Quantization of Relativistic Free Fields

In Chapter 2 we have shown that quantum mechanics of the \( N \)-body Schrödinger equation can be replaced by a Schrödinger field theory, where the fields are operators satisfying canonical equal-time commutation rules. They were given in Sections 2.8 and 2.10 for bosons and fermions, respectively. A great advantage of this reformulation of Schrödinger theory was that the field quantization leads automatically to symmetric or antisymmetric \( N \)-body wave functions, which in Schrödinger theory must be imposed from the outside in order to explain atomic spectra and Bose-Einstein condensation. Here we shall generalize the procedure to relativistic particles by quantizing the free relativistic fields of the previous section following the above general rules. For each field \( \phi(x, t) \) in a classical Lagrangian \( L(t) \), we define a canonical field momentum as the functional derivative (2.156)

\[
\pi(x, t) = p_x(t) = \frac{\delta L(t)}{\delta \dot{\phi}(x, t)},
\]

(7.1)

The fields are now turned into field operators by imposing the canonical equal-time commutation rules (2.143):

\[
[\phi(x, t), \phi(x', t)] = 0,
\]

(7.2)

\[
[\pi(x, t), \pi(x', t)] = 0,
\]

(7.3)

\[
[\pi(x, t), \phi(x', t)] = -i\delta^{(3)}(x - x').
\]

(7.4)

For fermions, one postulates corresponding anticommutation rules.

In these equations we have omitted the hat on top of the field operators, and we shall do so from now on, for brevity, since most field expressions will contain quantized fields. In the few cases which do not deal with field operators, or where it is not obvious that the fields are classical objects, we shall explicitly state this. We shall also use natural units in which \( c = 1 \) and \( \hbar = 1 \), except in some cases where CGS-units are helpful.

Before implementing the commutation rules explicitly, it is useful to note that all free Lagrangians constructed in the last chapter are local, in the sense that they are given by volume integrals over a Lagrangian density

\[
L(t) = \int d^3 x \mathcal{L}(x, t),
\]

(7.5)
where $\mathcal{L}(\mathbf{x}, t)$ is an ordinary function of the fields and their first spacetime derivatives. The functional derivative (7.1) may therefore be calculated as a partial derivative of the density $\mathcal{L}(x) = \mathcal{L}(\mathbf{x}, t)$:

$$\pi(x) = \frac{\partial \mathcal{L}(x)}{\partial \dot{\phi}(x)}. \quad (7.6)$$

Similarly, the Euler-Lagrange equations may be written down using only partial derivatives

$$\partial^\mu \frac{\partial \mathcal{L}(x)}{\partial [\partial^\mu \phi(x)]} = \frac{\partial \mathcal{L}(x)}{\partial \phi(x)}. \quad (7.7)$$

Here and in the sequel we employ a four-vector notation for all spacetime objects such as $x = (x^0, \mathbf{x})$, where $x^0 = ct$ coincides with the time $t$ in natural units.

The locality is an important property of all present-day quantum field theories of elementary particles. The spacetime derivatives appear usually only in quadratic terms. Since a derivative measures the difference of the field between neighboring spacetime points, the field described by a local Lagrangian propagates through spacetime via a “nearest-neighbor hopping”. In field-theoretic models of solid-state systems, such couplings are useful lowest approximations to more complicated short-range interactions. In field theories of elementary particles, the locality is an essential ingredient which seems to be present in all fundamental theories. All nonlocal effects arise from higher-order perturbation expansions of local theories.

### 7.1 Scalar Fields

We begin by quantizing the field of a spinless particle. This is a scalar field $\phi(x)$ that can be real or complex.

#### 7.1.1 Real Case

The action of the real field was given in Eq. (4.167). As we shall see in Subsec. 7.1.6, and even better in Subsec. 8.11.1, the quanta of the real field are neutral spinless particles. For the canonical quantization we must use the real-field Lagrangian density (4.169) in which only first derivatives of the field occur:

$$\mathcal{L}(x) = \frac{1}{2} \{[\partial \phi(x)]^2 - M^2 \phi^2(x)\}. \quad (7.8)$$

The associated Euler-Lagrange equation is the Klein-Gordon equation (4.170):

$$(-\partial^2 - M^2)\phi(x) = 0. \quad (7.9)$$

#### 7.1.2 Field Quantization

According to the rule (7.6), the canonical momentum of the field is

$$\pi(x) = \frac{\partial \mathcal{L}(x)}{\partial \dot{\phi}(x)} = \partial^0 \phi(x) = \dot{\phi}(x). \quad (7.10)$$
7 Quantization of Relativistic Free Fields

As in the case of non-relativistic fields, the quantization rules (7.2)–(7.4) will eventually render a multiparticle Hilbert space. To find it we expand the field operator $\phi(x,t)$ in normalized plane-wave solutions (4.180) of the field equation (7.9):

$$\phi(x) = \sum_p \left[ f_p(x)a_p + f^*_p(x)a^\dagger_p \right].$$  \hfill (7.11)

Observe that in contrast to a corresponding expansion (2.209) of the nonrelativistic Schrödinger field, the right-hand side contains operators for the creation and annihilation of particles. This is an automatic consequence of the fact that the quantizations of $\phi(x)$ and $\pi(x)$ lead to Hermitian field operators, and this is natural for quantum fields in the relativistic setting, which necessarily allow for the creation and annihilation of particles. Inserting for $f_p(x)$ and $f^*_p(x)$ the explicit wave functions (4.180), the expansion (7.11) reads

$$\phi(x) = \sum_p \frac{1}{\sqrt{2p_0V}} (e^{-ipx}a_p + e^{ipx}a_p^\dagger).$$  \hfill (7.12)

Since the zeroth component of the four-momentum $p^\mu$ in the exponent is on the mass shell, we should write $px$ more explicitly as $px = \omega_p t - px$. However, the notation $px$ is shorter, and there is little danger of confusion. If there is such a danger, we shall explicitly state the off- or on-shell property $p^0 = \omega_p$.

The expansions (7.11) and (7.12) are complete in the Hilbert space of free particles. They may be inverted for $a_p$ and $a_p^\dagger$ with the help of the orthonormality relations (4.177), which lead to the scalar products

$$a^\dagger_p = (f_p, \phi)_t, \quad a_p = -(f^*_p, \phi).$$  \hfill (7.13)

More explicitly, these can be written as

$$\left\{ \begin{array}{c} a_p \\ a^\dagger_p \end{array} \right\} = e^{\pm ip^0x_0} \frac{1}{\sqrt{2Vp^0}} \int d^3x e^{\mp ipx} \left[ \pm i\pi(x) + p^0 \phi(x) \right],$$  \hfill (7.14)

where we have inserted $\pi(x) = \dot{\phi}(x)$.

Using the canonical field commutation rules (7.2)–(7.4) between $\phi(x)$ and $\pi(x)$, we find for the coefficients of the plane-wave expansion (7.12) the commutation relations

$$\begin{array}{c} [a_p, a_{p'}] = [a^\dagger_p, a^\dagger_{p'}] = 0, \\
[a_p, a^\dagger_{p'}] = \delta_{p,p'}. \end{array}$$  \hfill (7.15)

making them creation and annihilation operators of particles of momentum $p$ on a vacuum state $|0\rangle$ defined by $a_p|0\rangle = 0$. Conversely, reinserting (7.15) into (7.12), we verify the original local field commutation rules (7.2)–(7.4).
7.1 Scalar Fields

In an infinite volume one often uses, in (7.11), the wave functions (4.181) with continuous momenta, and works with a plane-wave decomposition in the form of a Fourier integral

\[
\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \left( e^{-ipx}a_p + e^{ipx}a_p^\dagger \right),
\]

(7.16)

with the identification [recall (2.217)]

\[
a_p = \sqrt{2p^0}a_p, \quad a_p^\dagger = \sqrt{2p^0}a_p^\dagger.
\]

(7.17)

The inverse of the expansion is now

\[
\begin{pmatrix} a_p \\ a_p^\dagger \end{pmatrix} = e^{\pm ip^0x_0} \int d^3x e^{\pm ipx} \left[ \pm i\pi(x) + p^0\phi(x) \right].
\]

(7.18)

This can of course be written in the same form as in (7.13), using the orthogonality relations (4.182) for continuous-momentum wave functions (4.181):

\[
a_p^\dagger = (f_p, \phi), \quad a_p = -(f_p^*, \phi).
\]

(7.19)

In the expansion (7.11), the commutators (7.15) are valid after the replacement

\[
\delta_p, p' \rightarrow 2p^0 \delta^{(3)}(p - p') = 2p^0(2\pi\hbar)^3\delta^{(3)}(p - p'),
\]

(7.20)

which is a consequence of (1.190) and (1.196).

In the following we shall always prefer to work with a finite volume and sums over \( p \). If we want to find the large-volume limit of any formula, we can always go over to the continuum limit by replacing the sums, as in (2.257), by phase space integrals:

\[
\sum_p \rightarrow \int d^3p \frac{V}{(2\pi\hbar)^3},
\]

(7.21)

\[
\delta_p, p' \rightarrow \frac{(2\pi\hbar)^3}{V} \delta^{(3)}(p - p').
\]

(7.22)

After the replacement (7.17), the volume \( V \) disappears in all physical quantities.

A single-particle state of a fixed momentum \( p \) is created by

\[
|p\rangle = a_p^\dagger |0\rangle.
\]

(7.23)

Its wave function is obtained from the matrix elements of the field

\[
\langle 0|\phi(x)|p\rangle = \frac{1}{\sqrt{2Vp^0}} e^{-ipx} = f_p(x),
\]

\[
\langle p|\phi(x)|0\rangle = \frac{1}{\sqrt{2Vp^0}} e^{ipx} = f_p^*(x).
\]

(7.24)
As in the nonrelativistic case, the second-quantized Hilbert space is obtained by applying the particle creation operators $a_p^\dagger$ to the vacuum state $|0\rangle$, defined by $a_p^\dagger |0\rangle = 0$. This produces basis states (2.219):

$$|n_{p_1} n_{p_2} \ldots n_{p_k}\rangle = \mathcal{N}^{S,A} (\hat{a}_{p_1}^\dagger)^{n_{p_1}} \cdots (\hat{a}_{p_k}^\dagger)^{n_{p_k}} |0\rangle. \quad (7.25)$$

Multiparticle wave functions are obtained by matrix elements of the type (2.221).

The single-particle states are

$$|p\rangle = a_p^\dagger |0\rangle. \quad (7.26)$$

They satisfy the orthogonality relation

$$(p'|p) = 2 p^0 \delta^{(3)}(p' - p). \quad (7.27)$$

Their wave functions are

$$(0|\phi(x)|p) = e^{-ipx} = f_p(x),$$

$$(p|\phi(x)|0) = e^{ipx} = f_p^*(x). \quad (7.28)$$

**Energy of Free Neutral Scalar Particles**

On account of the local structure (7.5) of the Lagrangian, also the Hamiltonian can be written as a volume integral

$$H = \int d^3x \mathcal{H}(x). \quad (7.29)$$

The Hamiltonian density is the Legendre transform of the Lagrangian density

$$\mathcal{H}(x) = \pi(x) \partial^0 \phi(x) - \mathcal{L}(x)$$

$$= \frac{1}{2} [\partial^0 \phi(x)]^2 + \frac{1}{2} [\partial_x \phi(x)]^2 + \frac{M^2}{2} \phi^2(x). \quad (7.30)$$

Inserting the expansion (7.12), we obtain the Hamilton operator

$$H = \sum_{p,p'} \left\{ \frac{1}{2 \sqrt{p^0 p'^0}} \delta_{p,p'} \left( p^0 p'^0 + p \cdot p' + M^2 \right) \left( a_{p'}^\dagger a_p + a_p a_{p'}^\dagger \right) \right.$$  
$$+ \frac{1}{2 \sqrt{p^0 p'^0}} \delta_{-p,-p'} \left( -p^0 p'^0 - p \cdot p' + M^2 \right) \left[ a_{p'}^\dagger a_p e^{i(p^0 + p'^0)t} + a_p a_{p'} e^{-i(p^0 + p'^0)t} \right]\right\}$$

$$= \sum_{p} p^0 \left( a_{p}^\dagger a_p + \frac{1}{2} \right). \quad (7.31)$$

The vacuum state $|0\rangle$ has an energy

$$E_0 \equiv \langle 0|H|0\rangle = \frac{1}{2} \sum_{p} p^0 = \frac{1}{2} \sum_{p} \omega_p. \quad (7.32)$$
It contains the sum of the energies $\frac{1}{2}\hbar \omega_p$ of the zero-point oscillations of all “oscillator quanta” in the second quantization formalism. It is the result of all the so-called vacuum fluctuations of the field. In the limit of large volume, the momentum sum turns into an integral over the phase space according to the usual rule (7.21).

This energy is infinite. Fortunately, in most circumstances this infinity is unobservable. It is therefore often subtracted out of the Hamiltonian, replacing $H$ by

$$H := H - \langle 0|H|0 \rangle.$$  

(7.33)

The double dots to the left-hand side define what is called the normal product form of $H$. It is obtained by the following prescription for products of operators $a, a^\dagger$. Given an arbitrary product of creation and annihilation operators enclosed by double dots

$$:a^\dagger \cdots a \cdots a^\dagger \cdots a:,$$  

(7.34)

the product is understood to be re-ordered in a way that all creation operators stand to the left of all annihilation operators. For example,

$$:a_p^\dagger a_p + a_p a_p^\dagger : = 2 a_p^\dagger a_p,$$  

(7.35)

so that the normally-ordered free-particle Hamiltonian is

$$H := \sum_p \omega_p a_p^\dagger a_p.$$  

(7.36)

A prescription like this is necessary to make sure that the vacuum is invariant under time translations.

When following this ad hoc procedure, care has to be taken that one is not dealing with phenomena that are sensitive to the omitted zero-point oscillations. Gravitational interactions, for example, couple to zero-point energy. The infinity creates a problem when trying to construct quantum field theories in the presence of a classical gravitational field, since the vacuum energy gives rise to an infinite cosmological constant, which can be determined experimentally from solutions of the cosmological equations of motion, to have a finite value. A possible solution of this problem will appear later in Section 7.4 when quantizing the Dirac field.

We shall see in Eq. (7.248) that for a Dirac field, the vacuum energy has the same form as for the scalar field, but with an opposite sign. In fact, this opposite sign is a consequence of the Fermi-Dirac statistics of the electrons. As will be shown in Section 7.10, all particles with half-integer spins obey Fermi-Dirac statistics and require a different quantization than that of the Klein-Gordon field. They all give a negative contribution $-\hbar \omega_p / 2$ to the vacuum energy for each momentum and spin degree of freedom. On the other hand, all particles with integer spins are bosons as the scalar particles of the Klein-Gordon field and will be quantized in a similar way, thus contributing a positive vacuum energy $\hbar \omega_p / 2$ for each momentum and spin degree of freedom.
The sum of bosonic and fermionic vacuum energies is therefore

\[ E_{\text{vac}}^{\text{tot}} = \frac{1}{2} \sum_{p, \text{bosons}} \omega_p - \frac{1}{2} \sum_{p, \text{fermions}} \omega_p. \]  
(7.37)

Expanding \( \omega_p \) as

\[ \omega_p = \sqrt{p^2 + M^2} = |p| \left( 1 + \frac{1}{2} \frac{M^2}{p^2} - \frac{1}{8} \frac{M^4}{p^4} + \ldots \right), \]  
(7.38)

we see that the vacuum energy can be finite if the universe contains as many Bose fields as Fermi fields, and if the masses of the associated particles satisfy the sum rules

\[ \sum_{\text{bosons}} M^2 = \sum_{\text{fermions}} M^2, \]
\[ \sum_{\text{bosons}} M^4 = \sum_{\text{fermions}} M^4. \]  
(7.39)

The higher powers in \( M \) contribute with finite momentum sums in (7.37).

If these cancellations do not occur, the sums are divergent and a cutoff in momentum space is needed to make the sum over all vacuum energies finite. Summing over all momenta inside a momentum sphere \( |p| \) of radius \( \Lambda \) will lead to a divergent energy proportional to \( \Lambda^4 \).

Many authors have argued that all quantum field theories may be valid only if the momenta are smaller than the Planck momentum \( P_P \equiv m_P c = \hbar / \lambda_P \), where \( m_P \) is the Planck mass (4.356) and \( \lambda_P \) the associated Compton wavelength

\[ \lambda_P = \sqrt{\hbar G/3^3} = 1.616252(81) \times 10^{-33} \text{ cm}, \]  
(7.40)

which is also called Planck length. Then the vacuum energy would be of the order \( m_P^4 \approx 10^{76} \text{ GeV}^4 \). From the present cosmological reexpansion rate one estimates a cosmological constant of the order of \( 10^{-47} \text{ GeV}^4 \). This is smaller than the vacuum energy by a factor of roughly \( 10^{123} \). The authors conclude that, in the absence of cancellations of Bose and Fermi vacuum energies, field theory gives too large a cosmological constant by a factor \( 10^{123} \). This conclusion is, however, definitely false. As we shall see later when treating other infinities of quantum field theories, divergent quantities may be made finite by including an infinity in the initial bare parameters of the theory. Their values may be chosen such that the final result is equal to the experimentally observed quantity [1]. The cutoff indicates that the quantity depends on ultra-short-distance physics which we shall never know. For this reason one must always restrict one’s attention to theories which do not depend on ultra-short-distance physics. These theories are called renormalizable theories. The quantum electrodynamics to be discussed in detail in Chapter 12 was historically the first theory of this kind. In this theory, which is experimentally the most accurate theory ever, the situation of the vacuum is precisely the same as for the mass of the
electron. Also here the mass emerging from the interactions needs a cutoff to be finite. But all cutoff dependence is absorbed into the initial bare mass parameter so that the final result is the experimentally observed quantity (see also the later discussion on p. 1512).

If boundary conditions cause a modification of the sums over momenta, the vacuum energy leads to finite observable effects which are independent of the cutoff. These are the famous Casimir forces on conducting walls, or the van der Waals forces between different dielectric media. Both will be discussed in Section 7.12.

### 7.1.3 Propagator of Free Scalar Particles

The free-particle propagator of the scalar field is obtained from the vacuum expectation

\[ G(x, x') = \langle 0|T\phi(x)\phi(x')|0 \rangle. \]  

(7.41)

As for nonrelativistic fields, the propagator coincides with the Green function of the free-field equation. This follows from the explicit form of the time-ordered product

\[ T\phi(x)\phi(x') = \Theta(x_0 - x'_0)\phi(x)\phi(x') + \Theta(x'_0 - x_0)\phi(x')\phi(x). \]  

(7.42)

Multiplying \( G(x, x') \) by the operator \( \partial^2 \), we obtain

\[ \partial^2 T\phi(x)\phi(x') = T[\partial^2 \phi(x)]\phi(x') + 2\delta(x_0 - x'_0) \left[ \partial_0 \phi(x), \phi(x') \right] + \delta'(x_0 - x'_0) \left[ \phi(x), \phi(x') \right]. \]  

(7.43)

The last term can be manipulated according to the usual rules for distributions: It is multiplied by an arbitrary smooth test function of \( x_0 \), say \( f(x_0) \), and becomes after a partial integration:

\[ \int dx_0 f(x_0)\delta'(x_0 - x'_0)[\phi(x), \phi(x')] = \]  

\[ - \int dx_0 f'(x_0)\delta(x_0 - x'_0)[\phi(x), \phi(x')] - \int dx_0 f(x_0)\delta(x_0 - x'_0)[\dot{\phi}(x), \phi(x')]. \]  

(7.44)

The first term on the right-hand side does not contribute, since it contains the commutator of the fields only at equal-times, \( [\phi(x), \phi(x')]_{x_0=x'_0} \), where it vanishes. Thus, dropping the test function \( f(x_0) \), we have the equality between distributions

\[ \delta'(x_0 - x'_0)[\phi(x), \phi(x')] = -\delta(x_0 - x'_0)[\dot{\phi}(x), \phi(x')], \]  

(7.45)

and therefore with (7.4), (7.10), (7.41)–(7.43):

\[ (-\partial^2 - M^2)G(x, x') = i\delta(x_0 - x'_0)\delta^{(3)}(x - x') = i\delta^{(4)}(x - x'). \]  

(7.46)
To calculate \( G(x, x') \) explicitly, we insert (7.12) and (7.42) into (7.41), and find

\[
G(x, x') = \Theta(x_0 - x'_0) \frac{1}{2V} \sum_{p, p'} \frac{1}{\sqrt{p^0 p^0}} e^{-ip \cdot (p' \cdot x')} \langle 0 | a_p a_{p'}^\dagger | 0 \rangle \\
+ \Theta(x'_0 - x_0) \frac{1}{2V} \sum_{p, p'} \frac{1}{\sqrt{p^0 p^0}} e^{ip \cdot (p' \cdot x)} \langle 0 | a_{p'} a_{p}^\dagger | 0 \rangle \\
= \Theta(x_0 - x'_0) \frac{1}{2V} \sum_p \frac{1}{p^0} e^{-ip(x - x')} + \Theta(x'_0 - x_0) \frac{1}{2V} \sum_p \frac{1}{p^0} e^{ip(x' - x')}. \tag{7.47}
\]

It is useful to introduce the functions

\[
G^{(\pm)}(x, x') = \sum_p \frac{1}{2p^0 V} e^{\mp ip(x - x')}. \tag{7.48}
\]

They are equal to the commutators of the positive- and negative-frequency parts of the field \( \phi(x) \) defined by

\[
\phi^{(\pm)}(x) = \sum_p \frac{1}{\sqrt{2p^0 V}} e^{-ipx} a_p, \quad \phi^{(-)}(x) = \sum_p \frac{1}{\sqrt{2p^0 V}} e^{ipx} a_p^\dagger. \tag{7.49}
\]

They annihilate and create free single-particle states, respectively. In terms of these,

\[
G^{(+)}(x, x') = [\phi^{(+)}(x), \phi^{(-)}(x')], \quad G^{(-)}(x, x') = -[\phi^{(-)}(x), \phi^{(+)}(x')] = G^{(-)}(x', x). \tag{7.50}
\]

The commutators on the right-hand sides can also be replaced by the corresponding expectation values. They can also be rewritten as Fourier integrals

\[
G^{(\pm)}(x, x') = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} e^{\mp ip(x - x')} G^{(\pm)}(p, t - t'), \tag{7.51}
\]

with the Fourier components

\[
G^{(\pm)}(p, t - t') \equiv e^{\mp ip^0(t - t')}. \tag{7.52}
\]

These are equal to the expectation values

\[
G^{(+)}(p, t - t') = \langle 0 | a_{p \phi}(t) a_{p \phi}^\dagger(t') | 0 \rangle, \\
G^{(-)}(p, t - t') = \langle 0 | a_{p \phi}^\dagger(t') a_{p \phi}^\dagger(t) | 0 \rangle \tag{7.53}
\]

of Heisenberg creation and annihilation operators [recall the definition (1.286), and (2.132)], whose energy is \( p^0 = \omega_p \):

\[
a_{p \phi}(t) \equiv e^{iHt} a_p e^{-iHt} = e^{-ip^0 t} a_p, \quad a_{p \phi}^\dagger(t) \equiv e^{iHt} a_p^\dagger e^{-iHt} = e^{ip^0 t} a_p^\dagger. \tag{7.54}
\]

In the infinite-volume limit, the functions (7.48) have the Fourier representation

\[
G^{(\pm)}(x, x') = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} e^{\mp ip(x - x')}. \tag{7.55}
\]
The full commutator $[\phi(x), \phi(x')]$ receives contributions from both functions. It is given by the difference:

$$[\phi(x), \phi(x')] = G^{(+)}(x - x') - G^{(-)}(x - x') \equiv C(x - x'). \quad (7.56)$$

The right-hand side defines the commutator function, which has the Fourier representation

$$C(x - x') = -i \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} e^{ip(x-x')} \sin[p^0(x^0 - x'^0)]. \quad (7.57)$$

This representation is convenient for verifying the canonical equal-time commutation rules (7.2) and (7.4), which imply that

$$C(x - x') = 0, \quad \dot{C}(x - x') = -i\delta^{(3)}(\mathbf{x} - \mathbf{x'}), \quad \text{for } x^0 = x'^0. \quad (7.58)$$

Indeed, for $x^0 = x'^0$, the integrand in (7.57) is zero, so that the integral vanishes. The time derivative of $C(x - x')$ removes $p^0$ in the denominator, leading at $x^0 = x'^0$ directly to the Fourier representation of the spatial $\delta$-function.

Another way of writing (7.57) uses a four-dimensional momentum integral

$$C(x - x') = \int \frac{d^4p}{(2\pi)^4} 2\pi \Theta(p^0) \delta(p^2 - M^2) e^{-ip(x-x')} . \quad (7.59)$$

In contrast to the Feynman propagator which satisfies the inhomogeneous Klein-Gordon equation, the commutator function (7.56) satisfies the homogeneous equation:

$$(-\partial^2 - M^2)C(x - x') = 0. \quad (7.60)$$

The Fourier representation (7.59) satisfies this equation since multiplication from the left by the Klein-Gordon operator produces an integrand proportional to $(p^2 - M^2)\delta(p^2 - M^2) = 0$.

The equality of the two expressions (7.56) and (7.59) follows directly from the property of the $\delta$-function

$$\delta(p^2 - M^2) = \delta(p^0 - \omega_p^2) = \frac{1}{2\omega_p} [\delta(p^0 - \omega_p) + \delta(p^0 + \omega_p)]. \quad (7.61)$$

Due to $\Theta(p^0)$ in (7.59), the integral over $p^0$ runs only over positive $p^0 = \omega_p$, thus leading to (7.57).

In terms of the functions $G^{(\pm)}(x, x')$, the propagator can be written as

$$G(x, x') = G(x, x') = \Theta(x_0 - x'_0)G^{(+)}(x, x') + \Theta(x'_0 - x_0)G^{(-)}(x, x'). \quad (7.62)$$

As follows from (7.55) and (7.62), all three functions $G(x, x')$, $G^{(+)}(x, x')$, and $G^{(-)}(x, x')$ depend only on $x - x'$, thus exhibiting the translational invariance of the vacuum state. In the following, we shall therefore always write one argument $x - x'$ instead of $x, x'$. 

As in the nonrelativistic case, it is convenient to use the integral representation (1.319) for the Heaviside functions:

\[
\Theta(x_0 - x'_0) = \int_{-\infty}^{\infty} \frac{i}{2\pi} \frac{E - p^0 + i\eta}{E^2 - p^2 - M^2} e^{-i(E-p^0)(x_0-x'_0)},
\]
\[
\Theta(x'_0 - x_0) = -\int_{-\infty}^{\infty} \frac{i}{2\pi} \frac{E + p^0 - i\eta}{E^2 - p^2 - M^2} e^{-i(E+p^0)(x_0-x'_0)},
\]
(7.63)

with an infinitesimal parameter \(\eta > 0\). This allows us to reexpress \(G(x,x')\) more compactly using (7.55), (7.52), and (7.62) as

\[
G(x - x') = \int \frac{dE}{2\pi} \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \left( \frac{i}{E - p^0 + i\eta} - \frac{i}{E + p^0 - i\eta} \right) e^{-iE(x_0-x'_0)+ip(x-x')}.
\]
(7.64)

Here we have used the fact that \(p^0\) is an even function of \(p\), thus permitting us to change the integration variables \(p\) to \(-p\) in the second integral. By combining the denominators, we find

\[
G(x - x') = \int \frac{dE}{2\pi} \frac{d^3p}{(2\pi)^3} \frac{i}{E^2 - p^2 - M^2 + i\eta} e^{-iE(x_0-x'_0)+ip(x-x')}.
\]
(7.65)

This has a relativistically invariant form in which it is useful to rename \(E\) as \(p^0\), and write

\[
G(x - x') = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - M^2 + i\eta} e^{-ip(x-x')}.
\]
(7.66)

Note that in this expression, \(p^0\) is integrated over the entire \(p^0\)-axis. In contrast to all earlier formulas in this section, the energy variable \(p^0\) is no longer constrained to satisfy the mass shell conditions \((p^0)^2 = p^0 = p^2 + M^2\).

The integral representation (7.66) shows very directly that \(G(x,x')\) is the Green function of the free-field equation (7.9). An application of the differential operator \((-\partial^2 - M^2)\) cancels the denominator in the integrand and yields

\[
(-\partial^2 - M^2)G(x - x') = \int \frac{d^4p}{(2\pi)^4} (p^2 - M^2) \frac{i}{p^2 - M^2 + i\eta} e^{-ip(x-x')} = i\delta^{(4)}(x - x').
\]
(7.67)

This calculation gives rise to a simple mnemonic rule for calculating the Green function of an arbitrary free-field theory. We write the Lagrangian density (7.8) as

\[
\mathcal{L}(x) = \frac{1}{2} \phi(x) L(i\partial) \phi(x),
\]
(7.68)

with the differential operator

\[
L(i\partial) \equiv (-\partial^2 - M^2).
\]
(7.69)
The Euler-Lagrange equation (7.9) can then be expressed as

$$L(i\partial)\phi(x) = 0.$$  (7.70)

And the Green function is the Fourier transform of the inverse of $L(i\partial)$:

$$G(x - x') = \int \frac{d^4p}{(2\pi)^4} \frac{i}{L(p)} e^{-ip(x-x')}.$$  (7.71)

with an infinitesimal $-i\eta$ added to the mass of the particle. This rule can directly be generalized to all other free-field theories to be discussed in the sequel.

### 7.1.4 Complex Case

Here we use the Lagrangian (4.165):

$$L(x) = \partial^\mu \varphi^*(x) \partial_\mu \varphi(x) - M^2 \varphi^*(x) \varphi(x),$$  (7.72)

whose Euler-Lagrange equation is the same as in the real case, i.e., it has the form (7.9). We shall see later in Subsec. 8.11.1, that the complex scalar field describes charged spinless particles.

#### Field Quantization

According to the canonical rules, the complex fields possess complex canonical momenta

$$\pi_\varphi(x) \equiv \pi(x) \equiv \left. \frac{\partial L(x)}{\partial \partial^\mu \varphi(x)} \right| = \partial^\mu \varphi^*(x),$$

$$\pi_{\varphi^*}(x) \equiv \pi^*(x) \equiv \left. \frac{\partial L(x)}{\partial \partial^\mu \varphi^*(x)} \right| = \partial^\mu \varphi(x).$$  (7.73)

The associated operators are required to satisfy the equal-time commutation rules

$$[\pi(x, t), \varphi(x', t)] = -i\delta^{(3)}(x - x'),$$

$$[\pi^\dagger(x, t), \varphi^\dagger(x', t)] = -i\delta^{(3)}(x - x').$$  (7.74)

All other equal-time commutators vanish. We now expand the field operator into its Fourier components

$$\varphi(x) = \sum_p \frac{1}{\sqrt{2Vp^0}} \left( e^{-ipx} a_p + e^{ipx} b_p^\dagger \right),$$  (7.75)

where, in contrast to the real case (7.12), $b_p^\dagger$ is no longer equal to $a_p^\dagger$. The reader may wonder why we do not just use another set of annihilation operators $d_{-p}$ rather than creation operators $b_p^\dagger$. A negative sign in the momentum label would be appropriate since the associated wave function $f_p(x) = e^{ipx}/\sqrt{2Vp^0}$ has a negative momentum.
One formal reason for using $b^\dagger_p$ instead of $d_{-\mathbf{p}}$ is that this turns the expansion (7.75) into a straightforward generalization of the expansion (7.12) of the real field. A more physical reason will be seen below when discussing Eq. (7.88).

By analogy with (7.14), we invert (7.75) to find

$$\left\{ \begin{array}{c} a_p \\ b^\dagger_p \end{array} \right\} = e^{\pm ip_0 x_0} \sqrt{\frac{1}{2V_p}} \int d^3x \ e^{\mp ipx} \left[ \pm i \pi^+ (x) + p^0 \varphi (x) \right]. \quad (7.76)$$

Corresponding equations hold for Hermitian-adjoint operators $a^\dagger_p$ and $b_p$. From these equations we find that the operators $a_p$, $a^\dagger_p$, $b_p$, and $b^\dagger_p$ all commute with each other, except for

$$[a_p, a^\dagger_{p'}] = \delta_{p,p'}, \quad [b_p, b^\dagger_{p'}] = \delta_{p,p'}.$$ \quad (7.77)

Thus there are two types of particle states, created as follows:

$$|\mathbf{p}\rangle \equiv a^\dagger_p |0\rangle, \quad |\bar{\mathbf{p}}\rangle \equiv b^\dagger_p |0\rangle,$$ \quad (7.78)

$$\langle \mathbf{p}| \equiv \langle 0| a_p, \quad \langle \bar{\mathbf{p}}| \equiv \langle 0| b_p.$$ \quad (7.79)

They are referred to as particle and antiparticle states, respectively, and have the wave functions

$$\langle 0| \varphi (x)|\mathbf{p}\rangle = \frac{1}{\sqrt{2V_p}} e^{-ipx} = f_p (x), \quad \langle 0| \varphi^\dagger (x)|\bar{\mathbf{p}}\rangle = \frac{1}{\sqrt{2V_p}} e^{-ipx} = f_p (x),$$ \quad (7.80)

$$\langle \bar{\mathbf{p}}| \varphi (x)|0\rangle = \frac{1}{\sqrt{2V_p}} e^{ipx} = f_p^* (x), \quad \langle \mathbf{p}| \varphi^\dagger (x)|0\rangle = \frac{1}{\sqrt{2V_p}} e^{ipx} = f_p^* (x),$$ \quad (7.81)

which by analogy with (2.212) may be abbreviated as

$$\langle 0| \varphi (x)|\mathbf{p}\rangle = \langle 0| \mathbf{p}\rangle, \quad \langle 0| \varphi^\dagger (x)|\bar{\mathbf{p}}\rangle = \langle \mathbf{x}| \bar{\mathbf{p}}\rangle,$$ \quad (7.82)

$$\langle \bar{\mathbf{p}}| \varphi (x)|0\rangle = \langle \bar{\mathbf{p}}| \mathbf{x}\rangle, \quad \langle \mathbf{p}| \varphi^\dagger (x)|0\rangle = \langle \mathbf{p}| \mathbf{x}\rangle.$$ \quad (7.83)

We are now prepared to justify why we associated a creation operator for antiparticles $b^\dagger_p$ with the second term in the expansion (7.75), instead of an annihilation operator for particles $d_{-\mathbf{p}}$. First, this is in closer analogy with the real field in (7.12), which contains a creation operator $a^\dagger_p$ accompanied by the negative-frequency wave function $e^{ipx}$. Second, only the above choice leads to commutation rules (7.77) with the correct sign between $b_p$ and $b^\dagger_p$. The choice $d_p$ would have led to the commutator $[d_p, d^\dagger_{p'}] = -\delta_{p,p'}$. For the second-quantized Hilbert space, this would imply a negative norm for the states created by $d^\dagger_p$. Third, the above choice ensures that the wave functions of incoming states $|\mathbf{p}\rangle$ and $|\bar{\mathbf{p}}\rangle$ are both $e^{-ipx}$, i.e., they both oscillate in time with a positive frequency like $e^{-ip_{0t}}$, thus having a positive energy. With annihilation operators $d_p$ instead of $b^\dagger_p$, the energy of a state created by $d^\dagger_p$ would have been negative. This will also be seen in the calculation of the second-quantized energy to be performed now.
Multiparticle states are formed in the same way as in the real-field case in Eq. (7.25), except that creation operators \( a_p^\dagger \) and \( b_p^\dagger \) have to be applied to the vacuum state \(|0\rangle\):

\[
|n_p, n_{p_2} \ldots n_{p_k}; \bar{n}_{p_1}, \bar{n}_{p_2} \ldots \bar{n}_{p_k}\rangle = \mathcal{N}^{S,A}(a_{p_1}^\dagger)^{n_{p_1}} \ldots (a_{p_k}^\dagger)^{n_{p_k}} (b_{p_1}^\dagger)^{\bar{n}_{p_1}} \ldots (b_{p_k}^\dagger)^{\bar{n}_{p_k}} |0\rangle. \tag{7.84}
\]

As in the real-field case, we shall sometimes work in an infinite volume and use the single-particle states

\[
|p\rangle = a_p^\dagger |0\rangle, \quad |\bar{p}\rangle = b_p^\dagger |0\rangle, \tag{7.85}
\]

with the vacuum \(|0\rangle\) defined by \( a_p |0\rangle = 0 \) and \( b_p |0\rangle = 0 \). These states satisfy the same orthogonality relation as those in (7.27), and possess wave functions normalized as in Eq. (7.28).

### 7.1.5 Energy of Free Charged Scalar Particles

The second-quantized energy density reads

\[
\mathcal{H}(x) = \pi \phi^\prime_0(x) + \pi^\prime_0(x) \phi^\prime_0(x) - L(x) = \partial_0 \phi(x) \phi^\prime(x) - L(x) = \partial_0 \phi(x) \phi_0 \phi^\prime(x) \phi(x) M^2 \phi^\prime(x) \phi(x) \tag{7.86}
\]

Inserting the expansion (7.75), we find the Hamilton operator [analogous to (7.31)]

\[
H = \int d^3x \mathcal{H}(x) = \sum_p \left[ \frac{p^\prime_0^2 + p^2 + M^2}{2p^0} (a_p^\dagger a_p + b_p^\dagger b_p) + \frac{1}{2p^0} \left( a_p^\dagger b_{-p} e^{2ip^0 t} + b_{-p} a_p e^{-2ip^0 t} \right) \right] \tag{7.87}
\]

Note that the mixed terms in the second line appear with opposite momentum labels, since \( H \) does not change the total momentum. Had we used \( d_p \) rather than \( b_p^\dagger \) in the field expansion, the energy would have been

\[
H = \int d^3x \mathcal{H}(x) = \sum_p \left[ \frac{p^\prime_0^2 + p^2 + M^2}{2p^0} (a_p^\dagger a_p + d_p^\dagger d_p) + \frac{1}{2p^0} \left( a_p^\dagger d_{-p} e^{2ip^0 t} + d_{-p} a_p e^{-2ip^0 t} \right) \right] \tag{7.88}
\]
Here the mixed terms have subscripts with equal momenta. Since the commutation rule of $d_p, d_p^\dagger$ have the wrong sign, the eigenvalues of $d_p^\dagger d_p$ take negative integer values implying the energy of the particles created by $d_p^\dagger$ to be negative. For an arbitrary number of such particles, this would have implied an energy spectrum without lower bound, which is unphysical (since it would allow one to build a perpetuum mobile).

Taking the vacuum expectation value of the positive-definite energy (7.87), we obtain, as in Eq. (7.32),

$$E_0 \equiv \langle 0|H|0 \rangle = \sum_p p^0 = \sum_p \omega_p. \quad (7.89)$$

In comparison with Eq. (7.32), there is a factor 2 since the complex field has twice as many degrees of freedom as a real field. As in the real case, we may subtract this infinite vacuum energy to obtain finite expressions via the earlier explained normal-ordering prescription. Only the gravitational interaction is sensitive to the vacuum energy.

What is the difference between the particle states $|p\rangle$ and $|\bar{p}\rangle$ created by $a_p^\dagger$ and $b_p^\dagger$? We shall later see that they couple with opposite sign to the electromagnetic field. Here we only observe that they have the same intrinsic properties under the Poincaré group, i.e., the same mass and spin. They are said to be antiparticles of each other.

**Propagator of Free Charged Scalar Particles**

The propagator of the complex scalar field can be calculated in the same fashion as for real-fields with the result

$$G(x, x') \equiv G(x - x') = \langle 0|T\varphi(x)\varphi^\dagger(x')|0 \rangle$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - M^2 + i\eta} e^{-ip(x-x')}. \quad (7.90)$$

As before, the propagator is equal to the Green function of the free-field equation (7.9). It may again be obtained following the mnemonic rule on p. 484, by Fourier transforming the inverse of the differential operator $L(i\partial)$ in the field equation [recall (7.365) and (7.71)].

**7.1.6 Behavior under Discrete Symmetries**

Let us now see how the discrete operations of space inversion, time reversal, and complex conjugation of Subsecs. 4.5.2–4.5.4 act in the Hilbert space of quantized scalar fields.

