial \dot{q}} \dot{q} \tag{8.33}$$

and

$$\frac{\partial L^\epsilon}{\partial \epsilon} = \frac{\partial L}{\partial q} \dot{q} + \frac{\partial L}{\partial \dot{q}} \epsilon \ddot{q} = \frac{d}{dt} L. \tag{8.34}$$

This shows that time translations fulfill the symmetry condition (8.23), and that the Noether charge (8.24) coincides with the Hamiltonian found in Eq. (8.10).

8.3 Momentum and Angular Momentum

While the conservation law of energy follows from the symmetry of the action under time translations, conservation laws of momentum and angular momentum are found if the action is invariant under translations and rotations.

Consider a Lagrangian of a point particle in a euclidean space

$$L = L(x^i(t), \dot{x}^i(t), t). \tag{8.35}$$

In contrast to the previous discussion of time translation invariance, which was applicable to systems with arbitrary Lagrange coordinates $q(t)$, we denote the coordinates here by x^i to emphasize that we now consider cartesian coordinates. If the Lagrangian does depend only on the velocities $\dot{x}^i$ and not on the coordinates x^i themselves, the system is *translationally invariant*. If it depends, in addition, only on $\dot{\mathbf{x}}^2 = \dot{x}^i\dot{x}^i$, it is also rotationally invariant.

The simplest example is the Lagrangian of a point particle of mass m in euclidean space:

$$L = \frac{m}{2}\dot{\mathbf{x}}^2. \tag{8.36}$$

It exhibits both invariances, leading to conserved Noether charges of momentum and angular momentum, as we now demonstrate.

8.3.1 Translational Invariance in Space

Under a spatial translation, the coordinates x^i change to

$$x'^i = x^i + \epsilon^i, \tag{8.37}$$

where ϵ^i are small numbers. The infinitesimal translations of a particle path are [compare (8.4)]

$$\delta_{\rm s} x^i(t) = \epsilon^i. \tag{8.38}$$

Under these, the Lagrangian changes by

$$\begin{aligned} \delta_{\rm s} L &= L(x'^i(t), \dot{x}'^i(t), t) - L(x^i(t), \dot{x}^i(t), t) \\ &= \frac{\partial L}{\partial x^i}\delta_{\rm s} x^i = \frac{\partial L}{\partial x^i}\epsilon^i = 0. \end{aligned} \tag{8.39}$$

By assumption, the Lagrangian is independent of x^i, so that the right-hand side vanishes. This has to be compared with the symmetry variation of the Lagrangian around the classical orbit, calculated via the chain rule, and using the Euler-Lagrange equation:

$$\begin{aligned} \delta_{\rm s} L &= \left(\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}\right)\delta_{\rm s} x^i + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}^i}\delta_{\rm s} x^i\right] \\ &= \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}^i}\right]\epsilon^i. \end{aligned} \tag{8.40}$$

This has the form (8.6), from which we extract a conserved Noether charge (8.7) for each coordinate x^i:

$$p^i = \frac{\partial L}{\partial \dot{x}^i}. \tag{8.41}$$

These are simply the canonical momenta of the system.

8.3.2 Rotational Invariance

Under rotations, the coordinates x^i change to

$$x'^i = R^i{}_j x^j, \tag{8.42}$$

where $R^i{}_j$ is an orthogonal 3×3 -matrix. Infinitesimally, this can be written as

$$R^i{}_j = \delta^i{}_j - \omega_k \epsilon_{kij}, \tag{8.43}$$

where $\boldsymbol{\omega}$ is an infinitesimal rotation vector. The corresponding rotation of a particle path is

$$\delta_s x^i(t) = x'^i(t) - x^i(t) = -\omega^k \epsilon_{kij} x^j(\tau). \tag{8.44}$$

It is useful to introduce the antisymmetric infinitesimal rotation tensor

$$\omega_{ij} \equiv \omega_k \epsilon_{kij}, \tag{8.45}$$

in terms of which

$$\delta_s x^i = -\omega_{ij} x^j. \tag{8.46}$$

Then we can write the change of the Lagrangian under $\delta_s x^i$,

$$\begin{aligned} \delta_s L &= L(x'^i(t), \dot{x}'^i(t), t) - L(x^i(t), \dot{x}^i(t), t) \\ &= \frac{\partial L}{\partial x^i}\delta_s x^i + \frac{\partial L}{\partial \dot{x}^i}\delta_s \dot{x}^i, \end{aligned} \tag{8.47}$$

as

$$\delta_s L = -\left(\frac{\partial L}{\partial x^i}x^j + \frac{\partial L}{\partial \dot{x}^i}\dot{x}^j\right)\omega_{ij} = 0. \tag{8.48}$$

If the Lagrangian depends only on the rotational invariants $\mathbf{x}^2, \dot{\mathbf{x}}^2, \mathbf{x}\cdot\dot{\mathbf{x}}$, and on powers thereof, the right-hand side vanishes on account of the antisymmetry of ω_{ij}. This ensures the rotational symmetry.

We now calculate once more the symmetry variation of the Lagrangian via the chain rule and find, using the Euler-Lagrange equations,

$$\begin{aligned} \delta_s L &= \left(\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}\right)\delta_s x^i + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}^i}\delta_s x^i\right] \\ &= -\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}^i}x^j\right]\omega_{ij} = \frac{1}{2}\frac{d}{dt}\left[x_i \frac{\partial L}{\partial \dot{x}^j} - (i \leftrightarrow j)\right]\omega_{ij}. \end{aligned} \tag{8.49}$$

The right-hand side yields the conserved Noether charges of type (8.7), one for each antisymmetric pair i, j:

$$L^{ij} = x^i \frac{\partial L}{\partial \dot{x}^j} - x^j \frac{\partial L}{\partial \dot{x}^i} \equiv x^i p^j - x^j p^i. \tag{8.50}$$

These are the antisymmetric components of angular momentum.

Had we worked with the original vector form of the rotation angles ω^k, we would have found the angular momentum in the more common form:

$$L_k = \frac{1}{2}\epsilon_{kij} L^{ij} = (\mathbf{x} \times \mathbf{p})^k. \tag{8.51}$$

The quantum-mechanical operators associated with these, after replacing $p^i \to -i\partial/\partial x^i$, have the well-known commutation rules

$$[\hat{L}_i, \hat{L}_j] = i\epsilon_{ijk}\hat{L}_k. \tag{8.52}$$

In the tensor notation (8.50), these become

$$[\hat{L}_{ij}, \hat{L}_{kl}] = -i\left(\delta_{ik}\hat{L}_{jl} - \delta_{il}\hat{L}_{jk} + \delta_{jl}\hat{L}_{ik} - \delta_{jk}\hat{L}_{il}\right). \tag{8.53}$$

8.3.3 Center-of-Mass Theorem

Consider now the transformations corresponding to a uniform motion of the coordinate system. We shall study the behavior of a set of free massive point particles in euclidean space described by the Lagrangian

$$L(\dot{x}^i) = \sum_n \frac{m_n}{2}\dot{\mathbf{x}}_n^2. \tag{8.54}$$

Under Galilei transformations, the spatial coordinates and the time are changed to

$$\begin{aligned} \acute{x}^i(t) &= x^i(t) - v^i t, \\ t' &= t, \end{aligned} \tag{8.55}$$

where v^i is the relative velocity along the ith axis. The infinitesimal symmetry variations are

$$\delta_s x^i(t) = \acute{x}^i(t) - x^i(t) = -v^i t, \tag{8.56}$$

which change the Lagrangian by

$$\delta_s L = L(x^i - v^i t, \dot{x}^i - v^i) - L(x^i, \dot{x}^i). \tag{8.57}$$

Inserting the explicit form (8.54), we find

$$\delta_s L = \sum_n \frac{m_n}{2}\left[(\dot{x}_n^i - v^i)^2 - (\dot{x}_n{}^i)^2\right]. \tag{8.58}$$

This can be written as a total time derivative:

$$\delta_s L = \frac{d}{dt}\Lambda = \frac{d}{dt}\sum_n m_n\left[-\dot{x}_n^i v^i + \frac{v^2}{2}t\right], \tag{8.59}$$

proving that Galilei transformations are symmetry transformations in the Noether sense. By assumption, the velocities v^i in (8.55) are infinitesimal, so that the second term can be ignored.

By calculating $\delta_s L$ once more via the chain rule with the help of the Euler-Lagrange equations, and by equating the result with (8.59), we find the conserved Noether charge

$$\begin{aligned} Q &= \sum_n \frac{\partial L}{\partial \dot{x}^i}\delta_s x^i - \Lambda \\ &= \left(-\sum_n m_n \dot{x}_n^i t + \sum_n m_n x_n^i\right) v^i. \end{aligned} \tag{8.60}$$

Since the direction of the velocity v^i is arbitrary, each component is separately a constant of motion:

$$N^i = -\sum_n m_n \dot{x}^i t + \sum_n m_n {x_n}^i = \text{const.} \tag{8.61}$$

This is the well-known *center-of-mass theorem* [2]. Indeed, introducing the center-of-mass coordinates

$$x^i_{\text{CM}} \equiv \frac{\sum_n m_n {x_n}^i}{\sum_n m_n}, \tag{8.62}$$

and the associated velocities

$$v^i_{\text{CM}} = \frac{\sum_n m_n {\dot{x}_n}^i}{\sum_n m_n}, \tag{8.63}$$

the conserved charge (8.61) can be written as

$$N^i = \sum_n m_n(-v^i_{\text{CM}} t + x^i_{\text{CM}}). \tag{8.64}$$

The time-independence of N^i implies that the center-of-mass moves with uniform velocity according to the law

$$x^i_{\text{CM}}(t) = x^i_{0\,\text{CM}} + v^i_{\text{CM}} t, \tag{8.65}$$

where

$$x^i_{0\,\text{CM}} = \frac{N^i}{\sum_n m_n} \tag{8.66}$$

is the position of the center of mass at $t = 0$.

Note that in non-relativistic physics, the center-of-mass theorem is a consequence of momentum conservation since momentum $\equiv$ mass $\times$ velocity. In relativistic physics, this is no longer true.

8.3.4 Conservation Laws Resulting from Lorentz Invariance

In relativistic physics, particle orbits are described by functions in spacetime

$$x^\mu(\tau), \tag{8.67}$$

where τ is an arbitrary Lorentz-invariant parameter. The action is an integral over some Lagrangian:

$$\mathcal{A} = \int d\tau L\left(x^\mu(\tau), \dot{x}^\mu(\tau), \tau\right), \tag{8.68}$$

where $\dot{x}^\mu(\tau)$ denotes the derivative with respect to the parameter τ. If the Lagrangian depends only on invariant scalar products $x^\mu x_\mu, x^\mu \dot{x}_\mu, \dot{x}^\mu \dot{x}_\mu$, then it is invariant under Lorentz transformations

$$x^\mu \to \dot{x}^\mu = \Lambda^\mu{}_\nu\, x^\nu, \tag{8.69}$$

where $\Lambda^\mu{}_\nu$ is a 4×4 matrix satisfying

$$\Lambda g \Lambda^T = g, \tag{8.70}$$

with the Minkowski metric

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}. \tag{8.71}$$

For a free massive point particle in spacetime, the Lagrangian is

$$L(\dot{x}(\tau)) = -Mc\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}. \tag{8.72}$$

It is reparametrization invariant under $\tau \to f(\tau)$, with an arbitrary function $f(\tau)$. Under translations

$$\delta_{\rm s} x^\mu(\tau) = x^\mu(\tau) - \epsilon^\mu(\tau), \tag{8.73}$$

the Lagrangian is obviously invariant, satisfying $\delta_{\rm s}\mathcal{L} = 0$. Calculating this variation once more via the chain rule with the help of the Euler-Lagrange equations, we find

$$\begin{aligned} 0 &= \int_{\tau_\mu}^{\tau_\nu} d\tau \left(\frac{\partial L}{\partial x^\mu}\delta_{\rm s} x^\mu + \frac{\partial L}{\partial \dot{x}^\mu}\delta_{\rm s}\dot{x}^\mu \right) \\ &= -\epsilon^\mu \int_{\tau_\mu}^{\tau_\nu} d\tau \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right). \end{aligned} \tag{8.74}$$

From this we obtain the Noether charges

$$p_\mu \equiv -\frac{\partial L}{\partial \dot{x}^\mu} = Mc\frac{\dot{x}_\mu(\tau)}{\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}} = Mcu^\mu, \tag{8.75}$$

which satisfy the conservation law

$$\frac{d}{d\tau}p_\mu(t) = 0. \tag{8.76}$$

They are the conserved four-momenta of a free relativistic particle. The quantity

$$u^\mu \equiv \frac{\dot{x}^\mu}{\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}} \tag{8.77}$$

is the dimensionless relativistic four-velocity of the particle. It has the property $u^\mu u_\mu = 1$, and it is reparametrization invariant. By choosing for τ the physical time $t = x^0/c$, we can express u^μ in terms of the physical velocities $v^i = dx^i/dt$ as

$$u^\mu = \gamma(1, v^i/c), \qquad \text{with} \qquad \gamma \equiv \sqrt{1 - v^2/c^2}. \tag{8.78}$$

Note the minus sign in the definition (8.75) of the canonical momentum with respect to the nonrelativistic case. It is necessary to write Eq. (8.75) covariantly. The derivative with respect to $\dot{x}^\mu$ transforms like a covariant vector with a subscript μ, whereas the physical momenta are p^μ.