**Space Inversion**

Under space inversion, defined by the transformation

$$x \xrightarrow{P} x' = \tilde{x} = (x_0, -\mathbf{x}), \quad (7.91)$$
a real or complex scalar field is transformed according to
\[ \varphi(x) \xrightarrow{P} \varphi_P(x) = \eta_P \varphi(\tilde{x}), \]
(7.92)
with
\[ \eta_P = \pm 1. \]
(7.93)

For a real-field operator \( \phi(x) \), this transformation can be achieved by assigning a transformation law
\[ a_p \xrightarrow{P} a'_p = \eta_P a_{-p}, \]
\[ a'_p \xrightarrow{P} a'^{\dagger}_p = \eta_P a'^{\dagger}_{-p}, \]
(7.94)
to the creation and annihilation operators \( a'_p \) and \( a_p \) in the expansion (7.12). Indeed, by inserting (7.94) into (7.12), we find
\[ \phi'_P(x) = \eta_P \frac{1}{\sqrt{2p^0V}} \sum_p (e^{-ipx}a_{-p} + e^{ipx}a'^{\dagger}_{-p}) = \eta_P \phi(\tilde{x}). \]
(7.95)

It is possible to find a parity operator \( \mathcal{P} \) in the second-quantized Hilbert space which generates the parity in the transformations (7.94), (7.95), i.e., it is defined by
\[ \mathcal{P} a_p \mathcal{P}^{-1} \equiv a'_p = \eta_P a_{-p}, \]
\[ \mathcal{P} a'^{\dagger}_p \mathcal{P}^{-1} \equiv a'^{\dagger}_p = \eta_P a'^{\dagger}_{-p}. \]
(7.96)

An explicit representation of \( \mathcal{P} \) is
\[ \mathcal{P} = e^{i\pi G_P/2}, \quad G_P = \sum_p \left[ a'^{\dagger}_{-p}a_p - \eta_P a'^{\dagger}_p a_p \right]. \]
(7.97)

In order to prove that this operator has the desired effect, we form the operators
\[ a'^{\dagger}_p \equiv a'^{\dagger}_{-p} + \eta_P a'^{\dagger}_{-p}, \quad a^{\dagger}_p \equiv a'^{\dagger}_{-p} - \eta_P a'^{\dagger}_{-p}, \]
(7.98)
which are invariant under the parity operation (7.96), with eigenvalues \( \pm 1 \):
\[ \mathcal{P} a'^{\dagger}_p \mathcal{P}^{-1} = \pm a^{\dagger}_p. \]
(7.99)

Then we calculate the same property with the help of the unitary operator (7.97), expanding \( e^{iG_P x/2}a'^{\dagger}_p e^{-iG_P x/2} \) with the help of Lie’s expansion formula (4.105). This turns out to be extremely simple since the first commutators on the right-hand side of (4.105) are
\[ [G_P, a'^{\dagger}_p] = 0; \quad [G_P, a^{\dagger}_p] = -2a^{\dagger}_p, \]
(7.100)
so that the higher ones are obvious. The Lie series yields therefore
\[
e^{iG_P\pi/2}a_p^{\dagger}e^{-iG_P\pi/2} = a_p^{\dagger},
\]
\[
e^{iG_P\pi/2}a_p^{-\dagger}e^{-iG_P\pi/2} = a_p^{-\dagger}\left(1 - i\pi + \frac{1}{2!}\pi^2 + \ldots\right) = a_p^{\dagger}e^{-i\pi} = -a_p^{\dagger},
\]
(7.101)
showing that the operators \(a_p^{\dagger}\) do indeed satisfy (7.99).

For a complex field operator \(\varphi(x)\), the operator \(G_P\) in (7.97) has to be extended by the same expression involving the operators \(b_p^{\dagger}\) and \(b_p\).

Let us denote the multiparticle states of the types (7.25) or (7.84) in the second-quantized Hilbert space by \(|\Psi\rangle\). In the space of all such states, the operator \(\mathcal{P}\) is unitary:
\[
\mathcal{P}^\dagger = \mathcal{P}^{-1},
\]
(7.102)
i.e., for any two states \(|\Psi_1\rangle\) and \(|\Psi_2\rangle\), the scalar product remains unchanged by \(\mathcal{P}\)
\[
P\langle\Psi'_1|\Psi'_2\rangle_p = \langle\Psi_1|\mathcal{P}^\dagger\mathcal{P}|\Psi_2\rangle = \langle\Psi_1|\Psi_2\rangle.
\]
(7.103)
With this operator, the transformation (7.92) is generated by
\[
\varphi(x) \xrightarrow{\mathcal{P}} \mathcal{P}\varphi(x)\mathcal{P}^{-1} = \varphi'_{\eta}(x) = \eta_P\varphi(\tilde{x}).
\]
(7.104)

As an application, consider a state of two identical spinless particles which move in the common center of mass frame with a relative angular momentum \(l\). Such a state is an eigenstate of the parity operation with the eigenvalue
\[
\eta_P = (-)^l.
\]
(7.105)
This follows directly by applying the parity operator \(\mathcal{P}\) to the wave function
\[
|\Psi_{lm}\rangle = \int_0^\infty dp R_l(p) \int d^2\hat{p} Y_{lm}(\hat{p}) a_p^{\dagger}a_{-p}^{\dagger}|0\rangle,
\]
(7.106)
where \(R_l(p)\) is some radial wave function of \(p = |p|\). Using (7.96), the commutativity of the creation operators \(a_p^{\dagger}\) and \(a_{-p}^{\dagger}\) with each other, and the well-known fact\(^1\) that the spherical harmonic for \(-\hat{p}\) and \(\hat{p}\) differ by a phase factor \((-1)^l\), lead to
\[
\mathcal{P}|\Psi_{lm}\rangle = (-1)^l|\Psi_{lm}\rangle.
\]
(7.107)

For two different bosons of parities \(\eta_{P_1}\) and \(\eta_{P_2}\), the right-hand side is multiplied by \(\eta_{P_1}\eta_{P_2}\).

\(^1\)This follows from the property of the spherical harmonics \(Y_{lm}(\pi - \theta, \phi) = (-1)^{l-m}Y_{lm}(\theta, \phi)\) together with \(e^{im(\phi + \pi)} = (-1)^m e^{im\phi}\).
7.1 Scalar Fields

By time reversal we understand the coordinate transformation
\[ x \xrightarrow{T} x' = -\tilde{x} = (-x_0, \mathbf{x}). \quad (7.108) \]

The operator implementation of time reversal is not straightforward. In a time-reversed state, all movements are reversed, i.e., for spin-zero particles all momenta are reversed (see Subsec. 4.5.3). In this respect, there is no difference to the parity operation, and the transformation properties of the creation and annihilation operators are just the same as in (7.96). The associated phase factor is denoted by \( \eta_T \):
\[ a_p \xrightarrow{T} a'_p = \eta_T a_{-p}, \]
\[ a^\dagger_p \xrightarrow{T} a'^\dagger_p = \eta_T a^\dagger_{-p}. \quad (7.109) \]

We have indicated before [see Eq. (4.226)] and shall see below, that in contrast to the phase factor \( \eta_P = \pm 1 \) of parity, the phase factor \( \eta_T \) will not be restricted to \( \pm 1 \) by the group structure. It is unmeasurable and can be chosen to be equal to unity.

If we apply the operation (7.109) to the Fourier components in the expansion (7.12) of the field, we find
\[ \phi'_{\mathcal{T}}(x) = \eta_T \sum_p \frac{1}{\sqrt{2p^0V}} \left( e^{-ipx} a_{-p} + e^{ipx} a^\dagger_{-p} \right) \]
\[ = \eta_T \sum_p \frac{1}{\sqrt{2p^0V}} \left( e^{-ip\tilde{x}} a_p + e^{-ip\tilde{x}} a^\dagger_p \right) \]
\[ = \eta_T \phi(\tilde{x}). \quad (7.110) \]

This, however, is not yet the physically intended transformation law that was specified in Eq. (4.206):
\[ \phi(x) \xrightarrow{T} \phi'_{\mathcal{T}}(x) = \eta_T \phi(x_T), \quad (7.111) \]

which amounts to
\[ \phi(x) \xrightarrow{T} \phi'_{\mathcal{T}}(x) = \eta_T \phi(-\tilde{x}). \quad (7.112) \]

The sign change of the argument is achieved by requiring the time reversal transformation to be an antilinear operator \( \mathcal{T} \) in the multiparticle Hilbert space. It is defined by
\[ \mathcal{T} a_p \mathcal{T}^{-1} \equiv a'_p = \eta_T a_{-p}, \]
\[ \mathcal{T} a^\dagger_p \mathcal{T}^{-1} \equiv a'^\dagger_p = \eta_T a^\dagger_{-p}. \quad (7.113) \]

in combination with the antilinear property
\[ \mathcal{T} (\alpha a + \beta a^\dagger_p) \mathcal{T}^{-1} = \alpha^* \mathcal{T} a_p \mathcal{T}^{-1} + \beta^* \mathcal{T} a^\dagger_p \mathcal{T}^{-1}. \quad (7.114) \]
Whenever the time reversal operation is applied to a combination of creation and annihilation operators, the coefficients have to be switched to their complex conjugates. Under this antilinear operation, we indeed obtain (7.111) rather than (7.110):

\[ \phi(x) \xrightarrow{T} T \phi(x) T^{-1} = \phi'_T(x) = \eta_T \phi(x_T). \]  

The same transformation law is found for complex scalar fields by defining for particles and antiparticles

\[ T a_p T^{-1} \equiv a'_p = \eta_T a_{-p}, \]
\[ T b^*_p T^{-1} \equiv b^*_p = \eta_T b^*_{-p}. \]  

(7.116)

In the second-quantized Hilbert space consisting of states (7.25) and (7.84), the antiunitarity has the consequence that all scalar products are changed into their complex conjugates:

\[ \langle \Psi_2 | \Psi_1 \rangle \xrightarrow{T} T \langle \Psi'_2 | \Psi'_1 \rangle = \langle \Psi_1 | \Psi_2 \rangle^*, \]  

(7.117)

thus complying with the property (4.209) of scalar products of ordinary quantum mechanics (to which the second-quantized formalism reduces in the one-particle subspace). This makes the operator \( T \) antiunitary.

To formalize operations with the antiunitary operator \( T \) in the Dirac bra-ket language, it is useful to introduce an antilinear unit operator \( 1_A \) which has the effect

\[ \langle \Psi_2 | 1_A | \Psi_1 \rangle = 1_A \langle \Psi_2 | \Psi_1 \rangle = \langle \Psi_2 | \Psi_1 \rangle^*. \]  

(7.118)

The antiunitarity of the time reversal transformation is then expressed by the equation

\[ T^\dagger = T^{-1} 1_A, \]  

(7.119)

the operator \( 1_A \) causing all differences with respect to the unitary operator (7.102).

Due to the antilinearity, the factor \( \eta_T \) in the transformation law of the complex field is an arbitrary phase factor. We have mentioned this before in Subsec. 4.5.3. By applying the operator \( T \) twice to the complex field \( \phi(x) \), we obtain from (7.114) and (7.115):

\[ T^2 \phi(x) T^{-2} = \eta_T \eta_T^* \phi(x), \]  

(7.120)

so that the cyclic property of time reversal \( T^2 = 1 \) fixes

\[ \eta_T \eta_T^* = 1. \]  

(7.121)

This is in contrast to the unitary operators \( \mathcal{P} \) and \( \mathcal{C} \) where the phase factors must be \( \pm 1 \).

For real scalar fields, this still holds, since the second transformation law (7.113) must be the complex-conjugate of the first, which requires \( \eta_T \) to be real, leaving only the choices \( \pm 1 \).
The antilinearity has the consequence that a Hamilton operator which is invariant under time reversal
\[ THT^{-1} = H, \] (7.122)
possesses a time evolution operator \[ U(t, t_0) = e^{-iH(t-t_0)/\hbar} \] that satisfies
\[ TU(t, t_0)T^{-1} = U(t_0, t) = U^\dagger(t, t_0), \] (7.123)
in which the time order is inverted.

The antiunitary nature of \( T \) makes the phase factor \( \eta_T \) of a charged field an unmeasurable quantity. This is why we were able to choose \( \eta_T = 1 \) after Eq. (7.121) [see also (4.226)]. Time reversal invariance does not produce selection rules in the same way as discrete unitary symmetries do, such as parity. This will be discussed in detail in Section 9.7, after having developed the theory of the scattering matrix.

### Charge Conjugation

The operator implementation of the transformation (4.227) on a complex scalar field operator
\[ \varphi(x) = \sum_p \frac{1}{\sqrt{2V_p}} (e^{-ipx} a_p + e^{ipx} b_p^\dagger) \] (7.124)
is quite simple. We merely have to define a charge conjugation operator \( C \) by
\[ Ca_p^\dagger C^{-1} = \eta_C b_p^\dagger, \quad Cb_p C^{-1} = \eta_C a_p, \quad Ca_p C^{-1} = \eta_C b_p, \quad Cb_p C^{-1} = \eta_C a_p, \] (7.125)
where \( \eta_C = \pm 1 \) is the charge parity of the field. Applied to the complex scalar field operator (7.124) the operator \( C \) produces the desired transformation corresponding to (4.227):
\[ \varphi(x) \xrightarrow{C} C\varphi(x)C^{-1} = \varphi_C'(x) = \eta_C\varphi^\dagger(x). \] (7.126)

For a real field, we may simply identify \( b_p \) with \( a_p \), thus making the transformation laws (7.125) trivial:
\[ Ca_p^\dagger C^{-1} = \eta_C a_p^\dagger, \quad Ca_p C^{-1} = \eta_C a_p. \] (7.127)

The antiparticles created by \( b_p^\dagger \) have the same mass as the particles created by \( a_p^\dagger \). But they have an opposite charge. The latter follows directly from the transformation law (4.229) of the classical local current. In the present second-quantized formulation it is instructive to calculate the matrix elements of the operator current density (4.171):
\[ j_\mu(x) = -i\varphi^\dagger \partial_\mu \varphi. \] (7.128)
Evaluating this between single-particle states created by \( a_p^\dagger \) and \( b_p^\dagger \), we find

\[
\langle p'| j^\mu(x) | p \rangle = \langle 0 | a_{p'} j^\mu(x) a_p^\dagger | 0 \rangle = (p'^\mu + p^\mu) \frac{1}{\sqrt{2p'p}} e^{-i(p-p')^2 x},
\]

\[
\langle \hat{p}'| j^\mu(x) | \hat{p} \rangle = \langle 0 | b_{p'} j^\mu(x) b_p^\dagger | 0 \rangle = -(p'^\mu + p^\mu) \frac{1}{\sqrt{2p'p}} e^{-i(p-p')^2 x}. \quad (7.129)
\]

The charge of the particle states \( |p\rangle = a_p^\dagger | 0 \rangle \) and \( |\hat{p}\rangle = b_p^\dagger | 0 \rangle \) is given by the diagonal matrix element of the charge operator \( Q = \int d^3 x j^0(x) \) between these states:

\[
\langle p'| Q | p \rangle = \int d^3 x \langle p'| j^0(x) | p \rangle = \delta_{p',p}. \quad \langle \hat{p}'| Q | \hat{p} \rangle = \int d^3 x \langle \hat{p}'| j^0(x) | \hat{p} \rangle = -\delta_{p',p}. \quad (7.130)
\]

The opposite sign of the charges of particles and antiparticles is caused by opposite exponentials accompanying the annihilation and creation operators \( a_p \) and \( b_p^\dagger \) in the field \( \phi(x) \). As far as physical particles are concerned, the states \( |p\rangle = a_p^\dagger | 0 \rangle \) may be identified with \( \pi^+\)-mesons of momentum \( p \), the states \( |\hat{p}\rangle = b_p^\dagger | 0 \rangle \) with \( \pi^-\)-mesons. The scalar particles associated with the negative frequency solutions of the wave equation are called antiparticles. In this nomenclature, the particles of a real field are antiparticles of themselves.

Note that the two matrix elements in (7.129) have the same exponential factors, which is necessary to make sure that particles and antiparticles have the same energies and momenta.

An explicit representation of the unitary operator \( C \) is

\[
C = e^{i\pi G_C/2}, \quad G_C = \sum_p \left[ b_p^\dagger a_p + a_p^\dagger b_p - \eta p (a_p a_p + b_p^\dagger b_p) \right]. \quad (7.131)
\]

The proof is completely analogous to the parity case in Eqs. (7.97)–(7.101).

A bound state of a boson and its antiparticle in a relative angular momentum \( l \) has a charge parity \((-1)^l\). This follows directly from the wave function

\[
|\Psi\rangle = \int_0^\infty dp \, R_l(p) \int d^2 \hat{p} \, Y_{lm}(\hat{p}) a_p^\dagger b_{-p}^\dagger |0\rangle, \quad (7.132)
\]

where \( R_l(p) \) is some radial wave function of \( p = |p| \). Applying \( C \) to this we have

\[
C a_p^\dagger b_{-p}^\dagger |0\rangle = b_p^\dagger a_p^\dagger |0\rangle. \quad (7.133)
\]

Note that we must interchange the order of \( b_p^\dagger \) and \( a_p^\dagger \), as well as the momenta \( p \) and \(-p\), to get back to the original state. This results in a sign factor \((-1)^l\) from the spherical harmonic. An example is the \( \rho \)-meson which is a \( p \)-wave resonance of a \( \pi^+ \) and a \( \pi^- \)-meson, and has therefore a charge parity \( \eta_C = -1 \).

### 7.2 Spacetime Behavior of Propagators

Let us evaluate the integral representations (7.66) and (7.90) for the propagators of real and complex scalar fields. To do this we use two methods which will be
applied many times in this text. The integrands in (7.66) and (7.90) have two poles
in the complex $p^0$-plane as shown in Fig. 7.1, one at $p^0 = \sqrt{p^2 + M^2} - i\eta$, the
other at $p^0 = -\sqrt{p^2 + M^2} + i\eta$, both with infinitesimal $\eta > 0$. The first is due
to the intermediate single-particle state propagating in the positive time direction,
the other comes from an antiparticle propagating in the negative time direction. A
propagator with such pole positions is called a *Feynman propagator*.

### 7.2.1 Wick Rotation

The special pole positions make it possible to rotate the contour of integration in
the $p^0$-plane without crossing a singularity so that it runs along the imaginary axis
$dp^0 = idp^4$. This procedure is called a *Wick rotation*, and is illustrated in Fig.
7.2. Volume elements $d^4p = dp^0dp^1dp^2dp^3$ in Minkowski space go over into volume
elements $idp^1dp^2dp^3dp^4 \equiv id^4p_E$ in euclidean space.
By this procedure, the integral representations (7.66) and (7.90) for the propagator become

\[ G(x - x') = \int \frac{d^4p_E}{(2\pi)^4} \frac{1}{p_E^2 + M^2} e^{-ip_E(x - x')E}, \]  

(7.134)

where \( p_E^\mu \) is the four-momentum

\[ p_E^\mu = (p^1, p^2, p^3, p^4 = -ip^0), \]  

(7.135)

and \( p_E^2 \) the square of the momentum \( p_E^\mu \) calculated in the euclidean metric, i.e.,

\[ p_E^2 \equiv p^1^2 + p^2^2 + p^3^2 + p^4^2. \]

The vector \( x_E^\mu \) is the corresponding euclidean spacetime vector

\[ x_E^\mu = (x^1, x^2, x^3, x^4 = -ix^0), \]  

(7.136)

so that \( px = -p_E x_E \). With \( p_E^2 + M^2 \) being strictly positive, the integral (7.134) is now well-defined. Moreover, the denominator possesses a simple integral representation in the form of an auxiliary integral over an auxiliary parameter \( \tau \) which, as we shall see later, plays the role of the proper time of the particle orbits:

\[ \frac{1}{p_E^2 + M^2} = \int_0^\infty d\tau e^{-\tau(p_E^2 + M^2)}. \]  

(7.137)

Since this way of reexpressing denominators in propagators is extremely useful in quantum field theory, it is referred to, after its author, as Schwinger’s proper-time formalism [2]. The \( \tau \)-integral has turned the integral over the denominator into a quadratic exponential function. The exponent can therefore be quadratically completed:

\[ e^{ip_E(x - x')E - \tau p_E^2} \rightarrow e^{-(x - x')_E^2/4\tau - p_E^2/2}, \]  

(7.138)

with \( p_E' = p_E - i(x - x')_E/2\tau \). Since the measure of integration is translationally invariant, and the integrand has become symmetric under four-dimensional rotations, we can replace

\[ \int d^4p_E = \int d^4p_E' = \pi^2 \int_0^\infty dp_E^2 p_E'^2, \]  

(7.139)

and integrate out the four-momentum \( p_E' \), yielding

\[ G(x - x') = \frac{1}{16\pi^2} \int_0^\infty \frac{d\tau}{\tau^2} e^{-(x - x')_E^2/4\tau - M^2\tau}. \]  

(7.140)

The integral is a superposition of nonrelativistic propagators of the type (2.433), with Boltzmann-like weights \( e^{-M^2\tau} \), and an additional weight factor \( \tau^{-2} \) which may be viewed as the effect of an entropy factor \( e^S = e^{-2\log\tau} \). This representation has an interesting statistical interpretation. After deriving Eq. (2.433) for a nonrelativistic propagator at imaginary times, we observed that this looks like the probability for a random walk of a fixed length proportional to \( \hbar\beta \) to go from \( x \) to \( x' \) in three
dimensions. Here we see that the relativistic propagator looks like a random walk of arbitrary length in four dimensions, with a length distribution ruled mainly by the above Boltzmann factor. This leads to the possibility of describing, by relativistic quantum field theory, ensembles of random lines of arbitrary length. Such lines appear in many physical systems in the form of vortex lines and defect lines [6, 8, 18].

By changing the variable of integration from $\tau$ to $\omega = 1/4\tau$, this becomes

$$G(x - x') = \frac{1}{4\pi^2} \int_0^\infty d\omega \, e^{-\frac{(x-x')^2}{4\omega} - M^2/4\omega}. \quad (7.141)$$

Now we use the integral formula

$$\int_0^\infty \frac{d\omega}{\omega} \omega^\nu e^{-a\omega - b/4\omega} = 2 \left(\frac{b}{4a}\right)^{\nu/2} K_\nu(\sqrt{ab}), \quad (7.142)$$

where $K_\nu(z)$ is the modified Bessel function,\footnote{I.S. Gradshteyn and I.M. Ryzhik, op. cit., Formula 3.471.9.} to find

$$G(x - x') = \frac{M^2}{4\pi^2} K_1\left( \frac{M\sqrt{(x-x')^2}}{E} \right). \quad (7.143)$$

The $D$-dimensional generalization of this is calculated in the same way. Equation (7.140) becomes

$$G(x - x') = \frac{1}{(4\pi)^{D/2}} \int_0^\infty d\tau \, e^{-\frac{(x-x')^2}{4\tau} - M^2\tau}, \quad (7.144)$$

yielding, via (7.142),

$$G(x - x') = \int \frac{d^Dq}{(2\pi)^D} \frac{1}{q^2 + M^2} e^{i\eta(x - x')}$$

$$= \frac{1}{(2\pi)^{D/2}} \left[ \frac{M}{\sqrt{(x - x')^2}} \right]^{D/2 - 1} K_{D/2 - 1}\left( \frac{M\sqrt{(x-x')^2}}{E} \right). \quad (7.145)$$

### 7.2.2 Feynman Propagator in Minkowski Space

Let us now do the calculation in Minkowski space. There the proper-time representation of the propagators (7.66) and (7.90) reads

$$G(x - x') = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - M^2 + i\eta} e^{-ip(x-x')} = i \int_0^\infty d\tau \int \frac{d^4p}{(2\pi)^4} e^{-i[p(x-x') - \tau(p^2 - M^2 + i\eta)]}. \quad (7.146)$$
The momentum integrations can be done, after a quadratic completion, with the help of Fresnel’s formulas

\[
\int \frac{dp^0}{2\pi} e^{ip^0 a} = \frac{1}{2\sqrt{\pi a/i}}, \quad \int \frac{dp^i}{2\pi} e^{ip^i a} = \frac{1}{2\sqrt{\pi a/i}}. \tag{7.147}
\]

In these, it does not matter (as it did in the above Gaussian integrals) that \(p^0^2\) and \(p^i^2\) appear with opposite signs in the exponent.

In continuing the evaluation of (7.146), it is necessary to make sure that the integrals over \(dp^\mu\) and \(d\tau\) can be interchanged. The quadratic completion generates a term \(e^{-i(x-x')^2/4\tau}\) which is integrable for \(\tau \to 0\) only if we replace \((x-x')^2\) by \((x-x')^2 - i\eta\). Changing again \(\tau\) to \(\omega = 1/4\tau\), we arrive at the integral representation for the Feynman propagator

\[
G(x - x') = -\frac{i}{4\pi^2} \int_0^\infty d\omega \, e^{-i[(x-x')^2-i\eta]\omega + (M^2-i\eta)/4\omega}. \tag{7.148}
\]

The two small imaginary parts are necessary to make the integral convergent at both ends. The integral can be done with the help of the formula

\[
\int_0^\infty d\omega \, \omega^\nu e^{i(a\omega+b/4\omega)} = 2 \left( \frac{b}{4a} \right)^{\nu/2} i^{\nu/2} e^{i\pi\nu/2} H_\nu^{(1)}(\sqrt{ab}), \tag{7.149}
\]

where \(H_\nu^{(1)}(z)\) is the Hankel function of the first kind. Upon taking the complex conjugate, using

\[
[H_\nu^{(1)}(z)]^* = H_\nu^{(2)}(z^*), \tag{7.150}
\]

and replacing \(a\) by \(a^*\) and \(b\) by \(b^*\), we obtain

\[
G(x - x') = i \frac{M^2}{8\pi} \frac{H_0^{(2)}(M\sqrt{(x-x')^2})}{M\sqrt{(x-x')^2}}. \tag{7.151}
\]

The Hankel functions are combinations of Bessel and Neumann functions

\[
H_\nu^{(1,2)}(z) = J_\nu(z) \pm iN_\nu(z). \tag{7.152}
\]

In order to check the correctness of this expression, we go to the nonrelativistic limit by letting \(c \to \infty\) in the argument of the Hankel function. Displaying proper CGS-units, for clarity, this becomes

\[
\frac{M}{\hbar} \sqrt{(x-x')^2} = \frac{M}{\hbar} \sqrt{c^2(t-t')^2 - (x-x')^2} \approx \frac{Mc^2}{\hbar} \frac{(t-t')}{2} - \frac{1}{2} \frac{M}{\hbar} \frac{(x-x')^2}{2(t-t')}. \tag{7.153}
\]

\(^3\)ibid., Formula 3.471.11, together with 8.476.9.

\(^4\)The last two identities are from ibid., Formulas 8.476.8 and 8.476.11.
For large arguments, the Hankel functions behave like
\[ H_{\nu}^{(1,2)}(z) \approx \frac{2}{\pi z} e^{\pm i(\nu \pi/2 - \pi/2)}, \] (7.154)
so that (7.151) becomes, for \( t > t' \), and taking into account the nonrelativistic limit (4.156) of the fields,
\[ G(x - x') \xrightarrow{c \to \infty} \frac{1}{2M} \frac{1}{\sqrt{2\pi \hbar(t - t')/M}} \exp \left[ \frac{i M}{\hbar} \frac{(x - x')^2}{2t} \right], \] (7.155)
in agreement with the nonrelativistic propagator (2.241).

In the spacelike regime \((x - x')^2 < 0\), we continue \( \sqrt{(x - x')^2} \) analytically to \(-i\sqrt{-(x - x')^2}\), and \( H_1^{(2)}(z) = H_1^{(1)}(-z) \) together with the relation\(^5\)
\[ \frac{\pi i}{2} \frac{H_1^{(1)}(iz)}{(iz)^\nu} = K_\nu(z), \] (7.156)
to rewrite (7.151) as
\[ G(x - x') = \frac{M}{4\pi^2} \frac{K_1(M \sqrt{-(x - x')^2})}{M \sqrt{-(x - x')^2}}, \] (7.157)
in agreement with (7.143).

It is instructive to study the massless limit \( M \to 0 \), in which the asymptotic behavior\(^6\) \( K(z) \to 1/z \) leads to
\[ G(x - x') \to \frac{1}{4\pi^2} \frac{1}{(x - x')^2 + i\eta}. \] (7.158)

To continue this back to Minkowski space where \((x - x')_E^2 = -(x - x')^2\), it is necessary to remember the small negative imaginary part on \( x^2 \) which was necessary to make the \( \tau \)-integral (7.148) converge. The correct massless Green function in Minkowski space is therefore
\[ G(x - x') = -\frac{1}{4\pi^2} \frac{1}{(x - x')^2 - i\eta}. \] (7.159)
The same result is obtained from the \( M \to 0 \)-limit of the Minkowski space expression (7.151), using the limiting property\(^7\)
\[ H_\nu^{(1)}(z) \approx -H_\nu^{(2)}(z) \approx \frac{i}{\pi} \frac{\Gamma(\nu)}{(z/2)^\nu}. \] (7.160)

\(^5\)ibid., Formula 8.407.1.
\(^7\)ibid., Formula 9.1.9.
7.2.3 Retarded and Advanced Propagators

Let us contrast the spacetime behavior of the Feynman propagators (7.151) and (7.159) with that of the retarded propagator of classical electrodynamics. A retarded propagator is defined for an arbitrary interacting field $\phi(x)$ by the expectation value

$$G_R(x - x') \equiv \Theta(x^0 - x'^0) \langle 0 | [\phi(x), \phi(x')] | 0 \rangle. \quad (7.161)$$

In general, one defines a commutator function $C(x - x')$ by

$$C(x - x') \equiv \langle 0 | [\phi(x), \phi(x')] | 0 \rangle, \quad (7.162)$$

leading to

$$G_R(x - x') = \Theta(x^0 - x'^0) C(x - x'). \quad (7.163)$$

For a free field $\phi(x)$, the commutator is a $c$-number, so that the vacuum expectation values can be omitted, and $C(x - x')$ is given by (7.56), leading to a retarded propagator

$$G_R(x - x') = \Theta(x^0 - x'^0) [G^{(+)}(x - x') - G^{(-)}(x - x')]. \quad (7.164)$$

The Heaviside function in front of the commutator function $C(x - x')$ ensures the causality of this propagator. It also has the effect of turning $C(x - x')$, which solves the homogeneous Klein-Gordon equation, into a solution of the inhomogeneous equation. In fact, by comparing (7.164) with (7.62) and using the Fourier representations (7.63) of the Heaviside functions, we find, for the retarded propagator, a representation very similar to that of the Feynman propagator in (7.64), except for a reversed $i\eta$-term in the negative-energy pole:

$$G_R(x - x') = \int \frac{dE}{2\pi} \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \left( \frac{i}{E - p^0 + i\eta} - \frac{i}{E + p^0 + i\eta} \right) e^{-iE(x^0 - x'^0) + ip(x - x')} \quad (7.165)$$

By combining the denominators we now obtain, instead of (7.65),

$$\int \frac{dE}{2\pi} \frac{d^3p}{(2\pi)^3} \frac{i}{(E + i\eta)^2 - p^2 - M^2} e^{-iE(x^0 - x'^0) + ip(x - x')}, \quad (7.166)$$

which may be written as an off-shell integral of the type (7.66):

$$G_R(x - x') = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^+_\pm - M^2 + i\eta} e^{-ip(x - x')}, \quad (7.167)$$

where the subscript of $p_+$ indicates that a small $i\eta$-term has been added to $p^0$.

Note the difference in the Fourier representation with respect to that of the commutator function in (7.59). Due to the absence of a Heaviside function, the commutator function solves the homogeneous Klein-Gordon equation.

It is important to realize that the retarded Green function could be derived from a time-ordered expectation value of second-quantized field operators if we assume
that the Fourier components with the negative frequencies are associated with anni-
hilation operators $d_\nu$ rather than creation operators $b_\nu^\dagger$ of antiparticles. Then
the propagator would vanish for $x^0 < x'^0$, implying both poles in the $p^0$-plane to lie
below the real energy axis.

For symmetry reasons, one also introduces an advanced propagator
\[ G_A(x - x') \equiv \Theta(x'^0 - x^0) \langle 0 | [\phi(x), \phi(x')] | 0 \rangle. \] (7.168)
It has the same Fourier representation as $G_R(x - x')$, except that the poles lie both
above the real axis:
\[ G_A(x - x') = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - M^2} e^{-ip(x-x')}. \] (7.169)
There will be an application of this Green function in Subsec. 12.12.1.

Let us calculate the spacetime behavior of the retarded propagator. We shall
first look at the massless case familiar from classical electrodynamics. For $M = 0$,
the integral representation (7.167) reads
\[ G_R(x - x') \equiv \int \frac{d^4 p}{(2\pi)^4} e^{-ip(x-x')} \frac{i}{p^2 + \omega_p^2}, \] (7.170)
but with the poles in the $p^0$-plane at
\[ p^0 = p^0 = \pm \omega_p = \pm |p|, \] (7.171)
both sitting below the real axis. This is indicated by writing $p^2_\perp$ in the denominator
rather than $p^2$, the plus sign indicating an infinitesimal $+i\eta$ added to $p^0$.

In the Feynman case, the $p^0$-integral is evaluated with the decomposition (7.64)
and the integral representation (7.63) of the Heaviside function as follows:
\[ \int \frac{dp^0}{2\pi} e^{-ip^0(x^0-x'^0)} i \frac{1}{2\omega_p} \left( \frac{1}{p^0 - \omega_p + i\eta} - \frac{1}{p^0 + \omega_p - i\eta} \right) \] (7.172)
\[ = \frac{1}{2\omega_p} \left[ \theta(x^0 - x'^0) e^{-i\omega_p(x^0-x'^0)} + \theta(x'^0 - x^0) e^{i\omega_p(x'^0-x^0)} \right] = \frac{1}{2\omega_p} e^{-i\omega_p|x^0-x'^0|}. \]

In contrast, the retarded expression reads
\[ \int \frac{dp^0}{2\pi} e^{-ip^0(x^0-x'^0)} i \frac{1}{2\omega_p} \left( \frac{1}{p^0 - \omega_p + i\eta} - \frac{1}{p^0 + \omega_p + i\eta} \right) \]
\[ = \Theta(x^0 - x'^0) \frac{1}{2\omega_p} [e^{-i\omega_p(x^0-x'^0)} - e^{i\omega_p(x^0-x'^0)}]. \] (7.173)
Thus we may write
\[ G(x - x') = \int \frac{d^3 p}{(2\pi)^3} e^{ip(x-x')} e^{-i\omega_p|x^0-x'^0|}, \] (7.174)
\[ G_R(x - x') = \Theta(x^0 - x'^0) \int \frac{d^3 p}{(2\pi)^3} e^{ip(x-x')} [e^{-i\omega_p(x^0-x'^0)} - e^{i\omega_p(x^0-x'^0)}]. \] (7.175)
In the retarded expression we recognize, behind the Heaviside prefactor $\Theta(x^0 - x'^0)$, the massless limit of the commutator function $C(x, x') = C(x - x')$ of Eq. (7.57), in accordance with the general relation (7.163).

The angular parts of the spatial part of the Fourier integral

$$\int \frac{d^3p}{(2\pi)^3} e^{ip(x-x')}$$

are the same in both cases, producing an integral over $|p|$:

$$\frac{1}{2\pi R} \int_0^\infty \frac{d|p|}{|p|} |p| \sin (|p|R),$$

where $R = |x - x'|$. The Feynman propagator has therefore the integral representation

$$G(x - x') = \frac{1}{4\pi R} \int_0^\infty \frac{d|p|}{|p|} \omega_p \sin (|p||R| e^{-i\omega_p |x^0 - x'^0|}).$$

In the massless case where $\omega_p = |p|$, we can easily perform the $|p|$-integration and recover the previous result (7.159).

To calculate the retarded propagator $G_R(x - x') = \Theta(x^0 - x'^0) C(x - x')$ in spacetime, we may focus our attention upon the commutator function, whose integral representation is now

$$C(x - x') = \frac{1}{4\pi R} \int_0^\infty \frac{dp}{\pi} \frac{p}{\omega_p} \sin (pR) [e^{-i\omega_p (x^0 - x'^0)} - e^{i\omega_p (x^0 - x'^0)}].$$

In the massless case where $\omega_p = p$, we decompose the trigonometric function into exponentials, and obtain

$$C(x - x') = -i \frac{1}{4\pi R} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left\{ e^{-i\omega(x^0 - x'^0) - R} - e^{-i\omega(x^0 - x'^0) + R} \right\},$$

which is equal to

$$C(x - x') = -i \frac{1}{4\pi R} \left[ \delta(x^0 - x'^0 - R) - \delta(x^0 - x'^0 + R) \right].$$

It is instructive to verify in this expression the canonical equal-time commutation properties (7.58). For this we multiply $\hat{C}(x)$ by a test function $f(r)$ and calculate, with $r = |x|$,

$$\int d^3x \hat{C}(x) f(r) = -i \int dr r^2 \frac{1}{r} \left[ \delta(x^0 - r) - \delta(x^0 + r) \right].$$

Since $\dot{\delta}(x^0 - r) - \dot{\delta}(x^0 + r) = -(d/dr)[\delta'(x^0 - r) + \delta'(x^0 + r)]$, we can perform a partial integration and find

$$-i \int dr \left[ \delta(x^0 - r) + \delta(x^0 + r) \right] [rf(r)]' = -i \frac{d}{dx^0} \left[ |x^0| f(|x^0|)) \right].$$
At $x^0 = 0$, this is equal to $-if(0)$ at $x^0 = 0$, as it should.

Using the property of the $\delta$-function (7.61) in spacetime

$$\delta(x^0^2 - r^2) = \frac{1}{2r}[\delta(x^0 - r) + \delta(x^0 + r)], \quad (7.184)$$

we can also write the commutator function as

$$C(x - x') = -i\epsilon(x^0 - x'^0)\frac{1}{2\pi}\delta((x - x')^2), \quad (7.185)$$

where $\epsilon(x^0 - x'^0)$ is defined by

$$\epsilon(x^0 - x'^0) \equiv \Theta(x^0 - x'^0) - \Theta(-x^0 + x'^0), \quad (7.186)$$

which changes the sign of the second term in the decomposition (7.184) of $\delta((x - x')^2)$ to the form required in (7.184).

Note that in the form (7.185), the canonical properties (7.58) of the commutator function cannot directly be verified, since $\dot{\epsilon}(x^0 - x'^0) = 2\delta(x^0 - x'^0)$ is too singular to be evaluated at equal times. In fact, products of distributions are in general undefined, and their use is forbidden in mathematics.\(^8\) We must read $\epsilon(x^0)\delta(x^2)$ as the difference of distributions

$$\epsilon(x^0)\delta(x^2) \equiv \frac{1}{2r}[\delta(x^0 - r) - \delta(x^0 + r)] \quad (7.187)$$

to deduce its consequences. Then with (7.185), the retarded propagator (7.163) becomes

$$G_R(x - x') = -i\Theta(x^0 - x'^0)\frac{1}{4\pi R} \left[\delta(x^0 - x'^0 - R) - \delta(x^0 - x'^0 + R)\right]$$

$$= -i\Theta(x^0 - x'^0)\frac{1}{2\pi}\delta((x - x')^2). \quad (7.188)$$

The Heaviside function allows only positive $x^0 - x'^0$, so that only the first $\delta$-function in (7.188) contributes, and we obtain the well-known expression of classical electrodynamics:

$$G_R(x - x') = -i\Theta(x^0 - x'^0)\frac{1}{4\pi R}\delta(x^0 - x'^0 - R). \quad (7.189)$$

This propagator exists only for a causal time order $x^0 > x'^0$, for which it is equal to the Coulomb potential between points which can be connected by a light signal.

The retarded propagator $G_R(x, x')$ describes the massless scalar field $\phi(x)$ caused by a local spacetime event $i\delta^{(4)}(x')$. For a general source $j(x')$, it serves to solve the inhomogeneous field equation

$$-\partial^2\phi(x) = j(x) \quad (7.190)$$

\(^8\)An extension of the theory of distributions that includes also their products is developed in the textbook H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets*, World Scientific, Singapore 2009 (klnrt.de/b5).
by superposition:
\[ \phi(x) = -i \int d^4x' G_R(x, x') j(x'). \] (7.191)

Inserting (7.189), and separating the integral into time and space parts, the time \( x_0' \) can be integrated out and the result becomes
\[ \phi(x, t) = -\int d^3x' \frac{1}{4\pi|x - x'|} j(x', t_R), \] (7.192)

where
\[ t_R = t - |x - x'| \] (7.193)
is the time at which the source has emitted the field which arrives at the spacetime point \( x \). Relation (7.192) is the basis for the derivation of the Liénard-Wiechert potential, which is recapitulated in Appendix 7B.