For small Lorentz transformations near the identity we write

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu, \tag{8.79}$$

where

$$\omega^\mu{}_\nu = g^{\mu\lambda}\omega_{\lambda\nu} \tag{8.80}$$

is an arbitrary infinitesimal antisymmetric matrix. An infinitesimal Lorentz transformation of the particle path is

$$\begin{aligned} \delta_s x^\mu(\tau) &= \dot{x}^\mu(\tau) - x^\mu(\tau) \\ &= \omega^\mu{}_\nu x^\nu(\tau). \end{aligned} \tag{8.81}$$

Under it, the symmetry variation of a Lorentz-invariant Lagrangian vanishes:

$$\delta_s L = \left(\frac{\partial L}{\partial x^\mu} x^\nu + \frac{\partial L}{\partial \dot{x}^\mu}\dot{x}^\nu \right) \omega^\mu{}_\nu = 0. \tag{8.82}$$

This has to be compared with the symmetry variation of the Lagrangian calculated via the chain rule with the help of the Euler-Lagrange equation

$$\begin{aligned} \delta_s L &= \left(\frac{\partial L}{\partial x^\mu} - \frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^\mu} \right) \delta_s x^\mu + \frac{d}{d\tau}\left[\frac{\partial L}{\partial \dot{x}^\mu}\delta_s x^\mu \right] \\ &= \frac{d}{d\tau}\left[\frac{\partial L}{\partial \dot{x}^\mu}\dot{x}^\nu \right] \omega^\mu{}_\nu \\ &= \frac{1}{2}\omega_\mu{}^\nu \frac{d}{d\tau}\left(x^\mu \frac{\partial L}{\partial \dot{x}_\nu} - x^\nu \frac{\partial L}{\partial \dot{x}_\mu} \right). \end{aligned} \tag{8.83}$$

By equating this with (8.82), we obtain the conserved rotational Noether charges [containing again a minus sign as in (8.75)]:

$$L^{\mu\nu} = -x^{\mu}\frac{\partial L}{\partial \dot{x}_{\nu}} + x^{\nu}\frac{\partial L}{\partial \dot{x}_{\mu}} = x^{\mu}p^{\nu} - x^{\nu}p^{\mu}. \tag{8.84}$$

They are four-dimensional generalizations of the angular momenta (8.50). The quantum-mechanical operators

$$\hat{L}^{\mu\nu} \equiv i(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}) \tag{8.85}$$

obtained after the replacement $p^{\mu} \to i\partial/\partial x_{\mu}$ satisfy the four-dimensional spacetime generalization of the commutation relations (8.53):

$$[\hat{L}^{\mu\nu}, \hat{L}^{\kappa\lambda}] = i\left(g^{\mu\kappa}\hat{L}^{\nu\lambda} - g^{\mu\lambda}\hat{L}^{\nu\kappa} + g^{\nu\lambda}\hat{L}^{\mu\kappa} - g^{\nu\kappa}\hat{L}^{\mu\lambda}\right). \tag{8.86}$$

The quantities L^{ij} coincide with the earlier-introduced angular momenta (8.50). The conserved components

$$L^{0i} = x^{0}p^{i} - x^{i}p^{0} \equiv M_i \tag{8.87}$$

yield the relativistic generalization of the center-of-mass theorem (8.61):

$$M_i = \text{const.} \tag{8.88}$$

8.4 Generating the Symmetry Transformations

As mentioned in the introduction to this chapter, the relation between invariances and conservation laws has a second aspect. With the help of Poisson brackets, the charges associated with continuous symmetry transformations can be used to generate the symmetry transformation from which they were derived. Explicitly,

$$\delta_{\mathrm{s}}\hat{x} = -i\epsilon[\hat{Q}, \hat{x}(t)]. \tag{8.89}$$

The charge derived in Section 7.2 from the invariance of the system under time displacement is the most famous example for this property. The charge (8.30) is by definition the *Hamiltonian*,

$$Q \equiv H,$$

whose operator version generates infinitesimal time displacements by the *Heisenberg equation of motion*:

$$\dot{\hat{x}}(t) = -i[\hat{H}, \hat{x}(t)]. \tag{8.90}$$

This equation is obviously the same as (8.89).

To quantize the system canonically, we may assume the Lagrangian to have the standard form

$$L(x, \dot{x}) = \frac{M}{2}\dot{x}^2 - V(x), \tag{8.91}$$

so that the *Hamiltonian operator* becomes, with the canonical momentum $p \equiv \dot{x}$:

$$\hat{H} = \frac{\hat{p}^2}{2M} + V(\hat{x}). \tag{8.92}$$

Equation (8.90) is then a direct consequence of the canonical equal-time commutation rules

$$[\hat{p}(t), \hat{x}(t)] = -i, \quad [\hat{p}(t), \hat{p}(t)] = 0, \quad [\hat{x}(t), \hat{x}(t)] = 0. \tag{8.93}$$

The charges (8.41), derived in Section 7.3 from translational symmetry, are another famous example. After quantization, the commutation rule (8.89) becomes, with (8.38),

$$\epsilon^j = i\epsilon^i[\hat{p}^i(t), \hat{x}^j(t)]. \tag{8.94}$$

This coincides with one of the canonical commutation relations (here it appears only for time-independent momenta, since the system is translationally invariant).

The relativistic charges (8.75) of spacetime generate translations via

$$\delta_{\rm s}\hat{x}^\mu = \epsilon^\mu = -i\epsilon^\nu[\hat{p}_\nu(t), \hat{x}^\mu(\tau)], \tag{8.95}$$

in agreement with the relativistic canonical commutation rules (29.27).

Similarly we find that the quantized versions of the conserved charges L_i in Eq. (8.51) generate infinitesimal rotations:

$$\delta_{\rm s}\hat{x}^j = -\omega^i\epsilon_{ijk}\hat{x}^k(t) = i\omega^i[\hat{L}_i, \hat{x}^j(t)], \tag{8.96}$$

whereas the quantized conserved charges N^i of Eq. (8.61) generate infinitesimal Galilei transformations, and that the charges M_i of Eq. (8.87) generate pure rotational Lorentz transformations:

$$\begin{aligned} \delta_{\rm s}\hat{x}^j &= \epsilon_i\hat{x}^0 = i\epsilon_i[M_i, \hat{x}^j], \\ \delta_{\rm s}\hat{x}^0 &= \epsilon_i\hat{x}^i = i\epsilon_i[M_i, \hat{x}^0]. \end{aligned} \tag{8.97}$$

Since the quantized charges generate the rotational symmetry transformations, they form a *representation* of the generators of the symmetry group. When commuted with each other, they obey the same commutation rules as the generators of the symmetry group. The charges (8.51) associated with rotations, for example, have the commutation rules

$$[\hat{L}_i, \hat{L}_j] = i\epsilon_{ijk}\hat{L}_j, \tag{8.98}$$

which are the same as those between the 3×3 generators of the three-dimensional rotations $(L_i)_{jk} = -i\epsilon_{ijk}$.

The quantized charges of the generators (8.84) of the Lorentz group satisfy the commutation rules (8.86) of the 4×4 generators (8.85)

$$[\hat{L}^{\mu\nu}, \hat{L}^{\mu\lambda}] = -ig^{\mu\mu}\hat{L}^{\nu\lambda}. \tag{8.99}$$

This follows directly from the canonical commutation rules (8.95) [i.e., (29.27)].

8.5 Field Theory

A similar relation between continuous symmetries and constants of motion holds in field theory.

8.5.1 Continuous Symmetry and Conserved Currents

Let $\mathcal{A}$ be the action of an arbitrary field $\varphi(x)$,

$$\mathcal{A} = \int d^4x \mathcal{L}(\varphi, \partial\varphi, x), \tag{8.100}$$

and suppose that a transformation of the field

$$\delta_{\rm s}\varphi(x) = \epsilon\Delta(\varphi, \partial\varphi, x) \tag{8.101}$$

changes the Lagrangian density $\mathcal{L}$ merely by a total derivative

$$\delta_{\rm s}\mathcal{L} = \epsilon\partial_\mu\Lambda^\mu, \tag{8.102}$$

or equivalently, that it changes the action $\mathcal{A}$ by a surface term

$$\delta_{\rm s}\mathcal{A} = \epsilon\int d^4x\, \partial_\mu\Lambda^\mu. \tag{8.103}$$

Then $\delta_{\rm s}\mathcal{L}$ is called a *symmetry transformation*.

Given such a symmetry transformation, we can find a current four-vector

$$j^\mu = \frac{\partial L}{\partial\partial_\mu\varphi}\Delta - \Lambda^\mu \tag{8.104}$$

that has no four-divergence

$$\partial_\mu j^\mu(x) = 0. \tag{8.105}$$

The expression on the right-hand side of (8.104) is called a *Noether current*, and (8.105) is referred to as the associated *current conservation law*. It is a *local conservation law*.

The proof of (8.105) is just as simple as that of the time-independence of the charge (8.10) associated with the corresponding symmetry of the mechanical action (8.1) in Section 8.1. We calculate the symmetry variation of $\mathcal{L}$ under the symmetry transformation in a similar way as in Eq. (8.11), and find

$$\begin{aligned}\delta_{\rm s}\mathcal{L} &= \left(\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\right)\delta_{\rm s}\varphi + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\delta_{\rm s}\varphi\right)\\ &= \epsilon\left(\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\right)\Delta + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\Delta\right).\end{aligned} \tag{8.106}$$

Then we invoke the Euler-Lagrange equation to remove the first term. Equating the second term with (8.102), we obtain

$$\partial_\mu j^\mu \equiv \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \Delta - \Lambda^\mu \right) = 0. \tag{8.107}$$

The relation between continuous symmetries and conservation is called *Noether's theorem* [1].

Assuming all fields to vanish at spatial infinity, we can derive from the local law (8.107) a *global conservation law* for the charge that is obtained from the spatial integral over the *charge density* j^0:

$$Q(t) = \int d^3x\, j^0(\mathbf{x}, t). \tag{8.108}$$

Indeed, we may write the time derivative of the charge as an integral

$$\frac{d}{dt} Q(t) = \int d^3x\, \partial_0 j^0(\mathbf{x}, t) \tag{8.109}$$

and adding on the right-hand side a spatial integral over a total three-divergence, which vanishes due to the boundary conditions, we find

$$\frac{d}{dt} Q(t) = \int d^3x\, \partial_0 j^0(\mathbf{x}, t) = \int d^3x\, [\partial_0 j^0(\mathbf{x}, t) + \partial_i j^i(\mathbf{x}, t)] = 0. \tag{8.110}$$

Thus the charge is conserved:

$$\frac{d}{dt} Q(t) = 0. \tag{8.111}$$

8.5.2 Alternative Derivation

There is again an alternative derivation of the conserved current that is analogous to Eqs. (8.15)–(8.24). It is based on a variation of the fields under symmetry transformations whose parameter ϵ is made artificially spacetime-dependent $\epsilon \to \epsilon(x)$, thus extending (8.15) to

$$\delta_s^x \varphi(x) = \epsilon(x) \Delta(\varphi(x), \partial_\mu \varphi(x), x). \tag{8.112}$$

As before in Eq. (8.17), we calculate the Lagrangian density for a slightly transformed field

$$\varphi^\epsilon(x) \equiv \varphi(x) + \delta_s^x \varphi(x), \tag{8.113}$$

calling it

$$L^\epsilon \equiv L(\varphi^\epsilon(t), \partial \varphi^\epsilon(t)). \tag{8.114}$$

The corresponding action differs from the original one by

$$\delta_s^x \mathcal{A} = \int dx \left\{ \left[\frac{\partial \mathcal{L}^\epsilon}{\partial \epsilon(x)} - \partial_\mu \frac{\partial \mathcal{L}^\epsilon}{\partial \partial_\mu \epsilon(x)} \right] \delta\epsilon(x) + \partial_\mu \left[\frac{\partial \mathcal{L}^\epsilon}{\partial \partial_\mu \epsilon(x)} \delta\epsilon(x) \right] \right\}. \tag{8.115}$$

From this we obtain the Euler-Lagrange-like equation

$$\frac{\partial \mathcal{L}^{\epsilon}}{\partial \epsilon(x)} - \partial_\mu \frac{\partial \mathcal{L}^{\epsilon}}{\partial \partial_\mu \epsilon(x)} = 0. \tag{8.116}$$

By assumption, the action is a pure surface term under x-independent transformations, implying that

$$\frac{\partial \mathcal{L}^{\epsilon}}{\partial \epsilon(x)} = \partial_\mu \Lambda^\mu. \tag{8.117}$$

Together with (8.116), we see that

$$j^\mu = \frac{\partial \delta_{\rm s}^x \mathcal{L}}{\partial \partial_\mu \epsilon(x)} - \Lambda^\mu \tag{8.118}$$

has no four-divergence. By the chain rule of differentiation we calculate

$$\delta_{\rm s}^t \mathcal{L} = \frac{\partial \mathcal{L}(x)}{\partial \varphi} \epsilon \Delta + \frac{\partial \mathcal{L}}{\partial \partial_\nu \varphi} \partial_\nu \epsilon \Delta, \tag{8.119}$$

and see that

$$\frac{\partial \mathcal{L}^{\epsilon}}{\partial \partial_\mu \epsilon(x)} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \Delta(\varphi, \partial\varphi, x), \tag{8.120}$$

so that the current (8.118) coincides with (8.104).

8.5.3 Local Symmetries

If we apply the alternative derivation of a conserved current to a local symmetry, such as a local gauge symmetry, the current density (8.118) vanishes identically. Let us illuminate the symmetry origin of this phenomenon.

To be specific, we consider directly the field theory of electrodynamics. The theory does have a conserved charge resulting from the global U(1)-symmetry of the matter Lagrangian. There is a conserved current which is the source of a massless particle, the photon. This is described by a gauge field which is minimally coupled to the conserved current. A similar structure exists for many internal symmetries giving rise to nonabelian versions of the photon, such as gluons, whose exchange causes the strong interactions, and W- and Z-vector mesons, which mediate the weak interactions. It is useful to reconsider Noether's derivation of conservation laws in such theories.

The conserved matter current in a locally gauge-invariant theory cannot be found any more by the rule (8.118), which was so useful in the globally invariant theory. For the gauge transformation of quantum electrodynamics, the derivative with respect to the local field transformation $\epsilon(x)$ would simply be given by

$$j_\mu = \frac{\delta \mathcal{L}}{\partial \partial_\mu \Lambda}. \tag{8.121}$$

This would be identically equal to zero, due to local gauge invariance. We may, however, subject *just* the matter field to a local gauge transformation at *fixed* gauge fields. Then we obtain the correct current

$$j_\mu \equiv \left.\frac{\partial \mathcal{L}}{\partial \partial_\mu \Lambda}\right|_{\text{em}}. \tag{8.122}$$

Since the complete change under local gauge transformations $\delta_s^x \mathcal{L}$ vanishes identically, we can alternatively vary *only* the gauge fields and keep the particle orbit fixed:

$$j_\mu = -\left.\frac{\partial \mathcal{L}}{\partial \partial_\mu \Lambda}\right|_{\text{m}}. \tag{8.123}$$

This is done most simply by forming the functional derivative with respect to the gauge field, thereby omitting the contribution of $\overset{\text{em}}{\mathcal{L}}$:

$$j_\mu = -\frac{\partial \overset{\text{m}}{\mathcal{L}}}{\partial \partial_\mu \Lambda}. \tag{8.124}$$

An interesting consequence of local gauge invariance can be found for the gauge field itself. If we form the variation of the pure gauge field action

$$\delta_s \overset{\text{em}}{\mathcal{A}} = \int d^4x \ \text{tr}\left[\delta_s^x A_\mu \frac{\overset{\text{em}}{\mathcal{A}}}{\delta A_\mu}\right], \tag{8.125}$$

and insert for $\delta_s^x A$ an infinitesimal pure gauge field configuration

$$\delta_s^x A_\mu = -i\partial_\mu \Lambda(x), \tag{8.126}$$

the right-hand side must vanish for all $\Lambda(x)$. After a partial integration this implies the local conservation law $\partial_\mu j^\mu(x) = 0$ for the current:

$$j^\mu(x) = -i\frac{\delta \overset{\text{em}}{\mathcal{A}}}{\delta A_\mu}. \tag{8.127}$$

In contrast to the earlier conservation laws derived for matter fields, which were valid only if the matter fields obey the Euler-Lagrange equations, the current conservation law for gauge fields is valid for *all* field configurations. It is an *identity* which we may call *Bianchi identity* due to its close analogy with the Bianchi identities in Riemannian geometry.

To verify the conservation of (12.63), we insert the Lagrangian (12.3) into (12.63) and find $j^\nu = \partial_\mu F^{\mu\nu}/2$. This current is trivially conserved for any field configuration due to the antisymmetry of $F^{\mu\nu}$.

8.6 Canonical Energy-Momentum Tensor

As an important example for the field theoretic version of the theorem, consider the usual case that the Lagrangian density does not depend explicitly on the spacetime coordinates x:

$$\mathcal{L} = \mathcal{L}(\varphi, \partial\varphi). \tag{8.128}$$

We then perform a translation along an arbitrary direction $\nu = 0, 1, 2, 3$ of spacetime

$$x'^{\mu} = x^{\mu} - \epsilon^{\mu}, \tag{8.129}$$

under which field $\varphi(x)$ transforms as

$$\varphi'(x') = \varphi(x). \tag{8.130}$$

This equation expresses the fact that the field has the same value at the same absolute point in space and time, which in one coordinate system is labeled by the coordinates x^{μ} and in the other by x'^{μ}.

Under an infinitesimal translation of the field configuration coordinate, the Lagrangian density undergoes the following symmetry variation

$$\begin{aligned} \delta_{\rm s}\mathcal{L} &\equiv \mathcal{L}(\varphi'(x), \partial\varphi'(x)) - \mathcal{L}(\varphi(x), \partial\varphi(x)) \\ &= \frac{\partial\mathcal{L}}{\partial\varphi(x)}\delta_{\rm s}\varphi(x) + \frac{\partial\mathcal{L}}{\partial\partial_{\mu}\varphi}\partial_{\mu}\delta_{\rm s}\varphi(x), \end{aligned} \tag{8.131}$$

where

$$\delta_{\rm s}\varphi(x) = \varphi'(x) - \varphi(x) \tag{8.132}$$

is the symmetry variation of the fields. For the particular transformation (8.130) the symmetry variation becomes simply

$$\delta_{\rm s}\varphi(x) = \epsilon^{\nu}\partial_{\nu}\varphi(x). \tag{8.133}$$

The Lagrangian density (8.128) changes by

$$\delta_{\rm s}\mathcal{L}(x) = \epsilon^{\nu}\partial_{\nu}\mathcal{L}(x). \tag{8.134}$$

Hence the requirement (8.103) is satisfied and $\delta_{\rm s}\varphi(x)$ is a symmetry transformation. The function Λ happens to coincide with the Lagrangian density

$$\Lambda = \mathcal{L}. \tag{8.135}$$

We can now define a set of currents $j_{\nu}{}^{\mu}$, one for each ϵ^{ν}. In the particular case at hand, the currents $j_{\nu}{}^{\mu}$ are denoted by $\Theta_{\nu}{}^{\mu}$, and read:

$$\Theta_{\nu}{}^{\mu} = \frac{\partial\mathcal{L}}{\partial\partial_{\mu}\varphi}\partial_{\nu}\varphi - \delta_{\nu}{}^{\mu}\mathcal{L}. \tag{8.136}$$

They have no four-divergence

$$\partial_\mu \Theta_\nu{}^\mu(x) = 0. \tag{8.137}$$

As a consequence, the *total four-momentum* of the system, defined by

$$P^\mu = \int d^3x \Theta^{\mu 0}(x), \tag{8.138}$$

is independent of time.

The alternative derivation of the currents goes as follows. Introducing

$$\delta_s^x \varphi(x) = \epsilon^\nu(x)\partial_\nu \varphi(x), \tag{8.139}$$

we see that

$$\delta_s^x \varphi(x) = \varphi^\nu(x)\partial_\nu \varphi(x). \tag{8.140}$$

On the other hand, the chain rule of differentiation yields

$$\delta_s^x \mathcal{L} \;=\; \frac{\partial \mathcal{L}}{\partial \varphi(x)}\epsilon^\nu(x)\partial_\nu\varphi(x) + \frac{\partial \mathcal{L}}{\partial\partial_\mu\varphi(x)}\left\{[\partial_\mu \epsilon^\nu(x)]\partial_\nu\varphi + \epsilon^\nu \partial_\mu\partial_\nu\varphi(x)\right\}. \tag{8.141}$$

Hence

$$\frac{\partial \mathcal{L}^\epsilon}{\partial\partial_\mu \epsilon^\nu(x)} = \frac{\partial \mathcal{L}}{\partial\partial_\mu\varphi}\partial_\nu\varphi, \tag{8.142}$$

and we obtain once more the energy-momentum tensor (8.136).

Note that (8.142) can also be written as

$$\frac{\partial \mathcal{L}^\epsilon}{\partial\partial_\mu \epsilon^\nu(x)} = \frac{\partial \mathcal{L}}{\partial\partial_\mu\varphi}\frac{\partial \delta_s^x\varphi}{\partial\epsilon^\nu(x)}. \tag{8.143}$$

Since ν is a contravariant vector index, the set of currents $\Theta_\nu{}^\mu$ forms a Lorentz tensor called the *canonical energy-momentum tensor*. The component

$$\Theta_0{}^0 = \frac{\partial \mathcal{L}}{\partial\partial_0\varphi}\partial_0\varphi - \mathcal{L} \tag{8.144}$$

is recognized to be the Hamiltonian density in the canonical formalism.

8.6.1 Electromagnetism

As an important physical application of the field theoretic Noether theorem, consider the free electromagnetic field with the action

$$\mathcal{L} = -\frac{1}{4c}F_{\lambda\kappa}F^{\lambda\kappa}, \tag{8.145}$$

where $F_{\lambda\kappa}$ are the components of the field strength $F_{\lambda\kappa} \equiv \partial_\lambda A_\kappa - \partial_\kappa A_\lambda$. Under a translation in space and time from x^μ to $x^\mu - \epsilon\delta^\mu_\nu$, the vector potential undergoes a similar change as in (8.130):

$$A'^\mu(x') = \Lambda^\mu(x). \tag{8.146}$$

As before, this equation expresses the fact that at the same absolute spacetime point, which in the two coordinate frames is labeled once by x' and once by x, the field components have the same numerical values. The equation transformation law (8.146) can be rewritten in an infinitesimal form as

$$\begin{aligned} \delta_s A^\lambda(x^\mu) &\equiv A'^\lambda(x^\mu) - A^\lambda(x^\mu) \\ &= A'^\lambda(x'^\mu + \epsilon\delta_\nu{}^\mu) - A^\lambda(x^\mu) \quad (8.147) \\ &= \epsilon\partial_\nu A^\lambda(x^\mu). \quad (8.148) \end{aligned}$$

Under it, the field tensor changes as follows

$$\delta_s F^{\lambda\kappa} = \epsilon\partial_\nu F^{\lambda\kappa}, \quad (8.149)$$

so that the Lagrangian density is a total four-divergence:

$$\delta_s \mathcal{L} = -\epsilon\frac{1}{2c}F_{\lambda\kappa}\partial_\nu F^{\lambda\kappa} = \epsilon\partial_\nu\mathcal{L} \quad (8.150)$$

Thus, the spacetime translations (8.148) are symmetry transformations, and the currents

$$\Theta_\nu{}^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu A^\lambda}\partial_\nu A^\lambda - \delta_\nu{}^\mu\mathcal{L} \quad (8.151)$$

are conserved:

$$\partial_\mu\Theta_\nu{}^\mu(x) = 0. \quad (8.152)$$

Using $\partial\mathcal{L}/\partial\partial_\mu A^\lambda = -F^\mu{}_\lambda$, the currents (8.151) become more explicitly

$$\Theta_\nu{}^\mu = -\frac{1}{c}\left(F^\mu{}_\lambda\partial_\nu A^\lambda - \frac{1}{4}\delta_\nu{}^\mu F^{\lambda\kappa}F_{\lambda\kappa}\right). \quad (8.153)$$

They form the *canonical energy-momentum tensor* of the electromagnetic field.