7.2.4 Comparison of Singular Functions

Let us compare the spacetime behavior (7.188), (7.189) of the retarded propagator with that of the massless Feynman propagator. The denominator in (7.159) can be decomposed into partial fractions, and we find a form very close to (7.188):
\[ G(x - x') = -\frac{1}{8\pi^2 R} \left[ \frac{1}{|x^0 - x^0| - R - i\eta} - \frac{1}{|x^0 - x^0| + R - i\eta} \right]. \] (7.194)

Feynman has found it useful to denote the function \( 1/(t - i\eta) \) by \( i\pi\delta_+(t) \). This has a Fourier representation which differs from that of a Dirac \( \delta \)-function by containing only positive frequencies:
\[ \delta_+(t) \equiv \int_{0}^{\infty} \frac{d\omega}{\pi} e^{-i\omega(t - i\eta)}. \] (7.195)
The integral converges at large frequencies only due to the \(-i\eta\)-term, yielding
\[ \delta_+(t) = -\frac{1}{\pi} \frac{i}{t - i\eta}. \] (7.196)
The pole term can be decomposed as\(^9\)
\[ \frac{1}{t - i\eta} = \frac{i\eta}{t^2 + \eta^2} + \frac{t}{t^2 + \eta^2} = i\pi \delta(t) + \mathcal{P} \frac{\eta}{t}. \] (7.197)

Recall that the decomposition concerns distributions which make sense only if they are used inside integrals as multipliers of smooth functions. The symbol \( \mathcal{P} \) in the

\(^9\)This is often referred to as Sochocki’s formula. It is the beginning of an expansion in powers of \( \eta > 0 \): \( 1/(x \pm i\eta) = \mathcal{P}/x \mp i\pi\delta(x) + \eta[\pi\delta'(x) \pm id\mathcal{P}/x] + \mathcal{O}(\eta^2) \).
second term means that the integral has to be calculated with the \textit{principal-value} prescription.\footnote{Due to the entirely different context, no confusion is possible with the second-quantized parity operator $P$ introduced in (7.96).} For the function $\delta_+(t)$, the decomposition reads
\begin{equation}
\delta_+(t) = \delta(t) - \frac{i}{\pi} \frac{\mathcal{P}}{t}.
\end{equation}

An important property of this function is that it satisfies a relation like $\delta(t)$ in (7.184):
\begin{equation}
\delta_+(t^2 - r^2) = \frac{1}{2r}[\delta_+(t - r) - \delta_+(t + r)].
\end{equation}

Below we shall also need the complex conjugate of the function $\delta_+(t)$:
\begin{equation}
\delta_-(t) = \frac{1}{\pi} \frac{i}{t + i\eta} = (\delta_+(t))^*.
\end{equation}

From (7.197) we see that the two functions are related by
\begin{equation}
\delta_+(t) + \delta_-(t) = 2\delta(t).
\end{equation}

Because of (7.199), the Feynman propagator (7.194) can be rewritten as
\begin{equation}
G(x - x') = -\frac{1}{8\pi^2 R} \left[ \delta_+((x^0 - x'^0) - R) - \delta_+((x^0 - x'^0) + R) \right] = -\frac{i}{4\pi} \delta_+((x - x')^2).
\end{equation}

These expressions look very similar to those for the retarded propagator in Eqs. (7.188) and (7.189).

It is instructive to see what becomes of the $\delta_+$-function in Feynman propagators in the presence of a particle mass. According to (7.148), a mass term modifies the integrand in
\begin{equation}
\delta_+((x - x')^2) = \int_0^\infty \frac{d\omega}{\pi} e^{-i\omega(x-x')^2},
\end{equation}
(in which we omit the $-i\eta$ term, for brevity) from $e^{-i\omega(x-x')^2}$ to $e^{-i\omega(x-x')^2 - iM^2/4\omega}$.

We may therefore define a massive version of $\delta_+((x - x')^2)$ by
\begin{equation}
\delta^M_+((x - x')^2) = \int_0^\infty \frac{d\omega}{\pi} e^{-i\omega(x-x')^2 - iM^2/2\omega} = \frac{M^2}{2} \frac{H_1^{(2)}(M\sqrt{(x - x')^2})}{M\sqrt{(x - x')^2}}.
\end{equation}

A similar generalization of the function $\delta((x - x')^2)$ in (7.185) to $\delta^M((x - x')^2)$ may be found by evaluating the commutator function $C(x - x')$ in Eq. (7.185) at a nonzero mass $M$. Its Fourier representation was given in Eq. (7.179) and may be written as
\begin{equation}
C(x - x') = \frac{i}{2\pi^2} \int_0^\infty \frac{d|p| |p|^2}{\sqrt{|p|^2 + M^2}} \frac{\sin(|p|R)}{|p|R} \sin[p^0(x^0 - x'^0)].
\end{equation}
This can be expressed as a derivative

\[ C(x - x') = -i \frac{1}{4\pi r} \frac{d}{dr} F(r, x^0 - x'^0) \]  

(7.206)
of the function

\[ F(r, t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dp}{\sqrt{p^2 + M^2}} \cos(pR) \sin \sqrt{p^2 + M^2 t} \]  

(7.207)
The integral yields [2]:

\[
F(r, t) = \begin{cases}
J_0(M\sqrt{t^2 - r^2}) & \text{for } t > r, \\
0 & \text{for } t \in (-r, r), \\
-J_0(M\sqrt{t^2 - r^2}) & \text{for } t < -r,
\end{cases}
\]  

(7.208)
where \( J_\mu(z) \) are Bessel functions. By carrying out the differentiation in (7.206), using \( J'_0(z) = -J_1(z) \), we may write

\[ C(x - x') = -i \epsilon(x^0 - x'^0) \delta^M((x - x')^2) \]  

(7.209)
with

\[ \delta^M(x^2) = \delta(x^2) - \Theta(x^2) \frac{M^2 J_1(M\sqrt{x^2})}{2M\sqrt{x^2}}. \]  

(7.210)
The function \( \Theta(x^2) \) enforces the vanishing of the commutator at spacelike distances, a necessity for the causality of the theory. Using (7.210), we can write the retarded propagator \( G_R(x - x') = \Theta(x^0 - x'^0) C(x - x') \) as

\[ G_R(x - x') = -i \Theta(x^0 - x'^0) \frac{1}{2\pi} \delta^M((x - x')^2). \]  

(7.211)
In the massless limit, the second term in (7.210) disappears since \( J_1(z) \approx z \) for small \( z \), and (7.211) reduces to (7.188).

Summarizing, we may list the Fourier transforms of the various propagators as follows:

\[
\begin{align*}
\text{Feynman propagator} : & \quad i \frac{\delta^2}{p^2 - M^2 + i\eta} = \pi \delta_-(p^2 - M^2) = \pi \delta_-(p^{02} - \omega_p^2); \\
& = \frac{\pi}{2\omega_p} [\delta_-(p^0 - \omega_p) + \delta_+(p^0 + \omega_p)]; \\
\text{retarded propagator} : & \quad i \frac{\delta^2}{p_+^2 - M^2} = \left[ \frac{\pi}{2\omega_p} [\delta_-(p^0 - \omega_p) - \delta_-(p^0 + \omega_p)] \right]; \\
\text{advanced propagator} : & \quad i \frac{\delta^2}{p_-^2 - M^2} = -\left[ \frac{\pi}{2\omega_p} [\delta_+(p^0 - \omega_p) - \delta_+(p^0 + \omega_p)] \right]; \\
\text{commutator} : & \quad 2\pi \epsilon(p^0) \delta(p^2 - M^2) = 2\pi \epsilon(p^0) \delta(p^{02} - \omega_p^2) \\
& = \frac{\pi}{\omega_p} [\delta_-(p^0 - \omega_p) - \delta_+(p^0 + \omega_p)].
\end{align*}
\]  

(7.212)
7.2 Spacetime Behavior of Propagators

Figure 7.3 Integration contours in the complex $p^0$-plane of the Fourier integral for various propagators: $C_F$ for the Feynman propagator, $C_R$ for the retarded propagator, and $C_A$ for the advanced propagator.

The corresponding integration contours in the complex energy plane of the representations of the spacetime propagators are indicated in Fig. 7.3. Some exercise with these functions is given in Appendix 7C.

We end this section by pointing out an important physical property of the euclidean Feynman propagator. When generalizing the proper-time representation of $G(x - x')$ to $D$ spacetime dimensions, it reads

$$ G(x - x') = \int_0^\infty d\tau \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{(x-x')^2}{4\tau} - M^2\tau}. \quad (7.213) $$

This reduces to (7.140) for $D = 4$. The integrand can be interpreted as the probability\(^{11}\) that a random world line of length $L = 2D\tau/a$, which is stiff over a length scale $a = 2D$, has the end-to-end distance $\sqrt{(x-x')^2}$. The world line can have any shape. Each configuration is weighted with a Boltzmann probability $e^{-M^2\tau}$ depending on the various lengths. In recent years this worldline interpretation of the Feynman propagator has been very fruitful by giving rise to a new type of quantum field theory, the so-called disorder field theory.\(^{[6]}\) This theory permits us to study phase transitions of a variety of different physical systems in a unified way. It is dual to Landau's famous theory of phase transitions in which an order parameter plays an essential role. In the dual disorder descriptions, the phase transitions have in common that they can be interpreted as a consequence of a sudden proliferation of line-like excitations. This is caused by an overwhelming configuration entropy which sets in at a temperature at which the configurational entropy outweighs the Boltzmann suppression due to the energy of the line-like excitations, the latter being proportional to the length of the excitations. Examples are polymers in solutions, vortex lines in superfluids, defect lines in crystals, etc. Whereas the traditional field

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quantization of relativistic free fields is based on the introduction of an order parameter and its spacetime version, an order field, the new description is based on a disorder field describing random fluctuations of line-like excitations. As a result of this different way of looking at phase transitions one obtains a field theoretic formulation of the statistical mechanics of line ensembles, that yields a simple explanation of the phase transitions in superfluids and solids.

7.3 Collapse of Relativistic Wave Function

As earlier in the nonrelativistic discussion on p. 120, a four-field Green function can be used to illustrate the relativistic version of the notorious phenomenon of the collapse of the wave function in quantum mechanics [5]. If we create a particle at some spacetime point \( x' = (x'_0, \mathbf{x}') \), we generate a Klein-Gordon wave function that covers the forward light cone of this point \( x' \). If we annihilate the particle at some different spacetime point \( x'' = (x''_0, \mathbf{x}'') \), the wave function disappears from spacetime.

Let us see how this happens about in the quantum field formalism. The creation and later annihilation process give rise to a Klein-Gordon field

\[
\langle \varphi(x) \varphi^\dagger(x') \rangle = \sum_p \frac{1}{2V \omega_p} e^{-ip(x-x')}.
\]  

If we measure the particle density at a spacetime \( x'' \) point that lies only slightly later than the later spacetime point \( t \) by inserting the current operator (7.128) in the above Green function, we find

\[
G(x'', x, x') = \langle 0 | \rho(x'') \varphi(x) \varphi^\dagger(x') | 0 \rangle = -\langle 0 | i \varphi^\dagger(x'') \partial_0 \varphi(x'') \varphi(x) \varphi^\dagger(x') | 0 \rangle.
\]  

The second and the fourth field operators yield a Green function \( \Theta(x''_0 - x'_0) G(x'' - x') \). This is multiplied by the Green function of the first and the third field operators which is \( \Theta(x - x'') G(x'' - x') \), and is thus equal to zero since \( x''_0 > x_0 \). Thus proves that by the time \( x''_0 > x_0 \), the wave function created at the initial spacetime point \( x' \) has completely collapsed. More interesting is the situation if we perform an analogous study for a state \( | \varphi(x'_1) \varphi(x'_2) \rangle \) that contains two particles. Here we look at the Green function

\[
G(x'', x_1, x'_1, x_2, x'_2) = \langle 0 | \rho(x'') \varphi(x_1) \varphi(x_2) \varphi^\dagger(x'_1) \varphi^\dagger(x'_2) | 0 \rangle = -\langle 0 | i \varphi^\dagger(x'') \partial_0 \varphi(x'') \varphi(x_1) \varphi(x_2) \varphi^\dagger(x'_1) \varphi^\dagger(x'_2) | 0 \rangle.
\]  

Depending on the various positions of \( x'' \) with respect to \( x_1 \) and \( x_2 \), we can observe the consequences of putting counter for one particle of the two-particle wave function, i.e., from which we learn of the collapse of a one-particle content of the two-particle wave function.\(^{12}\)

7.4 Free Dirac Field

We now turn to the quantization of the Dirac field obeying the field equation (4.500):

\[(i\gamma^\mu \partial_\mu - M)\psi(x) = 0. \tag{7.217}\]

Their plane-wave solutions were given in Subsec. 4.13.1, and just as in the case of a scalar field, we shall introduce creation and annihilation operators for particles associated with these solutions.

The classical Lagrangian density is [recall (4.501)]:

\[\mathcal{L}(x) = \bar{\psi}(x)i\gamma^\mu \partial_\mu \psi(x) - M\bar{\psi}(x)\psi(x). \tag{7.218}\]

7.4.1 Field Quantization

The canonical momentum of the \(\psi(x)\) field is

\[\pi(x) = \frac{\partial \mathcal{L}(x)}{\partial [\partial_0 \psi(x)]} = i\bar{\psi}(x)\gamma^0 = i\psi(x). \tag{7.219}\]

Up to the factor \(i\), this is equal to the complex-conjugate field \(\psi(x)\), as in the non-relativistic equation (7.5.1). Note that the field \(\bar{\psi}\) has no conjugate momentum, since

\[\frac{\partial \mathcal{L}(x)}{\partial [\partial_0 \bar{\psi}(x)]} = 0. \tag{7.220}\]

This zero is a mere artifact of the use of complex field variables. It is unrelated to a more severe problem in Section 7.5.1, where the canonical momentum of a component of the real electromagnetic vector field vanishes as a consequence of gauge invariance.

If \(\psi(x, t)\) were a Bose field, its canonical commutation rule would read

\[[\psi(x, t), \psi^\dagger(x', t)] = \delta^{(3)}(x - x'). \tag{7.221}\]

However, since electrons must obey Fermi statistics to produce the periodic system of elements, the fields have to satisfy anticommutation rules

\[\left\{\psi(x, t), \psi^\dagger(x', t)\right\} = \delta^{(3)}(x - x'). \tag{7.222}\]

Recall that in the nonrelativistic case, this modification of the commutation rules was dictated by the Pauli principle and the implied antisymmetric electronic wave functions. To ensure this, the relativistic theory can be correctly quantized only by anticommutation rules. Commutation rules (7.221) are incompatible either with microcausality or the positivity of the energy. This will be shown in Section 7.10.

We now expand the field \(\psi(x)\) into the complete set of classical plane-wave solutions (4.662). In a large but finite volume \(V\), the expansion reads

\[\psi(x) = \sum_{p, s_3} \left[ f_{p s_3}^e(x) a_{p, s_3} + f_{p s_3}^c(x) b_{p, s_3}^\dagger \right], \tag{7.223}\]
orthogonality relations \((4.665)\) for the bispinors \(u, v\) is reversed in both plane waves \(e\). As in the scalar equation \((7.12)\), we have associated the expansion coefficients of the terms of which \(Eqs. \,(7.226)\) are simply \(\int \sqrt{V} \rho / M \left[ e^{-i p x} u(p, s_3) a_{p, s_3} + e^{i p x} v(p, s_3) b_{p, s_3}^\dagger \right]. \quad (7.224)\) The results are \(\int d^3 x e^{-i p x} \psi(x) = \sum_{p, s_3} \frac{\sqrt{V}}{\sqrt{p^0 / M}} \left[ e^{-i p_0 x_0} u(p, s_3) a_{p, s_3} + e^{i p_0 x_0} v(-p, s_3) b_{p, s_3}^\dagger \right]. \quad (7.225)\) The results are

\[
a_{p, s_3} = e^{i p_0 x_0} \frac{1}{\sqrt{V p^0 / M}} u(p, s_3) \int d^3 x e^{-i p x} \psi(x),
\]

\[
b_{p, s_3}^\dagger = e^{-i p_0 x_0} \frac{1}{\sqrt{V p^0 / M}} v(p, s_3) \int d^3 x e^{i p x} \psi(x), \quad (7.226)\]

these being the analogs of the scalar equations \((7.76)\). The same results can, of course, be obtained from \((7.223)\) with the help of the scalar products \((4.664)\), in terms of which \(Eqs. \,(7.226)\) are simply

\[
a_{p, s_3} = (f_{s_3}, \psi)_t, \quad b_{p, s_3}^\dagger = (f_{p, s_3}^c, \psi)_t, \quad (7.227)\]

as in the scalar equations \((7.19)\).

From \((7.226)\) we derive that all anticommutators between \(a_{p, s_3}, a_{p', s'_3}^\dagger, b_{p, s_3}, b_{p', s'_3}^\dagger\) vanish, except for

\[
\{a_{p, s_3}, a_{p', s'_3}^\dagger\} = \sqrt{\frac{M M}{p^0 \rho p' \rho}} u(p, s_3) u(p', s_3) \delta_{p, p'} \delta_{s_3, s'_3}, \quad (7.228)\]

\[
\{b_{p, s_3}, b_{p', s'_3}^\dagger\} = \sqrt{\frac{M M}{p^0 \rho p' \rho}} v(p, s_3) v(p', s_3) \delta_{p, p'} \delta_{s_3, s'_3}. \quad (7.229)\]

The single-particle states \(|p, s_3\rangle = a_{p, s_3}^\dagger |0\rangle\) have the wave functions

\[
\langle 0 | \psi(x) | p, s_3 \rangle = \frac{1}{\sqrt{V p^0 / M}} e^{-i p x} u(p, s_3) = f_{p, s_3}(x)
\]

\[
\langle p, s_3 | \psi(x) | 0 \rangle = \frac{1}{\sqrt{V p^0 / M}} e^{i p x} \bar{u}(p, s_3) = \bar{f}_{p, s_3}(x). \quad (7.231)\]
The similar antiparticle states
\[ |\bar{p}, s_3\rangle = b^\dagger_{\bar{p}, s_3} |0\rangle, \] (7.232)
have the matrix elements
\[ \langle \bar{p}s_3 | \psi(x) | 0 \rangle = \frac{1}{\sqrt{Vp^0/M}} e^{ipx} v(p, s_3) = f^c_{p, s_3}(x), \]
\[ \langle 0 | \bar{\psi}(x) | \bar{p}, s_3 \rangle = \frac{1}{\sqrt{Vp^0/M}} e^{-ipx} \bar{u}(p, s_3) = \bar{f}^c_{p, s_3}(x). \] (7.233)

As in the scalar field expansion (7.75), the negative-frequency solution \( v(p, s_3) e^{ipx} \) of the Dirac equation is associated with a creation operator \( b^\dagger_{p, s_3} \) of an antiparticle rather than a second annihilation operator \( d_{-p, s_3} \). This ensures that the antiparticle state \( |\bar{p}s_3\rangle \) has the same exponential form \( e^{-ipx} \) as that of \( |p, s_3\rangle \), both exhibiting the same time dependence \( e^{-ip^0 t} \) with a positive energy \( p^0 = \omega_p \). The other assignment would have given a negative energy.

In an infinite volume, a more convenient field expansion makes use of the plane-wave solutions (4.666). It uses covariant creation and annihilation operators as in (7.16) to expand:
\[ \psi(x) = \int \frac{d^3p}{(2\pi)^3 p^0/M} \sum_{s_3} \left[ e^{-ipx} u(p, s_3) a_{p, s_3} + e^{ipx} v(p, s_3) b^\dagger_{p, s_3} \right]. \] (7.234)
The commutation rules between \( a_{p, s_3} \), \( b^\dagger_{p, s_3} \), \( b_{p, s_3} \), and \( b^\dagger_{p, s_3} \) are the same as in (7.228) and (7.229), but with a replacement of the Kronecker symbols by their invariant continuum version similar to (7.20):
\[ \delta_{p, p'} \rightarrow \frac{p^0}{M} \delta^{(3)}(p - p') = \frac{p^0}{M} (2\pi\hbar)^D \delta^{(3)}(p - p'). \] (7.235)
Note that this fermionic normalization is different from the bosonic one in (7.20), where the factor in front of the \( \delta \)-function was \( 2p^0 \). This is a standard convention throughout the literature.

In an infinite volume, we may again introduce single-particle states
\[ |p, s_3\rangle = a^\dagger_{p, s_3} |0\rangle, \quad |\bar{p}, s_3\rangle = b^\dagger_{p, s_3} |0\rangle, \] (7.236)
with the vacuum \( |0\rangle \) defined by \( a_{p, s_3}|0\rangle = 0 \) and \( b_{p, s_3}|0\rangle = 0 \). These states will satisfy the orthogonality relations
\[ (p', s'_3 | p, s_3) = \frac{p^0}{M} \delta^{(3)}(p' - p) \delta_{s'_3, s_3}, \quad (\bar{p}', s'_3 | \bar{p}, s_3) = \frac{p^0}{M} \delta^{(3)}(p' - p) \delta_{s'_3, s_3}, \] (7.237)
in accordance with the replacement (7.235) [and in contrast to the normalization (7.27) for scalar particles]. They have the wave functions
\[ \langle 0 | \psi(x) | p, s_3 \rangle = e^{-ipx} u(p, s_3) = f_{p, s_3}(x), \quad \langle p, s_3 | \bar{\psi}(x) | 0 \rangle = e^{ipx} \bar{u}(p, s_3) = \bar{f}^c_{p, s_3}(x). \]
\[ \langle \bar{p}, s_3 | \psi(x) | 0 \rangle = e^{ipx} v(p, s_3) = f^c_{p, s_3}(x), \quad \langle 0 | \bar{\psi}(x) | \bar{p}, s_3 \rangle = e^{-ipx} \bar{v}(p, s_3) = \bar{f}^c_{p, s_3}(x). \] (7.238)
We end this section by stating also the explicit form of the quantized field of massless left-handed neutrinos and their antiparticles, the right-handed antineutrinos:

$$\psi^\dagger(x) = \frac{1 - \gamma^5}{2} \psi(x) = \sum_p \frac{1}{\sqrt{2Vp^0}} \left[ e^{-ipx} u_L(p) a_{p, -\frac{1}{2}} + e^{ipx} v_R(p) b_{p, \frac{1}{2}}^\dagger \right], \quad (7.239)$$

where the operators $a_{p, -\frac{1}{2}}^\dagger$ and $b_{p, \frac{1}{2}}^\dagger$ carry helicity labels $\pm \frac{1}{2}$, and $p^0 = |p|$. The massless helicity bispinors are those of Eqs. (4.726) and (4.727). Remember the normalization (4.725):

$$u_L(p) u_L(p) = 2p^0, \quad u_R(p) u_R(p) = 2p^0,$$

$$v_L(p) v_L(p) = 2p^0, \quad v_R(p) v_R(p) = 2p^0, \quad (7.240)$$

which is the reason for the factor $1/\sqrt{2Vp^0}$ in the expansion (7.239) [as in the expansions (7.12) and (7.75) for the scalar mesons].

### 7.4.2 Energy of Free Dirac Particles

We now turn to the energy of the free quantized Dirac field. The energy density is given by the Legendre transform of the Lagrangian density

$$\mathcal{H}(x) = \pi(x) \dot{\psi}(x) - \mathcal{L}(x)$$

$$= \dot{\psi}^\dagger(x) \psi(x) - \mathcal{L}(x)$$

$$= \bar{\psi}(x)i\gamma^i \partial_i \psi(x) + M\bar{\psi}(x)\psi(x). \quad (7.241)$$

Inserting the expansion (7.224) and performing the spatial integral [recall the step from (7.30) to (7.31)], the double sum over momenta reduces to a single sum, and we obtain the second-quantized Hamilton operator

$$H = \sum_{p,s_3}{M \over 2p^0} \left[ a_{p,s_3}^\dagger a_{p,s_3} \bar{u}(p, s_3)(\gamma^i p^i + M) u(p, s_3) + b_{p,s_3}^\dagger b_{p,s_3} \bar{v}(p, s_3)(-\gamma^i p^i + M) v(p, s_3) + a_{p,s_3}^\dagger b_{-p,s_3} \bar{u}(p, s_3)(\gamma^i p^i + M) v(-p, s_3') e^{2ip^0 t} + b_{-p,s_3}^\dagger a_{p,s_3} \bar{v}(-p, s_3)(-\gamma^i p^i + M) u(p, s_3') e^{2ip^0 t} \right]. \quad (7.242)$$

We now use the Dirac equation, according to which

$$(\gamma^i p^i + M) u(p, s_3) = \gamma^0 p^0 u(p, s_3),$$

$$(\gamma^i p^i + M) v(p, s_3) = -\gamma^0 p^0 v(p, s_3), \quad (7.243)$$

and the orthogonality relations (4.696)–(4.699), and simplify (7.243) to

$$H = \sum_{p,s_3} p^0 \left( a_{p,s_3}^\dagger a_{p,s_3} - b_{-p,s_3} b_{-p,s_3}^\dagger \right). \quad (7.244)$$
With the help of the anticommutation rule (7.229), this may be rewritten as

$$H = \sum_{p,s_3} p^0 \left( a^\dagger_{p,s_3} a_{p,s_3} + b^\dagger_{p,s_3} b_{p,s_3} \right) - \sum_{p,s_3} p^0. \quad (7.245)$$

The total energy adds up all single-particle energies. In contrast to the second-quantized scalar field, the vacuum has now a negatively infinite energy due to the zero-point oscillations, as announced in the discussion of Eq. (7.37).

Note that if we had used here the annihilation operators of negative energy states $d_{-p,-s_3}$ instead of the creation operators $b^\dagger_{p,s_3}$, then the energy would have read

$$H = \sum_{p,s_3} p^0 \left( a^\dagger_{p,s_3} a_{p,s_3} - d^\dagger_{p,s_3} d_{p,s_3} \right), \quad (7.246)$$

so that the particles created by $d^\dagger_{p,s_3}$ would again have had negative energies. By going over from $d_{-p,-s_3}$ to $b^\dagger_{p,s_3}$, we have transformed missing negative energy states into states of antiparticles, thereby obtaining a positive sign for the energy of an antiparticle. This happens, however, at the expense of having a negatively infinite zero-point energy of the vacuum. This also explains why the spin orientation $s_3$ changes sign under the above replacement. A missing particle with spin down behaves like a particle with spin up.

Actually, the above statements about the exchange $d_{p,s_3} \rightarrow b^\dagger_{p,s_3}$ are correct only as far as the sum of all momentum states and spin indices $s_3$ in (7.246) is concerned. For a single state, we also have to consider the momentum operator and the spin.

The situation is the same as in the free-electron approximation to the electrons in a metal. At zero temperature, the electrons are in the ground state, forming a Fermi sea in which all single-particle levels below a Fermi energy $E_F$ are occupied. If all energies are measured from that energy, all occupied levels have a negative energy, and the ground state has a large negative value. If an electron is kicked out from one of the negative-energy states, an electron-hole pair is observed, both particle and hole carrying a positive energy with respect to the undisturbed Fermi liquid. The hole appears with a positive charge relative to the Fermi sea.

Note also that unlike the Bose case, the exchange $d_{-p,-s_3} \rightarrow b^\dagger_{p,s_3}$ maintains the correct sign of canonical anticommutation rules (7.229). In fact, the negative sign of the term $-d^\dagger_{p,s_3} d_{p,s_3}$ in the Hamilton operator (7.246) is somewhat less devastating than in the Bose case. It can actually be avoided by a mere redefinition of the vacuum as the state in which all momenta are occupied by a $d^\dagger_{p,s_3}$ particle:

$$|0\rangle_{\text{new}} = \prod_{p,s_3} d^\dagger_{p,s_3} |0\rangle. \quad (7.247)$$

This state has now the same negative infinite energy

$$E_{0\text{ new}} \equiv \langle 0 | H | 0 \rangle_{\text{new}} = - \sum_{p,s_3} p^0, \quad (7.248)$$

found in the correct quantization. Counting from this ground state energy, all other states have positive energies, obtained either by adding a particle with the
operator \( a_{p,s3}^\dagger \) or by removing a particle with \( d_{p,-s3} \). Of course, this description is just a reflection of the fact that if creation and annihilation operators satisfy anticommutation rules \(-d_{p,s3}^\dagger d_{p,s3} = d_{p,s3}d_{p,s3}^\dagger - 1\), a reinterpretation of the operators \( d_{p,-s3} \to b_{p,s3}^\dagger \) and \( d_{p,-s3}^\dagger \to b_{p,s3} \) makes the product \( d_{p,s3}d_{p,s3}^\dagger \) a positive operator. For commutators, the same reinterpretation is impossible since \(-d_{p,s3}^\dagger d_{p,s3} \) can have any negative eigenvalue and it is impossible to introduce a new “zero level” that would make all energy differences positive!

It was the major discovery of Dirac that the annihilation of a negative-energy particle may be viewed as a creation of a positive-energy particle with the same mass and spin, but with reversed directions of momentum and spin. Dirac called it the \textit{antiparticle}. When dealing with electrons, the antiparticle is a \textit{positron}. Dirac imagined all negative energy states in the world as being filled, forming a sea of negative energy states, just as the above-described Fermi sea in metals. The annihilation of a negative-energy particle in the sea would create a hole which would appear to the observer as a particle of positive energy with the opposite charge.

As in the scalar case of Eqs. (7.32)–(7.36), it is often possible to simply drop this infinite energy of the vacuum by introducing a normal product \( \mathcal{H} \). As before, the double dots mean: Order all operators such that all creators stand to the left of all annihilators. But in contrast to the boson case, every transmutation of two Fermi operators, done to achieve the normal order, is now accompanied by a phase factor \(-1\).

### 7.4.3 Lorentz Transformation Properties of Particle States

The behavior of the bispinors \( u(p, s_3) \) and \( v(p, s_3) \) under Lorentz transformations determines the behavior of the creation and annihilation operators of particles and antiparticles, and thus of the particle states created by them, in particular the single-particle states (7.230) and (7.232). Under a Lorentz transformation \( \Lambda \), the field operator \( \psi(x) \) transforms according to the law (4.521):

\[
\psi(x) \xrightarrow{\Lambda} \psi_{\Lambda}(x) = D(\Lambda)\psi(\Lambda^{-1}x) = e^{-i\frac{\omega}{\hbar}(p_\mu x_\mu)} \psi(\Lambda^{-1}x), \tag{7.249}
\]

On the right-hand side, we now insert the expansion (7.234), so that

\[
D(\Lambda)\psi(\Lambda^{-1}x) = \int \frac{d^3p}{(2\pi)^3} \sum_{s_3,s_3'} \left[ e^{-ip'_{\mu}x_\mu} u(p', s_3') W_{s_3's_3}(p', \Lambda, p) a_{p,s_3} + e^{ip'_{\mu}x_\mu} v(p', s_3') W^*_{s_3's_3}(p', \Lambda, p) b_{p,s_3}^\dagger \right],
\]

where we have set \( p' \equiv \Lambda p \), and used the fact that \( p\Lambda^{-1}x = p'x \). In order to derive the transformation laws for the creation and annihilation operators, we rewrite this expansion in the same form as the original (7.234), expressing it now in terms of primed momenta and spins:

\[
D(\Lambda)\psi(\Lambda^{-1}x) = \int \frac{d^3p'}{(2\pi)^3} \sum_{s_3,s_3'} \left[ e^{-ip'_{\mu}x_\mu} u(p', s_3') a_{p',s_3'} + e^{ip'_{\mu}x_\mu} v(p', s_3') b_{p',s_3'}^\dagger \right]. \tag{7.250}
\]
Because of the Lorentz invariance of the integration measure in momentum space observed in (4.184), we can replace in (7.250)

\[ \int \frac{d^3p}{(2\pi)^3p^0/M} \rightarrow \int \frac{d^3p}{(2\pi)^3p^0/M} \]  

(7.251)

Comparing now the coefficients, we find the transformation laws

\[ a_{\mathbf{p}', s_3'} \xrightarrow{\Lambda} a'_{\mathbf{p}', s_3'} = \sum_{s_3=-1/2}^{1/2} W_{s_3', s_3}(p', \Lambda, p) a_{\mathbf{p}, s_3}, \]

\[ b^\dagger_{\mathbf{p}', s_3'} \xrightarrow{\Lambda} b'^\dagger_{\mathbf{p}', s_3'} = \sum_{s_3=-1/2}^{1/2} W^*_{s_3', s_3}(p', \Lambda, p) b^\dagger_{\mathbf{p}, s_3}, \]

(7.252)

In contrast to the Lorentz transformations (7.249) of the field, the creation and annihilation operators are transformed unitarily under the Lorentz group. The right-hand sides define a unitary representation of the Lorentz transformations \( \Lambda \):

\[ a'_{\mathbf{p}', s_3'} = U^{-1}(\Lambda) a_{\mathbf{p}', s_3'} U(\Lambda) = \sum_{s_3=-1/2}^{1/2} W_{s_3', s_3}(p', \Lambda, p) a_{\mathbf{p}, s_3}, \]

\[ b'^\dagger_{\mathbf{p}', s_3'} = U^{-1}(\Lambda) b^\dagger_{\mathbf{p}', s_3'} U(\Lambda) = \sum_{s_3=-1/2}^{1/2} W^*_{s_3', s_3}(p', \Lambda, p) b^\dagger_{\mathbf{p}, s_3}, \]

(7.253)

with similar relations for \( a^\dagger_{\mathbf{p}, s_3} \) and \( b_{\mathbf{p}, s_3} \). For the single-particle states (7.236), this implies the transformation laws:

\[ U(\Lambda) |\mathbf{p}, s_3\rangle = U(\Lambda) a^\dagger_{\mathbf{p}, s_3} U^{-1}(\Lambda) |0\rangle = \sum_{s_3'=-1/2}^{1/2} |\mathbf{p}', s_3'\rangle W_{s_3', s_3}(p', \Lambda, p), \]

\[ U(\Lambda) |\mathbf{p}, s_3\rangle = U(\Lambda) b^\dagger_{\mathbf{p}, s_3} U^{-1}(\Lambda) |0\rangle = \sum_{s_3'=-1/2}^{1/2} |\mathbf{p}', s_3'\rangle W^*_{s_3', s_3}(p', \Lambda, p), \]

(7.254)

where we have used the Lorentz-invariance of the vacuum state

\[ U(\Lambda) |0\rangle = |0\rangle. \]

(7.255)

We are now ready to understand the reason for introducing the matrix \( c \) in the bispinors \( v(\mathbf{p}, s_3) \) of Eq. (4.684), and thus the sign reversal of the spin orientation. In (4.742) we found that the canonical spin indices of \( v(\mathbf{p}, s_3) \) transformed under rotations by the complex-conjugate \( 2 \times 2 \) Wigner matrices with respect to those of \( u(\mathbf{p}, s_3) \). This implies that the Wigner rotations mixing the spin components of the operators \( a_{\mathbf{p}, s_3} \) and \( b^\dagger_{\mathbf{p}, s_3} \) are complex conjugate to each other. As a consequence, \( b^\dagger_{\mathbf{p}, s_3} \) transforms in the same way as \( a^\dagger_{\mathbf{p}, s_3} \), so that antiparticles behave in precisely the same way as particles under Lorentz transformations.
For massless particles in the helicity representation, the creation and annihilation operators transform under Lorentz transformations merely by a phase factor, as discussed at the end of Section 4.15.3.

Under translations by a four-vector $a^\mu$, the particle and antiparticle states $|\mathbf{p}, s_3\rangle = \hat{a}^\dagger_{\mathbf{p}, s_3}|0\rangle$ and $|\bar{\mathbf{p}}, s_3\rangle = \hat{b}^\dagger_{\mathbf{p}, s_3}|0\rangle$ transform rather trivially. Since they have a definite momentum, they receive merely a phase factor $e^{ipa}$. This follows directly from applying the transformation law (4.524) on the Dirac field operator:

$$\psi(x) \rightarrow \psi'(x) = \psi(x - a).$$

(7.256)

Inserting the operator expansion (7.234) into the right-hand side, we see that the creation and annihilation operators transform as follows:

$$a_{\mathbf{p}, s_3} \rightarrow a'_{\mathbf{p}, s_3} = e^{ipa} a_{\mathbf{p}, s_3},$$

$$b^\dagger_{\mathbf{p}, s_3} \rightarrow b'^\dagger_{\mathbf{p}, s_3} = e^{-ipa} b^\dagger_{\mathbf{p}, s_3}.$$  

(7.257)

These transformation laws define a unitary operator of translations $U(a)$:

$$a'_{\mathbf{p}, s_3} = U^{-1}(a)a_{\mathbf{p}, s_3} U(a) = e^{ipa} a_{\mathbf{p}, s_3},$$

$$b'^\dagger_{\mathbf{p}, s_3} = U^{-1}(a)b^\dagger_{\mathbf{p}, s_3} U(a) = e^{-ipa} b^\dagger_{\mathbf{p}, s_3}.$$  

(7.258)

On the single-particle states, the operator $U(a)$ has the effect

$$U(a)|\mathbf{p}, s_3\rangle = U(a)a^\dagger_{\mathbf{p}, s_3} U^{-1}(a)|0\rangle = |\mathbf{p}, s_3\rangle e^{ipa},$$

$$U(a)|\bar{\mathbf{p}}, s_3\rangle = U(a)b^\dagger_{\mathbf{p}, s_3} U^{-1}(\Lambda)|0\rangle = |\bar{\mathbf{p}}, s_3\rangle e^{ipa}.$$  

(7.259)

One can immediately write down an explicit expression for this operator:

$$U(a) = e^{iaP},$$

(7.260)

where $P^0$ is the Hamilton operator (7.245), rewritten in the infinite-volume form as

$$P^0 \equiv H = \int \frac{d^3p}{(2\pi)^3 p^0/M} \left(\hat{a}^\dagger_{\mathbf{p}, s_3} P^0 a_{\mathbf{p}, s_3} + \hat{b}^\dagger_{\mathbf{p}, s_3} P^0 b_{\mathbf{p}, s_3} - p^0 P^0 \delta^{(3)}(0)\right).$$  

(7.261)

The last term is a formal infinite-volume expression for the finite-volume vacuum energy

$$P^0_{\text{vac}} = - \int \frac{d^3p V}{(2\pi)^3 p^0},$$

(7.262)

as we see from the Fourier representation of $\delta^{(3)}(\mathbf{p}) = \int d^3x e^{ipx}$ which is equal to $V$ for $\mathbf{p} = 0$. The operator of total momentum reads

$$P \equiv \int \frac{d^3p}{(2\pi)^3 p^0/M} \left(\hat{a}^\dagger_{\mathbf{p}, s_3} \mathbf{p} a_{\mathbf{p}, s_3} + \hat{b}^\dagger_{\mathbf{p}, s_3} \mathbf{p} b_{\mathbf{p}, s_3}\right).$$  

(7.263)
Together, they form the operator of total four-momentum \( P^\mu = (P^0, \mathbf{p}) \) which satisfies the commutation rules with the particle creation and annihilation operators

\[
[P^\mu, a_{p,s}^\dagger] = p^\mu a_{p,s}^\dagger, \quad [P^\mu, b_{p,s}^\dagger] = p^\mu b_{p,s}^\dagger.
\] (7.264)

These express the fact that by adding a particle of energy and momentum \( p^\mu \) to the system, the total energy-momentum \( P^\mu \) is increased by \( p^\mu \).

By combining Lorentz transformations and translations as in (4.526), we cover the entire Poincaré group, and the single-particle states form an irreducible representation space of this group. The invariants of the representation are the mass \( M = \sqrt{p^2} \) and the spin \( s \). Thus the complete specification of a representation state is

\[
|\mathbf{p}, s_3[M, s]|.
\] (7.265)

The Hilbert space of two-particle states \(|\mathbf{p}, s_3[M, s]; \mathbf{p}', s_3'[M', s']\rangle\) gives rise to a reducible representation of the Poincaré group. The reducibility is obvious from the fact that if both particles are at rest, the state \(|0, s_3[M, s]; 0', s_3'[M', s']\rangle\) is simply a product of two rotational states of spins \( s \) and \( s' \), and decomposes into irreducible representations of the rotation group with spins \( S = |s - s'|, |s - s'| + 1, \ldots, s + s' \).

If the two particles carry momenta, the combined state will also have all possible orbital angular momenta. In the center-of-mass frame of the two particles, the momenta are of equal size and point in opposite directions. The states are then characterized by the direction of one of the momenta, say \( \mathbf{p} \), and may be written as \(|\mathbf{p}, s_3, s_3'\rangle\). They satisfy the completeness relation

\[
\sum_{s_3 = -s}^{s} \sum_{s_3' = -s'}^{s'} \int_0^\pi d\theta \int_0^{2\pi} d\varphi \langle \mathbf{p}, s_3, s_3' | \mathbf{p}, s_3, s_3' \rangle = 1,
\] (7.266)

where \( \theta \) and \( \varphi \) are the spherical angles of the direction \( \mathbf{p} \).

In the absence of spin, it is then simple to find the irreducible contents of the rotation group. We merely expand the state \(|\mathbf{p}\rangle\) in partial waves:

\[
|\mathbf{p}\rangle = \sum_{l,m} |l, m|\{l, m|\mathbf{p}\} \equiv \sum_{l,m} |l, m\} Y_{l,m}(\theta, \phi),
\] (7.267)

with the spherical harmonics \( Y_{l,m}(\theta, \phi) \). The states \(|j, m\rangle\) are orthonormal and complete in this Hilbert space:

\[
\{j, m|j', m'\} = \delta_{j,j'}\delta_{m,m'}, \quad \sum_{j,m} |j, m\} \{j, m| = 1.
\] (7.268)

In the presence of spins \( s, s' \), the decomposition is simplest if the spin orientations are specified in the helicity basis. Then the two-particle states possess an azimuthal angular momentum of size \( h - h' \) around the direction \( \mathbf{p} \), and may be written as \(|\mathbf{p}, h - h'\rangle\). These have the same rotation properties as the wave functions of a
spinning top with an azimuthal angular momentum \( h - h' \) around the body axis. The latter functions are well known — they are just the representation functions

\[
D^j_{m,m'}(\alpha, \beta, \gamma) = e^{-i(m\alpha + m'\gamma)} d^j_{m,m'}(\beta)
\]  

(7.269)

denote the rotational functions of angular momentum \( j \) introduced in Eq. (4.865). These functions serve as wave functions of a spinning top with angular momentum \( j \) and magnetic quantum number \( m \), and with an azimuthal component \( m' \) of angular momentum around the body axis. They are the basis states of irreducible representations of the rotation group in the center-of-mass frame. Thus, extending (7.267), we can expand

\[
|\hat{p}, h - h'\rangle = \sum_{j,m} |j, m, h - h'| \{ j, m, h - h'|\hat{p}\}
\]

(7.270)

with an orthonormal and complete set of states \( |j, m, h - h'\rangle \) at a fixed \( h - h' \). The normalization factor is determined to comply with the orthonormality property (4.885) of the rotation functions.

Now we boost this expansion from the center-of-mass frame back to the initial frame in which the total momentum is \( P = \hat{p} + p' \), the energy \( P^0 = p^0 + p'^0 \), and the mass \( \mu = \sqrt{P^2} \). The direction of \( P \) may be chosen as a quantization axis for the angular momentum \( j \). Then the quantum number \( m \) is equal to the helicity of the combined state. The angular momentum \( j \) in the rest frame determines the spin of the combined state. The resulting Clebsch-Gordan-like expansion of the two-particle state into irreducible representations of the Poincaré group is [10]

\[
|p, s_3 [M, s] ; p', s'_3 [M', s'] \rangle = \sum_{j,m} \int_0^\infty \frac{d\mu}{2\pi} \int \frac{d^3 p}{(2\pi)^3 p^0/\mu} |p, m [\mu, j] \eta\rangle \times \langle p, m [\mu, j] \eta |p, s_3 [M, s] ; p', s'_3 [M', s']\rangle,
\]

(7.271)

with the expansion coefficients

\[
\langle p, m [\mu, j] \eta |p, h [M, s] ; p', h' [M', s']\rangle = \delta^{(4)}(P - p - p') N(\mu; M, M') \times \frac{2j + 1}{4\pi} D^j_{m, h - h'}(\varphi, \theta, 0),
\]

(7.272)

where \( N(\mu; M, M') \) is some normalization factor. The product representation is not simply reducible. It requires distinguishing the different irreducible biparticle states according to their spin \( S \). For this purpose, a degeneracy label \( \eta \) is introduced. It may be taken as the pair of helicity indices \( (h, h') \) of the individual particles which the biparticle is composed of. However, to describe different processes most efficiently, other linear combinations may be more convenient. One may, for example, combine first the individual spins to a total intermediate spin \( S \). After this one combines the states with spin \( S \) with the orbital angular momentum \( L \) (in the center-of-mass
frame) to states with a total angular momentum \( j \) of the biparticle (in that frame). This corresponds to the \( LS \)-coupling scheme in atomic physics. Explicitly, these states are

\[
|\mathbf{P}, m [\mu, j] (LS)\rangle = \langle j, h-h'|L, 0; S, h-h'|S, h; s', -h'|\mathbf{P}, m [\mu, j] (h, h')\rangle.
\]

(7.273)

A sum over all \( h, h' \) is implied. Since the quantization axis is the direction of \( \mathbf{p} \) in the center-of-mass frame, the orbital angular momentum has no \( L_3 \)-component.

\[
\text{Figure 7.4 Different coupling schemes for two-particle states of total angular momentum } j \text{ and helicity } m. \text{ The first is the LS-coupling, the second the JL-coupling scheme.}
\]

Another possibility of coupling the two particles corresponds to the multipole radiation in electromagnetism. Here one of the particles, say \([M, s]\), is singled out to carry off radiation, which is analyzed according to its total angular momentum \( J \) composed of \( L \) and \( s \). This total angular momentum is coupled with the other spin \( s' \) to the combined total angular momentum \( j \) of the biparticle. Here the states are

\[
|\mathbf{P}, m [\mu, j] (JL)\rangle = \langle j, h|s, h; L, 0\rangle\langle j, h-h'|L, h; s', -h'|\mathbf{P}, m [\mu, j] (h, h')\rangle.
\]

(7.274)

As in (7.273), a sum over all \( h, h' \) is implied. The \( (LS)\)- and \( (JL)\)-states are related to each other by Racah’s recoupling coefficients.\(^{13}\)

The integral over \( \mu \) can be performed after decomposing the \( \delta \)-function that ensures the conservation of energy and momentum as

\[
\delta^{(4)}(P - p - p') = \delta^{(3)}(P - p - p') \delta\left(\mu - \sqrt{(p + p')^2}\right) \frac{P^0}{\mu}.
\]

(7.275)

A suitable normalization of the irreducible states is

\[
\langle \mathbf{P}, m [\mu, j] \eta | \mathbf{P}', m' [\mu', j'] \eta' \rangle = \delta^{(3)}(\mathbf{P} - \mathbf{P}') \delta(\mu - \mu') \delta_{j,j'} \delta_{m,m'} \delta_{\eta,\eta'}.
\]

(7.276)

For more details on this subject see Notes and References.

\(^{13}\)For the general recoupling theory of angular momenta see Chapter VI of the textbook by Edmonds, cited in Notes and References of Chapter 4.
Propagator of Free Dirac Particles

Let us now use the quantized field to calculate the propagator of the free Dirac field. Thus we form the vacuum expectation value of the time ordered product

$$S(x, x') = \langle 0 | T \psi(x) \bar{\psi}(x') | 0 \rangle. \quad (7.277)$$

For, if we use the explicit decomposition

$$T \psi(x) \bar{\psi}(x') = \Theta(x_0 - x_0') \psi(x) \bar{\psi}(x') - \Theta(x_0' - x_0) \bar{\psi}(x') \psi(x), \quad (7.278)$$

and apply the Dirac equation (4.661), we obtain

$$\left( i \gamma^\mu \partial_\mu - M \right) T \psi(x) \bar{\psi}(x') = T \left( i \gamma^\mu \partial_\mu - M \right) \psi(x) \bar{\psi}(x') + i \delta(x_0 - x_0') \left\{ \gamma_0 \psi(x), \bar{\psi}(x') \right\}. \quad (7.279)$$

The right-hand side reduces indeed to $i \delta^{(0)}(x - x')$, due to the Dirac equation (7.217) and the canonical commutation relation (7.222).

Let us now insert the free-field expansion (7.224) into (7.277) and calculate, in analogy to (7.47),

$$S_{\alpha \beta}(x, x') = S_{\alpha \beta}(x - x')$$

$$= \Theta(x_0 - x_0') \frac{1}{V} \sum_{p, s_3; p', s_3} \frac{M}{\sqrt{p^0 p'^0}} e^{ipx_0 - p'x_0'} u_\alpha(p, s_3') \bar{u}_\beta(p', s_3') \langle 0 | a_{p, s_3} a_{p', s_3}^\dagger | 0 \rangle$$

$$- \Theta(x_0' - x_0) \frac{1}{V} \sum_{p, s_3; p', s_3} \frac{M}{\sqrt{p^0 p'^0}} e^{ipx_0 - p'x_0'} v_\alpha(p, s_3) \bar{v}_\beta(p', s_3') \langle 0 | b_{p', s_3'} b_{p, s_3}^\dagger | 0 \rangle$$

$$= \Theta(x_0 - x_0') \frac{1}{V} \sum_{p, s_3} M \int e^{-ip(x-x')} \sum_{s_3} u_\alpha(p, s_3) \bar{u}_\beta(p, s_3)$$

$$- \Theta(x_0' - x_0) \frac{1}{V} \sum_{p, s_3} M \int e^{ip(x-x')} \sum_{s_3} v_\alpha(p, s_3) \bar{v}_\beta(p, s_3). \quad (7.280)$$

We now recall the polarization sums (4.702) and (4.703), and obtain

$$S_{\alpha \beta}(x - x') = \Theta(x_0 - x_0') \frac{1}{2V} \sum_p \frac{1}{p^0} e^{-ip(x-x')} (\not{p} + M)$$

$$+ \Theta(x_0' - x_0) \frac{1}{2V} \sum_p \frac{1}{p^0} e^{ip(x-x')} (-\not{p} + M). \quad (7.281)$$

Let us mention here that this type of decomposition can be found for particles of any spin. In (4.705) we introduced the polarization sum

$$P(p) = \sum_{s_3} u(p, s_3) \bar{u}(p, s_3), \quad \bar{P}(p) = \sum_{s_3} v(p, s_3) \bar{v}(p, s_3) = -P(-p). \quad (7.282)$$
In terms of these, the propagator has the general form

\[ S_{\alpha\beta}(x - x') = \Theta(x_0 - x'_0) \frac{1}{2V} \sum_p \frac{1}{p^0} e^{-ip(x-x')} P(p) \]

\[ + \Theta(x'_0 - x_0) \frac{1}{2V} \sum_p \frac{1}{p^0} e^{ip(x-x')} P(-p). \] (7.283)

This structure is found for particles of any spin, in particular also for integer-valued ones. If the statistics is chosen properly, the inverted signs in relation (4.706) between the polarization sums of particles and antiparticles is cancelled by the sign change in the definition of the time-ordered product for bosons and fermions. This will be discussed in more detail in Section 7.10.

In an infinite volume, the momentum sums are replaced by integrals and yield the invariant functions \( G^+(x - x'), G^-(x - x') \) [recall (7.48) and (7.55)]. The sum containing a momentum factor \( / p \) may be calculated by taking it outside in the form of a spacetime derivative, i.e., by writing

\[ \frac{1}{2V} \sum_p \frac{1}{p^0} e^{\mp ip(x-x')}(\pm / p + M) = (\pm i\partial / + M)G^{(\pm)}(x - x'). \] (7.284)

Note that the zeroth component inside the sum comes from the polarization sum over spinors and their energies lie all on the mass shell, so that the Green functions \( G^+(x - x') \) and \( G^-(x - x') \) contain only wave functions with on-shell energies. We therefore find the propagator

\[ S_{\alpha\beta}(x - x') = \Theta(x_0 - x'_0)(i\partial / + M)G^+(x - x') + \Theta(x'_0 - x_0)(i\partial / + M)G^-(x - x'). \] (7.285)

This expression can be simplified further by moving the derivatives to the left of the Heaviside function. This gives

\[ S_{\alpha\beta}(x - x') = (i\partial / + M) \left[ \Theta(x_0 - x'_0)G^+(x - x') + \Theta(x'_0 - x_0)G^-(x - x') \right] \]

\[ - i\gamma^0 \delta(x_0 - x'_0) \left[ G^+(x - x') - G^-(x - x') \right]. \] (7.286)

The second term happens to vanish because of the property

\[ G^{(\pm)}(x, 0) = G^{(\mp)}(x, 0). \] (7.287)

The final result is therefore the simple expression

\[ S_{\alpha\beta}(x - x') = (i\partial / + M)G(x - x'), \] (7.288)

i.e., the propagator of the Dirac field reduces to that of the scalar field multiplied by the differential operator \( i\partial / + M \). Inserting on the right-hand side the Fourier representation (7.66) of the scalar propagator, we find for Dirac particles the representation

\[ S_{\alpha\beta}(x - x') = \int \frac{d^4p}{(2\pi)^4} (\partial / + M) \frac{i}{p^2 - M^2 + i\eta} e^{-ip(x-x')}. \] (7.289)
This has to be contrasted with the Fourier representation of the intermediate expression (7.285). If the Heaviside functions are expressed as in Eq. (7.62), we see that (7.285) can be written as

\[ S_{\alpha\beta}(x - x') = \gamma^0 \int \frac{d^4p}{(2\pi)^4} \frac{i}{2\omega_p} \left( \frac{\omega_p}{p^0 - \omega_p + i\eta} - \frac{-\omega_p}{p^0 + \omega_p - i\eta} \right) e^{-ip(x-x')}, \quad (7.290) \]

The difference between (7.290) and (7.289) is an integral

\[ -\int \frac{dp^0}{2\pi} e^{-ip^0 (x^0 - x'^0)} \frac{i}{2\omega_p} \left( \frac{p^0 - \omega_p}{p^0 - \omega_p + i\eta} - \frac{p^0 + \omega_p}{p^0 + \omega_p - i\eta} \right). \quad (7.291) \]

Here the two distributions in the integrand are of the form \( x/(x \pm i\eta) \), so that they cancel each other and (7.291) vanishes.

The integrand in (7.289) can be rewritten in a more compact way using the product formula

\[ (\gamma^\mu p^\nu - M^2) = p^2 - M^2. \quad (7.292) \]

This leads to the Fourier representation of the Dirac propagator

\[ S_{\alpha\beta}(x - x') = \int \frac{d^4p}{(2\pi)^4} \frac{i}{\gamma^\mu p^\nu - M^2} e^{-ip(x-x')}. \quad (7.293) \]

It is worth noting that while in Eq. (7.281) the particle energy \( p^0 \) in \( \phi \) lies on the mass shell, being equal to \( \sqrt{p^2 + M^2} \), those in the integral (7.289) lie off-shell. They are integrated over the entire \( p^0 \)-axis and have no relation to the spatial momenta \( p \).

As in the case of the scalar fields [see Eqs. (7.364)], the free-particle propagator is equal to the Green function of the free-field equation. Indeed, by writing (4.661) as

\[ L(i\partial)\psi(x) = 0 \quad (7.294) \]

with the differential operator

\[ L(i\partial) = i\partial - M, \quad (7.295) \]

we see that the propagator is the Fourier transform of the inverse of \( L(p) \):

\[ S(x - x') = \int \frac{d^4p}{(2\pi)^4} \frac{i}{L(p)} e^{-ip(x-x')} \quad (7.296) \]

It obviously satisfies the inhomogeneous Dirac equation

\[ (i\gamma^\mu \partial_\mu - M)S(x, x') = i\delta^{(4)}(x - x'). \quad (7.297) \]

For completeness, we also write down the commutator function of Dirac fields. From the expansion (7.224) and the canonical commutation rules (7.228) and (7.229), we find directly

\[ [\psi(x), \bar{\psi}(x')] \equiv C_{\alpha\beta}(x - x') \quad (7.298) \]
with the commutator function

\[ C_{\alpha\beta}(x - x') = \frac{1}{2V} \sum_p \frac{1}{p} e^{-ip(x-x')}(\not{p} + M) - \frac{1}{2V} \sum_p \frac{1}{p} e^{ip(x-x')}(-\not{p} + M) \]

\[ = (i\not{\partial} + M)C(x - x'). \quad (7.299) \]

### 7.4.4 Behavior under Discrete Symmetries

Let us conclude this section by studying the behavior of spin-\(\frac{1}{2}\) particles under the discrete symmetries \(P, C, T\).

#### Space Inversion

The space reflection

\[ \psi(x) \xrightarrow{P} \psi'(x) = \eta_P \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \psi(\tilde{x}) = \eta_P \gamma_0 \psi(\tilde{x}) \]

is achieved by defining the unitary parity operator \(P\) on the creation and annihilation operators as follows:

\[ P a_{p,s}^\dagger P^{-1} = a\dagger(-p,s), \]

\[ P b_{p,s}^\dagger P^{-1} = b\dagger(-p,s). \quad (7.301) \]

The opposite sign in front of \(b_{p,s}^\dagger\) is necessary since the spinors behave under parity as follows:

\[ \gamma_0 u(p,s) = u(-p,s), \]

\[ \gamma_0 v(p,s) = -v(-p,s). \quad (7.302) \]

This follows directly from the explicit representation (4.674) and (4.684):

\[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{M_p} \sqrt{p^3} M \\ -\sqrt{M_p} \sqrt{p^3} M \end{pmatrix} \chi(s_3) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{M_p} \sqrt{p^3} M \\ \sqrt{M_p} \sqrt{p^3} M \end{pmatrix} \chi(s_3), \]

\[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{M_p} \sqrt{p^3} M \\ \sqrt{M_p} \sqrt{p^3} M \end{pmatrix} \chi^c(s_3) = -\frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{M_p} \sqrt{p^3} M \\ -\sqrt{M_p} \sqrt{p^3} M \end{pmatrix} \chi^c(s_3). \quad (7.303) \]

It is easy to verify that with (7.301) and (7.302), the second-quantized field \(\psi(x)\) with the expansion (7.224) transforms as it should:

\[ P\psi(x)P^{-1} = \psi_P(x) = \eta_P \gamma_0 \psi(\tilde{x}). \quad (7.304) \]

The opposite phase factors of \(a\dagger(p,s)\) and \(b\dagger(p,s)\) under space inversion imply that in contrast to two identical scalar particles the bound state of a Dirac particle

\[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]
with its antiparticle in a relative orbital angular momentum $l$ carries an extra minus sign, i.e., it has a parity

$$\eta_P = (-)^{l+1}. \quad (7.305)$$

Therefore, the ground state of a positronium atom, which is an $s$-wave bound state of an electron and a positron, represents a pseudoscalar composite particle whose spins are coupled to zero. For the two different Dirac particles, the combined parity (7.305) carries an extra factor consisting of the two individual parties: $\eta_P \eta_P$.

**Charge Conjugation**

Charge conjugation transforms particles into antiparticles. We therefore define this operation on the creation and annihilation operators of the Dirac particles by

$$Ca_{p,s_3}^\dagger C^{-1} = a_{p,s_3}^\dagger = \eta C b_{p,s_3}^\dagger,$$

$$Cb_{p,s_3}^\dagger C^{-1} = b_{p,s_3}^\dagger = \eta C a_{p,s_3}^\dagger. \quad (7.306)$$

To find out how this operation changes the Dirac field operator, we observe that the charge conjugation matrix $C$ introduced in Eq. (4.603) has the property of changing $\bar{u}^T(p, s_3)$ into $v(p, s_3)$, and $\bar{v}^T(p, s_3)$ into $u(p, s_3)$:

$$v(p, s_3) = C\bar{u}^T(p, s_3), \quad u(p, s_3) = C\bar{v}^T(p, s_3). \quad (7.307)$$

The first property was proven in (4.680). The second is proven similarly. As a consequence, the operations (7.306) have the following effect upon the second-quantized field $\psi(x)$:

$$C\psi(x)C^{-1} = \psi_C^\dagger(x) = \eta C\bar{\psi}^T(x). \quad (7.308)$$

Let us now check the transformation property of the second-quantized Dirac current $j^\mu(x)$ under charge conjugation. In the first-quantized form we have found in Eq. (4.617) that the current remains invariant. After field quantization however, there is a minus sign arising from the need to interchange the order of the Dirac field operators when bringing the transformed current $\bar{\psi}^T(x)\gamma^\mu\bar{\psi}^T(x)$ in Eq. (4.618) back to the original operator ordering $\psi(x)\gamma^\mu\psi(x)$ in the current $j^\mu(x)$. Thus we obtain the second-quantized transformation law

$$j^\mu(x) \xrightarrow{C} j^\mu(x) = -j^\mu(x), \quad (7.309)$$

rather than (4.617). This is the law listed in Table 4.12.8.

A bound state of a particle and its antiparticle such as the positronium in a relative angular momentum $l$, with the spins coupled to $S$, has the charge parity

$$\eta_C = (-1)^{l+S}. \quad (7.310)$$

In order to see this we form the state

$$|\psi_{lm}^{SS_3}\rangle = \int_0^\infty dp \ R_l(p) \int d^2\tilde{p} \ Y_{lm}(\tilde{p})a_{p,s_3}^\dagger b_{-p,s_3}^\dagger |0\rangle\langle S, S_3|s, s_3; s, s_3\rangle, \quad (7.311)$$
where \( \langle S, S_3 | s, s_3; s', s'_3 \rangle \) are the Clebsch-Gordan coefficients coupling the two spins to a total spin \( S \) with a third component \( S_3 \) [recall Table 4.2].

Under charge conjugation, the product \( a_{p,s_3}^+ b_{p,s'}^+ \) goes over into \( b_{p,s_3}^+ a_{p,s'}^+ \). In order to bring this back to the original state we have to interchange the order of the two operators and the spin indices \( s_3 \) and \( s'_3 \), and invert \( p \) into \(-p\). The first gives a minus sign, the second a \((-)^l\) sign, and the third a \((-)^{(S-2s_3)}\) sign, since \( \langle S, S_3 | s, s_3; s', s'_3 \rangle = (-)^{S-s'-s} \langle S, S_3 | s', s'_3; s, s_3 \rangle \).

(7.312)

Altogether, this gives \((-)^{l+S}\).

### Time Reversal

Under time reversal, the Dirac equation is invariant under the transformation (4.620):

\[
\psi(x) \xrightarrow{T} \psi_T^*(x) = D(T)\psi^*(x_T),
\]

(7.313)

with the \(4 \times 4\)-representation matrix \( D(T) \) of Eq. (4.629):

\[
D(T) = \eta_T C \gamma_5.
\]

(7.314)

The second-quantized version of this transformation of the Dirac field operators reads

\[
\psi(x) \xrightarrow{T} \mathcal{T}^{-1}\psi(x)\mathcal{T} = \psi_T^*(x) = D(T)\psi(-\tilde{x}).
\]

(7.315)

The antilinearity of \( \mathcal{T} \) generates the complex conjugation of the classical Dirac field in (7.313). Indeed, inserting the expansion (7.224) according to creation and annihilation operators into \( \mathcal{T}^{-1}\psi(x)\mathcal{T} \), the antilinearity leads to complex-conjugate spinor wave functions:

\[
\mathcal{T}^{-1}\psi(x)\mathcal{T} = \sum_{p,s_3} \frac{1}{\sqrt{Vp^/M}} \left[ e^{ipx} u^*(p,s_3)\mathcal{T}^{-1}a_{p,s_3} \mathcal{T} + e^{-ipx} v^*(p,s_3)\mathcal{T}^{-1}b_{p,s_3}^\dagger \mathcal{T} \right] = D(T)\psi(-\tilde{x}).
\]

(7.316)

Expanding likewise the field operator \( \psi(-\tilde{x}) \) on the right-hand side, and comparing the Fourier coefficients, we find the equations

\[
u^*(-p, s_3)\mathcal{T}^{-1}a_{-p,s_3} = D(T)u(p, s_3)\nu_{p,s_3},
\]

\[
u^*(-p, s_3)\mathcal{T}^{-1}b_{-p,s_3}^\dagger = D(T)v(p, s_3)\nu_{p,s_3}^\dagger.
\]

(7.317)

To calculate the expressions on the right-hand sides we go to the chiral representations (4.674), where

\[
D(T) = \eta_T C \gamma_5 = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix},
\]

(7.318)
and we see that
\[
C \gamma_5 \frac{1}{\sqrt{2}} \left( \sqrt{\frac{\not{p}}{M}} \right) \chi(s_3) = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{\not{p}}{M}} \right) \chi(s_3), \quad (7.319)
\]
\[
C \gamma_5 \frac{1}{\sqrt{2}} \left( -\sqrt{\frac{\not{p}}{M}} \right) \chi^c(s_3) = \frac{1}{\sqrt{2}} \left( -\sqrt{\frac{\not{p}}{M}} \right) \chi^c(s_3).
\]

We now proceed as in the treatment of Eq. (4.681), using relation (4.683) to find
\[
C \gamma_5 \frac{1}{\sqrt{2}} \left( \sqrt{\frac{\not{p}}{M}} \right) \chi(s_3) = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{\not{p}}{M}} \right)^* \chi(-s_3)(-1)^{s-s_3}, \quad (7.320)
\]
\[
C \gamma_5 \frac{1}{\sqrt{2}} \left( -\sqrt{\frac{\not{p}}{M}} \right) \chi(s_3) = \frac{1}{\sqrt{2}} \left( -\sqrt{\frac{\not{p}}{M}} \right)^* \chi(-s_3)(-1)^{s-s_3},
\]
and therefore
\[
C \gamma_5 u(p, s_3) = u^*(-p, -s_3)(-1)^{s-s_3},
\]
\[
C \gamma_5 v(p, s_3) = v^*(-p, -s_3)(-1)^{s-s_3}. \quad (7.321)
\]

Comparing now the two sides of Eqs. (7.316), we obtain the transformation laws for the creation and annihilation operators
\[
\mathcal{T} a^\dagger_{p,s_3} \mathcal{T}^{-1} \equiv a^\dagger_{-p,-s_3} = \eta \tau a^\dagger_{p,s_3} c_{s_3} = \eta \tau (-1)^{s-s_3} a^\dagger_{-p,-s_3},
\]
\[
\mathcal{T} b^\dagger_{p,s_3} \mathcal{T}^{-1} \equiv b^\dagger_{-p,-s_3} = \eta \tau b^\dagger_{p,s_3} c_{s_3} = \eta \tau (-1)^{s-s_3} b^\dagger_{-p,-s_3}. \quad (7.322)
\]

The occurrence of the $c$-matrix accounts for the fact that the time-reversed state has the opposite internal rotational motion, with the phases adjusted so that the rotation properties of the transformed state remain correct.

The operator $\mathcal{T}$ is antilinear and antiunitary. This has the effect that just as the wave functions of the Schrödinger theory, the Dirac wave function of a particle of momentum $p$
\[
f_p(x) = u(p, s_3) e^{-ipx} \quad (7.323)
\]
receives a complex conjugation under time reversal:
\[
f_{p,s_3}(x) \xrightarrow{T} f_{p,s_3}^*(x_T) = D(T) f^*_{p,s_3}(x_T) = D(T) u^*(p, s_3) e^{-ipx}. \quad (7.324)
\]
The time-reversed wave function on the right-hand side satisfies again the Dirac equation
\[
(i \not{\phi} - M) f_{p,s_3}^*(x_T) = 0. \quad (7.325)
\]
In momentum space this reads
\[
(i \not{\phi} - M) D(T) u^*(-p, s_3) = 0. \quad (7.326)
\]
It follows directly from (4.625) and a complex conjugation. A similar consideration holds for a wave function

\[ f_{p,s}(x) = \psi(p, s)e^{ipx}, \]  

(7.327)

where

\[ (\phi + M)D(T)v^*(-p, s) = 0. \]  

(7.328)

As discussed previously on p. 314, parity is maximally violated. One may therefore wonder why the neutron, which may be described by a Dirac spinor as far as its Lorentz transformation properties are concerned, has an extremely small electric dipole moment. At present, one has found only an upper bound for this:

\[ d_{el} < 16 \times 10^{-25} e \cdot cm. \]  

(7.329)

It was Landau who first pointed out that this can be understood as being a consequence of the extremely small violation of time reversal invariance. The electric dipole moment is a vector operator \( \mathbf{d} = e \mathbf{x} \) where \( \mathbf{x} \) is the distance between the positive and negative centers of charge in the particle. This operator is obviously invariant under time reversal. Now, for a neutron at rest, \( \mathbf{d} \) must be proportional to the only other vector operator available, which is the spin operator \( \mathbf{\sigma} \), i.e.,

\[ \mathbf{d} = \text{const.} \times \mathbf{\sigma}. \]  

(7.330)

But under time reversal, the right-hand side changes its sign, whereas the left-hand side does not. The constant must therefore be zero.\(^{14}\)

Time reversal invariance ensures that a two-particle scattering amplitude for the process

\[ 1 + 2 \rightarrow 3 + 4 \]  

(7.331)

is the same as for the reversed process

\[ 3 + 4 \rightarrow 1 + 2. \]  

(7.332)

This will be discussed later in Section 9.7, after having developed a scattering theory. Also other consequences of time reversal invariance can be found there.

### 7.5 Free Photon Field

Let us now discuss the quantization of the electromagnetic field. The classical Lagrangian density is, according to (4.237),

\[ \mathcal{L}(x) = \frac{1}{2}[\mathbf{E}(x)^2 - \mathbf{B}(x)^2] = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \]  

(7.333)

where \( \mathbf{E}(x) \) and \( \mathbf{B}(x) \) are electric and magnetic fields, and the Euler-Lagrange equations are

\[ \partial_\mu \frac{\partial \mathcal{L}}{\partial [\partial_\mu A^\nu]} = \frac{\partial \mathcal{L}}{\partial A^\nu}, \]  

(7.334)

i.e.,

\[ (g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu)A^\nu(x) = 0. \]  

(7.335)

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\(^{14}\)This argument is due to L.D. Landau, Nucl. Phys. 3, 127 (1957).
7.5.1 Field Quantization

The spatial components $A^i(x)$ ($i = 1, 2, 3$) have the canonical momenta

$$\pi_i(x) = \frac{\partial L(x)}{\partial \dot{A}^i(x)} = -F_{0i}(x) = -E^i(x), \quad (7.336)$$

which are just the electric field components. The zeroth component $A^0(x)$, however, poses a problem. Its canonical momentum vanishes identically

$$\pi_0(x) = \frac{\partial L(x)}{\partial \dot{A}^0(x)} = 0, \quad (7.337)$$

since the action does not depend on the time derivative of $A^0(x)$. This property of the canonical momentum is a so-called primary constraint in Dirac’s classification scheme of Hamiltonian systems.\(^{15}\) As a consequence, the Euler-Lagrange equation for $A^0(x)$:

$$\partial_i \frac{\partial L}{\partial \dot{A}^i} = \frac{\partial L}{\partial A^0}, \quad (7.338)$$

is merely an equation of constraint

$$-\nabla \cdot E(x, t) = 0, \quad (7.339)$$

which is Coulomb’s law for free fields, the first of the field equations (4.247). This is a so-called secondary constraint in Dirac’s nomenclature. In the presence of a charge density $\rho(x, t)$, the right hand side is equal to $-\rho(x, t)$.

We have encountered a vanishing field momentum before in the nonhermitian formulation of the complex nonrelativistic and scalar field theories. There, however, this was an artifact of the complex formulation of the canonical formalism [recall the remarks after Eqs. (7.5.1) and (7.220)]. In contrast to the present situation, the field was fully dynamical.

Let us return to electromagnetism and calculate the Hamiltonian density:

$$\mathcal{H} = \sum_{i=1}^{3} \pi_i \dot{A}^i - L = \frac{1}{2}(E^2 + B^2) + E \cdot \nabla A^0. \quad (7.340)$$

In the Hamiltonian $H = \int d^3x \mathcal{H}$, the last term can be integrated by parts and we find, after neglecting a surface term $\int d^3x \nabla \cdot (E A^0)(x, t)$ and enforcing Coulomb’s law (7.339), that the total energy is simply

$$H = \frac{1}{2} \int d^3x (E^2 + B^2). \quad (7.341)$$

We now attempt to convert the canonical fields $A^i(x, t)$ and $E^i(x, t)$ into field operators by enforcing canonical commutation rules. There is, however, an immediate obstacle to this caused by Coulomb’s law. Proceeding naively, we would impose the equal-time commutators

$$[A^i(x, t), A^j(x', t)] = 0,$$  \hspace{0.5cm} (7.342)

$$[\pi_i(x, t), \pi_j(x', t)] = 0,$$  \hspace{0.5cm} (7.343)

$$[\pi_i(x, t), A^j(x', t)] = - \left[ E^i(x, t), A^j(x', t) \right] \frac{\partial}{\partial x^j} \delta^{(3)}(x - x').$$  \hspace{0.5cm} (7.344)

But the third equation (7.344) is incompatible with Coulomb’s law (7.339). The reason for this is clear: We have seen above that the zeroth component of the vector potential $A^0(x, t)$ is not even an operator. As a consequence, Coulomb’s law written out as

$$\nabla^2 A^0(x, t) = - \partial^0 \nabla \cdot A(x, t),$$  \hspace{0.5cm} (7.345)

implies that $\nabla \cdot A(x)$ cannot be an operator either. As such it must commute with both canonical field operators $A^i(x, t)$ and $E^i(x, t)$. The commutation with $A^i(x, t)$ follows directly from the first relation (7.342). In order to enforce also the commutation with $E^i(x, t)$, we must correct the canonical commutation rules (7.344) between $E^i(x, t)$ and $A^j(x', t)$. We require instead that

$$[\pi_i(x, t), A^j(x', t)] = - [E^i(x, t), A^j(x', t)] = -i \delta_{ij}^T(x - x'),$$  \hspace{0.5cm} (7.346)

where $\delta_{ij}^T(x - x')$ is the transverse $\delta$-function:

$$\delta_{ij}^T(x - x') \equiv \int \frac{d^3k}{(2\pi)^3} e^{ik(x-x')} \left( \delta^{ij} - \frac{k^i k^j}{k^2} \right).$$  \hspace{0.5cm} (7.347)

This guarantees the vanishing commutation rule

$$[E^i(x, t), \nabla \cdot A(x', t)] = 0,$$  \hspace{0.5cm} (7.348)

which ensures that $\nabla \cdot A(x', t)$ is a c-number field, and so is $A^0(x', t)$ via (7.345), thus complying with Coulomb’s law (7.345).

In the second-quantized theory, the fields $A^0(x, t)$ and $\nabla \cdot A(x, t)$ play a rather inert role. Both are determined by an equation of motion. The reason for this lies in the gauge invariance of the action, which makes the theory independent of the choice of $\nabla \cdot A(x, t)$, or $A^0(x, t)$. In Section 4.6.2 we have learned that we are free to choose the Coulomb gauge in which $\nabla \cdot A(x, t)$ vanishes identically, so that by (7.345) also $A^0(x, t) \equiv 0$.

In the Coulomb gauge, the canonical field momenta of the vector potential (7.336) are simply their time derivatives:

$$\pi_i(x, t) = -E^i(x, t) = \dot{A}^i(x, t),$$  \hspace{0.5cm} (7.349)

just as for scalar fields [see (7.1)].
We are now ready to expand each component of the vector potential into plane waves, just as previously the scalar field:

\[ A^{\mu}(x) = \sum_{k,\lambda} \frac{1}{\sqrt{2Vk^{0}}} \left[ e^{-ik^{0}c^{\mu}(k,\lambda)}a_{k,\lambda} + e^{ik^{0}c^{\mu}(k,\lambda)}a_{k,\lambda}^\dagger \right]. \] (7.350)

Here \( c^{\mu}(k,\lambda) \) are four-dimensional polarization vectors (4.319), (4.331) labeled by the helicities \( \lambda = \pm 1 \). They contain the transverse spatial polarization vectors

\[ \epsilon(k, \pm 1) = \mp \frac{1}{\sqrt{2}} \begin{pmatrix} \cos \theta \cos \phi \mp i \sin \phi \\ \cos \theta \sin \phi \pm i \sin \phi \\ - \sin \theta \end{pmatrix} \] (7.351)

as \( \epsilon^{\nu}(k, \lambda) \equiv (0, \epsilon(k, \lambda)). \) The four-dimensional polarization vectors satisfy the orthonormality condition (4.337), and possess the polarization sums (4.340) and (4.345).

Let us quantize the vector field (7.350). In order to impose the canonical commutation rules upon the creation and annihilation operators, we invert the field expansion (7.350), as in the scalar equation (7.14), and find:

\[ \left\{ a_{k,\lambda}, a_{k',\lambda'}^\dagger \right\} = e^{+ik^{0}x_{0}} \frac{1}{\sqrt{2Vk^{0}}} \left\{ \epsilon^{\ast}(k, \lambda), \epsilon(k, \lambda') \right\} \int d^{3}x e^{\mp ikx} \left[ \mp i\dot{A}(x, t) + k^{0}A(x, t) \right]. \] (7.352)

Using now the commutators (7.346), we find

\[ [a_{k,\lambda}, a_{k',\lambda'}^\dagger] = 0, \quad [a_{k,\lambda}^\dagger, a_{k',\lambda'}^\dagger] = 0, \] (7.353)

and

\[ \left[ a_{k,\lambda}, a_{k',\lambda'}^\dagger \right] = \frac{\sqrt{k^{0}k'^{0}}}{2V} \frac{1}{k^{0} + k'^{0}} e^{i(k^{0} - k'^{0})x_{0}} \times \int d^{3}x d^{3}x' e^{-i(kx - k'x')} \epsilon^{\ast}_{i}(k, \lambda) \epsilon_{j}(k', \lambda') \delta_{ij}(x - x'). \] (7.354)

Inserting the finite-volume version of the Fourier representation (7.347) of the transverse \( \delta \)-function (7.347), the spatial integrations can be done with the result

\[ \epsilon^{\ast}_{i}(k, \lambda) \epsilon_{j}(k', \lambda') \delta_{ij}(k - k')/k^{2}. \]

Due to the transversality condition (4.314), this reduces to

\[ \epsilon^{\ast}_{i}(k, \lambda) \epsilon_{j}(k', \lambda') \delta^{ij} \delta_{k,k'}. \]

Using the orthonormality relation (4.333), we therefore obtain

\[ [a_{k,\lambda}, a_{k',\lambda'}^\dagger] = \delta_{\lambda\lambda'} \delta_{k,k'}. \] (7.355)

Thus we have found the usual commutation rules for the creation and annihilation operators of the two transversely polarized photons existing for each momentum \( k \).
Energy of Free Photons

We can easily express the field energy (7.341) in terms of the field operators. In the Coulomb gauge with $\nabla \cdot \mathbf{A}(x, t) = 0$, and $\mathbf{E}(x, t) = -\dot{\mathbf{A}}(x, t)$, the field energy simplifies to

$$H = \int d^3x \frac{1}{2} \left( \partial_0 A^i \partial_0 A^i + \nabla A^i \cdot \nabla A^i \right). \quad (7.356)$$

This is a sum over the field energies of the three components $A^i(x, t)$, each of them being of the same form as in Eq. (7.29) for a scalar field of zero mass. The subsequent calculation is therefore the same as the one in that equation. Inserting the field expansion (7.350), we find the Hamilton operator

$$H = \sum_{k, \lambda = \pm 1} k^0 \left( a_{k, \lambda}^\dagger a_{k, \lambda} + \frac{1}{2} \right), \quad (7.357)$$

which contains the vacuum energy

$$E_0 \equiv \langle 0 | H | 0 \rangle = \frac{1}{2} \sum_{k, \lambda = \pm 1} k^0 = \sum_k \omega_k, \quad (7.358)$$

due to the zero-point oscillations. It consists of the sum of all energies $\omega_k$.

Propagator of Free Photons

Let us now calculate the Feynman propagator of the photon field, defined by the vacuum expectation value

$$G^{\mu\nu}(x, x') \equiv \langle 0 | T A^\mu(x) A^\nu(x') | 0 \rangle. \quad (7.359)$$

Inserting the field expansion (7.350), we calculate the vacuum expectation separately for $t > t'$ and $t < t'$ as in (7.47), and find

$$G^{\mu\nu}(x, x') = \Theta(x_0 - x'_0) \frac{1}{2V} \sum_k \frac{1}{\omega_k} e^{-ik_0(x-x')} \sum_\lambda \epsilon^\mu(k, \lambda) \epsilon^{*\nu}(k, \lambda) + \Theta(x'_0 - x_0) \frac{1}{2V} \sum_k \frac{1}{\omega_k} e^{ik_0(x-x')} \sum_\lambda \epsilon^{*\mu}(k, \lambda) \epsilon^{\nu}(k, \lambda). \quad (7.360)$$

The Heaviside functions are represented as in Eqs. (7.63). The polarization sums in the two terms are real and therefore the same, both being given by the projection matrices $P_T^{\mu\nu}(k)$ or $P_T^{\mu\nu}(k)$ of Eqs. (4.340) or (4.345), respectively. Since these are independent of $k^0$, we may proceed as in the derivation of (7.64) from (7.47), thus arriving at a propagator in an infinite spatial volume:

$$G^{\mu\nu}(x, x') = \int \frac{d^4k}{(2\pi)^4} e^{-ik_0(x-x')} \frac{i}{k^2 + i\eta} P_T^{\mu\nu}(k). \quad (7.361)$$

The independence of $P_T^{\mu\nu}(k)$ on $k^0$ permits us to assume $k^0$ in the $\eta$-dependent expression (4.345) to be off mass shell, with $k^\mu$ covering the entire four-dimensional
momentum space in the integral (7.361). Alternatively, it may be assumed to be equal to the \( k \)-dependent mass-shell values \( \omega_k \), i.e., we may use the polarization sum \( P_T^{\mu \nu}(\omega_k, k) \) rather than (4.345). The latter is, however, an unconventional option, since the off-shell form is more convenient for deducing the consequences of gauge invariance in the presence of charged particles [see the discussion after (12.101)].