8.6.2 Dirac Field

We now turn to the Dirac field which has the well-known action

$$\mathcal{A} = \int d^4x\,\mathcal{L}(x) = \int d^4x\bar{\psi}(x)(i\gamma^\mu \overleftrightarrow{\partial}_\mu - M)\psi(x), \quad (8.154)$$

where γ^μ are the Dirac matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix}. \quad (8.155)$$

Here σ^μ, $\tilde{\sigma}^\mu$ are four 2×2 matrices

$$\sigma^\mu \equiv (\sigma^0, \sigma^i).\tilde{\sigma}^\mu \equiv (\sigma^0, -\sigma^i), \quad (8.156)$$

whose zeroth component is the unit matrix

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{8.157}$$

and whose spatial components consist of the *Pauli spin matrices*

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8.158}$$

On behalf of the algebraic properties of the Pauli matrices

$$\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k, \tag{8.159}$$

the Dirac matrices (8.155) satisfy the anticommutation rules

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{8.160}$$

Under spacetime translations

$$x'^\mu = x^\mu - \epsilon^\mu, \tag{8.161}$$

the Dirac field transforms in the same way as the previous scalar and vector fields:

$$\psi'(x') = \psi(x), \tag{8.162}$$

or infinitesimally:

$$\delta_{\rm s}\psi(x) = \epsilon^\mu \partial_\mu \psi(x). \tag{8.163}$$

The same is true for the Lagrangian density, where

$$\mathcal{L}'(x') = \mathcal{L}(x), \tag{8.164}$$

and

$$\delta_{\rm s}\mathcal{L}(x) = \epsilon^\mu \partial_\mu \mathcal{L}(x). \tag{8.165}$$

Thus we obtain the Noether current

$$\Theta_\nu{}^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi^\lambda} \partial_\nu \psi^\lambda + \text{c.c.} - \delta_\nu{}^\mu \mathcal{L}, \tag{8.166}$$

with the local conservation law

$$\partial_\mu \Theta_\nu{}^\mu(x) = 0. \tag{8.167}$$

From (8.154), we see that

$$\frac{\partial \mathcal{L}}{\partial \partial_\mu \psi^\lambda} = \frac{1}{2}\bar{\psi}\gamma^\mu, \tag{8.168}$$

so that we obtain the *canonical energy-momentum tensor* of the Dirac field:

$$\Theta_\nu{}^\mu = \frac{1}{2}\bar{\psi}\gamma^\mu \partial_\nu \psi^\lambda + \text{c.c.} - \delta_\nu{}^\mu \mathcal{L} \tag{8.169}$$

8.7 Angular Momentum

Let us now turn to angular momentum in field theory. Consider first the case of a scalar field $\varphi(\dot{x})$. Under a rotation of the coordinates,

$$x'^i = R^i{}_j x^j, \tag{8.170}$$

the field does not change, if considered at the same space point, i.e.,

$$\varphi'(x'^i) = \varphi(x^i). \tag{8.171}$$

The infinitesimal symmetry variation is:

$$\delta_{\rm s}\varphi(x) = \varphi'(x) - \varphi(x). \tag{8.172}$$

Using the infinitesimal form (8.46) of (8.170),

$$\delta x^i = -\omega_{ij} x^j, \tag{8.173}$$

we see that

$$\begin{aligned} \delta_{\rm s}\varphi(x) &= \varphi'(x^0, x'^i - \delta x^i) - \varphi(x) \\ &= \partial_i \varphi(x) x^j \omega_{ij}. \end{aligned} \tag{8.174}$$

Suppose we are dealing with a Lorentz-invariant Lagrangian density that has no explicit x-dependence:

$$\mathcal{L} = \mathcal{L}(\varphi(x), \partial\varphi(x)). \tag{8.175}$$

Then the symmetry variation is

$$\begin{aligned} \delta_{\rm s}\mathcal{L} &= \mathcal{L}(\varphi'(x), \partial\varphi'(x)) - \varphi(\varphi(x), \partial\varphi(x)) \\ &= \frac{\partial\mathcal{L}}{\partial\varphi(x)}\delta_{\rm s}\varphi(x) + \frac{\partial\varphi}{\partial\partial\varphi(x)}\partial_\mu \delta_{\rm s}\varphi(x). \end{aligned} \tag{8.176}$$

For a Lorentz-invariant $\mathcal{L}$, the derivative $\partial\mathcal{L}/\partial\partial_\mu\varphi$ is a vector proportional to $\partial_\mu\varphi$.

For the Lagrangian density, the rotational symmetry variation Eq. (8.174) becomes

$$\begin{aligned} \delta_{\rm s}\mathcal{L} &= \left[\frac{\partial\mathcal{L}}{\partial\varphi}\partial_i\varphi\, x^j + \frac{\partial\mathcal{L}}{\partial_\mu\varphi}\partial_\mu(\partial_i\mathcal{L}x^j)\right]\omega_{ij} \\ &= \left[(\partial_i\mathcal{L})x^j + \frac{\partial\mathcal{L}}{\partial\partial_j\varphi}\partial_i\varphi\right]\omega_{ij} = \partial_i\left(\mathcal{L}x^j\omega_{ij}\right). \end{aligned} \tag{8.177}$$

The right-hand side is a total derivative. In arriving at this result, the antisymmetry of φ_{ij} has been used twice: first for dropping the second term in the brackets, which is possible since $\partial\mathcal{L}/\partial\partial_i\varphi$ is proportional to $\partial_i\varphi$ as a consequence of the assumed rotational invariance[1] of $\mathcal{L}$. Second it is used to pull x^j inside the last parentheses.

Calculating $\delta_s\mathcal{L}$ once more with the help of the Euler-Lagrange equations gives

$$\begin{aligned}\delta_s\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\mathcal{L}}\delta_s\varphi + \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\partial_\mu\delta_s\varphi \\ &= \left(\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\right)\delta_s\varphi + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\delta_s\varphi\right) \\ &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\partial_i\varphi\; x^j\right)\omega_{ij}.\end{aligned} \tag{8.178}$$

Thus the Noether charges

$$L^{ij,\mu} = \left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\partial_i\varphi x^j - \delta_i{}^\mu\mathcal{L}\; x^j\right) - (i \leftrightarrow j) \tag{8.179}$$

have no four-divergence

$$\partial_\mu L^{ij,\mu} = 0. \tag{8.180}$$

The associated charges

$$L^{ij} = \int d^3x L^{ij,\mu} \tag{8.181}$$

are called the *total angular momenta* of the field system. In terms of the canonical energy-momentum tensor

$$\Theta_\nu{}^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\partial_\nu\varphi - \delta_\nu{}^\mu\mathcal{L}, \tag{8.182}$$

the current density $L^{ij,\mu}$ can also be rewritten as

$$L^{ij,\mu} = x^i\Theta^{j\mu} - x^j\Theta^{i\mu}. \tag{8.183}$$

8.8 Four-Dimensional Angular Momentum

A similar procedure can be applied to pure Lorentz transformations. An infinitesimal boost to rapidity ζ^i produces a coordinate change

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu = x^\mu + \delta^\mu{}_i\zeta^i x^\nu + \delta^\mu{}_0\zeta^i x^i. \tag{8.184}$$

This can be written as

$$\delta x^\mu = \omega^\mu{}_\nu x^\nu, \tag{8.185}$$

[1]Recall the similar argument after Eq. (8.48)

where

$$\begin{aligned}\omega_{ij} &= 0,\\ \omega_{0i} = -\omega_{i0} &= \zeta^i.\end{aligned} \tag{8.186}$$

With the tensor $\omega^\mu{}_\nu$, the restricted Lorentz transformations and the infinitesimal rotations can be treated on the same footing. The rotations have the form (8.185) for the particular choice

$$\begin{aligned}\omega_{ij} &= \epsilon_{ijk}\omega^k,\\ \omega_{0i} &= \omega_{i0} = 0.\end{aligned} \tag{8.187}$$

We can now identify the symmetry variations of the field as being

$$\begin{aligned}\delta_{\rm s}\varphi(x) &= \varphi'(x'^\mu - \delta x^\mu) - \varphi(x)\\ &= -\partial_\mu\varphi(x)x^\nu\omega^\mu{}_\nu.\end{aligned} \tag{8.188}$$

Just as in (8.177), the Lagrangian density transforms as the total derivative

$$\delta_{\rm s}\varphi = -\partial_\mu(\mathcal{L}x^\nu)\omega^\mu{}_\nu, \tag{8.189}$$

and we obtain the Noether currents

$$L^{\mu\nu,\lambda} = -\left(\frac{\partial\mathcal{L}}{\partial\partial_\lambda\varphi}\partial^\lambda\varphi\, x^\nu - \delta^{\mu\lambda}\mathcal{L}\, x^\nu\right) + (\mu\leftrightarrow\nu). \tag{8.190}$$

The right-hand side can be expressed in terms of the canonical energy-momentum tensor (8.136), so that we find

$$\begin{aligned}L^{\mu\nu,\lambda} &= -\left(\frac{\partial\mathcal{L}}{\partial\partial_\lambda\varphi}\partial^\lambda\varphi\, x^\nu - \delta^{\mu\lambda}\mathcal{L}x^\nu\right) + (\mu\leftrightarrow\nu)\\ &= x^\mu\Theta^{\nu\lambda} - x^\nu\Theta^{\mu\lambda}.\end{aligned} \tag{8.191}$$

These currents have no four-divergence

$$\partial_\lambda L^{\mu\nu,\lambda} = 0. \tag{8.192}$$

The associated charges

$$L^{\mu\nu} \equiv \int d^3x\, L^{\mu\nu,0} \tag{8.193}$$

are independent of time.

For the particular form of $\omega_{\mu\nu}$ in (8.186), we find time-independent components L^{i0}. The components L^{ij} coincide with the previously-derived angular momenta.

The constancy of L^{i0} is the relativistic version of the *center-of-mass theorem* (8.65). Indeed, since

$$L^{i0} = \int d^3x\,(x^i\Theta^{00} - x^0\Theta^{i0}), \tag{8.194}$$

we can define the relativistic center of mass

$$x^i_{\rm CM} = \frac{\int d^3x\, \Theta^{00} x^i}{\int d^3x\, \Theta^{00}}, \tag{8.195}$$

and the average velocity

$$v^i_{\rm CM} = c\frac{d^3x\Theta^{i0}}{\int d^3x\, \Theta^{00}} = c\frac{P^i}{P^0}. \tag{8.196}$$

Since $\int d^3x\Theta^{i0} = P^i$ is the constant momentum of the system, also $v^i_{\rm CM}$ is a constant. Thus, the constancy of L^{0i} implies the center-of-mass moves with the constant velocity

$$x^i_{\rm CM}(t) = x^i_{0\rm CM} + v^i_{0\rm CM}t, \tag{8.197}$$

with $x^i_{0\rm CM} = L^{0i}/P^0$. The quantities $L^{\mu\nu}$ are referred to as four-dimensional orbital angular momenta.

It is important to point out that the vanishing divergence of $L^{\mu\nu,\lambda}$ makes $\Theta^{\nu\mu}$ symmetric:

$$\begin{aligned} \partial_\lambda L^{\mu\nu,\lambda} &= \partial_\lambda(x^\mu\Theta^{\nu\lambda} - x^\nu\Theta^{\mu\lambda}) \\ &= \Theta^{\nu\mu} - \Theta^{\nu\mu} = 0. \end{aligned} \tag{8.198}$$

Thus, translationally invariant field theories whose orbital angular momentum is conserved have always a symmetric canonical energy-momentum tensor.

$$\Theta^{\mu\nu} = \Theta^{\nu\mu}. \tag{8.199}$$

8.9 Spin Current

If the field $\varphi(x)$ is no longer a scalar but carries spin degrees of freedom, the derivation of the four-dimensional angular momentum becomes slightly more involved.

8.9.1 Electromagnetic Fields

Consider first the case of electromagnetism where the relevant field is the four-vector potential $A^\mu(x)$. When going to a new coordinate frame

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{8.200}$$

the vector field at the same point remains unchanged in absolute spacetime. However, since the components A^μ refer to two different basic vectors in the different frames, they must be transformed accordingly. Indeed, since A^μ is a vector and transforms like x^μ, it must satisfy the relation characterizing a *vector field*:

$$A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x). \tag{8.201}$$

For an infinitesimal transformation

$$\delta_{\rm s} x^\mu = \omega^\mu{}_\nu x^\nu, \tag{8.202}$$

this implies a symmetry variation

$$\begin{aligned}\delta_{\rm s} A^\mu(x) &= A'^\mu(x) - A^\mu(x) = A'^\mu(x-\delta x) - A^\mu(x) \\ &= \omega^\mu{}_\nu A^\nu(x) - \omega^\lambda{}_\nu x^\nu \partial_\lambda A^\mu.\end{aligned} \tag{8.203}$$

The first term is a *spin transformation*, the other an *orbital transformation*. The orbital transformation can also be written in terms of the generators $\hat{L}_{\mu\nu}$ of the Lorentz group defined in (8.84) as

$$\delta_{\rm s}^{\rm orb} A^\mu(x) = -i\omega^{\mu\nu}\hat{L}_{\mu\nu}A(x). \tag{8.204}$$

It is convenient to introduce 4×4 spin transformation matrices $L_{\mu\nu}$ with the matrix elements:

$$(L_{\mu\nu})_{\lambda\kappa} \equiv i\left(g_{\mu\lambda}g_{\nu\kappa} - g_{\mu\kappa}g_{\nu\lambda}\right). \tag{8.205}$$

They satisfy the same commutation relations (8.86) as the differential operators $\hat{L}_{\mu\nu}$ defined in Eq. (8.85). By adding together the two generators $\hat{L}_{\mu\nu}$ and $L_{\mu\nu}$, we form the operator of total four-dimensional angular momentum

$$\hat{J}_{\mu\nu} \equiv \hat{L}_{\mu\nu} + L_{\mu\nu}, \tag{8.206}$$

and can write the symmetry variation (8.203) as

$$\delta_{\rm s}^{\rm orb} A^\mu(x) = -i\omega^{\mu\nu}\hat{J}_{\mu\nu}A(x). \tag{8.207}$$

If the Lagrangian density involves only scalar combinations of four-vectors A^μ, and if it has no explicit x-dependence, it changes under Lorentz transformations like a scalar field:

$$\mathcal{L}'(x') \equiv \mathcal{L}(A'(x'), \partial' A'(x')) = \mathcal{L}(A(x), \partial A(x)) \equiv \mathcal{L}(x). \tag{8.208}$$

Infinitesimally, this makes the symmetry variation a pure gradient term:

$$\delta_{\rm s}\mathcal{L} = -(\partial_\mu \mathcal{L} x^\nu)\omega^\mu{}_\nu. \tag{8.209}$$

Thus Lorentz transformations are symmetry transformations in the Noether sense. Following Noether's construction (8.179), we calculate the *current of total four-dimensional angular momentum*:

$$J^{\mu\nu,\lambda} = \frac{\partial\mathcal{L}}{\partial\partial_\lambda A_\mu}A^\nu - \left(\frac{\partial\mathcal{L}}{\partial\partial_\lambda A^\kappa}\partial^\mu A^\kappa x^\nu - \delta^{\mu\lambda}\mathcal{L}x^\nu\right) - (\mu\leftrightarrow\nu). \tag{8.210}$$

The last two terms have the same form as the current $L^{\mu\nu,\lambda}$ of the four-dimensional angular momentum of the scalar field. Here they are the *currents of the four-dimensional orbital angular momentum:*.

$$L^{\mu\nu,\lambda} = -\left(\frac{\partial\mathcal{L}}{\partial\partial_\lambda A^\kappa}\partial^\mu A^\kappa x^\nu - \delta^{\mu\lambda}\mathcal{L}x^\nu\right) + (\mu \leftrightarrow \nu). \tag{8.211}$$

Note that this current has the form

$$L^{\mu\nu,\lambda} = -i\frac{\partial\mathcal{L}}{\partial\partial_\lambda A^\kappa}\hat{L}^{\mu\nu}A^\kappa + \left[\delta^{\mu\lambda}\mathcal{L}x^\nu - (\mu \leftrightarrow \nu)\right], \tag{8.212}$$

where $\hat{L}^{\mu\nu}$ are the differential operators of four-dimensional angular momentum in the commutation rules (8.86).

Just as the scalar case (8.191), the currents (8.211) can be expressed in terms of the canonical energy-momentum tensor as

$$L^{\mu\nu,\lambda} = x^\mu\Theta^{\nu\lambda} - x^\nu\Theta^{\mu\lambda}. \tag{8.213}$$

The first term in (8.210),

$$\Sigma^{\mu\nu,\lambda} = \left[\frac{\partial\mathcal{L}}{\partial\partial_\lambda A_\nu}A^\nu - (\mu \leftrightarrow \nu)\right], \tag{8.214}$$

is referred to as the *spin current*. It can be written in terms of the 4×4-generators (8.205) of the Lorentz group as

$$\Sigma^{\mu\nu,\lambda} = -i\frac{\partial\mathcal{L}}{\partial\partial_\lambda A^\kappa}(L^{\mu\nu})_{\kappa\sigma}A^\sigma. \tag{8.215}$$

The two currents together,

$$J^{\mu\nu,\lambda}(x) \equiv L^{\mu\nu,\lambda}(x) + \Sigma^{\mu\nu,\lambda}(x), \tag{8.216}$$

are conserved, satisfying $\partial_\lambda J^{\mu\nu,\lambda}(x) = 0$. Individually, they are not conserved.

The total angular momentum is given by the charge

$$J^{\mu\nu} = \int d^3x\, J^{\mu\nu,0}(x). \tag{8.217}$$

It is a constant of motion. Using the conservation law of the energy-momentum tensor we find, just as in (8.198), that the orbital angular momentum satisfies

$$\partial_\lambda L^{\mu\nu,\lambda}(x) = -\left[\Theta^{\mu\nu}(x) - \Theta^{\nu\mu}(x)\right]. \tag{8.218}$$

From this we find the divergence of the spin current

$$\partial_\lambda \Sigma^{\mu\nu,\lambda}(x) = -\left[\Theta^{\mu\nu}(x) - \Theta^{\nu\mu}(x)\right]. \tag{8.219}$$

For the charges associated with orbital and spin currents

$$L^{\mu\nu}(t) \equiv \int d^3x L^{\mu\nu,0}(x), \qquad \Sigma^{\mu\nu}(t) \equiv \int d^3x \Sigma^{\mu\nu,0}(x), \tag{8.220}$$

this implies the following time dependence:

$$\begin{aligned} \dot{L}^{\mu\nu}(t) &= -\int d^3x \left[\Theta^{\mu\nu}(x) - \Theta^{\nu\mu}(x)\right], \\ \dot{\Sigma}^{\mu\nu}(t) &= \int d^3x \left[\Theta^{\mu\nu}(x) - \Theta^{\nu\mu}(x)\right]. \end{aligned} \tag{8.221}$$

Thus fields with a nonzero spin density have always a non-symmetric energy momentum tensor.

In general, the current density $J^{\mu\nu,\lambda}$ of total angular momentum reads

$$J^{\mu\nu,\lambda} = \left(\frac{\partial \delta_{\rm s}^x \mathcal{L}}{\partial\partial_\lambda \omega_{\mu\nu}(x)} - \delta^{\mu\lambda}\mathcal{L}x^\nu\right) - (\mu \leftrightarrow \nu). \tag{8.222}$$

By the chain rule of differentiation, the derivative with respect to $\partial_\lambda\omega_{\mu\nu}(x)$ can come only from field derivatives, for a scalar field

$$\frac{\partial \delta_{\rm s}^x \mathcal{L}}{\partial\partial_\lambda \omega_{\mu\nu}(x)} = \frac{\partial \mathcal{L}}{\partial\partial_\lambda \varphi}\frac{\partial \delta_{\rm s}^x \varphi}{\partial \omega_{\mu\nu}(x)}, \tag{8.223}$$

and for a vector field

$$\frac{\partial \delta_{\rm s}^x \mathcal{L}}{\partial\partial_\lambda \omega_{\mu\nu}(x)} = \frac{\partial \mathcal{L}}{\partial\partial_\lambda A^\kappa}\frac{\partial \delta_{\rm s}^x A^\kappa}{\partial \omega_{\mu\nu}}. \tag{8.224}$$

The alternative rule of calculating angular momenta is to introduce spacetime-dependent transformations

$$\delta^x x = \omega^\mu{}_\nu(x)x^\nu, \tag{8.225}$$

under which the scalar fields transform as

$$\delta_{\rm s}\varphi = -\partial_\lambda\varphi\, \omega^\lambda{}_\nu(x)x^\nu, \tag{8.226}$$

and the Lagrangian density as

$$\delta_{\rm s}^x\varphi = -\partial_\lambda\mathcal{L}\, \omega^\lambda{}_\nu(x)x^\nu = -\partial_\lambda(x^\nu\mathcal{L})\omega^\lambda{}_\nu(x). \tag{8.227}$$

By separating spin and orbital transformations of $\delta_{\rm s}^x A^\kappa$, we find the two contributions $\sigma^{\mu\nu,\lambda}$ and $L^{\mu\nu,\lambda}$ to the current $J^{\mu\nu,\lambda}$ of the total angular momentum, the latter receiving a contribution from the second term in (8.222).

8.9.2 Dirac Field

We now turn to the Dirac field. Under a Lorentz transformation (8.200), this transforms according to the law

$$\psi(x') \xrightarrow{\Lambda} \psi'_\Lambda(x) = D(\Lambda)\psi(x), \tag{8.228}$$

where $D(\Lambda)$ are the 4×4 spinor representation matrices of the Lorentz group. Their matrix elements can most easily be specified for infinitesimal transformations. For an infinitesimal Lorentz transformation

$$\Lambda_\mu{}^\nu = \delta_\mu{}^\nu + \omega_\mu{}^\nu, \tag{8.229}$$

under which the coordinates are changed by

$$\delta_{\rm s} x^\mu = \omega^\mu{}_\nu x^\nu, \tag{8.230}$$

the spin components transform under the representation matrix

$$D(\delta_\mu{}^\nu + \omega_\mu{}^\nu) = \left(1 - i\frac{1}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right), \tag{8.231}$$

where $\sigma_{\mu\nu}$ are the 4×4 matrices acting on the spinor space

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]. \tag{8.232}$$

From the anticommutation rules (8.160), it is easy to verify that the spin matrices $S_{\mu\nu} \equiv \sigma_{\mu\nu}/2$ satisfy the same commutation rules (8.86) as the previous orbital and spin-1 generators $\hat{L}_{\mu\nu\mu}$ and $L_{\mu\nu}$ of Lorentz transformations.

The field has the symmetry variation [compare (8.203)]:

$$\begin{aligned} \delta_{\rm s}\psi(x) &= \psi'(x) - \psi(x) = D(\delta_\mu{}^\nu + \omega_\mu{}^\nu)\psi(x-\delta x) - \psi(x) \\ &= -i\frac{1}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\psi(x) - \omega^\lambda{}_\nu x^\nu \partial_\lambda \psi(x) \\ &= -i\frac{1}{2}\omega_{\mu\nu}\left(S^{\mu\nu} + \hat{L}^{\mu\nu}\right)\psi(x) \equiv -i\frac{1}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu}\psi(x), \end{aligned} \tag{8.233}$$

the last line showing the separation into spin and orbital transformation for a Dirac particle.

Since the Dirac Lagrangian is Lorentz-invariant, it changes under Lorentz transformations like a scalar field [compare (8.208)]:

$$\mathcal{L}'(x') = \mathcal{L}(x). \tag{8.234}$$

Infinitesimally, this amounts to

$$\delta_{\rm s}\mathcal{L} = -(\partial_\mu \mathcal{L} x^\nu)\omega^\mu{}_\nu. \tag{8.235}$$

With the Lorentz transformations being symmetry transformations in the Noether sense, we calculate the *current of total four-dimensional angular momentum* extending the formulas (8.191) and (8.210) for scalar field and vector potential. The result is

$$J^{\mu\nu,\lambda} = \left(-i\frac{\partial\mathcal{L}}{\partial\partial_\lambda\psi}\sigma^{\mu\nu}\psi - i\frac{\partial\mathcal{L}}{\partial\partial_\lambda\psi}\hat{L}^{\mu\nu}\psi + \text{c.c.}\right) + \left[\delta^{\mu\lambda}\mathcal{L}x^\nu - (\mu\leftrightarrow\nu)\right]. \tag{8.236}$$

As before in (8.211) and (8.191), the orbital part of (8.236) can be expressed in terms of the canonical energy-momentum tensor as

$$L^{\mu\nu,\lambda} = x^{\mu}\Theta^{\nu\lambda} - x^{\nu}\Theta^{\mu\lambda}. \tag{8.237}$$

The first term in (8.236) is the *spin current*

$$\Sigma^{\mu\nu,\lambda} = \frac{1}{2}\left(-i\frac{\partial\mathcal{L}}{\partial\partial_\lambda\psi}\sigma^{\mu\nu}\psi + \text{c.c.}\right). \tag{8.238}$$

Inserting (8.168), this becomes explicitly

$$\Sigma^{\mu\nu,\lambda} = -\frac{i}{2}\bar{\psi}\gamma^{\lambda}\sigma^{\mu\nu}\psi = \frac{1}{2}\bar{\psi}\gamma^{[\mu}\gamma^{\nu}\gamma^{\lambda\}]}\psi = \frac{1}{2}\epsilon^{\mu\nu\lambda\kappa}\bar{\psi}\gamma^{\kappa}\psi. \tag{8.239}$$

The spin density is completely antisymmetric in the three indices [3].