In any case, the free-photon propagator is a Green function of the field equations. Neglecting surface terms, we rewrite the action associated with the Lagrangian density (7.333), after a partial integration, as

\[
\mathcal{A} = \frac{1}{2} \int d^4x \, A^\mu(x) (g_{\mu \nu} \partial^2 - \partial_\mu \partial_\nu) A^\nu(x). \tag{7.362}
\]

It corresponds to a Lagrangian density

\[
\mathcal{L}(x) = \frac{1}{2} A^\mu(x) L_{\mu \nu}(i \partial) A^\nu(x), \tag{7.363}
\]

with the differential operator

\[
L_{\mu \nu}(i \partial) \equiv g_{\mu \nu} \partial^2 - \partial_\mu \partial_\nu. \tag{7.364}
\]

The Euler-Lagrange equation (7.335) can then be written as

\[
L_{\mu \nu}(i \partial) A^\nu(x) = 0. \tag{7.365}
\]

Ordinarily, we would define a Green function by the inhomogenous differential equation

\[
L_{\mu \nu}(i \partial) G^{\lambda \nu}(x - x') = i \delta_\mu^\lambda \delta^{(4)}(x - x'). \tag{7.366}
\]

Here, however, this equation cannot be solved since the Fourier transform of the differential operator \( L_{\mu \nu}(i \partial) \) is the \( 4 \times 4 \)-matrix

\[
L_{\mu \nu}(k) = -g_{\mu \nu} k^2 + k_\mu k_\nu, \tag{7.367}
\]

and this possesses an eigenvector with eigenvalue zero, the vector \( k^\nu \). Thus \( L_{\mu \nu}(k) \) cannot have an inverse, and Eq. (7.366) cannot have a solution. Instead, the nonzero spatial components of the propagator (7.361) satisfy the transverse equation

\[
L_{ij}(i \partial) G^{jk}(x - x') = -i \delta^{ik} (x - x') \delta(x^0 - x'^0), \tag{7.368}
\]

and the transversality condition

\[
\partial_i G^{ij}(x - x') = 0. \tag{7.369}
\]

Let us also calculate the commutator from the Feynman propagator according to the rules (7.212). It is most conveniently expressed in terms of the explicitly \( k^0 \)-independent projection matrix (4.340) as

\[
[A^\mu(x), A^\nu(x')] = \int \frac{d^4k}{(2\pi)^4} \epsilon(k^0) \delta(k^2) e^{-ik(x-x')} P_T^{\mu \nu}(k). \tag{7.370}
\]
7.5 Free Photon Field

Since the polarization sum depends only on the spatial momenta, it can be taken out of the integral replacing $k_i$ by $-i\partial_i$, yielding

$$[A^\mu(x), A^\nu(x')] = \left[ -g^{\mu\nu} + \left( \begin{array}{cc} 1 & 0 \\ 0 & -\partial^i \partial^j / \partial^2 \end{array} \right) \right] C(x - x'). \tag{7.371}$$

Here $C(x - x')$ is the commutator function of the scalar field defined in (7.56) with Fourier representations (7.57), (7.206), and (7.179). Taking the time derivative of one of the fields and using the property $\dot{C}(x - x') = -i\delta^{(3)}(x - x')$ for $x_0 = x'_0$ which follows from Eq. (7.58), we find the canonical equal-time commutator

$$[\dot{A}^\mu(x, t), A^\nu(x', t)] = -i \left[ -g^{\mu\nu} + \left( \begin{array}{cc} 1 & 0 \\ 0 & -\partial^i \partial^j / \partial^2 \end{array} \right) \right] \delta^{(3)}(x - x'). \tag{7.372}$$

Recall that this was the starting point (7.346) of the quantization procedure in the Coulomb gauge with $E(x, t) = -\dot{A}(x, t)$.

It is instructive to see how the quantization proceeds in the presence of a source term, where the action (7.362) reads

$$A = \frac{1}{2} \int d^4x A^\mu(x)(g_{\mu\nu}\partial^2 - \partial_\mu \partial_\nu)A^\nu(x) - \int d^4x A^\mu(x)j_\mu(x). \tag{7.373}$$

Going to the individual components $A^0$ and $A = (A^1, A^2, A^3)$ as in (4.249), this reads

$$A = \int d^4x \left\{ \frac{1}{2} \left[ A^0(x)(-\nabla^2)A^0(x) - 2A^0(x)\partial_\mu \nabla^i A^i(x) \\
- A(x)(\partial_0^2 - \nabla^2)A(x) - A^i(x)\nabla^i \nabla^i(x) \right] - \rho(x)A^0(x) + j(x)A(x) \right\}. \tag{7.374}$$

As we have observed in (4.249), the Coulomb gauge (4.274) ensures that the field equation for $A^0$ has no time derivative and can be solved by the instantaneous Coulomb potential of the charge density $\rho(x)$ by Eq. (4.273). The field equation for the spatial components, however, is hyperbolic and amounts to Ampère’s law:

$$\left( \partial_0^2 - \nabla^2 \right) A(x) = j(x). \tag{7.375}$$

This is solved in Eq. (5.21) with the help of the retarded Coulomb potential.

7.5.2 Covariant Field Quantization

There exists an alternative formalism in which the photon propagator is manifestly covariant. This can, however, be achieved only at the expense of extending the Hilbert space by a nonphysical sector from which the physical subspace is obtained by certain operator conditions. Such an approach can be followed consistently if the interaction does not mix physical states with unphysical ones. As a function of time,
physical states must remain physical, i.e., the physical subspace must be invariant under the dynamics of the system.

Gauge transformations modify only the unphysical states. It will turn out that half of them have a negative or zero norm and do not allow for a probabilistic interpretation. This, however, does not cause any problems since the physical subspace is positive-definite and dynamically invariant. On this space, the second-quantized time evolution operator $U = e^{-iHt}$ is unitary and the completeness sums between physical observables can be restricted to physical states.

The covariant quantization method is based on a modified Lagrangian in which $A^0(x)$ plays no longer a special role, so that every field component $A^\mu(x)$ possesses a canonically conjugate momentum. For this purpose one introduces an auxiliary field $D(x)$, and adds to the original Lagrangian a so-called gauge-fixing term

$$L(x) \rightarrow L'(x) = L(x) + L_{GF}(x),$$

which is defined by

$$L_{GF}(x) \equiv -D(x)\partial^\mu A_\mu(x) + \frac{\alpha}{2}D^2(x), \quad \alpha \geq 0. \tag{7.377}$$

Now, it is this auxiliary field $D(x)$, which has no canonical momentum:

$$\pi_D(x) = \frac{\partial L(x)}{\partial \dot{D}(x)} \equiv 0, \tag{7.378}$$

so that the Euler-Lagrange equation for $D(x)$ is not a proper equation of motion, but merely a constraint:

$$\alpha D(x) = \partial_\mu A^\mu(x). \tag{7.379}$$

In contrast to the earlier treatment, all four components of the vector potential $A^\mu(x)$ are here dynamical fields and obey a proper equation of motion:

$$\partial^\mu F_{\mu\nu}(x) = \partial^2 A^\mu(x) - \partial^\mu \partial_\nu A^\nu(x) = -\partial_\nu D(x). \tag{7.380}$$

Together with the constraint (7.379), we can write these field equations as

$$\partial^2 A^\mu(x) - \left(1 - \frac{1}{\alpha}\right) \partial^\mu \partial_\nu A^\nu(x) = 0. \tag{7.381}$$

From these we may derive, by one more differentiation, a field equation for $D(x)$:

$$\partial^2 D(x) = 0, \tag{7.382}$$

which shows that $D(x)$ is a massless Klein-Gordon field.

If we want to use the new extended Lagrangian for the description of electromagnetism, where the field equations are $\partial^\mu F_{\mu\nu} = 0$, we have to make sure that $D(x)$ is,

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17This modification was proposed by E. Fermi, Rev. Mod. Phys. 4, 125 (1932).
at the end, identically zero at all times. With \( D(x) \) satisfying the free Klein Gordon equation (7.382), this is guaranteed if \( D(x) \) satisfies the initial conditions

\[
D(x, t) \equiv 0, \quad t = t_0, \quad (7.383)
\]

\[
\dot{D}(x, t) \equiv 0, \quad t = t_0. \quad (7.384)
\]

In other words, if the auxiliary field is zero and does not move at some initial time \( t_0 \), for example at \( t_0 \to -\infty \), it will vanish everywhere at all times. With these initial conditions we can replace the original Lagrangian density by the extended version \( \mathcal{L}'(x) \). This has the desired property that all field components \( A^\mu(x) \) possess a proper canonically conjugate momentum:

\[
\pi_\mu(x) = \frac{\partial \mathcal{L}(x)}{\partial \dot{A}^\mu(x)} = -F_{0\mu}(x) - g_{0\mu}D(x). \quad (7.385)
\]

Thus, while the spatial field components \( A^i(x) \) have the electric fields \(-E^i(x)\) as their canonical momenta as before, the new auxiliary field \(-D(x)\) plays the role of a canonical momentum to the time component \( A^0(x) \). We can now quantize the fields \( A^\mu(x) \) and the associate field momenta \((-D(x), -E^i(x))\) by the canonical equal-time commutation rules:

\[
\begin{align*}
[A^\mu(x, t), A^{\prime \mu}(x', t)] &= 0, & [E^i(x, t), A^{\prime i}(x', t)] &= i\delta^{ij}\delta^{(3)}(x - x'), \\
[D(x, t), A^i(x', t)] &= 0, & [D(x, t), A^0(x', t)] &= i\delta^{(3)}(x - x'), \\
[E^i(x, t), E^{\prime i}(x', t)] &= 0, & [D(x, t), E^{\prime i}(x', t)] &= 0, \\
[E^i(x, t), A^0(x', t)] &= 0, & [D(x, t), D(x', t)] &= 0. \quad (7.386)
\end{align*}
\]

The Hamiltonian density associated with the Lagrangian density \( \mathcal{L}'(x) \) is

\[
\mathcal{H}' = \pi_\mu \dot{A}^\mu - \mathcal{L} = -E \cdot \dot{A} - D \dot{A}^0 - \frac{1}{2}(E^2 - B^2) + D \partial^\mu A_\mu - \frac{\alpha}{2} D^2. \quad (7.387)
\]

Expressing the electric field in terms of the vector potential, \( E^i = -F^{0i} = -\dot{A}^i - \partial_i A^0 \), this becomes

\[
\mathcal{H}' = \frac{1}{2} (E^2 + B^2) + E \cdot \nabla A^0 + D \nabla \cdot A - \frac{\alpha}{2} D^2. \quad (7.388)
\]

For a vanishing field \( D(x) \), this is the same Hamiltonian density as before in (7.340).

Let us now quantize this theory in terms of particle creation and annihilation operators.

**Feynman Gauge \( \alpha = 1 \)**

The field equations (7.381) become simplest if we take the special case \( \alpha = 1 \) called the *Feynman gauge*. Then (7.381) reduces to

\[
-\partial^2 A^\mu(x) = 0, \quad (7.389)
\]
so that the four components $A^\mu(x)$ simply obey four massless Klein-Gordon equations. The fields can then be expanded into plane-wave solutions as

$$A^\mu(x) = \sum_{k,\nu=0}^{3} \frac{1}{2V k^0} \left[ e^{-ikx} \epsilon^\mu(\nu) a_{k,\nu} + e^{ikx} \epsilon^\mu(\nu) a^\dagger_{k,\nu} \right],$$  \hspace{1cm} (7.390)

where $\epsilon^\mu(\nu)$ are now four momentum-independent polarization vectors

$$\epsilon^\mu(\nu) = g_{\mu\nu},$$  \hspace{1cm} (7.391)

satisfying the orthogonality and completeness relations

$$\epsilon^\mu(\nu) \epsilon^\mu(\nu') = g_{\nu\nu'},$$  \hspace{1cm} (7.392)

$$\sum_{\nu' \nu} g_{\nu'\nu} \epsilon^\mu(\nu') \epsilon^\mu(\nu') = g_{\mu\nu'}.$$  \hspace{1cm} (7.393)

Inverting the field expansion, we obtain

$$\left\{ \begin{array}{l} a_{k,\nu} \\ a^\dagger_{k,\nu} \end{array} \right\} = e^{\pm ik^0x^0} \sum_{\nu'=0}^{3} g^{\nu\nu'} \frac{1}{\sqrt{2V k^0}} \left\{ \epsilon^\dagger_{\nu'}(\nu') \right\} \int d^3x e^{\pm i kx} \left[ \pm i \dot{A}^\mu(x) + k^0 A^\mu(x) \right].$$  \hspace{1cm} (7.394)

Here we use the canonical commutation rules. From Eqs. (7.386) we see that $A^\mu(x, t)$ and $A^\nu(x', t)$ commute with each other at equal times:

$$[A^\mu(x, t), A^\nu(x', t)] = 0.$$  \hspace{1cm} (7.395)

Expressing the canonical momentum (7.385) in terms of the vector potential, $\pi_i(x) = -F^i_\nu(x) = -E^i(x) = \dot{A}^i(x) + \partial_i A^0(x)$, the canonical commutator $[\pi_i(x, t), A^j(x', t)] = -i\delta^{ij} \delta^{(3)}(x - x')$ implies

$$[\dot{A}^i(x, t), A^j(x', t)] = -i\delta^{ij} \delta^{(3)}(x - x') - \partial_i[A^0(x, t), A^j(x', t)].$$  \hspace{1cm} (7.396)

Now we use (7.395) to see that

$$[\dot{A}^0(x, t), A^j(x', t)] = -i\delta^{ij} \delta^{(3)}(x - x').$$  \hspace{1cm} (7.397)

In contrast to (7.346), the right-hand side is not restricted to the transverse $\delta$-function.

To find the commutation relations for $A^0(x, t)$, we take the canonical commutator

$$[D(x, t), A^0(x', t)] = i\delta^{(3)}(x - x')$$  \hspace{1cm} (7.398)

and express $D$ in the Feynman gauge as $\partial^\mu A_\mu = \partial_0 A^0 + \partial_i A^i$ [recalling (7.379)]. The commutators of $A^i(x)$ with $A^0(x)$ vanish so that (7.398) amounts to

$$[\dot{A}^0(x, t), A^0(x', t)] = i\delta^{(3)}(x - x').$$  \hspace{1cm} (7.399)
We now use the fact that the commutators of \( D(x, t) \) with the spatial components \( A^i(x', t) \) vanish, due to the third of the canonical commutation rules (7.386). Then the identity (7.379) leads to
\[
[\dot{A}^0(x, t), A^i(x', t)] = 0. 
\] (7.400)

Similarly we derive
\[
[\dot{A}^i(x, t), A^0(x', t)] = 0. \] (7.401)

This follows from the canonical commutator \([E^i(x, t), A^0(x', t)] = 0\) after expressing again \( E^i = -\dot{A}^i + \partial^\mu A^\mu \), or more directly by differentiating the commutator \([A^i(x, t), A^0(x', t)] = 0\) with respect to \( t \).

For the commutators between the time derivatives of the fields, we find similarly:
\[
[\dot{A}^i(x, t), \dot{A}^0(x', t)] = [E^i, E^0] + \{[E^i, \partial_j A^0] - (ij)\} + [\partial_i A^0, \partial_j A^0] = 0,
\]
\[
[\dot{A}^0(x, t), \dot{A}^i(x', t)] = [D - \partial_j A^i, -E^0 - \partial_i A^0] = 0,
\]
\[
[\dot{A}^0(x, t), \dot{A}^0(x', t)] = [D - \partial_j A^i, D - \partial_j A^i] = 0. \] (7.402)

On the right-hand sides, the equal-time arguments \( x, t \) and \( x', t \) have been omitted for brevity.

Summarizing, the commutators between the field components \( A^\mu(x', t) \) and their time derivatives are given by the manifestly covariant expressions
\[
[\dot{A}^\mu(x, t), A^\nu(x', t)] = ig^\mu\nu \delta^{(3)}(x - x'),
\]
\[
[A^\mu(x, t), A^\nu(x', t)] = 0,
\]
\[
[\dot{A}^\mu(x, t), \dot{A}^\nu(x', t)] = 0. \] (7.403)

They have the same form as if \( A^\mu(x, t) \) were four independent Klein-Gordon fields, except for the fact that the sign in the commutator between the temporal components of \( \dot{A}^\mu(x, t) \) is opposite to that between the spatial components. This will be the source of considerable complications in the subsequent discussion.

Using the covariant commutation rules (7.403), we find from (7.394) the commutation rules for the creation and annihilation operators:
\[
[a_{k,\nu}, a_{k',\nu}^\dagger] = [a_{k,\nu}^\dagger, a_{k',\nu}^\dagger] = 0,
\]
\[
[a_{k,\nu}, a_{k',\nu}^\dagger] = -\delta_{k,k'} g_{\nu\nu'}. \] (7.404)

It is useful to introduce the contravariant creation and annihilation operators
\[
a_{\mu k}^\dagger \equiv \sum_{\nu=0}^{3} e^\nu(\nu)a_{k,\nu}^\dagger, \quad a_{\mu k} \equiv \sum_{\nu=0}^{3} e^{\nu}(\nu)a_{k,\nu}. \] (7.405)

They satisfy the commutation rules
\[
[a_{\mu k}, a_{\nu k'}^\dagger] = [a_{\mu k}^\dagger, a_{\nu k'}^\dagger] = 0,
\]
\[
[a_{\mu k}, a_{\nu k'}^\dagger] = -\delta_{k,k'} g^{\mu\nu}. \] (7.406)
The opposite sign of the commutator $[\hat{A}^0(x, t), \hat{A}^0(x', t)]$ shows up in the commutators (7.404) and (7.406) between creation and annihilation operators with the polarization labels $\mu = 0$ or $\nu = 0$. This has the unpleasant consequence that the states created by the operators $a_{k_\mu}^0 = a_{k,0}$ have a negative norm:

$$\langle 0 | a_{k_\mu}^0 a_{k_\nu}^{0\dagger} | 0 \rangle \equiv \langle 0 | [a_{k_\mu}^0, a_{k_\nu}^{0\dagger}] | 0 \rangle = -\langle 0 | 0 \rangle = -1. \quad (7.407)$$

Such states cannot be physical. By applying any odd number of these creation operators to the vacuum, one obtains an infinite set of unphysical states.

Another infinite set of unphysical states is

$$|0\rangle_n \equiv (a_{k_\mu}^{0\dagger} \pm a_{k_\nu}^{3\dagger})^n |0\rangle, \quad n > 1. \quad (7.408)$$

These have a vanishing norm, as follows from the fact that $a_{k_\mu}^{0\dagger} \pm a_{k_\nu}^{3\dagger}$ commutes with its Hermitian conjugate. The label $3$ can of course be exchanged by $1$ or $2$.

**Fermi-Dirac Subsidiary Condition**

At first sight, this seems to make the usual probabilistic interpretation of quantum mechanical amplitudes impossible. However, as announced in the beginning, this problem can be circumvented. According to Eqs. (7.380)–(7.384), the classical equations of motion are satisfied correctly only if the auxiliary field $D(x)$ vanishes at some initial time, together with its velocity $\dot{D}(x)$. In the quantized theory we have to postulate an equivalent operator property. Exactly the same condition cannot be imposed upon the second-quantized field operator version of $D(x)$. That would be too stringent, since it would contradict the canonical quantization rule

$$[D(x, t), \hat{A}^0(x', t)] = i\delta^{(3)}(x - x'). \quad (7.409)$$

We must require that the equation $D(x, t) = 0$ is true only when applied to physical states. Thus we postulate that only those states in the Hilbert space are physical which satisfy the subsidiary conditions

$$D(x, t) |\psi_{phys}\rangle = 0, \quad \dot{D}(x, t) |\psi_{phys}\rangle = 0 \quad (7.410)$$

at some fixed initial time $t$. Because of the equation of motion (7.382) for $D(x)$, this is equivalent to requiring at all times:

$$D(x, t) |\psi_{phys}\rangle \equiv 0. \quad (7.411)$$

This is called the *Fermi-Dirac condition*.\(^{18}\)

In order to discuss its consequences it is useful to introduce the “vectors”

$$\hat{k}_\mu \equiv k_\mu / k^0, \quad \hat{k}_\nu \equiv \hat{k}_\mu / 2k^0, \quad \text{with} \quad k \equiv (k_0, k), \quad \tilde{k} \equiv (k^0, -k), \quad (7.412)$$

with \( k^0 = |k| \) on the light cone. The quotation marks are used in the same sense as in Section 4.9.6, where we introduced the “vector” (4.420). Due to the normalization factors, both \( \hat{k}_\mu^s \) and \( \hat{k}_\mu^\bar{s} \) do not transform like vectors under Lorentz transformations. Nevertheless, they do have Lorentz-invariant scalar products when multiplied with each other:

\[
\hat{k}_s \hat{k}_s = 0, \quad \hat{k}_\bar{s} \hat{k}_\bar{s} = 0, \quad \hat{k}_s \hat{k}_\bar{s} = 1.
\] (7.413)

By analogy with Eqs. (7.619) and (4.420), we shall span the four-dimensional vector space here by supplementing the transverse polarization vectors (7.619), (4.420) by the four-component objects

\[
\epsilon^\mu(k, s) = \hat{k}_\mu^s, \quad \epsilon^\mu(k, \bar{s}) = \hat{k}_\mu^{\bar{s}}.
\] (7.414)

These agree with the earlier-introduced scalar and antiscalar polarization vectors (7.619) and (4.420) in Section 4.9.6, except for a different normalization which will be more convenient in the sequel. We shall use the same symbols for these new objects. Since there is only little danger of confusion, this will help avoiding a proliferation of symbols. Together with the transverse polarization vectors (4.331), the four “vectors” satisfy the orthogonality relation

\[
\sum_{\sigma = \pm 1, s, \bar{s}} \epsilon^\mu(k, \sigma) \epsilon^{\mu*}(k, \sigma') g^{\sigma \sigma'} = \delta^{\mu \nu}.
\] (7.415)

where the index \( \sigma \) runs through \(+1, -1, s, \bar{s}\), and \( g^{\sigma \sigma'} \) is the metric

\[
g^{\sigma \sigma'} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}_{\sigma \sigma'}
\] (7.416)

The completeness relation reads, in a slight modification of (4.421),

\[
\sum_{\sigma = \pm 1, s, \bar{s}} \epsilon^\mu(k, \sigma) \epsilon^{\nu*}(k, \sigma') g^{\sigma \sigma'} = \delta^{\mu \nu}.
\] (7.417)

Multiplying the field expansion (7.390) with the left-hand side of this relation, and defining creation and annihilation operators with polarization label \( \sigma = \pm 1, s, \bar{s} \) by

\[
a_{k, \sigma} = \epsilon_\sigma^*(\sigma) \sum_{\nu=0}^3 \epsilon_\mu(\nu)a_{k, \nu}, \quad a_{k, \sigma}^\dagger = \epsilon_\mu(\sigma) \sum_{\nu=0}^3 \epsilon_\mu(\nu)a_{k, \nu}^\dagger,
\] (7.418)

we obtain the new field expansion

\[
A^\mu(x) = \sum_k \frac{1}{2Vk^0} \sum_{\sigma = \pm 1, s, \bar{s}} \left[ e^{-ikx} \epsilon^\mu(k, \sigma)a_{k, \sigma} + e^{ikx} \epsilon^{\mu*}(k, \sigma)a_{k, \sigma}^\dagger \right].
\] (7.419)

This expansion may be inverted by equations like (7.394), expressing the new creation and annihilation operators \( a_{k, \sigma}^\dagger \) and \( a_{k, \sigma} \) in terms of the fields \( A^\mu(x) \) and
\(\hat{A}^\mu(x)\). The result looks precisely the same as in (7.394), except that the metric \(g^{\mu\nu}\) is replaced by \(g^{\sigma\sigma'}\) of (7.416), and the polarization vectors \(\epsilon^\mu(\nu')\) by \(\epsilon^\mu(k, \sigma')\).

For the two transverse polarization labels \(\sigma = \pm 1\), the new creation and annihilation operators coincide with those introduced during the earlier noncovariant quantization in expansion (7.350), i.e., the operators \(a^\dagger_{k,\sigma}\) are equal to \(a^\dagger_{k,\lambda}\) for polarization labels \(\sigma = \lambda = \pm 1\). In addition, the new expansion (7.419) contains the creation and annihilation operators:

\[
a_{k,s} = \hat{k}_s^\mu \sum_{\nu=0}^3 \epsilon_{\mu}(\nu)a_{k,\nu}, \quad a^\dagger_{k,s} = \hat{k}_s^\mu \sum_{\nu=0}^3 \epsilon^*_{\mu}(\nu)a^\dagger_{k,\nu}, \quad (7.420)
\]

\[
a_{k,s} = \hat{k}_s^\mu \sum_{\nu=0}^3 \epsilon_{\mu}(\nu)a_{k,\nu}, \quad a^\dagger_{k,s} = \hat{k}_s^\mu \sum_{\nu=0}^3 \epsilon^*_{\mu}(\nu)a^\dagger_{k,\nu}. \quad (7.421)
\]

For the spatial momenta \(k\) pointing in the z-direction, these are

\[
a_{k,s} = a^0_k - a^3_k, \quad a^\dagger_{k,s} = a^0_k - a^3_k, \quad a^\dagger_{k,s} = a^0_k + a^3_k. \quad (7.422)
\]

Note that the previous physical polarization sum (7.403) may be written as

\[
P^\mu_\text{phys}(k) = \sum_{\sigma=\pm 1} \epsilon^\mu(k, \sigma)\epsilon^\nu(k, \sigma) = -g^{\mu\nu} + \hat{k}^\mu_s k^\nu_s + \hat{k}^\nu_s k^\mu_s. \quad (7.423)
\]

The commutation rules for the new creation and annihilation operators are

\[
[a_{k,\sigma}, a_{k',\sigma'}] = [a^\dagger_{k,\sigma}, a^\dagger_{k',\sigma'}] = 0,
[a_{k,\sigma}, a^\dagger_{k,\sigma'}] = -g_{\sigma\sigma'}\delta_{k,k'}. \quad (7.424)
\]

By Fourier-transforming the field \(D(x)\), we see that the Fermi-Dirac condition can be rewritten as

\[
a_{k,s}|\psi_\text{phys}\rangle = 0, \quad (7.425)
\]
\[
a^\dagger_{k,s}|\psi_\text{phys}\rangle = 0, \quad (7.426)
\]

i.e., the physical vacuum is annihilated by both the scalar creation and by the annihilation operators.

Let us calculate the energy of the free-photon system. For this we integrate the Hamiltonian density (7.388) over all space, insert \(D(x) = A^0(x) + \partial_\mu A^\mu(x), \quad E(x) = - - \hat{A}^i - \nabla A^0, \quad B(x) = \nabla \times A(x)\), and perform some partial integrations, neglecting surface terms, to obtain for \(\alpha = 1\) the simple expression

\[
H' = \int d^3x \frac{1}{2} (-\hat{A}^\mu \hat{A}_\mu - \nabla A^\mu \nabla A_\mu). \quad (7.427)
\]

This is precisely the sum of four independent Klein-Gordon energies [compare with (7.30) and contrast this with (7.356)], one for each spacetime component \(A^\mu(x)\).
Due to relativistic invariance, however, the energy of the component $A^0(x)$ has an opposite sign which is related to the above-observed opposite sign in the commutation relations (7.404) and (7.406) for the $\nu = 0$ and $\mu = 0$ polarizations of the creation and annihilation operators.

Inserting now the expansion (7.390) of the field $A^\mu(x)$ in terms of creation and annihilation operators, we obtain by the same calculation which led to Eqs. (7.87) and (7.357):

$$H' = \sum_k \frac{k^0}{2} \left[ -a^\dagger_{k,\nu} a_{k,\nu'} - a_{k,\nu} a^\dagger_{k,\nu'} \right] g^{\nu\nu'} = \sum_k \frac{k^0}{2} \left[ -a^\dagger_{k,\mu} a_k^\mu - a_k^\mu a^\dagger_{k,\mu} \right].$$  \hspace{1cm} (7.428)

This operator corresponds to an infinite set of four-dimensional oscillators, one for every spatial momentum and polarization state. Due to the indefinite metric $g^{\nu\nu'}$, the Hilbert space is not of the usual type, and the subsidiary conditions (7.425) and (7.426) are necessary to extract physically consistent results.

The expansion (7.419) in terms of transverse, scalar, and longitudinal creation and annihilation operators yields the Hamiltonian (7.428) in the form

$$H' = \sum_k \frac{k^0}{2} \left[ \sum_{\sigma = \pm 1} \left( a^\dagger_{k,\sigma} a_{k,\sigma} + a_{k,\sigma} a^\dagger_{k,\sigma} \right) - a_{k,s} a^\dagger_{k,s} - a_{k,s} a^\dagger_{k,s} - a_{k,s} a^\dagger_{k,s} - a_{k,s} a^\dagger_{k,s} \right].$$  \hspace{1cm} (7.429)

Let us bring this operator to normal order, i.e., to normal order with respect to the physical vacuum. For the transverse modes we move the annihilation operators to the right of the creation operators, as usual. For the modes $s$ and $l$, the physical normal order has the operators $a_{k,s}$ and $a^\dagger_{k,s}$ on the right. This allows immediate use of the subsidiary conditions (7.425) and (7.426) for the physical vacuum. Using the two commutation rules

$$[a_{k,s}, a^\dagger_{k',s}] = -\delta_{k,k'},$$

$$[a^\dagger_{k,s}, a_{k',s}] = \delta_{k,k'},$$  \hspace{1cm} (7.430)

the Hamiltonian takes the form:

$$H' = \sum_k \frac{k^0}{2} \left[ \sum_{\sigma = \pm 1} \left( a^\dagger_{k,\sigma} a_{k,\sigma} + \frac{1}{2} \right) - a_{k,s} a^\dagger_{k,s} - a_{k,s} a^\dagger_{k,s} \right].$$  \hspace{1cm} (7.431)

Note that the ordering of the $s$ and $l$ components produces no constant term due to the opposite signs on the right-hand sides of the commutation rules (7.430). The Dirac conditions (7.425) and (7.426) have the consequence that the last two terms vanish for all physical states. Thus the Hamilton operator contains only the energies of transverse photons and can be replaced by

$$H'_{\text{phys}} = \sum_k \frac{k^0}{2} \sum_{\sigma = \pm 1} \left( a^\dagger_{k,\sigma} a_{k,\sigma} + a_{k,\sigma} a^\dagger_{k,\sigma} \right).$$  \hspace{1cm} (7.432)
Finally, we bring also the transverse creation and annihilation operators to a normal order to obtain

\[ H'_{\text{phys}} = \sum_k k^0 \sum_{\sigma = \pm 1} \left( a^\dagger_{k,\sigma} a_{k,\sigma} + \frac{1}{2} \right). \]  

(7.433)

The second term gives the vacuum energy:

\[ E_0 \equiv \langle 0 | H' | 0 \rangle = \frac{1}{2} \sum_k k^0 = \sum_k \omega_k. \]  

(7.434)

In contrast to all other particles, this divergent expression has immediate experimental consequences in the laboratory — it is observable as the so-called Casimir effect, to be discussed in Section 7.12.

The photon number operator is obtained by dropping the factor \( k^0/2 \) in the normally ordered part of the Hamiltonian (7.429):

\[ N = \sum_k \left[ \left( \sum_{\sigma = \pm 1} a^\dagger_{k,\sigma} a_{k,\sigma} \right) - a^\dagger_{k,\bar{s}} a_{k,\bar{s}} - a^\dagger_{k,s} a_{k,s} \right]. \]  

(7.435)

With the Dirac conditions (7.425) and (7.426), the last two terms can be omitted between physical states, so that only the transverse photons are counted:

\[ N = \sum_k \sum_{\sigma = \pm 1} \left( a^\dagger_{k,\sigma} a_{k,\sigma} \right). \]  

(7.436)

Its action on the physical subspace is completely analogous to that of \( H' \).

**Physical Hilbert Space**

In the above calculations of the energies we have assumed that the physical vacuum \( |0_{\text{phys}}\rangle \) has a unit norm. This, however, cannot be true. A simple argument shows that the Fermi-Dirac condition (7.411) is inconsistent with the canonical commutation rule (7.398). For this one considers the diagonal element

\[ \langle 0_{\text{phys}} | [D(x, t), A^0(x', t)] | 0_{\text{phys}} \rangle = i \delta^{(3)}(x - x') \langle 0_{\text{phys}} || 0_{\text{phys}} \rangle. \]  

(7.437)

Writing out the commutator and taking \( D(x, t) \) out of the expectation values, the left-hand side apparently vanishes whereas the right-hand side is obviously nonzero. However, this problem is not fatal for the quantization approach. It occurs and can be solved in ordinary quantum mechanics. Consider the canonical commutator \([p, x] = -i\) between localized states \( |x\rangle\):

\[ \langle x | [p, x] | x \rangle_{\text{not}} = -i \langle x | x \rangle. \]  

(7.438)

The left-hand side gives zero, the right-hand side infinity, so the Heisenberg uncertainty principle seems violated. The puzzle is resolved by the fact that Eq. (7.438)
is meaningless since the states \(|x\rangle\) are not normalizable. They satisfy the orthogonality condition with a \(\delta\)-function \(\langle x'|x\rangle = \delta(x' - x)\), in which the right-hand side is a distribution. The equation must be multiplied by a smooth test function of \(x\) or \(x'\) to turn into a finite equation, as in the treatment of Eq. (7.43). Similarly, we can derive from the canonical commutator the equation

\[
\langle x'|[p,x]|x\rangle = -i\langle x'|x\rangle.
\]  

(7.439)

This is zero for \(x \neq x'\), and gives correct finite results after such a smearing procedure.

Another way to escape the contradiction is by abandoning the use of completely localized states \(|x\rangle\) and working only with approximately localized states, for example the lattice states \(|x^n\rangle\) with the proper orthogonality relations (1.138) and completeness relations (1.143). Another possibility is to use a sequence of wave packets in the form of narrow Gaussians

\[
\langle x'|x\rangle^\eta = \frac{1}{\sqrt{\pi \eta}}e^{-(x' - x)^2/2\eta},
\]  

(7.440)

which becomes more and more localized for \(\eta \to 0\).

The same subtleties exist for the commutation rule (7.398). The physical states in the Hilbert space are improper states. To see how this comes about let us go to the position representation of the Hamilton operator (7.428) which reads

\[
H = \sum_k \frac{k^0}{2} \left[ \left( \frac{\partial}{\partial \chi_k} \right)^2 - \chi_k^2 \right],
\]  

(7.441)

where

\[
\chi_k^2 = \chi_k^{02} - \chi_k^{12} - \chi_k^{22} - \chi_k^{32}.
\]  

(7.442)

The ground state is given by the product of harmonic wave functions

\[
\langle \chi|0\rangle = \prod_k \frac{1}{\pi^{1/4}}e^{\chi_k^2/2}.
\]  

(7.443)

The excited states are obtained by applying to this the creation operators

\[
d_{k\mu}^\dagger = \frac{1}{\sqrt{2}} \left( \chi_k^\mu + g^{\mu\nu} \frac{\partial}{\partial \chi_k^\nu} \right).
\]  

(7.444)

Due to the positive sign of \(\chi_k^{00}\) in the ground state (7.443), this procedure does not produce a traditional oscillator Hilbert space with the scalar product

\[
\langle \psi'|\psi\rangle = \prod_{k,\mu} \int d\chi_k^\mu \langle \psi'|\chi\rangle \langle \chi|\psi\rangle.
\]  

(7.445)

However, this is not a serious problem, since it can be resolved by rotating the contour of integration in \(\chi_k^0\) in the complex \(\chi_k^0\)-plane by 90°, so that it runs along the imaginary \(\chi_k^0\)-axis.
We now impose the subsidiary conditions (7.425) and (7.426) upon this Hilbert space. To simplify the discussion we shall consider only the problematic modes with polarization labels 0 and 3 at a fixed momentum $k$ in the $z$-direction, so that the momentum labels can be suppressed.

Then the conditions (7.425) and (7.426) take the simple form

\[ (a^0 - a^3)\langle \psi_{\text{phys}} \rangle = 0 \] (7.446)

and

\[ (a^{0\dagger} - a^{3\dagger})\langle \psi_{\text{phys}} \rangle = 0. \] (7.447)

In the position representation, these conditions amount to

\[ \left( \frac{\partial}{\partial \chi^0} + \frac{\partial}{\partial \chi^3} \right) \langle \chi | \psi_{\text{phys}} \rangle = 0; \] (7.448)

\[ (\chi^0 - \chi^3)\langle \chi | \psi_{\text{phys}} \rangle = 0. \] (7.449)

The first condition is fulfilled by wave functions which do not depend on $\chi^0 + \chi^3$, the second restricts the dependence on $\chi^0 - \chi^3$ to a $\delta$-function. The wave function of the physical vacuum is therefore (for particles of a fixed momentum running along the $z$-axis)

\[ \langle \chi | 0_{\text{phys}} \rangle = \delta(\chi^0 - \chi^3)\frac{1}{\pi^{1/4}}e^{-\left(\chi^1 + \chi^2\right)^2/2}. \] (7.450)

The complete set of physical states is obtained by applying to this any number of creation operators for transverse photons

\[ a^{1\dagger} = \frac{1}{\sqrt{2}} \left( \chi^1 - \frac{\partial}{\partial \chi^1} \right), \quad a^{2\dagger} = \frac{1}{\sqrt{2}} \left( \chi^2 - \frac{\partial}{\partial \chi^2} \right). \] (7.451)

The physical vacuum (7.450) displays precisely the unpleasant feature which caused the problems with the quantum mechanical equation (7.438). Due to the presence of the $\delta$-functions the wave function is not normalizable. The normalization problem of the wave function $\delta(\chi^0 - \chi^3)$ is completely analogous to that of ordinary plane waves, and we know how to do quantum mechanics with such generalized states. The standard Hilbert space is obtained by superimposing plane waves to wave packets. Normalizable wave packets can be constructed to approximate the positional wave functions $\delta(\chi^0 - \chi^0)$ in various ways. One may, for instance, discretize the field variables $\chi^\mu$ and work on a lattice in field space. Alternatively one may replace the $\delta$-function in (7.450) by a narrow Gaussian

\[ \delta(\chi^0 - \chi^3) \rightarrow \text{const} \times e^{-\left(\chi^0 - \chi^3\right)^2/2\eta}, \] (7.452)

with an appropriate normalization factor.

When performing calculations in the physical Hilbert space of the Dirac quantization scheme, we would like to proceed as in the case of Klein-Gordon and Dirac particles, assuming that the scalar product $\langle 0_{\text{phys}} | 0_{\text{phys}} \rangle$ is unity, as we did when evaluating the vacuum energy (7.434) (which was really illegal as we have just learned).
Only with a unit norm of the vacuum state can we find physical expectation values by simply bringing all creation and annihilation operators between the vacuum states to normal order, the desired result being the sum of the \(c\)-numbers produced by the commutators. Such a unit norm is achieved by introducing the analogs of the wave packets (7.440), which in this case are normalizable would-be vacuum states to be denoted by \(|0_{\text{phys}}\rangle\eta\) which approach the true unnormalizable vacuum state arbitrarily close for \(\eta \rightarrow 0\). These would-be vacuum states are most easily constructed algebraically.

Considering only the problematic creation and annihilation operators \(a^0\dagger\), \(a^0\) and \(a^3\dagger\), \(a^3\), we define an auxiliary state \(|0_{\text{aux}}\rangle\) to be the one that is annihilated by \(a^0\dagger\) and \(a^3\):

\[
a^0\dagger|0_{\text{aux}}\rangle = 0, \quad a^3|0_{\text{aux}}\rangle = 0. \tag{7.453}
\]

From this we construct the physical vacuum state as a power series

\[
|0_{\text{phys}}\rangle = \sum_{n,m=0}^{\infty} c_{n,m}(a^3\dagger)^n(a^0)^m|0_{\text{aux}}\rangle. \tag{7.454}
\]

The subsidiary conditions (7.425) and (7.426) are now

\[
a_s|0_{\text{phys}}\rangle = (a^0 - a^3)|0_{\text{phys}}\rangle = 0, \quad a_s\dagger|0_{\text{phys}}\rangle = (a^{0\dagger} - a^{3\dagger})|0_{\text{phys}}\rangle = 0. \tag{7.455}
\]

These are obviously fulfilled by the coherent state:

\[
|0_{\text{phys}}\rangle = e^{a^3\dagger a^0}|0_{\text{aux}}\rangle. \tag{7.456}
\]

To verify this we merely note that \(a^3\) acts on this state like a differential operator \(\partial/\partial a^3\dagger\) producing a factor \(a^0\), and the same holds with \(a^3\dagger\) and \(a^3\) replaced by \(a^{0\dagger}\) and \(a^0\), respectively.