The conservation properties of the three currents are the same as in Eqs. (8.217)–(8.221).

Due to the presence of spin, the energy-momentum tensor is nonsymmetric.

8.10 Symmetric Energy-Momentum Tensor

Since the presence of spin is the cause of asymmetry of the canonical energy-momentum tensor, it is suggestive that an appropriate use of the spin current should help to construct a new modified momentum tensor

$$T^{\mu\nu} = \Theta^{\mu\nu} + \Delta\Theta^{\nu\mu}, \tag{8.240}$$

that is symmetric, while still having the fundamental property of $\Theta^{\mu\nu}$ that its spatial integral $P^{\mu} = \int d^3x\, T^{\mu 0}$ yields the total energy-momentum vector of the system. This is ensured by the fact that $\Delta\Theta^{\mu 0}$ being a three-divergence of a spatial vector. Such a construction was found in 1939 by Belinfante [4]. He introduced the tensor

$$T^{\mu\nu} = \Theta^{\mu\nu} - \frac{1}{2}\partial_{\lambda}(\Sigma^{\mu\nu,\lambda} - \Sigma^{\nu\lambda,\mu} + \Sigma^{\lambda\mu,\nu}), \tag{8.241}$$

whose symmetry is manifest, due to (8.219) and the symmetry of the last two terms under the exchange $\mu \leftrightarrow \nu$. Moreover, the relation (8.241) for the $\mu 0$-components of (8.241),

$$T^{\mu 0} = \Theta^{\mu 0} - \frac{1}{2}\partial_{\lambda}(\Sigma^{\mu 0,\lambda} - \Sigma^{0\lambda,\mu} + \Sigma^{\lambda\mu,0}), \tag{8.242}$$

ensures that the spatial integral over $J^{\mu\nu,0} \equiv x^{\mu}T^{\nu 0} - x^{\nu}T^{\mu 0}$ leads to the same total angular momentum

$$J^{\mu\nu} = \int d^3x\, J^{\mu\nu,0} \tag{8.243}$$

as the canonical expression (8.216). Indeed, the zeroth component of (8.242) is

$$x^\mu \Theta^{\nu 0} - x^\nu \Theta^{\mu 0} - \frac{1}{2}\left[\partial_k(\Sigma^{\mu 0,k} - \Sigma^{0k,\mu} + \Sigma^{k\mu,0})x^\nu - (\mu \leftrightarrow \nu)\right]. \tag{8.244}$$

Integrating the second term over d^3x and performing a partial integration gives, for $\mu = 0, \nu = i$:

$$-\frac{1}{2}\int d^3x \left[x^0\partial_k(\Sigma^{i0,k} - \Sigma^{0k,i} + \Sigma^{ki,0}) - x^i\partial_k(\Sigma^{00,k} - \Sigma^{0k,0} + \Sigma^{k0,0})\right] = \int d^3x\, \Sigma^{0i,0}, \tag{8.245}$$

and for $\mu = i, \nu = j$:

$$-\frac{1}{2}\int d^3x \left[x^i\partial_k(\Sigma^{j0,k} - \Sigma^{0k,j} + \Sigma^{kj,0}) - (i \leftrightarrow j)\right] = \int d^3x\, \Sigma^{ij,0}. \tag{8.246}$$

The right-hand sides are the contributions of spin to the total angular momentum.

For the electromagnetic field, the spin current (8.214) reads explicitly

$$\Sigma^{\mu\nu,\lambda} = -\frac{1}{c}\left[F^{\lambda\mu}A^\nu - (\mu \leftrightarrow \nu)\right]. \tag{8.247}$$

From this we calculate the Belinfante correction

$$\begin{aligned}\Delta\Theta^{\mu\nu} &= \frac{1}{2c}[\partial_\lambda(F^{\lambda\mu}A^\nu - F^{\lambda\nu}A^\mu) - \partial_\lambda(F^{\mu\nu}A^\lambda - F^{\mu\lambda}A^\nu) + \partial_\lambda(F^{\nu\lambda}A^\mu - F^{\nu\mu}A^\lambda)] \\ &= \frac{1}{c}\partial_\lambda(F^{\nu\lambda}A^\mu).\end{aligned} \tag{8.248}$$

Adding this to the canonical energy-momentum tensor (8.153)

$$\Theta^{\mu\nu} = -\frac{1}{c}(F^\nu{}_\lambda\partial^\mu A^\lambda - \frac{1}{4}g^{\mu\nu}F^{\lambda\kappa}F_{\lambda\kappa}), \tag{8.249}$$

we find the symmetric energy-momentum tensor

$$T^{\mu\nu} = -\frac{1}{c}(F^\nu{}_\lambda F^{\mu\lambda} - \frac{1}{4}g^{\mu\nu}F^{\lambda\kappa}F_{\lambda\kappa}) + \frac{1}{c}(\partial_\lambda F^{\nu\lambda})A^\mu. \tag{8.250}$$

The last term vanishes due to the free Maxwell field equations, $\partial_\lambda F^{\mu\nu} = 0$. Therefore it can be dropped. Note that the proof of the symmetry of $T^{\mu\nu}$ involves the field equations via the divergence equation (8.219).

It is useful to see what happens to Belinfante's energy-momentum tensor in the presence of an external current, i.e., if $T^{\mu\nu}$ is calculated from the Lagrangian

$$\mathcal{L} = -\frac{1}{4c}F_{\mu\nu}F^{\mu\nu} - \frac{1}{c^2}j^\mu A_\mu, \tag{8.251}$$

with an external current. The energy-momentum tensor is

$$\Theta^{\mu\nu} = \frac{1}{c}\left(F^\nu{}_\lambda\partial^\mu A^\lambda - \frac{1}{4}g^{\mu\nu}F^{\lambda\kappa}F_{\lambda\kappa}\right) + \frac{1}{c^2}g^{\mu\nu}j^\lambda A_\lambda, \tag{8.252}$$

generalizing (8.25).

The spin current is the same as before, and we find Belinfante's energy-momentum tensor [4]:

$$\begin{aligned} T^{\mu\nu} &= \Theta^{\mu\nu} + \frac{1}{c}\partial_\lambda(F^{\nu\lambda}A^\mu) \qquad (8.253)\\ &= -\frac{1}{c}(F^\nu{}_\lambda F^{\mu\lambda} - \frac{1}{4}g^{\mu\nu}F^{\lambda\kappa}F_{\lambda\kappa}) + \frac{1}{c^2}g^{\mu\nu}j^\lambda A_\lambda + \frac{1}{c}(\partial_\lambda F^{\nu\lambda})A^\mu. \end{aligned}$$

Using Maxwell's equations $\partial_\lambda F^{\nu\lambda} = -j^\lambda$, the last term can also be rewritten as

$$-\frac{1}{c}j^\nu A^\mu. \qquad (8.254)$$

This term prevents $T^{\mu\nu}$ from being symmetric, unless the current j^λ vanishes.

8.10.1 Gravitational Field

The derivation of the canonical energy-momentum tensor $\Theta^{\mu\nu}$ for the gravitational field is similar to that for the electromagnetic field in Subsection 8.9.1. We start from the quadratic action of the gravitational field (4.372),

$$\overset{\mathrm{f}}{\mathcal{A}} = -\frac{1}{8\kappa}\int d^4x(\partial_\mu h^{\nu\lambda}\partial^\mu h_{\nu\lambda} - 2\partial_\lambda h_{\mu\nu}\partial^\mu h^{\nu\lambda} + 2\partial_\mu h^{\mu\nu}\partial_\nu h - \partial_\mu h\partial^\mu h), \qquad (8.255)$$

and identify the canonically conjugate field $\pi_{\lambda\mu\nu}$,

$$\pi_{\lambda\mu\nu} \equiv \frac{\partial \overset{\mathrm{f}}{\mathcal{L}}}{\partial\partial^\lambda h^{\mu\nu}}, \qquad (8.256)$$

as being

$$\pi_{\lambda\mu\nu}=\frac{1}{8\kappa}[(\partial_\lambda h_{\mu\nu}-\partial_\mu h_{\lambda\nu})\,(\eta_{\lambda\nu}\partial_\mu h - \eta_{\mu\nu}\partial_\lambda h)-\eta_{\lambda\nu}\partial^\sigma h_{\sigma\mu}+\eta_{\mu\nu}\partial^\sigma h_{\sigma\lambda}]+(\mu\leftrightarrow\nu). \qquad (8.257)$$

It is antisymmetric under the exchange $\lambda \leftrightarrow \mu$, and symmetric under $\mu \leftrightarrow \nu$. From the integrand in (8.255) we calculate, according to the general expression (8.136),

$$\begin{aligned} \overset{\mathrm{f}}{\Theta}{}^\mu{}_\nu &= \frac{\partial\mathcal{L}}{\partial\partial^\nu h^{\lambda\kappa}}\partial^\mu h^{\lambda\kappa} - \eta^{\mu\nu}\mathcal{L} = \pi_{\nu\lambda\kappa}\partial^\mu h^{\lambda\kappa} - \eta^\mu{}_\nu\mathcal{L}\\ &= \frac{1}{2\kappa}(\partial_\nu h_{\lambda\kappa} - \partial_\kappa h_{\delta\lambda} + \eta_{\nu\kappa}\partial_\lambda h - \eta_{\nu\lambda}\partial_\kappa h - \eta_{\nu\kappa}\partial^\sigma h_{\sigma\lambda} + \eta_{\nu\kappa}\partial^\sigma h_{\sigma\nu})\partial^\mu h^{\lambda\kappa}\\ &- \frac{\eta^\mu{}_\nu}{8\kappa}(\partial_\kappa h^{\sigma\lambda}\partial^\kappa h_{\sigma\lambda} - 2\partial_\lambda h_{\sigma\nu}\partial^\sigma h^{\nu\lambda} + 2\partial_\sigma h^{\sigma\nu}\partial_\nu h - \partial_\sigma h\partial^\sigma h). \qquad (8.258) \end{aligned}$$

In order to find the symmetric energy-momentum tensor $\overset{\mathrm{f}}{T}{}^{\mu\nu}$, we follow Belinfante's construction rule. The spin current density is calculated as in Subsection 8.9.1, starting from the substantial derivative of the tensor field

$$\delta_s h^{\mu\nu} = \omega^\mu{}_\kappa h^{\kappa\nu} + \omega^\nu{}_\kappa h^{\mu\kappa}. \qquad (8.259)$$

Following the Noether rules, we find, as in (8.215),

$$\Sigma^{\mu\nu,\lambda} = 2\frac{\partial\phi}{\partial\partial_\lambda h_{\mu\kappa}} - (\mu \leftrightarrow \nu) = 2\left[\pi_{\lambda\mu\kappa}h^{\nu\kappa} - (\mu \leftrightarrow \nu)\right]. \tag{8.260}$$

Combining the two results according to Belinfante's formula (4.57), we obtain the symmetric energy-momentum tensor

$$\overset{\mathrm{f}}{T}{}^{\mu\nu} = \pi^{\nu\lambda\kappa}\partial^\mu h_{\lambda\kappa} - \partial_c(\pi^{\lambda\mu\kappa}h^\nu{}_\kappa - \pi^{\lambda\nu\kappa}h^\mu{}_\kappa - \pi^{\mu\nu\kappa}h^c{}_d + \pi^{\mu\lambda\kappa}h^\nu{}_\kappa\pi^{\nu\lambda\kappa}h^\mu{}_\kappa - \pi^{\nu\mu\kappa}h_{\lambda\kappa}) - \eta^{\mu\nu}\mathcal{L}. \tag{8.261}$$

Using the field equation $\partial_\mu\pi^{\mu\nu\lambda} = 0$ and the Hilbert gauge (4.399) with $\partial_\mu\phi^{\mu\nu} = 0$, this takes the simple form in $\phi^{\mu\nu}$:

$$\overset{\mathrm{f}}{T}{}^{\mu\nu} = \frac{1}{8\kappa}\left[2\partial^\mu\phi^{\lambda\kappa}\partial^\nu\phi_{\lambda\kappa} - \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\left(\partial_\lambda\phi^{\kappa\sigma}\partial^\lambda\phi_{\kappa\sigma} - \frac{1}{2}\partial_\lambda\phi\partial^\lambda\phi\right)\right]. \tag{8.262}$$

8.11 Internal Symmetries

In quantum field theory, an important role is played by *internal symmetries*. They do not involve any change in the spacetime coordinate of the fields, whose symmetry transformations have the simple form

$$\phi'(x) = e^{-i\alpha G}\phi(x), \tag{8.263}$$

where G are the generators of some Lie group and α the associated transformation parameters. The field ϕ may have several indices on which the generators G act as a matrix. The symmetry variation associated with (8.263) is obviously

$$\delta_s\phi'(x) = -i\alpha G\phi(x). \tag{8.264}$$

The most important example is that of a complex field ϕ and a generator $G = 1$, where (8.263) is simply a multiplication by a constant phase factor. One also speaks of U(1)-symmetry. Other important examples are those of a triplet or an octet of fields ϕ_i with G being the generators of an SU(2) vector representation or an SU(3) octet representation (the adjoint representations of these groups). The first case is associated with charge conservation in electromagnetic interactions, the other two with isospin and SU(3) invariance in strong interactions. The latter symmetries are, however, not exact.

8.11.1 U(1)-Symmetry and Charge Conservation

Suppose that a Lagrangian density $\mathcal{L}(x) = \mathcal{L}(\phi(x), \partial\phi(x), x)$ depends only on the absolute squares $|\phi|^2, |\partial\phi|^2, |\phi\partial\phi|$. Then $\mathcal{L}(x)$ is invariant under U(1)-transformations

$$\delta_s\phi(x) = -i\phi(x). \tag{8.265}$$

Indeed:

$$\delta_s \mathcal{L} = 0. \tag{8.266}$$

On the other hand, we find by the chain rule of differentiation:

$$\delta_s \mathcal{L} = \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial_\mu \phi} \right) \delta_s \phi + \left[\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right] \delta_s \phi = 0. \tag{8.267}$$

The Euler-Lagrange equation removes the first part of this, and inserting (8.265) we find by comparison with (8.266) that

$$j_\mu = -\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \phi \tag{8.268}$$

is a conserved current.

For a free relativistic complex scalar field with a Lagrangian density

$$\mathcal{L}(x) = \partial_\mu \varphi^* \partial_\mu \varphi - m^2 \varphi^* \varphi \tag{8.269}$$

we have to add the contributions of real and imaginary parts of the field ϕ in formula (8.268). Then we obtain the conserved current

$$j_\mu = -i \varphi^* \overleftrightarrow{\partial}_\mu \varphi \tag{8.270}$$

where $\varphi^* \overleftrightarrow{\partial}_\mu \varphi$ denotes the left-minus-right derivative:

$$\varphi^* \overleftrightarrow{\partial}_\mu \varphi \equiv \varphi^* \partial_\mu \varphi - (\partial_\mu \varphi^*) \varphi. \tag{8.271}$$

For a free Dirac field, we find from (8.268) the conserved current

$$j^\mu(x) = \bar{\psi}(x) \gamma^\mu \psi(x). \tag{8.272}$$

8.11.2 SU(N)-Symmetry

For more general internal symmetry groups, the symmetry variations have the form

$$\delta_s \varphi = -i \alpha_i G_i \varphi, \tag{8.273}$$

and the conserved currents are

$$j_i^\mu = -i \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} G_i \varphi. \tag{8.274}$$

8.11.3 Broken Internal Symmetries

The physically important symmetries SU(2) of isospin and SU(3) are not exact. The Lagrange density is not strictly zero. In this case we remember the alternative derivation of the conservation law from (8.116). We introduce the spacetime-dependent parameters $\alpha(x)$ and conclude from the extremality property of the action that

$$\partial_\mu \frac{\partial \mathcal{L}^\epsilon}{\partial \partial_\mu \alpha_i(x)} = \frac{\partial \mathcal{L}^\epsilon}{\partial \alpha_i(x)}. \tag{8.275}$$

This implies the divergence law for the above derived current

$$\partial_\mu j_i^\mu(x) = \frac{\partial \mathcal{L}^\epsilon}{\partial \alpha_i}. \tag{8.276}$$

8.12 Generating the Symmetry Transformations of Quantum Fields

As in quantum mechanical systems, the charges associated with the conserved currents of the previous section can be used to generate the transformations of the fields from which they were derived. One merely has to invoke the canonical field commutation rules.

As an important example, consider the currents (8.274) of an internal U(N)-symmetry. Their charges

$$Q^i = -i \int d^3x \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} G_i \varphi \tag{8.277}$$

can be written as

$$Q^i = -i \int d^3x \pi G_i \varphi, \tag{8.278}$$

where $\pi(x) \equiv \partial \mathcal{L}(x)/\partial \partial_\mu \varphi(x)$ is the canonical momentum of the field $\varphi(x)$. After quantization, these fields satisfy the canonical commutation rules:

$$\begin{aligned} [\pi(\mathbf{x},t), \varphi(\mathbf{x}',t)] &= -i\delta^{(3)}(\mathbf{x}-\mathbf{x}'), \\ [\varphi(\mathbf{x},t), \varphi(\mathbf{x}',t)] &= 0, \\ [\pi(\mathbf{x},t), \pi(\mathbf{x}',t)] &= 0. \end{aligned} \tag{8.279}$$

From this we derive directly the commutation rule between the quantized charges (8.278) and the field $\varphi(x)$:

$$[Q^i, \hat{\varphi}(x)] = -\alpha^i G_i \varphi(x). \tag{8.280}$$

We also find that the commutation rules among the quantized charges are

$$[\hat{Q}^i, \hat{Q}^j] = [G^i, G^j]. \tag{8.281}$$

Since these coincide with those of the matrices G_i, the operators Q^i are seen to form a representation of the generators of the symmetry group in the Fock space.

It is important to realize that the commutation relations (8.280) and (8.281) remain also valid in the presence of symmetry breaking terms, as long as these do not contribute to the canonical momentum of the theory. Such terms are called *soft symmetry breaking terms*. The charges are no longer conserved, so that we must attach a time argument to the commutation relations (8.280) and (8.281). All times in these relations must be the same, in order to invoke the equal-time canonical commutation rules.

The most important example is the canonical commutation relation (8.95) itself, which holds also in the presence of any potential $V(q)$ in the Hamiltonian. This breaks translational symmetry, but does not contribute to the canonical momentum $p = \partial L/\partial \dot{q}$. In this case, the relation generalizes to

$$\epsilon^j = i\epsilon^i[\hat{p}^i(t), \hat{x}^j(t)], \tag{8.282}$$

which is correct thanks to the validity of the canonical commutation relations (8.93) at arbitrary equal times, also in the presence of a potential.

Other important examples are the commutation rules of the conserved charges associated with the Lorentz generators (8.237):

$$J^{\mu\nu} \equiv \int d^3x J^{\mu\nu,0}(x), \tag{8.283}$$

which are the same as those of the 4×4-matrices (8.205), and those of the quantum mechanical generators (8.85):

$$[\hat{J}^{\mu\nu}, \hat{J}^{\mu\lambda}] = -ig^{\mu\mu}\hat{J}^{\nu\lambda}. \tag{8.284}$$

The generators $J^{\mu\nu} \equiv \int d^3x J^{\mu\nu,0}(x)$ are sums $J^{\mu\nu} = L^{\mu\nu}(t) + \Sigma^{\mu\nu}(t)$ of charges (8.220) associated with orbital and spin rotations. According to (8.221), the individual charges are time-dependent. Only their sum is conserved. Nevertheless, they both generate Lorentz transformations: $L^{\mu\nu}(t)$ on the spacetime argument of the fields, and $\Sigma^{\mu\nu}(t)$ on the spin indices. As a consequence, they both satisfy the commutation relations (8.284):

$$\begin{aligned} [\hat{L}^{\mu\nu}, \hat{L}^{\mu\lambda}] &= -ig^{\mu\mu}\hat{L}^{\nu\lambda}, \\ [\hat{\Sigma}^{\mu\nu}, \hat{\Sigma}^{\mu\lambda}] &= -ig^{\mu\mu}\hat{\Sigma}^{\nu\lambda}. \end{aligned} \tag{8.285}$$

The commutators (8.281) have played an important role in developing a theory of strong interactions, where they first appeared in the form of a *charge algebra* of the broken symmetry SU(3) $\times$ SU(3) of weak and electromagnetic charges. This symmetry will be discussed in more detail in Chapter 10.

8.13 Energy Momentum Tensor of a Relativistic Massive Point Particle

If we want to study energy and momentum of charged relativistic point particles in an electromagnetic field, it is useful to consider the action (8.68) with (8.72) as a spacetime integral over a *Lagrangian density*:

$$\mathcal{A} = \int d^4x\, \mathcal{L}(x), \qquad \text{with} \quad \mathcal{L}(x) = \int_{\tau_\mu}^{\tau_b} L(\dot{x}^\mu(\tau))\delta^{(4)}(x - x(\tau)). \tag{8.286}$$

We can then derive for point particles local conservation laws that look very similar to those for fields. Instead of doing this from scratch, however, we shall simply take the already known global conservation laws and convert them into the local ones by inserting appropriate δ-functions with the help of the trivial identity

$$\int d^4x\, \delta^{(4)}(x - x(\tau)) = 1. \tag{8.287}$$

Consider for example the conservation law (8.74) for the momentum (8.75). With the help of (8.287) this becomes

$$0 = -\int d^4x \int_{-\infty}^{\infty} d\tau \left[\frac{d}{d\tau} p_\lambda(\tau)\right] \delta^{(4)}(x - x(\tau)). \tag{8.288}$$

Note that the boundaries of the four volume in this expression contain the information on initial and final times. We now perform a partial integration in τ, and rewrite (8.288) as

$$0 = -\int d^4x \int_{-\infty}^{\infty} d\tau \frac{d}{d\tau} \left[p_\lambda(\tau)\delta^{(4)}(x - x(\tau))\right] + \int d^4x \int_{-\infty}^{\infty} d\tau\, p_\lambda(\tau)\partial_\tau \delta^{(4)}(x - x(\tau)). \tag{8.289}$$

The first term vanishes if the orbits come from and disappear into infinity. The second term can be rewritten as

$$0 = -\int d^4x\, \partial_\nu \left[\int_{-\infty}^{\infty} d\tau\, p_\lambda(\tau)\dot{x}^\nu(\tau)\delta^{(4)}(x - x(\tau))\right]. \tag{8.290}$$

This shows that the tensor

$$\Theta^{\lambda\nu}(x) \equiv \int_{-\infty}^{\infty} d\tau\, p^\lambda(\tau)\dot{x}^\nu(\tau)\delta^{(4)}(x - x(\tau)) \tag{8.291}$$

satisfies the local conservation law

$$\partial_\nu \Theta^{\lambda\nu}(x) = 0. \tag{8.292}$$

This is the conservation law of the energy-momentum tensor of a massive point particle.

The total momenta are obtained from the spatial integrals over $\Theta^{\lambda 0}$:

$$P^{\mu}(t) \equiv \int d^3x\, \Theta^{\lambda 0}(x). \tag{8.293}$$

For point particles, they coincide with the canonical momenta $p^{\mu}(t)$. If the Lagrangian depends only on the velocity $\dot{x}^{\mu}(t)$ and not on the position $x^{\mu}(t)$, the momenta $p^{\mu}(t)$ are constants of motion: $p^{\mu}(t) \equiv p^{\mu}$.

The Lorentz invariant quantity

$$M^2 = P^2 = g_{\mu\nu} P^{\mu} P^{\nu} \tag{8.294}$$

is called the *total mass* of the system. For a single particle it coincides with the mass of the particle.