The norm of this state is infinite:

\[
\langle 0_{\text{phys}}|0_{\text{phys}}\rangle = \sum_{n,m=0}^{\infty} \frac{1}{nm!}\langle 0_{\text{aux}}|(a^3)^n(a^{0\dagger})^m(a^{3\dagger})^m(a^0)^m|0_{\text{aux}}\rangle = \sum_{n=0}^{\infty} 1 = \infty. \tag{7.457}
\]

We now approximate this physical vacuum by a sequence of normalized states \(|0_{\text{phys}}\rangle^\eta\) defined by

\[
|0_{\text{phys}}\rangle^\eta \equiv \sqrt{\eta(2 - \eta)} e^{(1 - \eta)a^3\dagger a^0}\sqrt{\eta(2 - \eta)} \sum_{n=0}^{\infty} \frac{1}{n!}(1 - \eta)(a^3\dagger a^0)^n|0_{\text{aux}}\rangle. \tag{7.458}
\]

It is easy to check that these have a unit norm:

\[
\eta\langle 0_{\text{phys}}|0_{\text{phys}}\rangle^\eta = \eta(2 - \eta) \sum_{n=0}^{\infty} (1 - \eta)^{2n} = 1. \tag{7.459}
\]

On this sequence of states, the subsidiary conditions (7.455) are not exactly satisfied. Using the above derivative rules, we see that

\[ a^3 |0_{\text{phys}}\rangle^\eta = \frac{\partial}{\partial a^3} |0_{\text{phys}}\rangle^\eta = (1 - \eta) a^0 |0_{\text{phys}}\rangle^\eta, \]
\[ a^0 |0_{\text{phys}}\rangle^\eta = \frac{\partial}{\partial a^0} |0_{\text{phys}}\rangle^\eta = (1 - \eta) a^{3\dagger} |0_{\text{phys}}\rangle^\eta, \] (7.460)
corresponding to the approximate subsidiary conditions

\[ a_s |0_{\text{phys}}\rangle^\eta = \eta a^0 |0_{\text{phys}}\rangle^\eta, \]
\[ a_s^\dagger |0_{\text{phys}}\rangle^\eta = \eta a^{3\dagger} |0_{\text{phys}}\rangle^\eta. \] (7.461)

Let us calculate the norm of the states on the right-hand side. Using the power series expansion (7.458), we find

\[ \eta \langle 0_{\text{phys}} | a^3 a^{3\dagger} |0_{\text{phys}}\rangle^\eta \]
\[ = \eta(2 - \eta) \sum_{n,m=0}^{\infty} \frac{1}{n!m!} \langle 0_{\text{aux}} | (1 - \eta)^n (a^3)^n (a^{3\dagger})^n a^3 a^{3\dagger} (1 - \eta)^m (a^{3\dagger})^m (a^0)^m |0_{\text{aux}}\rangle \]
\[ = \eta(2 - \eta) \sum_{n=0}^{\infty} (n + 1)(1 - \eta)^{2n}, \] (7.462)
and thus

\[ \eta \langle 0_{\text{phys}} | a^3 a^{3\dagger} |0_{\text{phys}}\rangle^\eta = \frac{1}{\eta(2 - \eta)}. \] (7.463)

Similarly we derive

\[ \eta \langle 0_{\text{phys}} | a^0 a^{0\dagger} |0_{\text{phys}}\rangle^\eta = \frac{1}{\eta(2 - \eta)}, \] (7.464)
and via the commutation rules

\[ a^0 a^{0\dagger} = -1 + a^{0\dagger} a^0, \quad a^{3\dagger} a^3 = -1 + a^3 a^{3\dagger} \] (7.465)
the expectation values

\[ \eta \langle 0_{\text{phys}} | a^0 a^{0\dagger} |0_{\text{phys}}\rangle^\eta = \eta \langle 0_{\text{phys}} | a^{3\dagger} a^3 |0_{\text{phys}}\rangle^\eta = \frac{(1 - \eta)^2}{\eta(2 - \eta)}. \] (7.466)

These results can actually be obtained directly by trivial algebra from Eqs. (7.461) using the commutation relations (7.465).

Equations (7.463) and (7.464) show that the norm of the states on the right-hand side of Eqs. (7.461) is \( \sqrt{\eta/(1 - \eta)} \), the normalized states \( |0_{\text{phys}}\rangle^\eta \) satisfy the desired constraints (7.455) in the limit \( \eta \to 0 \), thus converging towards the physical vacuum state.
To be consistent, all calculations in the Dirac quantization scheme have to be done in one of the would-be vacuum states \([0_{\text{phys}}]^{\eta}\), with the limit taken at the end. Only in this way the earlier-observed apparent contradiction in (7.437) can be avoided. In the present context, it is observed when taking the expectation value of the commutation relation \([a_s, a_s^\dagger] = -1\):

\[
\langle 0_{\text{phys}} | a_s a_s^\dagger | 0_{\text{phys}} \rangle - \langle 0_{\text{phys}} | a_s^\dagger a_s | 0_{\text{phys}} \rangle = - \langle 0_{\text{phys}} | 0_{\text{phys}} \rangle. \tag{7.467}
\]

Since \(a_s^\dagger\) annihilates the physical vacuum to its right and to its left, the left-hand side should be zero, contradicting the right-hand side. For the normalized finite-\(\eta\) would-be vacuum, on the other hand, the equation becomes

\[
\eta \langle 0_{\text{phys}} | a_s a_s^\dagger | 0_{\text{phys}} \rangle - \eta \langle 0_{\text{phys}} | a_s^\dagger a_s | 0_{\text{phys}} \rangle = -1. \tag{7.468}
\]

From (7.463)–(7.464), we find

\[
\eta \langle 0_{\text{phys}} | a_s a_s^\dagger | 0_{\text{phys}} \rangle = -1, \quad \eta \langle 0_{\text{phys}} | a_s^\dagger a_s | 0_{\text{phys}} \rangle = -1, \tag{7.469}
\]

so that (7.468) is a correct equation. Thus, although the states (7.461) have a small norm of order \(\sqrt{\eta}\), the products \(a_s a_s^\dagger\) and \(a_s^\dagger a_s\) have a unit expectation value.

We can easily see the reason for this by calculating

\[
a_s | 0_{\text{phys}} \rangle^\eta = (2 - \eta) a_s^0 | 0_{\text{phys}} \rangle^\eta,
\]

\[
a_s^\dagger | 0_{\text{phys}} \rangle^\eta = (2 - \eta) a_s^3 | 0_{\text{phys}} \rangle^\eta, \tag{7.470}
\]

and by observing that the norm of the states on the right-hand sides is, by (7.463) and (7.464), equal to \((2 - \eta)/\sqrt{\eta(2 - \eta)} = \sqrt{(2 - \eta)/\eta}\), which diverges like \(1/\sqrt{\eta}\) for \(\eta \to 0\) [being precisely the inverse of the norm of the states (7.461)]. Thus we can only drop safely expectation values in which \(a_s\) or \(a_s^\dagger\) act upon the physical vacuum to its right, if there is no operator \(a_s^\dagger\) or \(a_s\) doing the same thing to its left. This could be a problem with respect to the vacuum energy of the unphysical modes in (7.429), which is proportional to

\[
- \eta \langle 0_{\text{phys}} | a_s a_s^\dagger + a_s^\dagger a_s + a_s^\dagger a_s^\dagger a_s | 0_{\text{phys}} \rangle^\eta. \tag{7.471}
\]

Fortunately, this expectation value does vanish, as it should, since by (7.463)–(7.466):

\[
\eta \langle 0_{\text{phys}} | a_s a_s^\dagger | 0_{\text{phys}} \rangle^\eta = \eta \langle 0_{\text{phys}} | a_s^3 a_s^\dagger | 0_{\text{phys}} \rangle^\eta = 1/2,
\]

\[
\eta \langle 0_{\text{phys}} | a_s^\dagger a_s | 0_{\text{phys}} \rangle^\eta = \eta \langle 0_{\text{phys}} | a_s^\dagger a_s | 0_{\text{phys}} \rangle^\eta = -1/2, \tag{7.472}
\]

so that the energy of the unphysical modes in the normalized would-be vacuum state \(|0_{\text{phys}}\rangle^\eta\) is indeed zero for all \(\eta\), and the true vacuum energy in the limit \(\eta \to 0\) contains only the zero-point oscillations of the physical transverse modes.

In this context it is worth emphasizing that the normally ordered operator \(a^0 a^0 - a^3 a^3\) is not normally ordered with respect to the physical vacuum \(|0_{\text{phys}}\rangle\) and thus
does not yield zero when sandwiched between two such states. This is obvious by rewriting $a^0 a^0 - a^3 a^3$ as $a_s a_s + a_s a_s$, and the fact that the annihilation operators $a_s$ and $a_s$ on the right-hand side do not annihilate the physical vacuum: $a_{k,s} |0_{\text{phys}}\rangle \neq 0$, $a_{k,s} |0_{\text{phys}}\rangle \neq 0$.

In axiomatic quantum field theory, the vacuum is always postulated to be a proper discrete state with a unit norm. Without this property, the discussion of symmetry operations in a Hilbert space becomes quite subtle, since infinitesimal transformations can produce finite changes of a state. The Dirac vacuum $|0_{\text{phys}}\rangle$ is not permitted by this postulate, whereas the pseudophysical vacuum $|0_{\text{phys}'}\rangle$ is.

There are other problems within axiomatic quantum field theory, however, due to the non-uniqueness of this vacuum, and due to the fact that the vacuum is not separated from all other states by an energy gap, an important additional postulate. That latter postulate is actually unphysical, due to the masslessness of the photon and the existence of many other massless particles in nature. But it is necessary for the derivation of many “rigorous results” in that somewhat esoteric discipline.

Because of the subtleties with the limiting procedure of the vacuum state, it will not be convenient to work with the Dirac quantization scheme. Instead we shall use a simplified procedure due to Gupta and Bleuler, to be introduced below. First, however, we shall complete the present discussion by calculating the photon propagator within the Dirac scheme.

**Propagator in Dirac Quantization Scheme**

Let us calculate the propagator of the $A_\mu$-field in the Dirac quantization scheme. We take the expansion of the field operator (7.419) and evaluate the expectation value

$$G^{\mu\nu}(x,x') = \frac{1}{\eta} \langle 0_{\text{phys}} | T A^\mu(x) A^\nu(x') | 0_{\text{phys}} \rangle \eta,$$

in which we take the limit $\eta \to 0$ at the end. For $x_0 > x'_0$, we obtain a contribution proportional to

$$\Theta(x_0 - x'_0) \frac{1}{V} \sum_k \frac{1}{2k_0} e^{-ik(x-x')}$$

multiplied by the sum of the expectation values of

$$\epsilon^{\mu*}(k, +1) \epsilon^\nu(k, +1) a_{k+1} a_{k+1}^\dagger, \quad \epsilon^{\mu*}(k, -1) \epsilon^\nu(k, -1) a_{k-1} a_{k-1}^\dagger,$$

and

$$\epsilon^{\mu*}(k, s) \epsilon^\nu(k, s) a_{k,s} a_{k,s}^\dagger, \quad \epsilon^{\mu*}(k, \bar{s}) \epsilon^\nu(k, \bar{s}) a_{k,\bar{s}} a_{k,\bar{s}}^\dagger,$$

and a contribution proportional to

$$\Theta(x_0 - x'_0) \frac{1}{V} \sum_k \frac{1}{2k_0} e^{ik(x-x')}$$
multiplied by the sum of the expectation values of

\[ \epsilon^\mu(k, +1) \epsilon^{\nu*}(k, +1) a_{k,+1}^\dagger a_{k,+1}, \quad \epsilon^\mu(k, -1) \epsilon^{\nu*}(k, -1) a_{k,-1}^\dagger a_{k,-1}, \quad (7.478) \]

and

\[ \epsilon^\mu(k, \bar{s}) \epsilon^{\nu*}(k, s) a_{k,s}^\dagger a_{k,s}, \quad \epsilon^\mu(k, s) \epsilon^{\nu*}(k, \bar{s}) a_{k,s}^\dagger a_{k,s}. \quad (7.479) \]

The first two terms containing transverse photons produce a polarization sum

\[ \epsilon^{\mu*}(k, +1) \epsilon^\nu(k, +1) + \epsilon^{\mu*}(k, -1) \epsilon^\nu(k, -1). \quad (7.480) \]

The expectation values of the normally ordered transverse photon terms (7.478) vanish. The remaining terms in (7.476) and (7.479) are evaluated with the help of the matrix elements (7.472) of opposite signs. They cancel each other, due to the evenness of \( \epsilon^\mu(k, s) \epsilon^{\nu*}(k, \bar{s}) = k^{\mu 2} - k^{\bar{s} 2} \) as functions of \( k^\mu \). Such a function, appearing as a factor inside the momentum sums (7.474) and (7.477), guarantees their equality and ensures the cancellation of the contributions from (7.476) and (7.479).

For \( x_0 < x_0' \), we obtain once more the same expression with spacetime and Lorentz indices interchanged: \( x \leftrightarrow x', \mu \leftrightarrow \nu \).

The propagator becomes therefore

\[
G^{\mu\nu}(x, x') = n\langle 0_{\text{phys}} | T A^{\mu}(x) A^{\nu}(x') | 0_{\text{phys}} \rangle n = \Theta(x_0 - x_0') \frac{1}{V} \sum_k \frac{1}{2k^0} e^{-ik(x - x')} \sum_{\nu = 1}^2 \epsilon^{\mu*}(k, \nu) \epsilon^\nu(k, \nu) + \Theta(x_0' - x_0) \frac{1}{V} \sum_k \frac{1}{2k^0} e^{ik(x - x')} \sum_{\nu = 1}^2 \epsilon^{\mu*}(k, \nu) \epsilon^\nu(k, \nu).
\]

This is the same propagator as in Eq. (7.361), where it was obtained in the manifestly noncovariant quantization scheme, in which only the physical degrees of freedom of the vector potential were made operators. Thus, although the field operators have been quantized with the covariant commutation relations (7.386), the selection procedure of the physical states has produced the same noncovariant propagator.

It is worth pointing out the difference between the propagator (7.361) and the retarded propagator used in classical electrodynamics. The classical one is given by the same Fourier integral as \( G(x - x') \):

\[ G_R(x - x') \equiv \int \frac{d^4k}{(2\pi)^4} e^{-ik(x - x')} \frac{i}{k^2}, \]

except that the poles at \( k^0 \equiv \pm \omega = |k| \) are both placed below the real axis. The quantum-field-theoretic \( k^0 \)-integral

\[ \int \frac{dk^0}{2\pi} e^{-ik_0(x_0 - x_0')} i \frac{1}{|k|} \left( \frac{1}{k^0 - |k| + i\eta} - \frac{1}{k^0 + |k| - i\eta} \right) \]

\[ = \frac{1}{2\omega} [\Theta(x^0 - x_0') e^{-i\omega(x^0 - x_0')} + \Theta(x^0' - x_0) e^{i\omega(x^0 - x_0')}] = \frac{1}{2\omega} e^{-i\omega|x^0 - x_0'|} \]
with \( \omega \) on the light cone, \( \omega = |k| \), has to be compared with the retarded expression

\[
\int \frac{dk^0}{2\pi} e^{-ik^0(x^0 - x'^0)} \frac{i}{2|k|} \left( \frac{1}{k^0 - |k| + i\eta} - \frac{1}{k^0 + |k| + i\eta} \right) = \Theta(x^0 - x'^0) \frac{1}{2\omega} \sin[\omega(x^0 - x'^0)].
\]

The angular part of the spatial integral

\[
\int \frac{d^3k}{(2\pi)^3} e^{ikx}
\]

can be done in either case in the same way leading to

\[
\frac{1}{2\pi^2R} \int_0^\infty d\omega \sin(\omega R).
\]

Thus we find

\[
G_R(x - x') = i\Theta(x^0 - x'^0) \int_{-\infty}^{\infty} d\omega \{ e^{-i\omega|x^0 - x'^0|} - e^{i\omega|x^0 - x'^0|} \}.
\]

Since \( R \) and \( x^0 - x'^0 \) are both positive, the result is

\[
G_R(x - x') = i\Theta(x^0 - x'^0) \frac{1}{4\pi R} \delta(x^0 - x'^0 - R).
\]

The retarded propagator exists only for causal times \( x^0 > x'^0 \) and it is equal to the Coulomb potential if the end points can be connected by a light signal. In contrast to this, the propagator of the quantized electromagnetic field, in which the creation operator of antiparticles accompanies the negative energy solutions of the wave equation, is

\[
G(x - x') = i\frac{1}{4\pi R} \delta_+(|x^0 - x'^0| - R) + i\frac{1}{4\pi R} \delta_+(-|x^0 - x'^0| - R),
\]

where

\[
\delta_+(t) = \int \frac{d\omega}{\pi} e^{-i\omega t}
\]

is equal to twice the positive-frequency part of the Fourier decomposition of the \( \delta \)-function. With the help of an infinitesimal imaginary part in the time argument, the integral can be done by closing the contour in the upper or lower half-plane, and the results of both contours can be expressed as

\[
\delta_+(t) = \frac{1}{\pi i} \frac{1}{t + i\eta}.
\]

This, in turn, can be decomposed as follows:

\[
\delta_+(t) = \frac{1}{\pi i} \left( \frac{i\eta}{t^2 + \eta^2} + \frac{t}{t + \eta^2} \right) = \delta(t) + \frac{1}{\pi i} \mathcal{P} \frac{t}{t^2 + \eta^2}.
\]
The second term selects the principal value in any integral involving \( \delta_+(t) \).

Incidentally, both \( \delta_+(t) \) and \( \delta(t) \) satisfy
\[
\delta(t^2 - R^2) = \frac{1}{2R}[\delta(t - R) + \delta(t + R)].
\]
(7.493)

As a consequence, the propagator \( G(x - x') \) in (7.489) can also be written as
\[
G(x - x') = i\frac{1}{2\pi}\delta_+((x - x')^2),
\]
(7.494)
to be compared with the retarded propagator of (7.488):
\[
G_R(x - x') = \Theta(x^0 - x'^0)i\frac{1}{2\pi}\delta((x - x')^2).
\]
(7.495)

In configuration space, the massive propagators are quite simply related to the massless ones. One simply writes the \( \delta \)-functions \( \delta((x - x')^2) \) and \( \delta_+((x - x')^2) \) in \( G_R(x - x') \) and \( G(x - x') \) as Fourier integrals
\[
\delta((x - x')^2) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega(x - x')^2}, \quad \delta_+((x - x')^2) = \int_{0}^{\infty} \frac{d\omega}{\pi} e^{i\omega(x - x')^2}
\]
(7.496)
and replaces \( e^{-i\omega (x - x')^2} \) by \( e^{-i\omega (x - x')^2 - \frac{M^2}{\omega}} \) as in (7.141). With the additional term in the exponent, the \( \omega \)-integrals yield Bessel functions:
\[
\delta(x^2) \rightarrow \delta(x^2) + \Theta(x^2)\frac{M^2 J_1(M\sqrt{x^2})}{2M\sqrt{x^2}},
\]
(7.497)
\[
\delta_+(x^2) \rightarrow \delta(x^2) + \mathcal{P}\frac{M^2 J_1(M\sqrt{x^2}) - iN_1(M\sqrt{x^2})}{2M\sqrt{x^2}}.
\]
(7.498)
For \( M \to 0 \), we use the limiting behavior \( J_1(z) \to z/2 \), \( N_1(z) \to -2/\pi z \) and see that the mass term disappears in the first equation, whereas in the second term it becomes \( \mathcal{P}/\pi ix^2 \).

According to the wave equation (7.381), the polarization vectors have to satisfy
\[
k^2\epsilon^{\mu}(k, \nu) - \left(1 - \frac{1}{\alpha}\right) k^{\mu} k^{\nu} \epsilon^{\nu}(k, \nu) = 0.
\]
(7.499)

The propagator of the fields \( A^\mu \) in momentum space is obtained by inverting the inhomogeneous version of this equation: Correspondingly, the polarization vectors in the field expansion (7.390) will now, instead of (7.392)–(7.393), satisfy the completeness and orthogonality relations
\[
\sum_{\nu\nu'} g^{\nu\nu'} \epsilon^{\mu}(k, \nu) \epsilon^{\nu\nu'}(k, \nu') = g^{\mu\nu'} - \left(1 - \frac{1}{\alpha}\right) \frac{k^{\mu} k^{\nu'}}{k^2},
\]
(7.500)
\[
\left[g^{\mu\nu'} - \left(1 - \frac{1}{\alpha}\right) \frac{k^{\mu} k^{\nu'}}{k^2}\right] \epsilon_{\mu}(k, \nu) \epsilon_{\nu'}(k, \nu') = g^{\nu\nu'}.
\]
(7.501)
7.5.3 Gupta-Bleuler Subsidiary Condition

One may wonder whether it is possible to avoid the awesome limiting procedure \( \eta \to 0 \) and find a way of working with an ordinary vacuum state. This is indeed possible if one is only interested in physical matrix elements containing at least one particle. Such matrix elements can be calculated using only the condition (7.425), while discarding the condition (7.426). Thus, we select what we shall call pseudo-physical states by requiring only

\[
a_{k,s} |\psi_{\text{phys}}\rangle = 0.
\]  

(7.502)

The restriction to this condition is the basis of the Gupta-Bleuler approach to quantum electrodynamics [11]. Take, for instance, the number operator (7.435) and insert it between two pseudo-physical states selected in this way. It yields

\[
\langle \psi_{\text{phys}}' | N | \psi_{\text{phys}} \rangle = \langle \psi_{\text{phys}}' | a_{k,+1} \delta_{k} a_{k,-1} | \psi_{\text{phys}} \rangle - \langle \psi_{\text{phys}}' | [a_{k,s} a_{k,s} + a_{k,s} a_{k,s}] | \psi_{\text{phys}} \rangle.
\]  

(7.503)

The second line vanishes due to the condition (7.448) and its Hermitian adjoint. Thus the number operator counts only the number of transverse photons.

The same mechanism ensures correct particle energies. With the help of the first commutation rule in (7.430), we bring the Hamilton operator (7.431) to the normally ordered form in the pseudo-physical vacuum state \(|0_{\text{phys}}\rangle\):

\[
H' = \sum_{k} k^{0} \left[ \sum_{\sigma = \pm 1} \left( a_{k,\sigma} a_{k,\sigma} + \frac{1}{2} \right) + 1 - a_{k,s} a_{k,s} - a_{k,s} a_{k,s} \right].
\]  

(7.504)

The last two terms vanish because of the Gupta-Bleuler condition (7.502). Thus the Hamilton operator counts only the energies of the transverse photons.

So why did we have to impose the second Dirac condition (7.426) at all? It is needed to ensure only one property of the theory: the correct vacuum energy. In the Gupta-Bleuler approach this energy comes out wrong by a factor two, showing that the unphysical degrees of freedom have not been completely eliminated. The normal ordering of the Hamilton operator (7.431) with respect to the pseudo-physical vacuum state \(|0_{\text{phys}}\rangle\) has produced an extra vacuum energy

\[
E_{0}^{\text{extra}} = \sum_{k} k^{0} = \sum_{k} \omega_{k},
\]  

(7.505)

which previously did not appear because of the second Dirac condition (7.426). It is the vacuum energy of two unphysical degrees of freedom for each momentum \(k\). This wrong vacuum energy may also be seen directly by bringing the Hamilton operator (7.428) to normal order using the commutation rule (7.406), yielding

\[
H = \sum_{k} k^{0} \left( -a_{k,\mu} a_{k,\mu} + \frac{1}{2} \right),
\]  

(7.506)

the second term showing the vacuum energy of four polarization degrees of freedom, rather than just the physical ones.

A heuristic remedy to the wrong vacuum energy in the Gupta-Bleuler formalism will be discussed below, to be followed by a more satisfactory one in Section 14.16.
7.5 Free Photon Field

Photon Propagator in Gupta-Bleuler Approach

We now calculate the propagator of the $A_\mu$-field in the Gupta-Bleuler quantization scheme, which is given by the expectation value in the pseudo-physical vacuum state

$$G^{\mu\nu}(x, x') = \langle 0_{\text{phys}} | T A^\mu(x) A^\nu(x') | 0_{\text{phys}} \rangle. \quad (7.507)$$

Expanding the vector field $A^\mu$ as in (7.419), and inserting it into (7.507), we obtain for $x_0 > x'_0$ the same contributions as in Eqs. (7.474)–(7.479), except that the expectation values are to be taken in the pseudo-physical vacuum state $|0_{\text{phys}}\rangle$, where only $a_{k,s} |0_{\text{phys}}\rangle = 0$ and $\langle 0_{\text{phys}} | a_{k,s}^\dagger = 0$. As a consequence, there are contributions from the expectation values of the terms (7.476), yielding a polarization sum

$$\epsilon^{\mu\nu}(k, \bar{s}) \epsilon^{\nu}(k, s) + \epsilon^{\mu\nu}(k, s) \epsilon^{\nu}(k, \bar{s}),$$

which, together with the transverse sum (7.480), adds up to

$$\sum_{\sigma = \pm 1} \epsilon^{\mu\sigma}(k, \sigma) \epsilon^{\nu}(k, \sigma) - \epsilon^{\mu\sigma}(k, s) \epsilon^{\nu}(k, \bar{s}) - \epsilon^{\mu\sigma}(k, \bar{s}) \epsilon^{\nu}(k, s) = -g^{\mu\nu}. \quad (7.508)$$

For $x_0 < x'_0$, the same expression is found with spacetime and Lorentz indices interchanged: $x \leftrightarrow x'$, $\mu \leftrightarrow \nu$, so that the propagator becomes

$$G^{\mu\nu}(x, x') = \langle 0_{\text{phys}} | T A^\mu(x) A^\nu(x') | 0_{\text{phys}} \rangle \quad (7.509)$$

$$= -g^{\mu\nu} \left[ \Theta(x_0 - x'_0) \frac{1}{V} \sum_k \frac{1}{2k^0} e^{-ik(x-x')} + \Theta(x'_0 - x_0) \frac{1}{V} \sum_k \frac{1}{2k^0} e^{ik(x-x')} \right].$$

Up to the prefactor $-g^{\mu\nu}$, this agrees with Eq. (7.47) for the propagator $G(x, x')$ of the Klein-Gordon field for zero mass, which we write as in (7.62) in the form

$$G^{\mu\nu}(x, x') = \langle 0 | T A^\mu(x) A^\nu(x') | 0 \rangle$$

$$= -g^{\mu\nu} \left[ \Theta(x_0 - x'_0) G^{(+)}(x, x') + \Theta(x'_0 - x_0) G^{(-)}(x, x') \right] \quad (7.510)$$

$$= -g^{\mu\nu} G(x, x').$$

This photon propagator is much simpler than the previous one in (7.361), making it much easier to perform calculations, which is the main virtue of the Gupta-Bleuler quantization scheme. A second advantage of the propagator (7.510) is that, in contrast to the noncovariant propagator (7.366), it is a proper Green function associated with the wave equation (7.389) in the $\alpha = 0$ -gauge, satisfying the inhomogeneous field equation

$$-\partial^2 G^{\mu\nu}(x - x') = -g^{\mu\nu} i \delta^{(4)}(x - x'). \quad (7.511)$$

A further special feature of the Gupta-Bleuler quantization procedure that is worth pointing out is that the condition $a_{k,s} |\psi_{\text{phys}}\rangle = 0$ does not uniquely fix the pseudo-physical vacuum state $|0_{\text{phys}}\rangle$. The vacuum state $|0_{\text{phys}}\rangle$ is only one possible choice, but infinitely many other states are equally good candidates, as long as their physical photon number is zero. Focussing attention only on states of a fixed
photon momentum parallel to the $z$-axis, the zero-norm states (7.408) all satisfy the condition $a_{k,s}|\psi^\text{phys}\rangle = 0$. As a consequence, any superposition of such states

$$|\tilde{0}\rangle = |0\rangle + c^{(1)}_k|0\rangle_1 + c^{(2)}_k|0\rangle_2 + \ldots$$

(7.512)

has a unit norm. It contains no transverse photon and satisfies the condition $a_{k,s}|\tilde{0}\rangle = 0$, and is thus an equally good pseudo-vacuum state of the system.

The expectation value of the vector potential $A^\mu$ in this vacuum is nonzero. Its $k$th component is equal to the coefficient $c^{(1)}$ associated with this momentum, i.e., it is some function $c(k)$. Because of the second of the commutation rules (7.430), its $k$th Fourier component $\epsilon^\mu(k,\nu)a_{k,\nu}$ has an expectation $\epsilon^\mu(k, s)$, with a coefficient $c_k = c^{(1)}_k$:

$$\langle \tilde{0}|A^\mu|\tilde{0}\rangle = c_k\epsilon^\mu(k, s),$$

(7.513)

i.e., it is proportional to the four-momentum $k^\mu$. In $x$-space, the expectation value of such a field is equal to a pure gradient $\partial^\mu\Lambda(x)$, so that it carries no electromagnetic field.

The condition (7.502) implies that

$$\langle \tilde{0}|a^\dagger_{k,0}a_{k,0}|\tilde{0}\rangle = \langle \tilde{0}|a^\dagger_{k,3}a_{k,3}|\tilde{0}\rangle.$$  

(7.514)

The two sides are equal to some function of $k$, say $k^0\chi(k)$. In a covariant way, the vacuum is thus characterized by the expectations

$$\begin{align*}
\langle \tilde{0}|a^\dagger_{k,\mu}a_{k',\nu}|\tilde{0}\rangle &= k_{\mu}^\nu\epsilon^\mu(k, \nu)k_{\nu}^\nu\epsilon^\nu(k, \nu')\chi(k)\delta_{k,k'} \\
\langle \tilde{0}|a_{k,\mu}a^\dagger_{k',\nu}|\tilde{0}\rangle &= [-\delta^\mu_{\nu'} + k_{\mu}^\nu\epsilon^\mu(k, \nu)k_{\nu'}^\nu\epsilon^\nu(k, \nu')\chi(k)]\delta_{k,k'}.
\end{align*}$$

(7.515)

If we use any of these vacua and do not impose the condition (7.502), the propagator

$$G_0^{\mu\nu}(x - x') = \langle \tilde{0}|TA^{\mu}(x)A^{\nu}(x')|\tilde{0}\rangle$$

(7.516)

is a gauge-transformed version of the original one. Let us calculate it explicitly. Using (7.515), we obtain

$$
\begin{align*}
G_0^{\mu\nu}(x - x') &= \langle \tilde{0}|TA^{\mu}(x)A^{\nu}(x')|\tilde{0}\rangle \\
&= \frac{1}{V} \sum_k \frac{1}{2k^0} e^{-ik(x-x')} \left[-\Theta(x_0 - x'_0)g^{\mu\nu'} + k_{\mu'}^\nu\epsilon^\nu_\epsilon(k, \nu)k_{\nu'}^\nu\epsilon^\nu(k, \nu')\chi(k)\right] \\
&\quad \times \epsilon^\mu(k, \nu)\epsilon^\nu(k, \nu') \\
&= \frac{1}{V} \sum_k \frac{1}{2k^0} e^{ik(x-x')} \left[\Theta(x'_0 - x_0)g^{\mu\nu'} + k_{\mu}^\nu\epsilon^\mu(k, \nu)k_{\nu'}^\nu\epsilon^\nu(k, \nu')\chi(k)\right] \\
&\quad \times \epsilon^{\mu'}(k, \nu')\epsilon^{\nu'}(k, \nu') \\
&= \frac{1}{V} \sum_k \frac{1}{2k^0} e^{-ik(x-x')} \left[-\Theta(x_0 - x'_0)g^{\mu\nu} + k^\mu k^\nu\chi(k)\right] \\
&\quad + \frac{1}{V} \sum_k \frac{1}{2k^0} e^{ik(x-x')} \left[-\Theta(x'_0 - x_0)g^{\mu\nu} + k^\mu k^\nu\chi(k)\right].
\end{align*}
$$

(7.517)
The $g^{\mu\nu}$-terms are again equal to $G^{\mu\nu}$ of Eq. (7.510). The $k^\mu k^\nu$ terms may be placed outside the integrals where they become spacetime derivatives $-\partial^\mu \partial^\nu$ of a function

$$F(x - x') = \frac{1}{V} \sum_k \frac{1}{2k^0} \left[ e^{-ik(x-x')} + e^{ik(x-x')} \right] \chi(k).$$  \hspace{1cm} (7.518)

Thus we find

$$G_0^{\mu\nu}(x - x') = G_0^{\mu\nu}(x - x') - \partial^\mu \partial^\nu F(x - x').$$  \hspace{1cm} (7.519)

Physical observables must, of course, be independent of $F(x - x')$. This will be proven in Chapter 12 after Eq. (12.101).

Note that in contrast to $G^{\mu\nu}(x - x')$, the Fourier components of $F(x - x')$ are restricted to the light cone. This is seen more explicitly by rewriting the sums in (7.518) as integrals, and introducing the $\delta$-function

$$\delta(k^2) = \frac{1}{2\omega_k} \left[ \delta(k^0 - \omega_k) + \delta(k^0 + \omega_k) \right],$$  \hspace{1cm} (7.520)

so that $F(x - x')$ is equal to the four-dimensional integral

$$F(x - x') = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-x')} 2\pi \delta(k^2) \chi(k).$$  \hspace{1cm} (7.521)

The function $F(x - x')$ is therefore a solution of the Klein-Gordon equation $\partial^2 F(x - x') = 0$. This is the same type of freedom we are used to from all Green functions that satisfy the same inhomogeneous differential equation

$$-\partial^2 G^{\mu\nu} = -ig^{\mu\nu} \delta^{(4)}(x - x').$$  \hspace{1cm} (7.522)

All solutions of this equation differ from each other by solutions of the homogeneous differential equation, and so do the propagators $G_0^{\mu\nu}(x - x')$ and $G_0^{\mu\nu}(x - x')$ coming from different vacuum states.

**Remedy to Wrong Vacuum Energy in Gupta-Bleuler Approach**

In the Hamilton operators (7.504) and (7.506) we found an important failure of the Gupta-Bleuler quantization scheme. The energy contains also the vacuum oscillations of the two unphysical degrees of freedom oscillating in the $0$- and the $k$-direction. Due to the opposite commutation rules of the associated creation and annihilation operators, there is no cancellation of the two contributions. The Gupta-Bleuler approach eliminates the unphysical modes from all multiparticle states except for the vacuum. The elimination of this vacuum energy in the covariant approach was achieved almost two decades later by Faddeev and Popov [12]. Their method was developed for the purpose of quantizing nonabelian gauge theories. In these theories, the elimination of the unphysical gauge degrees of freedom cannot be
achieved with the help of a Gupta-Bleuler type of subsidiary condition (7.502). Essential to their approach is the use of the functional-integral formulation of quantum field theory. In it, all operator calculations are replaced by products of infinitely many integrals over classical fluctuating fields. This formulation will be introduced in Chapter 14. When fixing a covariant gauge in the gauge-invariant quantum partition function of the electromagnetic fields, one must multiply the functional integral by a factor that removes the unphysical modes from the partition function. At zero temperature, this factor subtracts, in the abelian gauge theory of QED, precisely the vacuum energies of the two unphysical modes. The removal requires an equal amount of negative vacuum energy. In Eq. (7.245) we have observed that fermions have negative vacuum energies. The subtraction of the unphysical vacuum energies can therefore be achieved by introducing two fictitious Fermi fields, a ghost field \( C(x) \) and an anti-ghost field \( \bar{C}(x) \). They are completely unphysical objects with spin zero, so that they do not appear in the physical Hilbert space, i.e., the physical state vectors satisfy the conditions

\[
C^\dagger(x)|0_{\text{phys}}\rangle = 0, \quad \bar{C}^\dagger(x)|0_{\text{phys}}\rangle = 0.
\]

These fields are called Faddeev-Popov ghosts. In QED with gauge \( \partial^\mu A^\mu(x) = 0 \), their only contribution lies in the negative vacuum energy

\[
E_0^{\text{ghosts}} = -2 \sum_k \frac{k^0}{2}
\]

which cancels the excessive vacuum energy (7.505) in the Gupta-Bleuler approach.

In quantum electrodynamics, the effect of the ghosts is trivial so that their introduction is superfluous for all purposes except for calculating the vacuum energy. This is the reason why their role was recognized only late in the development [13]. In nonabelian gauge theories, the ghost fields have nontrivial interactions with the physical particles, which must be included in all calculations to make the theory consistent.

In the light of this development, the correct way of writing the Lagrangian density of the free-photon field is

\[
\mathcal{L}_{\text{tot}} = \mathcal{L} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{ghost}} = -\frac{1}{4} F_{\mu\nu}^2 - D(x)\partial^\mu A_\mu(x) + \alpha D^2(x)/2 - i\partial_\mu \bar{C} \partial^\mu C.
\]

We shall see that this Lagrangian has an interesting new global symmetry between the Bose fields \( A^\mu, D \) and the Fermi fields \( C, \bar{C} \), called supersymmetry, which will be discussed in Chapter 25. In this context it is called BRS symmetry [?].

The Gupta-Bleuler gauge condition plus the subsidiary ghost conditions (7.523) eliminating the unphysical degrees of freedom are equivalent to requiring that the charges generating all global symmetry transformations annihilate the physical vacuum. Although structurally somewhat complicated, this is the most satisfactory way of formulating the covariant quantization procedure of the photon field.

\[20\] For this reason, the failure to display the spin-statistics connection to be derived in Section 7.10 does not produce any causality problems.
The pseudophysical vacuum $|0_{\text{phys}}\rangle$ of the Gupta-Bleuler approach has a unit norm, which is an important advantage over the physical vacuum $|0_{\text{phys}}\rangle$ of the Dirac approach. In axiomatic quantum field theory it can be proved that for any local operator $O(x)$, one has

$$O(x)|0\rangle = 0 \text{ if and only if } O(x) = 0. \quad (7.526)$$

This is the reason why the Gupta-Bleuler subsidiary condition involves necessarily nonlocal operators. Recall that the operator in (7.502) contains only the positive-frequency part of the local field $D(x)$. The negative frequencies would be needed to make the field local. This will become clear in the proof of the spin-statistic theorem in Section 7.10.

**Arbitrary Gauge Parameter $\alpha$**

A quantization of the electromagnetic field is also possible for different values of the gauge parameter $\alpha$ in the Lagrangian density (7.377) which so far has been set equal to unity, for simplicity. For an arbitrary value of $\alpha$, the field equation (7.381) can be written as

$$L_{\mu\nu}(i\partial)A^\nu(x) = 0, \quad (7.527)$$

with the differential operator

$$L_{\mu\nu}(i\partial) \equiv -\partial^2 g_{\mu\nu} + \left(1 - \frac{1}{\alpha}\right) \partial_\mu \partial_\nu. \quad (7.528)$$

Multiplying Eq. (7.527) by $\partial_\mu$, we find that the vector potential cannot have any four-divergence:

$$\partial_\mu A^\mu(x) = 0. \quad (7.529)$$

In a gauge-invariant formulation, this property can be chosen as the Lorenz gauge condition. If the Lagrangian contains a gauge-fixing part like (7.377), the Lorenz condition $\partial_\mu A^\mu(x) = 0$ is a consequence of the equations of motion. For such a field, the equations of motion reduce to four Klein-Gordon equations, just as before in the Feynman case $\alpha = 1$ [see (7.389)]:

$$-\partial^2 A^\mu(x) = 0. \quad (7.530)$$

We shall not go through the entire quantization procedure and the derivation of the propagator in this case. What can be given without much work is the Green function defined by the inhomogeneous field equation

$$L^{\mu\nu}(i\partial)G_{\nu\kappa}(x - x') = -i\delta^{(4)}(x - x'). \quad (7.531)$$

In momentum space, it reads

$$L^{\mu\nu}(k)G_{\nu\kappa}(k) = -i\delta^{(4)}_\kappa. \quad (7.532)$$
with the $4 \times 4$-matrix

$$L^{\mu\nu}(k) = k^2 \left( g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) + \frac{k^2}{\alpha} \frac{k^\mu k^\nu}{k^2}. \quad (7.533)$$

This matrix has an eigenvector $k^\mu$ with an eigenvalue $k^2/\alpha$, and three eigenvectors orthogonal to it with eigenvalues $k^2$. For a finite $\alpha$, the matrix has an inverse, which is most easily found by decomposing $L^{\mu\nu}(k)$ as

$$L^{\mu\nu}(k) = k^2 P_T^{\mu\nu}(k) + \frac{k^2}{\alpha} P_L^{\mu\nu}(k), \quad (7.534)$$

with

$$P_T^{\mu\nu}(k) = g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2},$$

$$P_L^{\mu\nu}(k) = \frac{k^\mu k^\nu}{k^2}. \quad (7.535)$$

These matrices are projections. Indeed, they both satisfy the defining relation

$$P^{\mu\nu} P^\nu_{\nu} = P^{\mu\nu}. \quad (7.536)$$

In the decomposition (7.534), the inverse is found by inverting the coefficients in front of the projections $P_T(k)$ and $P_L(k)$:

$$L^{-1\mu\nu}(k) = \frac{1}{k^2} P_T^{\mu\nu}(k) + \frac{\alpha}{k^2} P_L^{\mu\nu}(k) = \frac{1}{k^2} \left( g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) + \frac{\alpha}{k^2} \frac{k^\mu k^\nu}{k^2}. \quad (7.537)$$

After this, we can solve (7.532) by

$$G^{\mu\nu}(k) = -i L^{-1\mu\nu}(k). \quad (7.538)$$

The propagator has therefore the Fourier representation

$$G^{\mu\nu}(x - x') = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-x')} \frac{i}{k^2 + i\eta} \left[ -g^{\mu\nu} + (1 - \alpha) \frac{k^\mu k^\nu}{k^2 + i\eta} \right]. \quad (7.539)$$

As usual, we have inserted imaginary parts $i\eta$ to regularize the integral in the same way as in the propagator of the scalar field. For the term $k^\mu k^\nu/(k^2 + i\eta)$ the prescription how to place the additional poles is not immediately obvious. The best way to derive it is by giving the photon a small mass and setting it later equal to zero. The reader is referred to the next section where the massive vector meson is treated. He may be worried about the physical meaning of the double-pole in the integrand of the propagator (7.539). A pole in the propagator is associated with a particle state in Hilbert space. How does a double pole manifest itself in Hilbert space? Heisenberg called the associated state a dipole ghost [16]. Instead of satisfying a Schrödinger equation

$$(E - H)|\psi\rangle = 0, \quad (7.540)$$
such a state satisfies the two equations

\[
(E - H)|\psi\rangle \neq 0, \quad (E - H)^{2}|\psi\rangle = 0. \quad (7.541)
\]

It is unphysical. In the present context, such states do not cause any harm. They are an artifact of the gauge fixing procedure and do not contribute to any observable quantity, due to gauge invariance. A similar situation will arise from moving charges in Chapter 12.