Subjecting the orbits $x^{\mu}(\tau)$ to Lorentz transformations according to the rules of the last section we find the currents of the total angular momentum

$$L^{\mu\nu,\lambda} \equiv x^{\mu} \Theta^{\nu\lambda} - x^{\nu} \Theta^{\mu\lambda}, \tag{8.295}$$

to satisfy the conservation law:

$$\partial_{\lambda} L^{\mu\nu,\lambda} = 0. \tag{8.296}$$

A spatial integral over the zeroth component of the current $L^{\mu\nu,\lambda}$ yields the conserved charges:

$$L^{\mu\nu}(t) \equiv \int d^3x\, L^{\mu\nu,0}(x) = x^{\mu} p^{\nu}(t) - x^{\nu} p^{\mu}(t). \tag{8.297}$$

8.14 Energy Momentum Tensor of a Massive Charged Particle in a Maxwell Field

Let us consider an important combination of a charged point particle and an electromagnetic field Lagrangian

$$\mathcal{A} = -mc \int_{\tau_{\mu}}^{\tau_{\nu}} d\tau \sqrt{g_{\mu\nu} \dot{x}^{\mu}(\tau) \dot{x}^{\nu}(\tau)} - \frac{1}{4c} \int d^4x F_{\mu\nu} F^{\mu\nu} - \frac{e}{c^2} \int_{\tau_{\mu}}^{\tau_{\nu}} d\tau \dot{x}^{\mu}(\tau) A_{\mu}(x(\tau)). \tag{8.298}$$

By varying the action in the particle orbits, we obtain the Lorentz equation of motion

$$\frac{dp^{\mu}}{d\tau} = \frac{e}{c} F^{\mu}{}_{\nu} \dot{x}^{\nu}(\tau). \tag{8.299}$$

We now vary the action in the vector potential, and find the Maxwell-Lorentz equation

$$-\partial_{\nu} F^{\mu\nu} = \frac{e}{c} \dot{x}^{\nu}(\tau). \tag{8.300}$$

The action (8.298) is invariant under translations of the particle orbits and the electromagnetic fields. The first term is obviously invariant, since it depends only on the derivatives of the orbital variables $x^\mu(\tau)$. The second term changes under translations by a pure divergence [recall (8.134)]. The interaction term also changes by a pure divergence, which is seen as follows: Since the symmetry variation changes the coordinates as $x^\nu(\tau) \to x^\nu(\tau) - \epsilon^\nu$, and the field $A_\mu(x^\nu)$ is transformed as follows:

$$A_\mu(x^\nu) \to A'_\mu(x^\nu) = A_\mu(x^\nu + \epsilon^\nu) = A_\mu(x^\nu) + \epsilon^\nu \partial_\mu A_\mu(x^\nu), \tag{8.301}$$

we have altogether the symmetry variation

$$\delta_s \mathcal{L} = \epsilon^\nu \partial_\nu \overset{\mathrm{m}}{\mathcal{L}}. \tag{8.302}$$

We now calculate the same variation once more using the equations of motion. This gives

$$\delta_s \mathcal{A} = \int d\tau \frac{d}{d\tau} \frac{\partial L^{\mathrm{m}}}{\partial x'^\mu} \delta_s x^\mu + \int d^4x \frac{\partial \overset{\mathrm{em}}{\mathcal{L}}}{\partial \partial_\lambda A^\mu} \delta_s A^\mu. \tag{8.303}$$

The first term can be treated as in (8.289)–(8.290), after which it acquires the form

$$\begin{aligned} -\int_{\tau_\mu}^{\tau_\nu} d\tau \frac{d}{d\tau}\left(p_\mu + \frac{e}{c}A_\mu\right) &= -\int d^4x \int_{-\infty}^{\infty} d\tau \frac{d}{d\tau}\left[\left(p_\mu + \frac{e}{c}A_\mu\right)\delta^{(4)}(x - x(\tau)\right] \\ &\quad + \int d^4x \int_{-\infty}^{\infty} d\tau \left(p_\mu + \frac{e}{c}A_\mu\right) \frac{d}{d\tau}\delta^{(4)}(x - x(\tau)), \end{aligned} \tag{8.304}$$

and thus, after dropping boundary terms,

$$-\int_{\tau_\mu}^{\tau_\nu} d\tau \frac{d}{d\tau}(p_\mu + \frac{e}{c}A_\mu) = \partial_\lambda \int d^4x \int_{-\infty}^{\infty} d\tau \left(p_\mu + \frac{e}{c}A_\mu\right) \frac{dx^\lambda}{d\tau}\delta^{(4)}(x - x(\tau)). \tag{8.305}$$

The electromagnetic part is the same as before, since the interaction contains no derivative of the gauge field. In this way we find the canonical energy-momentum tensor

$$\begin{aligned} \Theta^{\mu\nu}(x) &= \int d\tau \left(p^\mu + \frac{e}{c}A^\mu\right) \dot{x}^\nu(\tau)\delta^{(4)}(x - x(\tau)) \\ &\quad - \frac{1}{c}\left(F^\nu{}_\lambda \partial^\mu A^\lambda - \frac{1}{4}g^{\mu\nu}F^{\lambda\kappa}F_{\lambda\kappa}\right). \end{aligned} \tag{8.306}$$

Let us check its conservation by calculating the divergence:

$$\begin{aligned} \partial_\nu \Theta^{\mu\nu}(x) &= \int d\tau \left(p + \frac{e}{c}A_\mu\right) \dot{x}^\nu(\tau)\partial_\nu \delta^{(4)}(x - x(\tau)) \\ &\quad - \frac{1}{c}\partial_\nu F^\nu{}_\lambda \partial^\mu A^\lambda - \frac{1}{c}\left[F^\nu{}_\lambda \partial_\nu \partial^\mu A^\lambda - \frac{1}{4}\partial^\mu (F^{\lambda\kappa}F_{\lambda\kappa})\right]. \end{aligned} \tag{8.307}$$

The first term is, up to a boundary term, equal to

$$-\int d\tau \left(p^\mu + \frac{e}{\tau}A^\mu\right) \frac{d}{d\tau}\delta^{(4)}(x - x(\tau)) = \int d\tau \left[\frac{d}{d\tau}\left(p^\mu + \frac{e}{c}A^\mu\right)\right] \delta^{(4)}(x - x(\tau)). \tag{8.308}$$

Using the Lorentz equation of motion (8.299), this becomes

$$\frac{e}{c}\int_{-\infty}^{\infty} d\tau \left(F^{\mu}{}_{\nu}\dot{x}^{\nu}(\tau) + \frac{d}{d\tau}A^{\mu}\right)\delta^{(4)}(x-x(\tau)). \tag{8.309}$$

Inserting the Maxwell equation

$$\partial_{\nu}F^{\mu\nu} = -e\int d\tau (dx^{\mu}/d\tau)\delta^{(4)}(x-x(\tau)), \tag{8.310}$$

the second term in Eq. (8.307) can be rewritten as

$$-\frac{e}{c}\int_{-\infty}^{\infty} d\tau \frac{dx_{\lambda}}{d\tau}\partial^{\mu}A^{\lambda}\delta^{(4)}(x-x(\tau)), \tag{8.311}$$

which is the same as

$$-\frac{e}{c}\int d\tau \left(\frac{dx_{\mu}}{d\tau}F^{\mu\lambda} + \frac{dx_{\lambda}}{d\tau}\partial^{\lambda}A^{\mu}\right)\delta^{(4)}(x-x(\tau)), \tag{8.312}$$

thus canceling (8.309). The third term in (8.307) is finally equal to

$$-\frac{1}{c}\left[F^{\nu}{}_{\lambda}\partial^{\mu}F_{\nu}{}^{\lambda} - \frac{1}{4}\partial^{\mu}(F^{\lambda\kappa}F_{\lambda\kappa})\right], \tag{8.313}$$

due to the antisymmetry of $F^{\nu\lambda}$. With the help of the homogeneous Maxwell equation we verify the *Bianchi identity*

$$\partial_{\lambda}F_{\mu\nu} + \partial_{\mu}F_{\nu\lambda} + \partial_{\nu}F_{\lambda\mu} = 0. \tag{8.314}$$

is identically guaranteed.

It is easy to construct from (8.306) Belinfante's symmetric energy-momentum tensor. We merely observe that the spin density comes entirely from the vector potential, and is hence the same as before in (8.247). Hence the additional piece to be added to the canonical energy-momentum tensor is again [see (8.248)]

$$\begin{aligned} \Delta\Theta^{\mu\nu} &= \frac{1}{c}\partial_{\lambda}(F^{\mu\nu}A^{\mu}) \\ &= \frac{1}{2}(\partial_{\lambda}F^{\nu\lambda}A^{\mu} + F^{\nu\lambda}\partial_{\lambda}A^{\mu}). \end{aligned} \tag{8.315}$$

The second term in this expression serves to symmetrize the electromagnetic part of the canonical energy-momentum tensor and brings it to the Belinfante form:

$$\overset{\mathrm{em}}{T}{}^{\mu\nu} = -\frac{1}{c}\left(F^{\nu}{}_{\lambda}F^{\mu\lambda} - \frac{1}{4}g^{\mu\nu}F^{\lambda\kappa}F_{\lambda\kappa}\right). \tag{8.316}$$

The first term in (8.315), which in the absence of charges vanishes, is now just what is needed to symmetrize the matter part of $\Theta^{\mu\nu}$. Indeed, using once more Maxwell's equation, it becomes

$$-\frac{e}{c}\int d\tau \dot{x}^{\nu}(\tau)A^{\mu}\delta^{(4)}(x-x(\tau)), \tag{8.317}$$

thus canceling the corresponding term in (8.306). In this way we find that the total energy-momentum tensor of charged particles plus electromagnetic fields is simply the sum of the two symmetric energy-momentum tensors:

$$\begin{aligned} T^{\mu\nu} &= \overset{\rm m}{T}{}^{\mu\nu} + \overset{\rm em}{T}{}^{\mu\nu} \\ &= \frac{1}{m}\int_{-\infty}^{\infty} d\tau\, u^\mu u^\nu \delta^{(4)}(x-x(\tau)) - \frac{1}{c}\left(F^\nu{}_\lambda F^{\mu\lambda} - \frac{1}{4}g^{\mu\nu}F^{\lambda\kappa}F_{\lambda\kappa}\right). \end{aligned} \tag{8.318}$$

For completeness, let us also cross-check its conservation:

$$\partial_\nu T^{\mu\nu} = 0. \tag{8.319}$$

Indeed, forming the divergence of the first term gives [in contrast to (8.309)]

$$\frac{e}{c}\int d\tau\, \dot{x}^\nu(\tau) F^\mu{}_\nu(x(\tau)), \tag{8.320}$$

which is canceled by the divergence in the second term [in contrast to (8.312)]

$$-\frac{1}{c}\partial_\nu F^\nu{}_\lambda F^{\mu\lambda} = -\frac{e}{c}\int d\tau\, \dot{x}_\lambda(\tau) F^{\mu\lambda}(x(\tau)). \tag{8.321}$$

Notes and References

For more details see
L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, Addison-Wesley, Reading, Mass., 1951;
S. Schweber, *Relativistic Quantum Fields*, Harper and Row, New Yoerk, N.Y., 1961;
A.O. Barut, *Electrodynamics and Classical Theory of Fields and Particles*, MacMillan, New York, N.Y. 1964;
J.D. Jackson, *Classical Electrodynamics*, John Wiley & Sons, New York, N.Y., 1975;
H. Ohanian, *Classical Electrodynamics*, Allyn and Bacon, Boston, Mass., 1988.

The individual citations refer to:

[1] E. Noether, Nachr. d. vgl. Ges. d. Wiss. Göttingen, Math-Phys. Klasse, **2**, 235 (1918);
See also
E. Bessel-Hagen, Math. Ann. **84**, 258 (1926);
L. Rosenfeld, Me. Acad. Roy. Belg. **18**, 2 (1938);
F. Belinfante, Physica **6**, 887 (1939).

[2] S. Coleman and J.H. VanVleck, Phys. Rev. **171**, 1370 (1968).

[3] This property is important for being able to construct a consistent quantum mechanics in spacetime with torsion. See the textbook
H. Kleinert, *Path Integrals in Quantum Mechanics Statistics, Polymer Physics, and Financial Markets*, World Scientific, Singapore 2008 (`http://klnrt.de/b5`).

[4] The Belinfante energy-momentum tensor is discussed further in
H. Kleinert, *Gauge Fields in Condensed Matter*, Vol. II *Stresses and Defects*, World Scientific Publishing, Singapore 1989, pp. 744–1443 (`http://klnrt.de/b2`).
H. Kleinert, *Multivalued Fields in Condensed Matter, Electromagnetism, and Gravitation*, World Scientific, Singapore 2009, pp. 1–497 (`http://klnrt.de/b11`).