Since the \(D(x)\)-field is now equal to \(\partial_{\mu}A^{\mu}(x) / \alpha\) [recall (7.379)], the gauge parameter \(\alpha\) enters in various commutation relations involving \(A^{\mu}(x, t)\) and \(\dot{A}^{\mu}(x, t)\). Proceeding as in Eqs. (7.395)–(7.403), we find that the gauge parameter enters into the commutation rules among \(A^{\mu}(x, t)\) and \(\dot{A}^{\mu}(x, t)\) of Eqs. (7.399)–(7.402) as follows:

\[
\begin{align*}
[\dot{A}^{0}(x, t), A^{0}(x', t)] &= \alpha[D(x, t), A^{0}(x', t)] = i\alpha \delta^{(3)}(x - x'), \quad (7.542) \\
[\dot{A}^{i}(x, t), \dot{A}^{i}(x', t)] &= 0, \quad (7.543) \\
\left[\dot{A}^{0}(x, t), \dot{A}^{i}(x', t)\right] &= -(1 - \alpha)i \partial_{i} \delta^{(3)}(x - x'), \quad (7.544) \\
\left[\dot{A}^{0}(x, t), \dot{A}^{0}(x', t)\right] &= 0. \quad (7.545)
\end{align*}
\]

The other commutators (7.397), (7.400), and (7.401) remain unchanged.

These commutators are needed to verify that the vacuum expectation of the time-ordered product

\[
G^{\mu\nu}(x - x') \equiv \langle 0|TA^{\mu}(x)A^{\nu}(x')|0\rangle \quad (7.546)
\]

satisfies the field equation with a \(\delta\)-function source (7.656). The proof of this is nontrivial:

\[
\begin{align*}
&- \left[\partial^{2}g_{\mu\nu} - \left(1 - \frac{1}{\alpha}\right) \partial_{\mu}\partial_{\nu}\right] G^{\mu\nu}(x - x') = \\
&-\langle 0|T \left[\partial^{2}g_{\mu\nu} - \left(1 - \frac{1}{\alpha}\right) \partial_{\mu}\partial_{\nu}\right] A^{\mu}(x)A^{\nu}(x')|0\rangle - \delta(x^{0} - x'^{0})g_{\mu\nu}\langle 0|[\dot{A}^{\nu}(x), A^{\nu}(x')]|0\rangle \\
&+ \left(1 - \frac{1}{\alpha}\right) g_{\mu\nu}\delta(x^{0} - x'^{0})\langle 0|[\partial_{\nu}A^{\nu}(x), A^{\nu}(x')]|0\rangle = -i\delta_{\mu\nu}\delta^{(4)}(x - x'), \quad (7.547)
\end{align*}
\]

the right-hand side requiring the commutation relations (7.542)–(7.545).

It is useful to check the consistency of the propagator with the canonical commutation rules. This is done in Appendix 7C.

### 7.5.4 Behavior under Discrete Symmetries

Under P,C,T, the vector potential has the transformation properties [recall Section 4.6]:

\[
\begin{align*}
A^{\mu}(x) &\xrightarrow{P} A^{\mu}_{p}(x) = \dot{A}^{\mu}(x), \quad (7.548) \\
A^{\mu}(x) &\xrightarrow{T} A^{\mu}_{t}(x) = \dot{A}^{\mu}(xT), \quad (7.549) \\
A^{\mu}(x) &\xrightarrow{C} A^{\mu}_{c}(x) = -A^{\mu}(x). \quad (7.550)
\end{align*}
\]
The photon has a negative charge parity which coincides with that of the vector current \( V^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x) \) formed from Dirac fields (see Table 4.12.8), to which it will later be coupled (see Chapter 12).

These properties are implemented in the second-quantized Hilbert space by transforming the photon creation operators as follows:

\[
P a_{k,\nu}^\dagger P^{-1} = a_{-k,-\nu}^\dagger, \tag{7.551}
\]

\[
T a_{k,\nu}^\dagger T^{-1} = a_{-k,\nu}^\dagger, \tag{7.552}
\]

\[
C a_{k,\nu}^\dagger C^{-1} = -a_{k,\nu}^\dagger. \tag{7.553}
\]

The annihilation operators transform by the Hermitian-adjoint relations.

### 7.6 Massive Vector Bosons

Besides the massless photons, there exist also massive vector bosons. The most important examples are the fundamental vector bosons mediating the weak interactions, to be discussed in detail in Chapter 27. They can be electrically neutral or carry an electric charge of either sign: \( \pm e \). There also exist strongly interacting vector particles of a composite nature. They are short-lived and only observable as resonances in scattering experiments, the most prominent being a resonance in the two-pion scattering amplitude. These are the famous \( \rho \)-mesons which will be discussed in Chapter 24. They also carry charges \( 0, \pm e \). The actions of a neutral and charged massive vector field \( V^\mu(x) \) are:

\[
\mathcal{A} = \int d^4x \mathcal{L}(x) = \int d^4x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} M^2 V_\mu V^\mu \right), \tag{7.554}
\]

and

\[
\mathcal{A} = \int d^4x \mathcal{L}(x) = \int d^4x \left( -\frac{1}{2} F_{\mu\nu}^* F^{\mu\nu} + M^2 V_\mu^* V^\mu \right), \tag{7.555}
\]

with the equations of motion

\[
\partial_\mu F^{\mu\nu} + M^2 V^\nu = 0, \tag{7.556}
\]

or

\[
[(-\partial^2 - M^2)g_{\mu\nu} + \partial_\mu \partial_\nu]V^\mu(x) = 0. \tag{7.557}
\]

These imply that the vector field \( V^\mu(x) \) has no four-divergence:

\[
\partial_\mu V^\mu(x) = 0, \tag{7.558}
\]

and that each component satisfies the Klein-Gordon equation

\[
(-\partial^2 - M^2)V^\mu(x) = 0. \tag{7.559}
\]
It will sometimes be convenient to view the photon as an $M \to 0$ limit of a massive vector meson. For this purpose we have to add a gauge fixing term to the Lagrangian to allow for a proper limit. The extended action reads

$$\mathcal{A} = \int d^4 x \mathcal{L}(x) = \int d^4 x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} M^2 V_\mu V^\mu - \frac{1}{2\alpha} (\partial_\mu V^\mu)^2 \right\},$$

resulting in the field equation

$$\partial_\mu F^{\mu\nu} + M^2 V^\nu + \frac{1}{\alpha} \partial^\nu \partial_\mu V^\mu = 0,$$

which reads more explicitly

$$\left[ (-\partial^2 - M^2) g_{\mu\nu} + \left( 1 - \frac{1}{\alpha} \right) \partial_\mu \partial_\nu \right] V^\mu(x) = 0.$$

Multiplying (7.561) with $\partial_\nu$ from the left gives, for the divergence $\partial_\nu V^\nu$, the Klein-Gordon equation

$$(\partial^2 + \alpha M^2) \partial_\nu V^\nu(x) = 0,$$

from which the constraint (7.558) follows in the limit of large $\alpha$.

### 7.6.1 Field Quantization

The canonical field momenta are

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial (\partial_0 V^\mu)} = -F^{\ast}_{\mu0},$$

where the complex conjugation is, of course, irrelevant for a real field. As for electromagnetic fields, the zeroth component $V^0(x)$ has no canonical field momentum. The vanishing of $\pi_0$ is a primary constraint. It expresses the fact that the Euler-Lagrange equation (4.811) for $V^0(x)$ is not a dynamical field equation.

For the spatial components $V^i(x)$, there are nonvanishing canonical equal-time commutation rules:

$$[\pi_i(x, t), V^j(x', t)] = -i\delta_i^j \delta(3)(x - x'),$$

$$[\pi^i(x, t), V^{j\dagger}(x', t)] = -i\delta^i_j \delta(3)(x - x').$$

From these we can calculate commutators involving $V^0(x, t)$ by using relation (4.811). The results are

$$[V^{0\dagger}(x, t), V^i(x', t)] = \frac{i}{M^2} \partial_\delta(3)(x - x'),$$

and

$$[V^0(x, t), V^i(x', t)] = 0.$$
To exhibit the particle content in the second-quantized fields, we now expand $V^\mu(x)$ into the complete set of classical solutions of Eqs. (7.556) and (7.558) in a large but finite volume $V$:

$$\sum_{k,s_3=0,\pm1} \frac{1}{\sqrt{2Vk^0}} \left[ e^{-ikx} e^\mu(k, s_3)a_{k,s_3} + e^{ikx} e^\mu_*(k, s_3)b_{k,s_3}^\dagger \right].$$

(7.568)

If the vector fields are real, the same expansion holds with particles and antiparticles being identified. Dealing with massive vector particles, we may label the polarization states by the third components $s_3$ of angular momentum in the particle’s rest frame. The polarization vectors were given explicitly in Eq. (4.815).

As for scalar and Dirac fields, we have associated the expansion coefficients of the plane waves of negative energy $e^{-ikx}$ with a creation operator $b_{k,s_3}^\dagger$ rather than a second type of annihilation operator $d_{k,s_3}$. In principle, the polarization vector associated with the antiparticles could have been some charge-conjugate polarization vector $e^\mu_*(k, s_3)$, by analogy with the spinors $v^*(p, s_3) = u^{c*}(p, s_3)$ [recall (4.678)]. In the present case of a vector field, that is multiplied with a Lorentz transformation by a real 4 × 4 Lorentz matrix $A^\mu_\nu$. Charge conjugation exchanges the polarization vector (or tensor) by its complex conjugate. This is a direct consequence of the charge conjugation property (7.309) of the four-vector currents to which the standard massive vector fields are coupled in the action. For the vector potential $V^\mu(x) \equiv A^\mu(x)$ of electromagnetism this will be seen most explicitly in Eq. (12.54).

Alternatively we may expand the field $V^\mu(x)$ in terms of helicity polarization states as

$$V^\mu(x) = \sum_k \frac{1}{\sqrt{2Vk^0}} \sum_{\lambda=\pm1,0} e^{-ipx} e^\mu_H(k, \lambda)a_{k,\lambda} + e^{ipx} e^\mu_*(k, \lambda)b_{k,\lambda}^\dagger,$$

(7.569)

where $e^\mu_H(k, \lambda)$ are the polarization vectors (4.822).

The quantization of the massive vector field with the action (7.560) proceeds most conveniently by introducing a transverse field

$$V^\mu_T(x) \equiv V^\mu + \frac{1}{\alpha M^2} \partial_\mu \partial_\nu V^\nu(x),$$

(7.570)

which is divergenceless, as a consequence of (7.563):

$$\partial^\mu V^\mu_T(x) = 0.$$

(7.571)

The full vector field is then a sum of a purely transverse field and the gradient of a scalar field $\partial_\nu V^\nu(x)$:

$$V^\mu(x) = V^\mu_T(x) - \frac{1}{\alpha M^2} \partial_\mu \partial_\nu V^\nu(x).$$

(7.572)

It can be expanded in terms of three creation and annihilation operators $a_{k,\lambda}^\dagger$, $a_{k,\lambda}$ for the three physical polarization states with helicities $\lambda = 0, \pm1$, and an extra pair of operators $a_{k,s}^\dagger$, $a_{k,s}$ for the scalar degree of freedom. The commutation rules are

$$[a_{k,\lambda}, a_{k',\lambda'}^\dagger] = [a_{k,\lambda}^\dagger, a_{k',\lambda'}] = 0,$$

$$[a_{k,\lambda}, a_{k',\lambda'}^\dagger] = -\delta_{k,k'} g_{\mu\nu},$$

(7.573)
where \( \lambda \) runs through the helicities 0, ±1, and \( s \). The metric \( g_{\lambda\lambda'} \) has a negative sign for \( \lambda = 0, \pm 1 \), and a positive one for \( \nu = s \). These commutation rules have the same form as those in (7.404), except for the different meaning of the labels \( \nu \). The properly quantized field is then

\[
V^\mu(x) = \sum_k \sum_{\lambda=0,\pm1} \frac{1}{\sqrt{2V_k^0}} \left[ e^{-ikx} \epsilon^\mu(k,\lambda) a_{k,\lambda} + \epsilon^{\mu*}(k,\lambda) a_{k,\lambda}^\dagger \right] + \frac{1}{\sqrt{2V_k^0}} \left[ e^{-ikx} \epsilon^\mu(k,s) a_{k,s} + \epsilon^{\mu*}(k,s) a_{k,s}^\dagger \right].
\]

(7.574)

### 7.6.2 Energy of Massive Vector Particles

The energy can be calculated by close analogy with that of photons. We shall consider a real field, and admit an additional coupling to an external current density \( j^\mu(x) \), with an interaction

\[
L_{\text{int}} = -j^\mu(x)V^\mu(x).
\]

(7.575)

This causes only minor additional labor but will be useful to understand a special feature of the massive propagator to be calculated in the next section. The current term changes the equations of motion (7.556) to

\[
\partial_\mu F^\mu_\nu + M^2 V_\nu - j^\nu = 0,
\]

(7.576)

implying, for the non-dynamical zeroth component of \( V^\mu(x) \), the relation

\[
V^0(x) = \frac{1}{M^2} \left[ \nabla \cdot \dot{\mathbf{V}}(x) + \nabla^2 V^0(x) - j^0(x) \right],
\]

(7.577)

rather than (4.811). As in electromagnetism, we define the negative vector of the canonical field momenta (7.564) as the “electric” field strength of the massive vector field:

\[
\mathbf{E}(x) \equiv -\pi(x) = \mathbf{\dot{V}}(x) - \nabla V^0(x).
\]

(7.578)

Combining this with (7.577), we obtain

\[
\mathbf{\dot{V}}(x) = -\mathbf{E}(x) - \nabla V^0(x) = -\mathbf{E}(x) + \frac{1}{M^2} \nabla[\nabla \cdot \mathbf{E}(x) - j^0(x)].
\]

(7.579)

We now form the Hamiltonian density (7.340) as the Legendre transform

\[
\mathcal{H} = \pi \mathbf{\dot{V}} - L_0 - L_{\text{int}}.
\]

(7.580)

Inserting (7.579), we obtain

\[
\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}},
\]

(7.581)

with a free Hamiltonian density

\[
\mathcal{H}_0 = \frac{1}{2} \left[ \mathbf{E}^2 + \mathbf{B}^2 + \frac{V_0}{M^2} (\nabla \cdot \mathbf{E}) + M^2 \mathbf{V}^2 \right].
\]

(7.582)
and an interaction energy density

\[ H_{\text{int}} = j \cdot V - \frac{1}{M^2} j^0 \nabla \cdot E + \frac{1}{M^2} j^0 v^2. \]  

(7.583)

Here we have introduced the field

\[ B(x) = \nabla \times V(x) \]  

(7.584)

by analogy with the magnetic field of electromagnetism [compare (7.340)].

Discarding again the external current, we form the free Hamiltonian

\[ H_0 = \int d^3x \{ -\frac{1}{2} [\dot{V}^\mu \dot{V}^\mu - \nabla V^\mu \cdot \nabla V^\mu - M^2 V^\mu V^\mu] - \nabla \cdot (E V^0) \}. \]  

(7.585)

The last term is a surface term that can be omitted in an infinite volume. Inserting the expansion (7.569) for the field operator, and proceeding as in the case of the photon energy (7.356), we obtain the energy

\[ H_0 = \sum_{k, \lambda = 0, \pm 1} k^0 \left( a_{k, \lambda}^\dagger a_{k, \lambda} + \frac{1}{2} \right). \]  

(7.586)

This contains a vacuum energy

\[ E_0 \equiv \langle 0 | H_0 | 0 \rangle = \frac{1}{2} \sum_{k, \lambda = 0, \pm 1} k^0 = \frac{3}{2} \sum_k \omega_k, \]  

(7.587)

due to the zero-point oscillations. The factor 3 accounts for the different polarization states.

### 7.6.3 Propagator of Massive Vector Particles

Let us now calculate the propagator of a massive vector particle from the vacuum expectation value of the time-ordered product of two vector fields. As in (7.360) we find

\[ G_{\mu\nu}(x, x') = \Theta(x_0 - x'_0) \frac{1}{2V} \sum_k \frac{1}{k^0} e^{-i k(x - x')} \sum_{\lambda} e^{i\lambda(x, \lambda)} \epsilon_{\mu\nu}(x, \lambda) \]

\[ + \Theta(x'_0 - x_0) \frac{1}{2V} \sum_k \frac{1}{k^0} e^{i k(x - x')} \sum_{\lambda} e^{i\lambda(x, \lambda)} \epsilon_{\mu\nu}(x, \lambda), \]  

(7.588)

and insert the completeness relation (4.820) to write

\[ G_{\mu\nu}(x, x') = \Theta(x_0 - x'_0) \frac{1}{2V} \sum_k \frac{1}{k^0} \left( -g^{\mu\nu} + \frac{k^\mu k^\nu}{M^2} \right) \]  

\[ + \Theta(x'_0 - x_0) \frac{1}{2V} \sum_k \frac{1}{k^0} e^{i k(x - x')} \left( -g^{\mu\nu} + \frac{k^\mu k^\nu}{M^2} \right). \]  

(7.589)
This expression has precisely the general form (7.283), with the polarization sums of particles and antiparticles being equal to one another. The common polarization sum can be pulled in front of the Heaviside functions, yielding

\[
G^{\mu\nu}(x, x') = \left( -g^{\mu\nu} - \frac{\partial^{\mu}\partial^{\nu}}{M^2} \right) \left[ \Theta(x_0 - x'_0)G^{(+)}(x - x') + \Theta(x'_0 - x_0)G^{(-)}(x - x') \right] + \frac{1}{M^2} \left\{ [\partial^{\mu}\partial^{\nu}, \Theta(x^0 - x'^0)]G^{(+)}(x - x') + [\partial^{\mu}\partial^{\nu}, \Theta(x'^0 - x^0)]G^{(-)}(x - x') \right\}.
\]

The bracket in the first expression is equal to the Feynman propagator of the scalar field. The second term only contributes if both \( \mu \) and \( \nu \) are 0. For \( \mu = i \) and \( \nu = j \), it vanishes trivially. For \( \mu = 0 \) and \( \nu = i \), it gives

\[
\frac{1}{M^2} \left[ \partial^{i}\delta(x^0 - x'^0)G^{(+)}(x - x') + \partial^{i}\delta(x'^0 - x^0)G^{(-)}(x - x') \right].
\]

Since

\[
\delta(x^0 - x'^0)G^{(+)}(x - x') = \delta(x'^0 - x^0)G^{(-)}(x - x'),
\]

these components vanish. If both components \( \mu \) and \( \nu \) are zero, we use the trivial identities

\[
[\partial_0 \partial_0, \Theta(x^0 - x'^0)] = \partial_0 \delta(x^0 - x'^0) + 2\delta(x^0 - x'^0)\partial_0,
\]

\[
[\partial_0 \partial_0, \Theta(x'^0 - x^0)] = -\partial_0 \delta(x'^0 - x^0) - 2\delta(x^0 - x'^0)\partial_0,
\]

and the relation

\[
\partial_0 G^{(\pm)}(x - x') \bigg|_{x_0 = x'_0 = 0} = \pm \frac{i}{2} \delta^{(3)}(x - x'),
\]

(7.591)

to obtain \(-2i\delta^{(4)}(x - x')\) from the last two terms in (7.590). By a calculation like (7.44), we see that the first terms remove a factor 2 from this, so that we find for the Feynman propagator of the massive vector field the expression

\[
G^{\mu\nu}(x, x') = -\int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-x')}}{k^2 - M^2 + i\eta} \left( g^{\mu\nu} - \frac{k^\mu k^\nu}{M^2} \right) - \frac{i}{M^2} \delta_{\mu0} \delta_{\nu0} \delta^{(4)}(x - x').
\]

(7.592)

Only the first term is a Lorentz tensor. The second term destroys the covariance. It is known as a Schwinger term.

Let us trace its origin in momentum space. If we replace the Heaviside functions in (7.588) by their Fourier integral representations (7.63), we find a representation of the type (7.290) for the propagator:

\[
G^{\mu\nu}(x - x') = \int \frac{d^4k}{(2\pi)^4} \frac{i}{2\omega_p} \left[ \frac{P^{\mu\nu}(\omega_k, k)}{k^0 - \omega_k + i\eta} - \frac{P^{\mu\nu}(-\omega_k, k)}{k^0 + \omega_k - i\eta} \right] e^{-ip(x - x')},
\]

(7.593)
with the on-shell projection matrices

$$
P^{\mu\nu}(\omega_k, k) \equiv P^{\mu\nu}(k^0, k) \bigg|_{k^0 = \omega_k} \equiv - \left( g^{\mu\nu} - \frac{k\mu k\nu}{M^2} \right) \bigg|_{k^0 = \omega_k}. \tag{7.594}
$$

Expression (7.593) is a tensor only if the on-shell energy $k^0 = \omega_k$ in the argument of $P^{\mu\nu}(k^0, k)$ is replaced by the off-shell integration variable $k^0$. The difference between (7.594) and such a covariant version has two contributions: one from the linear term in $k^0$, and one from the quadratic term. The first vanishes by the same mechanism as in the Dirac discussion of (7.290). The quadratic term contributes only for $(\mu, \nu) = (0, 0)$, where

$$
P^{00}(\omega_k, k) - P^{00}(k^0, k) = \frac{\omega_k^2 - k^{02}}{M^2}. \tag{7.595}
$$

The combination $\omega_k^2 - k^{02} = M^2 - k^2$ cancels the denominators in (7.593), thus producing precisely Schwinger’s $\delta$-function term (7.592).

The covariant part of the propagator coincides with the Green function of the field equation (7.557). Indeed, the differential operator on the left-hand side can be written by analogy with (7.534) as

$$
L_{\mu\nu}(i\partial)G^{\nu\kappa}(x - x') = -i\delta_{\mu}^{\kappa}\delta^{(4)}(x - x'), \tag{7.596}
$$

so that the calculation of the Green function requires inverting the matrix in momentum space [the analog of (7.534)]

$$
L_{\mu\nu}(k) = (k^2 - M^2)P_{T\mu\nu}(k) - M^2P_{L\mu\nu}(k). \tag{7.597}
$$

This has the inverse

$$
L^{-1\mu\nu}(k) = \frac{1}{k^2 - M^2} P^{\mu\nu}(k) - \frac{1}{M^2} P^{\mu\nu}(k) = \frac{1}{k^2 - M^2} \left( g^{\mu\nu} - \frac{k\mu k\nu}{M^2} \right), \tag{7.598}
$$

yielding the solution of (7.596) in momentum space:

$$
G^{\mu\nu}(k) = -iL^{-1\mu\nu}(k). \tag{7.599}
$$

This is precisely the Fourier content in the first term of (7.592). The mass contains an infinitesimal $-i\eta$ to ensure that particles and antiparticles decay both at positive infinite times.

For a massive vector meson whose action contains a gauge fixing term as in (7.560), the matrix (7.597) reads

$$
L_{\mu\nu}(k) = (k^2 - M^2)P_{T\mu\nu}(k) + \left( \frac{k^2}{\alpha} - M^2 \right) P_{L\mu\nu}(k). \tag{7.600}
$$
It has an inverse (7.598):

\[
L^{-1 \mu \nu}(k) = \frac{1}{k^2 - M^2}P_L^{\mu \nu}(k) + \frac{\alpha}{k^2 - \alpha M^2}P_L^{\mu \nu}(k)
\]

\[
= \frac{1}{k^2 - M^2} \left[ g^{\mu \nu} - \frac{k^\mu k^\nu}{k^2 - \alpha M^2} \right].
\]

Due to the Schwinger term, the propagator is different from the Green function associated with the field equation (7.557). In Chapters 9, 10, and 14, we shall see that in the presence of interactions there are two ways of evaluating the physical consequences. One of them is based on the interaction picture of quantum mechanics and the Schwinger-Dyson perturbation expansion for scattering amplitudes derived in Section 1.6. The other makes use of functional integrals and eventually leads to similar results. The first is based on a Hamiltonian approach in which time-ordered propagators play a central role, and interactions are described with the help of an interaction Hamiltonian. The second is centered on a spacetime formulation of the action, where covariant Green functions are relevant, rather than time-ordered propagators. Eventually, the results will be the same in both cases, as we shall see. The cancellation of all Schwinger terms will be caused by the last term in the Hamiltonian density (7.583).

The commutator of two vector fields is

\[
[V^\mu(x), V^\nu(x')] = C_{\mu \nu}(x - x'),
\]

with

\[
C^\mu \nu(x - x') = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-x')/2\pi \epsilon(k^0)} \delta(k^2 - M^2) \left( -g^{\mu \nu} + \frac{k^\mu k^\nu}{M^2} \right)
\]

\[= \left( -g^{\mu \nu} - \frac{\partial^\mu \partial^\nu}{M^2} \right) C(x - x').
\]

At equal times, we rewrite in the integrand

\[
\delta(k^2 - M^2) = \frac{1}{2\omega_k} [\delta(k^0 - \omega_k) + \delta(k^0 + \omega_k)],
\]

with \(\omega_k \equiv \sqrt{k^2 + M^2}\), and see that the oddness of \(\epsilon(k^0)\) makes the commutator vanish, except for \((\mu, \nu) = (0, i)\), where \(k^0 \epsilon(k^0) = |k^0|\) produces at equal times an even integral

\[
\int d^4k e^{ik(x-x')} \delta(k^2-M^2)k^0k^i = \int d^3k^i e^{ik(x-x')} = -i\partial^i \delta^{(3)}(x-x') = i\partial_i \delta^{(3)}(x-x'),
\]

thus verifying the canonical commutation relation (7.566).

Let us end this section by justifying the earlier-used \(i\eta\)-prescription in the propagator (7.539) of the photon field. For this we add a mass term to the electromagnetic
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Lagrangian (7.376) with the gauge-fixing Lagrangian, which leads to a massive vector photon Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} M^2 A_\mu A^\mu - D(x) \partial^\mu A_\mu(x) + \frac{\alpha}{2} D^2(x), \quad \alpha \geq 0.$$  \hfill (7.604)

The Euler-Lagrange equations are now [compare (7.381)]

$$\left[-\partial^2 - M^2\right] g_{\mu\nu} + \left(1 - \frac{1}{\alpha} \right) \partial_\mu \partial_\nu A^\nu(x) = 0,$$  \hfill (7.605)

so that the propagator in momentum space obeys the equation

$$\left[(k^2 - M^2) g_{\mu\nu} - \left(1 - \frac{1}{\alpha} \right) k_\mu k_\nu\right] G^{\mu\nu}(k) = -i \delta^\mu_\kappa.$$  \hfill (7.606)

The matrix on the left-hand side is decomposed into transverse and longitudinal parts as

$$L_{\mu\nu}(k) = (k^2 - M^2) P^T_{\mu\nu}(k) + \frac{1}{\alpha} \left(k^2 - \alpha M^2\right) P_L_{\mu\nu}(k),$$  \hfill (7.607)

from which we obtain the inverse

$$L^{-1}_{\mu\nu}(k) = \frac{1}{k^2 - M^2} P^T_{\mu\nu}(k) + \frac{\alpha}{k^2 - \alpha M^2} P_L^{\mu\nu}(k),$$

$$= \frac{1}{k^2 - M^2} \left( g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) + \frac{\alpha}{k^2 - \alpha M^2} \frac{k^\mu k^\nu}{k^2},$$  \hfill (7.608)

which can be rearranged to

$$L^{-1}_{\mu\nu}(k) = \frac{g^{\mu\nu}}{k^2 - M^2} + \left(\alpha - 1\right) \frac{k^\mu k^\nu}{(k^2 - \alpha M^2)(k^2 - M^2)},$$  \hfill (7.609)

so that (7.547) is solved by the matrix

$$G^{\mu\nu}(k) = -i L^{-1}_{\mu\nu}(k).$$  \hfill (7.610)

The condition $\alpha > 0$ in the Lagrangian (7.604) ensures that both poles in the second denominator lie at a physical mass square.

Adding to the mass an infinitesimal imaginary part $-i\eta$ with $\alpha > 0$, and letting $M \to 0$, we obtain precisely the $i\eta$-prescription of Eq. (7.539).

### 7.7 Wigner Rotation of Spin-1 Polarization Vectors

Let us verify that the polarization vectors $\epsilon^\mu(k, s_3)$ and their complex-conjugates $\epsilon^{\mu*}(k, s_3)$ lead to the correct Wigner rotations for the annihilation and creation operators $a_{k, s_3}$ and $b^\dagger_{k, s_3}$. These have to be the spin-1 generalizations of (4.736) and (4.742), or (4.737) and (4.738). The simplest check are the analogs of Eqs. (4.743).
For this we form the $4 \times 3$-matrices corresponding to $u(0)$ and $v(0)$ in (4.743) [recall (4.669) and (4.670)]

$$e^\mu(k) \equiv \mp \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ i & 0 & -i \\ 0 & 0 & 0 \end{pmatrix}, \quad e^{\mu*}(k) \equiv \mp \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ -i & 0 & i \\ 0 & 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (7.611)

Multiplying these by the $4 \times 4$-generators [recall (4.54)–(4.56)]

$$L_3 = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_1 = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad L_2 = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$  \hspace{1cm} (7.612)

from the left, we find the same result as by applying the $3 \times 3$ spin-1 representation matrices [recall (4.848)–(4.849)]:

$$D^j(L_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad D^j(L_1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad D^j(L_2) = -i \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$  \hspace{1cm} (7.613)

from the right.

For finite Lorentz transformation, the polarization vectors undergo the Wigner rotations

$$e^\mu(k, s_3) \xrightarrow{\Lambda} e'\mu(k, s_3) = \Lambda^\mu_\nu e^\nu(k, s_3) = \sum_{s_3'=-1}^{1} e^\mu(p, s_3') W^{(1)}_{s_3', s_3}(p', \Lambda, p),$$  \hspace{1cm} (7.614)

$$e^{\mu*}(k, s_3) \xrightarrow{\Lambda} e^{\mu*}(k, s_3) = \Lambda^\nu_\mu e^{\nu*}(k, s_3) = \sum_{s_3'=-1}^{1} e^{\mu*}(p, s_3') W_{s_3', s_3}(p', \Lambda, p),$$  \hspace{1cm} (7.615)

where $W^{(1)}_{s_3', s_3}$ is the spin-1 representation of the $2 \times 2$ Wigner rotations (4.736). The second transformation law follows from the first as a consequence of the reality of the Lorentz transformations $\Lambda^\nu_\mu$.

In closer analogy with (4.678), we can obtain the charge-conjugate polarization vectors by forming

$$e_c^\mu(k, s_3) \equiv -e^\mu(k, s_3') c^{(1)}_{s_3 s_3'},$$  \hspace{1cm} (7.616)

where $c^{(s)}_{s_3 s_3'}$ denotes the spin-$s$ representation of the rotation matrix $c = e^{-i\pi/2}$ introduced in Eq. (4.900). It is easy to verify that $e_c^\mu(k, s_3)$ coincides with $e^{\mu*}(k, s_3)$. The matrix $c^{(s)}_{s_3 s_3'}$ turns the spin into the opposite direction:

$$c^{(s)}_{s_3 s_3'} = \langle s, s_3 | e^{-iL_2\pi/2} | s', s_3' \rangle = d_{s_3 s_3'}^{s} (\pi) = (-)^{s+s_3} \delta_{s_3, -s_3'},$$  \hspace{1cm} (7.617)
and has the property [recall (4.901)]:

$$D^j(R)c^{(j)} = c^{(j)}D^j(R),$$

(7.618)

which confirms that $e^\mu_c(k, s_3)$ transforms with the complex-conjugate of the Wigner rotation of $e^\mu(k, s_3)$.

In principle, there exists also a fourth vector which may be called scalar polarization vector to be denoted by

$$e^\mu(k, s) \equiv k^\mu.$$

(7.619)

This polarization vector will be of use later in Subsecs. 4.9.6 and 7.5.2. It corresponds to a pure gauge degree of freedom since in $x$-space it has the form $\partial^\mu \Lambda$. As such, it transforms under an extra independent and irreducible representation of the Lorentz group describing a scalar particle degree of freedom and no longer forms part of the vector particle. It certainly does not contribute to the gauge-invariant electromagnetic action.

Observe that for a very small mass where $\omega_k \to |k|$, the scalar polarization vector has the limit

$$M e^\mu(k, s) \xrightarrow{M \to 0} |k| \begin{pmatrix} 1 \\ i k \end{pmatrix} = k^\mu,$$

(7.620)

i.e., it goes over into the unphysical scalar degree of freedom. This is why the longitudinal polarization does not contribute to the action of a massless vector particle.

### 7.7.1 Behavior under Discrete Symmetry Transformations

Under the discrete transformations $P, T, \text{and } C$, a massive vector field transforms in the same way as the vector potential of electromagnetism in Eqs. (7.548)–(7.552), except for different possible phases $\eta_P, \eta_T, \eta_C$.

In the second-quantized Hilbert space, these amount to the following transformation laws for the creation and annihilation operators of particles of helicity $\lambda$.

Under parity we have:

$$\mathcal{P} a_{k,\lambda}^\dagger \mathcal{P}^{-1} \equiv a_{k,\lambda}^\dagger = \eta_P a_{-k,-\lambda}^\dagger,$$

(7.621)

with $\eta_P = \pm 1$. Under time reversal:

$$\mathcal{T} a_{k,\lambda}^\dagger \mathcal{T}^{-1} \equiv a_{k,\lambda}^\dagger = \eta_T a_{-k,\lambda}^\dagger,$$

(7.622)

with an arbitrary phase factor $\eta_T$, and under charge conjugation

$$\mathcal{C} a_{k,\lambda}^\dagger \mathcal{C}^{-1} \equiv a_{k,\lambda}^\dagger = \eta_C a_{k,\lambda}^\dagger,$$

(7.623)

with $\eta_C = \pm 1$.

Vector mesons such as the $\rho$-meson of mass $m_\rho \approx 759$ MeV and the $\omega$-meson of mass $m_\omega \approx 782$ MeV transform with the three phase factors $\eta_P, \eta_T, \eta_C$ being equal.
to those of a photon. This is a prerequisite for the theory of vector meson dominance of electromagnetic interactions which approximate all electromagnetic interactions of strongly interacting particles by assuming the photon to become a mixture of a neutral $\rho$-meson and an $\omega$-meson, which then participate in strong interactions.

We omit the calculation of the Hamilton operator since the result is an obvious generalization of the previous expression (7.586) for neutral vector bosons:

$$H = \sum_{\mathbf{p}} p^0 \sum_{\lambda = \pm 1, 0} (a_{p, \lambda}^\dagger a_{p, \lambda} - b_{p, \lambda}^\dagger b_{p, \lambda}) = \sum_{\mathbf{p}} p^0 \sum_{s_3 = \pm 1, 0} (a_{p, s_3}^\dagger a_{p, s_3} - b_{p, s_3}^\dagger b_{p, s_3}).$$

(7.624)

### 7.8 Spin-3/2 Fields

The Rarita-Schwinger field of massive spin-3/2 particles is expanded just like (7.224) as follows:

$$\psi_\mu(x) = \sum_{\mathbf{p}, S_3} \frac{1}{\sqrt{V p^0/M}} \left[ e^{-ipx} u_\mu(p, S_3) a_{p, s_3} + e^{ipx} v_\mu(p, S_3) b_{p, s_3}^\dagger \right] u_{p, s_3},$$

(7.625)

where the spinors $u_\mu(p, S_3)$ and $v_\mu(p, S_3)$ are solutions of equations analogous to (4.663):

$$L^{\mu\nu}(p) u_\nu(p, S_3) = 0, \quad L^{\mu\nu}(-p) v_\nu(p, S_3) = 0.$$  

(7.626)

By analogy with the Dirac matrices ($\sl{p} - M$) and ($\sl{p} - M$) in those equations, the matrices $L^{\mu\nu}(p)$ and $L^{\mu\nu}(-p)$ are the Fourier-transformed differential operators (4.955) with positive mass shell energy $p^0 = \sqrt{p^2 + M^2}$. The Rarita-Schwinger spinors $u_\mu(p, S_3)$ and $v_\mu(p, S_3)$ can be constructed explicitly from those of spin 1/2 and the polarization vectors (7.616) of spin 1 with the help of the Clebsch-Gordan coefficients in Table 4.2:

$$u_\mu(p, S_3) = \sum_{s_3, s_3'} \langle \frac{3}{2}, S_3 | 1, s_3; \frac{1}{2}, s_3' \rangle \epsilon_\mu(p, s_3) u(p, s_3'),$$

(7.627)

$$v_\mu(p, S_3) = \sum_{s_3, s_3'} \langle \frac{3}{2}, S_3 | 1, s_3; \frac{1}{2}, s_3' \rangle \epsilon_\mu(p, s_3) v(p, s_3').$$

(7.628)

The calculation of the propagator

$$S_{\mu\nu}(x, x') = \langle 0 | T \psi_\mu(x) \bar{\psi}_\nu(x') | 0 \rangle$$

(7.629)

is now somewhat involved. As in the case of spin-zero, spin-1/2, and massive spin-1 fields, the result can be derived most directly by inverting the matrix $L^{\mu\nu}(p)$. For this we make use of the fact that, for reasons of covariance, $L^{-1}_{\mu\nu}$ may be expanded into the tensors $g_{\mu\nu}, \gamma_\mu \gamma_\nu, \gamma_{\mu p_\nu}, \gamma_{\nu p_\mu}, p_\mu p_\nu$ with coefficients of the form $A + B \sl{p}$. When
multiplying this expansion by $L^\nu\lambda(p)$ and requiring the result to be equal to $\delta^\nu_\lambda$, we find $L^{-1}_{\mu\nu}$ to be equal to the $4 \times 4$ spinor matrix

$$L^{-1}_{\mu\nu} = \left\{ \frac{\not{p} + M}{p^2 - M^2} \left[ -g_{\mu\nu} + \frac{1}{3} \gamma_\mu \gamma_\nu + \frac{1}{3M} (\gamma_\mu p_\nu - \gamma_\nu p_\mu) + \frac{2}{3M} p_\mu p_\nu \right] \right. \right.$$

$$\left. \left. - \frac{1}{6M^2 (2w + 1)^2} [(4w + 2) \gamma_\mu p_\nu + (w + 1) \not{p} \gamma_\mu + 2wp_\mu \gamma_\nu + 2wM \gamma_\mu \gamma_\nu] \right\}.$$

For simplicity, we have stated this matrix only for the case of a real $w$. By analogy with (7.289), the propagator is equal to the Fourier transform of $L^{-1}_{\mu\nu}(p)$:

$$S_{\mu\nu}(x, x') = \int \frac{d^4 p}{(2\pi)^4} L^{-1}_{\mu\nu}(p)e^{-ip(x-x')}.$$

(7.631)

The wave equation has a nontrivial solution where $L^{-1}(p)$ has a singularity. From (7.630) we see that this happens only on the positive- and negative-energy mass shells at

$$p^0 = \pm \sqrt{p^2 + M^2}.$$

(7.632)

By writing the prefactor in (7.630) as

$$\frac{\not{p} + M}{p^2 - M^2} = \frac{1}{\not{p} - M},$$

(7.633)

we see that only solutions of

$$(\not{p} - M)_{\alpha'} \psi_{\mu\nu'}(p) = 0, \quad p^0 = \pm \sqrt{p^2 + M^2}$$

(7.634)

can cause this singularity. These are the spinors $\psi_{\mu}(p, S_3)$ and $\psi_{\mu}(-p, S_3)$, respectively. The residue matrix accompanying the singularity is independent of the parameter $c$. It must have the property of projecting precisely into the subspace of $\psi_{\mu\alpha}(p)$ in which the wave equation $L^\mu\nu(p)\psi_\nu(p) = 0$ is satisfied. Let

$$L^{-1}_{-\mu\nu}(p) = \frac{2M}{p^2 - M^2} P^{\mu\nu}(p) + \text{regular piece}.$$

(7.635)

After a little algebra, we find

$$P^{\mu\nu}(p) = \frac{\not{p} + M}{2M} \left[ -g_{\mu\nu} + \frac{1}{3} \gamma_\mu \gamma_\nu + \frac{1}{3M} (p_\mu \gamma_\nu - p_\nu \gamma_\mu) + \frac{2}{3M^2} p_\mu p_\nu \right].$$

(7.636)

The residue matrix $P^{\mu\nu}(p)$ is uniquely fixed only for $p_0$ on the upper and lower mass shells $p_0 = \pm \sqrt{p^2 + M^2}$. There it has the form $P^{\mu\nu}_\pm(p)$, respectively. The expression (7.636) is a common covariant off-shell extension of these two matrices.
These matrices and their extension \( P_{\mu\nu}(p) \) are independent of the parameter \( c \). They are projection matrices, satisfying
\[
P_{\mu\nu} g_{\nu\kappa} P_{\kappa\lambda} = P_{\mu\lambda}. \tag{7.637}
\]

It is easy to verify that for \( p^2 = M^2 \), the matrix \( P_{\mu\nu}(p) \) satisfies the same equations as the Rarita-Schwinger spinors \( u_{\mu}(p, S_3) \):
\[
(\not{\dot{p}} - M) P_{\mu\nu}(p) = 0, \quad P_{\mu\nu}(p) p^\nu = 0, \quad P_{\mu\nu}(p) \gamma^\nu = 0. \tag{7.638, 7.639, 7.640}
\]

For this reason, \( P_{\mu\nu}(\pm p) \) with \( p^0 = \omega_p \) can be expanded as
\[
P_{\mu\nu}(p) = \sum_{s_3} u_{\mu}(p, S_3) \bar{u}_\nu(p, S_3), \quad P_{\mu\nu}(-p) = -\sum_{s_3} v_{\mu}(p, S_3) \bar{v}_\nu(p, S_3). \tag{7.641, 7.642}
\]

The projection matrix is equal to the polarization sums of the spinors \( u_{\mu}(p, S_3) \) and \( v_{\mu}(p, S_3) \), except for a minus sign accounting for the fact that \( \bar{v}(p, S_3) v(p, S_3) = -\bar{u}(p, S_3) u(p, S_3) \), just as in the Dirac case [recall the discussion after Eq. (4.705)].

The behavior under discrete symmetry transformations of the creation and annihilation operators of a Rarita-Schwinger field is the same as that of a product of a Dirac operator and a massive vector meson operator, i.e., it is given by a combination of the transformation laws (7.301), (7.322), (7.306) and (7.621), (7.622), (7.623).

### 7.9 Gravitons

In order to quantize the gravitational field we first have to rewrite the action as squares of first derivative terms. After a partial integration, it reads
\[
A = \frac{1}{8\kappa} \int d^4x \left[ \partial_\lambda h^{\mu\nu} \partial^\lambda h_{\mu\nu} - 2 \partial^\rho h^{\nu\lambda} \partial_\lambda h_{\mu\rho} + 2 \partial_\mu h \partial_\nu h^{\mu\rho} - \partial_\mu h \partial_\nu h \right], \tag{7.643}
\]
where \( h \equiv h^\mu_\mu \). A few further partial integrations bring this to the alternative form
\[
A = \int d^4x \mathcal{L}(x) = \frac{1}{2} \int d^4x \pi_{\lambda\mu} \partial_\lambda h_{\mu\nu}, \tag{7.644}
\]
where
\[
\pi_{\lambda\mu} \equiv \frac{1}{8\kappa} \left[ \partial_\lambda h_{\mu\nu} - \partial_\mu h_{\lambda\nu} + g_{\lambda\nu} \partial_\mu h - g_{\mu\nu} \partial_\lambda h + g_{\mu\nu} \partial^\rho h^{\kappa\lambda} - g_{\lambda\nu} \partial^\rho h^{\kappa\mu} \right] + (\mu \leftrightarrow \nu). \tag{7.645}
\]
is defined by
\[ \pi_{\lambda\mu\nu}(x) \equiv \frac{\partial L(x)}{\partial[\partial_\lambda h_{\mu\nu}(x)]}. \] (7.646)

It is antisymmetric in \( \lambda\mu \) and symmetric in \( \mu\nu \). The components \( \pi_{0\mu\nu} \) play the role of the canonical field momenta. Similar to the electromagnetic case, four of the six independent components \( \pi_{0ij} \) vanish, as a consequence of the invariance of the action under gauge transformations (4.380):
\[ h_{\mu\nu}(x) \rightarrow h_{\mu\nu}(x) + \partial_\mu \Lambda_\nu(x) + \partial_\nu \Lambda_\mu(x). \] (7.647)

Going over to the field \( \phi_{\mu\nu}(x) \) of Eq. (4.403) and the Hilbert gauge (4.403), we expand the field into plane waves \( e^{-ikx} \) with \( k^0 = |k| \) and two transverse polarization tensors (4.403):
\[ \phi_{\mu\nu}(x) = \frac{1}{\sqrt{2V^0}} \sum_k \sum_{\lambda=\pm2} \left[ a_{k,\lambda}\epsilon^{\mu\nu}(k,\lambda)e^{-ikx} + a^\dagger_{k,\lambda}\epsilon^{\mu\nu}(k,\lambda)e^{ikx} \right]. \] (7.648)

The behavior under discrete symmetry transformations of the creation and annihilation operators of a free graviton field is the same as that of a product of two photon operators in Eqs. (7.548)–(7.550).

### 7.10 Spin-Statistics Theorem

When quantizing Klein-Gordon and Dirac fields we directly used commutation rules for spin 0 and anticommutation rules for spin 1/2. In nonrelativistic quantum field theory this procedure was dictated by experimental facts. It is one of the important successes of relativistic quantum field theory that this connection between spin and statistics is a necessity if one wants to quantize free fields canonically. Let us first look at the real scalar field theory, and consider the field commutator expanded as in (7.11):
\[ i[\partial^0\phi(x,t),\phi(x',t)] = \sum_{p,p'} \frac{1}{\sqrt{2V^02V^0'}} \left( e^{i(p_0x-p'_0x')}p^0[a_p,a^\dagger_{p'}] + e^{-i(p_0x-p'_0x')}(-p^0)[a^\dagger_p,a_{p'}] \right). \] (7.649)

Here we have left out terms with vanishing commutators \( [a_p,a_{p'}] \) and \( [a^\dagger_p,a^\dagger_{p'}] \). Inserting the commutators (7.15), we see that both contributions just add up correctly to give a \( \delta \)-function, as they should, in order to satisfy the canonical field commutation rules (7.2)–(7.4). Suppose now for a moment that the particles obeyed Fermi statistics. Then the symbol \( [\partial^0\phi(x,t),\phi(x',t)] \) would stand for anticommutators, and the two contributions would have to be subtracted from each other, giving zero. Thus a real relativistic scalar field cannot be quantized according to the (wrong) Fermi statistics. The situation is similar for a complex scalar field, where we obtain
\[ i[\partial^0\phi(x,t),\phi^\dagger(x',t)] = \sum_{p,p'} \frac{1}{\sqrt{2V^02V^0'}} \left( e^{i(p_0x-p'_0x')}p^0[a_p,a^\dagger_{p'}] + e^{-i(p_0x-p'_0x')}(-p^0)[b^\dagger_p,b_{p'}] \right). \] (7.650)
Here all vanishing commutators have been omitted. Again, the two terms add up correctly to give a $\delta^{(3)}$-function, while the use of anticommutators would have led to a vanishing of the right-hand side.

These observations are related to another remark made earlier when we expanded the free complex field into the solutions of the wave equation, and where we wrote down immediately creation operators $b_p^\dagger$ for the negative energy solutions $e^{ipx}$ as a generalization of $a_p^\dagger$ for the real field. At that place we could, in principle, have had the option of using $b_p$. But looking at (7.650), we realize that this would not have led to canonical commutation rules in either statistics.

Consider now the case of spin-1/2 fields. Here we calculate the anticommutator expanded via (7.223):

$$\{\psi(x, t), \psi^\dagger(x', t)\} = \sum_{p, p'} \frac{1}{\sqrt{V p^0/M} \sqrt{V p'^0/M}} \sum_{s_3, s_3'} \sum_{\pm=1/2} e^{ip(x-p')} \left\{ a_{p, s_3}, a_{p', s_3'}^\dagger \right\} u(p, s_3) u^\dagger(p', s_3')$$

$$+ e^{-i(p-x')} \left\{ b_{p, s_3}, b_{p', s_3'}^\dagger \right\} v(p, s_3) v^\dagger(p', s_3'). \quad (7.651)$$

Inserting here the anticommutation rules (7.228) and (7.229),

$$\left\{ a_{p, s_3}, a_{p', s_3'}^\dagger \right\} = \delta_{p, p'} \delta_{s_3, s_3'},$$

$$\left\{ b_{p, s_3}, b_{p', s_3'}^\dagger \right\} = \delta_{p, p'} \delta_{s_3, s_3'}, \quad (7.652)$$

and the polarization sums (4.702) and (4.703),

$$\sum_{s_3} u(p, s_3) u^\dagger(p, s_3) = \frac{p + M}{2M} \gamma_0,$$

$$\sum_{s_3} v(p, s_3) v^\dagger(p, s_3) = \frac{p - M}{2M} \gamma_0, \quad (7.653)$$

we find

$$\{\psi(x, t), \psi^\dagger(x', t)\} = \sum_p \frac{1}{V p^0/M} \left( e^{-ip(x-x')} \frac{p + M}{2M} \gamma_0 + e^{ip(x-x')} \frac{p - M}{2M} \gamma_0 \right)$$

$$= \sum_p \frac{1}{V} e^{ip(x-x')} = \delta^{(3)}(x - x'). \quad (7.654)$$

The function $\delta^{(3)}(x - x')$ arises from the $p^0 \gamma^0$-terms after a cancellation of the terms with $M$ and $p \gamma$ in the numerator.

Let us see what would happen if we use commutators for quantization and thus the wrong particle statistics. Then the $[b_{p, s_3}, b_{p, s_3}]$-term would change its sign and fail to produce the desired $\delta^{(3)}$-function. At first sight, one might want to correct this by using, in the expansion of $\psi(x, t)$, annihilation operators $d_{p, s_3}$ rather than creators $b_{p, s_3}^\dagger$. Then the commutator $[b_{p, s_3}, b_{p, s_3}]$ would be replaced by $[d_{p, s_3}, d_{p, s_3}^\dagger]$, and give indeed a correct sign after all!
However, the problem would then appear at a different place. When calculating the energy in (7.624), we found

$$H = \sum_{p,s} p^0 \left( a_{p,s}^\dagger a_{p,s} - b_{p,s}^\dagger b_{p,s} \right) ,$$

(7.655)

independent of statistics. Changing $b_{p,s}^\dagger$ to $d_{p,s}^\dagger$ in the field expansion, would give instead

$$H = \sum_{p,s} p^0 \left( a_{p,s}^\dagger a_{p,s} - d_{p,s}^\dagger d_{p,s} \right) .$$

(7.656)

But this is an operator whose eigenvalues can take arbitrarily large negative values on states containing a large number of creation operators $d_{p,s}^\dagger$ applied to the vacuum state $|0\rangle$. Such an energy is unphysical since it would imply the existence of a perpetuum mobile.

The spin-statistics relation can be extended in a straightforward way to vector and tensor fields, and further to fields of any spin $s$. For each of these fields, quantization with the wrong statistics would imply either a failure of locality or of the positivity of the energy.

We shall sketch the general proof only for the spinor field $\xi(x)$ introduced in Section 4.19. This is expanded into plane waves (4.19) as before, assigning creation operators of antiparticles to the Fourier components of the negative-energy solutions:

$$\xi(x) = \sum_p \frac{1}{\sqrt{2 V p^0}} \left[ e^{-ipx} w(p, s_3) a_{p,s_3} + e^{ipx} w^c(p, s_3) b_{p,s_3}^\dagger \right] .$$

(7.657)

Here

$$w^c(p, s_3) = c^{(2s)} w^*(p, s_3) = w(p, s_3') c^{(s)}_{s_3', s_3} = w(p, -s_3)(-1)^{s-s_3}$$

(7.658)

are the charge-conjugate spinors. The matrices $c^{(2s)}$ have the labels $c^{(2s)}_{n_1, n_2; n_1', n_2'}$. They are the spin-2s equivalent of the matrices $c^{(s)}_{s_3, s_3'}$ reversing the spin direction by a rotation around the $y$-axis by an angle $\pi$ [recall the defining equation (4.900)]. The relation between their indices $n_1, n_2$ and $s_3$ is the same as in the rest spinors (4.938). These matrices have the property

$$c^{(2s)} c^{(2s)\dagger} = 1, \quad \left( c^{(2s)} \right)^2 = (-1)^{2s} .$$

(7.659)

The commutation or anticommutation relation between two such spinor fields has the form

$$i[\phi(x, t), \phi^\dagger(x', t')]_{\mp} = \sum_p \frac{1}{2 V p^0} \left[ e^{-ip(x-x')} \sum_{s_3} w(p, s_3) w^*(p, s_3) \right. \left. \pm e^{ip(x-x')} \sum_{s_3} w^c(p, s_3) w^{c*}(p, s_3) \right] .$$

(7.660)
Recalling the polarization sum (4.940)
\[ P(p) \equiv \sum_{s_3} w^{(2s)}(p, s_3) w^{(2s)*}(p, s_3) = \left( \frac{p \sigma}{M} \right)^{2s}, \tag{7.661} \]
and noting that the charge-conjugate spinors have the same sum, this becomes
\[ i[\phi(x, t), \phi^\dagger(x', t') \right)_\mp = \sum_p \frac{1}{2Vp^0} P(p) \left[ e^{-ip(x-x')} \mp e^{ip(x-x')} \right]. \tag{7.662} \]

The polarization sum \( P(p) \) is a homogenous polynomial in \( p^0 \) and \( p \) of degree \( 2s \). It has therefore the symmetry property \( P(-p) = (-1)^{2s} P(p) \). The energy \( p^0 \) lies on the mass shell \( \omega_p = \sqrt{p^2 + M^2} \). Replacing all even powers \( \omega_p^{2n} \) by \( (p^2 + M^2)^n \), and all odd powers \( \omega_p^{2n+1} \) by \( \omega_p (p^2 + M^2)^n \), we obtain for \( P(p) \) the following generic dependence on the spatial momenta
\[ P(p) = P_0(p) + \omega p P_1(p), \tag{7.663} \]
where \( P_0(p) \) and \( P_1(p) \) are polynomials of \( p \) with the reflection properties
\[ P_0(-p) = (-1)^{2s} P_0(p), \quad P_1(-p) = (-1)^{2s} P_1(p). \tag{7.664} \]
Thus we can write
\[ i[\phi(x, t), \phi^\dagger(x', t') \right)_\mp \]
\[ = \sum_p \frac{1}{2Vp^0} \left\{ [P_0(p) + p^0 P_1(p)] e^{-ip(x-x')} \mp (-1)^{2s} \left[ P_0(-p) - p^0 P_1(-p) \right] e^{ip(x-x')} \right\}. \tag{7.665} \]

After replacing \( \pm p \) by spatial derivatives \( -i \nabla \), the momentum-dependent factors can be taken outside the momentum sums, and we obtain
\[ i[\phi(x, t), \phi^\dagger(x', t') \right)_\mp = P_0(-i \nabla) \sum_p \frac{1}{2Vp^0} \left[ e^{-ip(x-x')} \mp (-1)^{2s} e^{ip(x-x')} \right] 
+ P_1(-i \nabla) \sum_p \frac{1}{2V} \left[ e^{-ip(x-x')} \pm (-1)^{2s} e^{ip(x-x')} \right]. \tag{7.666} \]

In the limit of infinite volume, the right-hand side becomes [recall (7.47)]
\[ P_0(-i \nabla) \left[ G^{(+)}(x - x') \mp (-1)^{2s} G^{(+)}(x - x') \right] 
+ P_1(-i \nabla) \left[ G^{(+)}(x - x') \pm (-1)^{2s} G^{(+)}(x - x') \right]. \tag{7.667} \]

At equal times, this reduces to
\[ P_0(-i \nabla)[1 \mp (-1)^{2s}] G^{(+)}(x - x', 0) 
+ P_1(-i \nabla)[1 \pm (-1)^{2s}] \delta^{(3)}(x - x'). \tag{7.668} \]
Locality requires that the commutator at spacelike distances vanishes. Since $G^{(+)}(x - x', 0) \neq 0$, the prefactor of the first term must be zero, and hence

$(-1)^{2s} = \begin{cases} 1 & \text{for bosonic commutators,} \\ -1 & \text{for fermionic anticommutators.} \end{cases}$

(7.669)

This proves the spin-statistics relation for spin-$s$ spinor fields $\xi(x)$.

Note that the locality of the free fields always forces particles and antiparticles to have the same mass and spin.

### 7.11 CPT-Theorem

All of the above local free-field theories have a universal property under the discrete symmetry operations $C$, $P$, and $T$. When being subjected to a product $CPT$ of the three operations, the Lagrangian is invariant. We shall see later in Chapter 27 that this remains true also in the presence of interactions that violate $CP$- or $P$-invariance, as long as these are local and Lorentz-invariant.

When introducing local interactions between local fields they will always have the property of being invariant under the operation $CPT$. This is the content of the so-called CPT-theorem. This property guarantees that the equality of masses and spins of particles and antiparticles remains true also in the presence of interactions. If there is a mass difference in nature, it must be extremely small. Direct measurements of masses of electrons and protons and their antiparticles are too insensitive to detect a difference with present experiments.

### 7.12 Physical Consequences of Vacuum Fluctuations — Casimir Effect

For each of the above relativistic quantum fields we have found an infinite vacuum energy due to the vacuum fluctuations of the field. For a real scalar field, this energy is [see (7.32)]

$$E_0 = \frac{1}{2} \sum_k k^0 = \frac{1}{2} \sum_k \omega_k = \frac{\hbar c}{2} \sum_k \sqrt{k^2 + c^2 M^2 / \hbar^2},$$

(7.670)

where $k = p / \hbar$ are the wave vectors. In many calculations, this energy is irrelevant and can be discarded. There are, however, some important physical phenomena where this energy matters. One of them is the cosmological constant as discussed on p. 479. The other appears in electromagnetism and is observable in the form of van der Waals forces between different dielectric media.

In the present context of free fields in a vacuum, the most relevant phenomenon is the Casimir effect. Vacuum fluctuations of electromagnetic fields cause an attraction between two parallel closely spaced silver plates in an otherwise empty space. The basic electromagnetic property of a silver plate is to enforce the vanishing of the
parallel electric field at its surface, due to the high conductivity. Let us study a specific situation and suppose the silver plates to be parallel to the $xy$-plane, one at $z = 0$ and the other at $z = d$. To have discrete momenta the whole system is imagined to be enclosed in a very large conducting box

$$x \in (-L_x/2, L_x/2), \quad y \in (-L_y/2, L_y/2), \quad z \in (-L_z/2, L_z/2). \quad (7.671)$$

The plates divide the box into a thin slice of volume $L_x L_y d$, and two large pieces of volumes $L_y L_y L_z/2$ below, and $L_x L_y (L_z/2 - d)$ above the slice (see Fig. 7.5). These are three typical resonant cavities of classical electrodynamics. For waves of frequency $\omega$, the Maxwell equations $\partial_\mu F^{\mu\nu} = 0$ read, in natural units with $c = 1$ [recall (4.247)],

$$\nabla \times \mathbf{B} = \dot{\mathbf{E}} = -i \omega \mathbf{E}, \quad \nabla \cdot \mathbf{E} = 0. \quad (7.672)$$

Since the field components parallel to the conducting surface vanish, the first equation implies that the same is true for the magnetic field components normal to the surface. Combining the Maxwell equations (7.672) with the equations

$$\nabla \times \mathbf{E} = i \omega \mathbf{B}, \quad \nabla \cdot \mathbf{B} = 0. \quad (7.673)$$

which follow from the Bianchi identity $\partial_\mu \tilde{F}^{\mu\nu} = 0$ [recall (4.246)], the fields $\mathbf{E}$ and $\mathbf{B}$ are seen to satisfy the *Helmholtz equations*

$$\left(\nabla^2 + \omega^2\right) \begin{cases} \mathbf{E}(x) \\ \mathbf{B}(x) \end{cases} = 0. \quad (7.674)$$

They are solved by the plane-wave ansatz

$$\begin{cases} \mathbf{E}(x) \\ \mathbf{B}(x) \end{cases} = \begin{cases} \mathbf{E}_0(x_\perp) \\ \mathbf{B}_0(x_\perp) \end{cases} e^{ikz}, \quad (7.675)$$

![Figure 7.5 Geometry of the plates for the calculation of the Casimir effect.](image-url)
where \( x_\perp \equiv (x, y) \) are the components of \( x \) parallel to the plates. The fields \( E_0(x_\perp), B_0(x_\perp) \) satisfy the differential equation
\[
(\nabla_\perp^2 + \omega^2 - k_z^2) \begin{cases} E_0(x_\perp) \\ B_0(x_\perp) \end{cases} = 0,
\]
where \( \nabla_\perp \) is the transverse derivative
\[
\nabla_\perp \equiv (\nabla_x, \nabla_y, 0).
\]

We now split the original fields \( E(x) \) and \( B(x) \) into a component parallel and orthogonal to the unit vector \( \hat{z} \) along the \( z \)-axis:
\[
E = E_z \hat{z} + E_\perp \\
B = B_z \hat{z} + B_\perp.
\]

Now we observe that the Maxwell equations (7.672) can be solved for \( E_\perp, B_\perp \) in terms of \( E_z, B_z \):
\[
B_\perp = \frac{1}{\omega^2 + \nabla^2} \left[ \nabla_\perp \partial_z B_z + i \omega \hat{z} \times \nabla_\perp E_z \right], \\
E_\perp = \frac{1}{\omega^2 + \nabla^2} \left[ \nabla_\perp \partial_z E_z - i \omega \hat{z} \times \nabla_\perp B_z \right].
\]

Therefore we only have to find \( E_z, B_z \). Let us see what boundary conditions the field components satisfy at the conducting surfaces, where we certainly have
\[
E_\perp = 0, \quad B_z = 0.
\]

Due to (7.679), this implies \( \nabla_\perp \partial_z E_z |_{\text{surface}} = 0 \). The Maxwell equations admit two types of standing-wave solutions with these conditions:

1) Transverse magnetic waves
\[
B_z \equiv 0, \\
E_z = \varphi_E(x_\perp) \cos \frac{\pi z}{d} n \quad n = 0, 1, 2, \ldots.
\]

2) Transverse electric waves
\[
E_z \equiv 0, \\
B_z = \varphi_B(x_\perp) \sin \frac{\pi z}{d} n \quad n = 1, 2, 3, \ldots,
\]

where \( \varphi_{E,B}(x_\perp) \) solve the transverse Helmholtz equation
\[
(\nabla_\perp^2 + \omega^2 - k_z^2) \varphi_{E,B}(x_\perp) = 0.
\]

If the dimensions of the box along \( x \)- and \( y \)-axes are much larger that \( d \), the quantization of the transverse momenta becomes irrelevant. Any boundary condition
will yield a nearly continuous set of states with the transverse density of states \( L_x L_y / (2\pi)^2 \). Thus we may record that there are two standing waves for each momentum

\[
\mathbf{k} = \left( k_x, k_y, k_z^n = \frac{\pi}{d} n \right), \quad (7.684)
\]

except for the case \( n = 0 \) when there is only one wave, namely the transverse magnetic wave. Using the discrete momenta in the sum (7.670), the energy between the plates may therefore be written as

\[
E_d = \frac{\hbar c}{2} L_x L_y \int \frac{dk_x dk_y}{(2\pi)^2} \sum_{k_z} \sqrt{k_x^2 + k_y^2 + k_z^2}. \quad (7.685)
\]

We have reinserted the proper fundamental constants \( \hbar \) and \( c \). The sum carries a prime which is supposed to record the presence of an extra factor \( \frac{1}{2} \) for \( n = 0 \). In the two semi-infinite regions outside the two plates, also the momentum \( k_z \) may be taken as a continuous variable so that there are the additional vacuum energies

\[
E_{\text{outside}} = \frac{\hbar c}{2} L_x L_y L_z \int \frac{dk_x dk_y dk_z}{(2\pi)^3} 2\sqrt{k_x^2 + k_y^2 + k_z^2}
\]

\[
+ \frac{\hbar c}{2} L_x L_y \left( \frac{L_z}{2} - d \right) \int \frac{dk_x dk_y dk_z}{(2\pi)^3} 2\sqrt{k_x^2 + k_y^2 + k_z^2}. \quad (7.686)
\]

Let us now compare this energy with the corresponding expression in the absence of the plates

\[
E_0 = \frac{\hbar c}{2} L_x L_y L_z \int \frac{dk_x dk_y dk_z}{(2\pi)^3} 2\sqrt{k_x^2 + k_y^2 + k_z^2}. \quad (7.687)
\]

By subtracting this from \( E_d + E_{\text{outside}} \), we find the change of the energy due to the presence of the plates

\[
\Delta E = \hbar c L_x L_y \int \frac{dk_x dk_y}{(2\pi)^2} \left( \sum_{k_z^n} \sqrt{k_x^2 + k_y^2 + k_z^2} - \int_0^{\infty} \frac{dk_z d}{\pi} \right) \sqrt{k_x^2 + k_y^2 + k_z^2}. \quad (7.688)
\]

It is caused by the difference in the particle spectra, once with discrete \( k_z^n \) between the plates and once with continuous \( k_z \) without plates. The evaluation of (7.688) proceeds in two steps. First we integrate over the \( k_\perp = (k_x, k_y) \) variables, but enforce the convergence by inserting a cutoff function \( f(k_\perp^2 + k_z^2) \). It is identical to unity up to some large \( k_\perp^2 = k_x^2 + k_y^2 \), say \( k_\perp^2 \leq \Lambda^2 \). For \( k_\perp^2 \gg \Lambda^2 \), the cutoff function decreases rapidly to zero. Later we shall take the limit \( \Lambda^2 \to \infty \). The cutoff function is necessary to perform the mathematical operations. In the physical system a function of this type is provided by the finite thickness and conductivity of the plates. This will have the effect that, for wavelengths much smaller than the thickness, the plates become transparent to the electromagnetic waves. Thus they are no longer able to enforce the boundary conditions which are essential for the discreteness of the spectrum. Since the energy difference comes mainly from long wavelengths, the precise way in which the short wavelengths are cut off is irrelevant.
We then proceed by considering the remaining function of \( k_z \)

\[
g(k_z) \equiv \int dk_\perp \sqrt{k_\perp^2 + k_z^2} f(k_\perp^2 + k_z^2),
\]  
(7.689)

in terms of which the energy difference is given by

\[
\Delta E = \hbar c L_x L_y \frac{1}{4\pi^2} \left( \sum_{k_z = k_n^z} g(k_z) \right).
\]

To evaluate this expression it is convenient to change the variables from \( k_n^z \) to \( n = k_z d/\pi \) and from \( k_\perp^2 \) to \( \nu^2 = k_\perp^2 d^2/\pi^2 \), and to introduce the auxiliary function

\[
G(n) \equiv \frac{d^3}{\pi^4} g(\pi n/d) = \frac{d^3}{\pi^3} \int_0^\infty dk_\perp^2 k f(k^2)
\]

\[
= \int_0^\infty d\nu^2 \nu f \left( \pi^2 \nu^2/d^2 \right).
\]

Then \( \Delta E \) can be written as

\[
\Delta E = \hbar c L_x L_y \frac{\pi^2}{4d^3} \left[ \sum_{n=0} G(n) - \int_0^\infty dn G(n) \right].
\]

(7.692)

The difference between a sum and an integral over the function \( G(n) \) is given by the well-known *Euler-Maclaurin formula* which reads [see Eq. (7A.20) in Appendix 7A]

\[
\frac{1}{2} G(0) + G(1) + G(2) + \ldots + G(n-1) + \frac{1}{2} G(n) - \int_0^n dn G(n)
\]

\[
= \sum_{p=1}^\infty \frac{B_{2p}}{(2p)!} \left[ G^{(2p-1)}(n) - G^{(2p-1)}(0) \right].
\]

(7.693)

Here \( B_p \) are the Bernoulli numbers, defined by the Taylor expansion [see (7A.15)]

\[
\frac{t}{e^t - 1} = \sum_{p=0}^\infty B_p \frac{t^p}{p!},
\]

(7.694)

whose lowest values are [see (7A.6)]

\[
B_0 = 1, \ B_1 = -\frac{1}{2}, \ B_2 = \frac{1}{6}, \ B_3 = 0, \ B_4 = -\frac{1}{30}, \ B_5 = 0, \ B_6 = \frac{1}{42}, \ldots
\]

(7.695)

The primed sum in (7.685) is therefore obtained from the expansion

\[
\sum_{n=0}^\infty G(n) - \int_0^\infty dn G(n) = -\frac{B_2}{2!} [G'(0) - G'(\infty)]
\]

\[
-\frac{B_4}{4!} [G'''(0) - G'''(\infty)]
\]

\[
-\frac{B_6}{6!} [G^{(5)}(0) - G^{(5)}(\infty)]
\]

(7.696)
Since the cutoff function \( f(k^2) \) vanishes exponentially fast at infinity, the function \( G(n) \) and all its derivatives vanish in the limit \( n \to \infty \) (so that the factor \( 1/2 \) at the upper end of the primed sum is irrelevant). For \( n = 0 \), only one expansion coefficient turns out to be nonzero. To see this, we denote \( f(\pi^2 n^2/d^2) \) by \( \tilde{f}(n^2) \) and calculate

\[
G'(0) = -[2n^2 \tilde{f}(n^2)]_{n=0} = 0, \\
G''(0) = -[4n \tilde{f}(n^2) + 4n^3 \tilde{f}'(n^2)]_{n=0} = 0, \\
G'''(0) = -[4 \tilde{f}(n^2) + 18n^2 \tilde{f}'(n^2) + 8n^4 \tilde{f}''(n^2)]_{n=0} = -4, \\
G^{(l)}(0) = 0, \quad l > 3. 
\] (7.697)

The higher derivatives contain an increasing number of derivatives of \( \tilde{f}(\nu^2) \). Since \( \tilde{f}(\nu^2) \) starts out being unity up to large arguments, all derivatives vanish at \( \nu = 0 \). Thus we arrive at

\[
\sum_{n=0}^{\infty} G(n) - \int_0^{\infty} dn \, G(n) = 4 \frac{B_4}{4!} = -\frac{1}{180}. 
\] (7.698)

This yields the energy difference

\[
\Delta E = -L_x L_y \frac{\pi^2}{720d^3}. 
\] (7.699)

There is an *attractive* force between silver plates decreasing with the inverse forth power of the distance:

\[
F = -L_x L_y \frac{\pi^2}{240d^4}. 
\] (7.700)

Between steel plates of an area 1 cm\(^2\) at a distance of 0.5 \( \mu \)m, the force is 0.2 dyne/cm\(^2\). The existence of this force was verified experimentally [17].

Experiments are often done by bringing a conducting sphere close to a plate. Then the force (7.701) is modified by a factor \( 2\pi R d/3 \) to

\[
F = -\frac{2\pi R d}{3} \frac{\pi^2}{240d^4}. 
\] (7.701)

In dielectric media, a similar calculation renders the important van der Waals forces between two plates of different dielectric constants.\(^2\)

It is also interesting to study the force between an electrically conducting plate and a magnetically conducting plate. The transverse electric modes have the form (7.681), but with \( n \) running through the half-integer values \( n = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \). These values ensure that the electric field is maximal at the second plate so that the transverse magnetic field vanishes. The transverse magnetic modes have the form (7.681)

with \( n \) running through the same half-integer values. The relevant modification of the Euler-MacLaurin formula is

\[
G(\frac{1}{2}) + G(\frac{3}{2}) + \ldots + G(\frac{n}{2}) - \int_{0}^{n} dx \, G(x)
= \sum_{p=1}^{\infty} \frac{1}{p!} B_p(\omega) \left[ G'^{(2p-1)}(n) - G'^{(2p-1)}(0) \right]. \tag{7.702}
\]

It is a special case of the general formula [see Eq. (7A.25) in Appendix 7A]

\[
\sum_{n=0}^{m-1} G(a + nh + \omega h) - \int_{a}^{b} dx \, G(x)
= \sum_{p=1}^{h^{p-1}} \frac{1}{p!} B_p(\omega) \left[ G'^{(2p-1)}(b) - G'^{(2p-1)}(a) \right], \tag{7.703}
\]

where \( b = a + mh \). Here \( B_p(\omega) \) are the Bernoulli functions defined by the generalization of formula (7.694):

\[
\frac{te^{\omega t}}{e^t - 1} = \sum_{p=0}^{\infty} B_p(\omega) \frac{t^p}{p!}, \tag{7.704}
\]

They are related to the Bernoulli numbers \( B_p \) by

\[
B_n(\omega) = \sum_{p=1}^{n} \binom{n}{p} B_p \omega^{n-p}. \tag{7.705}
\]

For instance

\[
B_n(\frac{1}{2}) = -(1 - 2^{1-n})B_n, \tag{7.706}
\]

so that

\[
B_0(\frac{1}{2}) = 1, \quad B_1(\frac{1}{2}) = 0, \quad B_2(\frac{1}{2}) = -\frac{1}{12}, \quad B_3(\frac{1}{2}) = 0, \quad B_4(\frac{1}{2}) = \frac{7}{360}, \quad B_5(\frac{1}{2}) = 0, \ldots. \tag{7.707}
\]

For \( h = 1 \) and \( \omega = \frac{1}{2} \) this is reduced to (7.702). The right-hand side is a modified version of (7.696):

\[
\sum_{n=1}^{\infty} G'^{(\frac{n}{2})} - \int_{0}^{\infty} dx \, G(x) = -\frac{B_2(\frac{1}{2})}{2!} [G'(0) - G'(\infty)]
- \frac{B_4(\frac{1}{2})}{4!} [G''(0) - G''(\infty)]
- \frac{B_6(\frac{1}{2})}{6!} [G^{(5)}(0) - G^{(5)}(\infty)] \tag{7.708}
\]

and the previous result (7.698) is replaced by

\[
\sum_{n=1}^{\infty} G'^{(\frac{n}{2})} - \int_{0}^{\infty} dx \, G(x) = 4 \frac{B_4(\frac{1}{2})}{4!}. \tag{7.709}
\]
Since $B_4(\frac{1}{2}) = \frac{7}{240} = -\frac{7}{8} B_4$, the energy difference (7.699) is modified by a factor $-7/8$:

$$\Delta E = L_x L_y \hbar c \frac{7}{8} \frac{\pi^2}{720 d^3}. \tag{7.710}$$

The force is now repulsive and a little weaker than the previous attraction [17].

### 7.13 Zeta Function Regularization

There exists a more elegant method of evaluating the energy difference (7.688) without using a cutoff function $f(k^2)$. This method has become popular in recent years in the context of the field theories of critical phenomena near second-order phase transitions. The method will be discussed in detail when calculating the properties of interacting particles in Chapter 11. The Casimir effect presents a good opportunity for a short introduction. We observe that we can rewrite an arbitrary negative power of a positive quantity $a$ as an integral

$$a^{-z} = \frac{1}{\Gamma(z)} \int_0^\infty \frac{d\tau}{\tau} \tau^z e^{-\tau a}. \tag{7.711}$$

This follows by a simple rescaling from the integral representation for the Gamma function\textsuperscript{22}

$$\Gamma(z) = \int_0^\infty \frac{d\tau}{\tau} \tau^z e^{-\tau}, \tag{7.712}$$

which converges for $\text{Re } z > 0$. Then we rewrite the energy difference (7.688) as

$$\Delta E = \hbar c L_x L_y \int \frac{dk_x dk_y}{(2\pi)^2} \left( \sum_{k_z = k_z}^\prime - \int_0^\infty \frac{dk_z d\tau}{\pi} \right) (k_x^2 + k_y^2 + k_z^2)^{-z}, \tag{7.713}$$

and evaluate this exactly for sufficiently large $z$, where sum and integrals converge. At the end we continue the result analytically to $z = -1/2$.

If we do not worry about the convergence at each intermediate step, this procedure is equivalent to using formula (7.714) for $z = -1/2$

$$\sqrt{a} = \frac{1}{\Gamma(-\frac{1}{2})} \int_0^\infty \frac{d\tau}{\tau} \tau^{-1/2} e^{-\tau a}, \tag{7.714}$$

and by rewriting the energy difference (7.688) as

$$\Delta E = \hbar c L_x L_y \frac{1}{\Gamma(-\frac{1}{2})} \int_0^\infty \frac{d\tau}{\tau} \tau^{-1/2} \int \frac{dk_x dk_y}{(2\pi)^2} \left( \sum_{k_z = k_z}^\prime - \int_0^\infty \frac{dk_z d\tau}{\pi} \right) e^{-\tau(k_x^2 + k_y^2 + k_z^2)}, \tag{7.715}$$

\textsuperscript{22}I.S. Gradshteyn and I.M. Ryzhik, op. cit., Formula 8.310.1.
Being Gaussian, the integrals over \( k_x, k_y \) can immediately be done, yielding

\[
\Delta E = \hbar c L_x L_y \frac{1}{4\pi \Gamma(-\frac{1}{2})} \int_0^\infty \frac{d\tau}{\tau} \tau^{-3/2} \left( \sum_{k_z=k_0}^{'} \int_0^\infty \frac{dk_zd}{\pi} \right) e^{-\tau k_z^2}. \tag{7.716}
\]

The remaining sum-minus-integral over \( k_z \) is evaluated as follows. First we write

\[
\left( \sum_{k_z=k_0}^{'} - \int_0^\infty \frac{dk_zd}{\pi} \right) e^{-\tau k_z^2} = \frac{1}{2} \left( \sum_{n=-\infty}^{\infty} - \int_{-\infty}^{\infty} dn \right) e^{-\tau \pi n^2/\alpha^2}. \tag{7.717}
\]

Then we make use of the Poisson summation formula (1.205):

\[
\sum_{m=-\infty}^{\infty} e^{2\pi i\mu m} = \sum_{n=-\infty}^{\infty} \delta(\mu - n). \tag{7.718}
\]

This allows us to express the sum over \( n \) as an integral over \( n \), which is restricted to the original sum with the help of an extra sum over integers \( m \):

\[
\frac{1}{2} \sum_{n=-\infty}^{\infty} e^{-\alpha n^2} = \frac{1}{2} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dn e^{-\alpha n^2 + 2\pi i\mu m}. \tag{7.719}
\]

The Gaussian integral over \( n \) can then be performed after a quadratic completion. It gives

\[
\frac{1}{2} \sum_{n=-\infty}^{\infty} e^{-\alpha n^2} = \frac{1}{2} \sum_{m=-\infty}^{\infty} \sqrt{\frac{\pi}{\alpha}} e^{-4\pi^2 m^2/\alpha}. \tag{7.720}
\]

The two sides are said to be the dual transforms of each other. The left-hand side converges fast for large \( \alpha \), the right-hand side does so for small \( \alpha \). This duality transformation is fundamental to the theoretical description of many phase transitions where the partition functions over fundamental excitations can be brought to different forms by such transformations, one converging fast for low temperatures, the other for high temperatures.\(^{23}\)

Now, subtracting from the sum (7.720) over \( n \) the integral over \( dn \) removes precisely the \( m = 0 \) term from the auxiliary sum over \( m \) on the right-hand side, so that we find the difference between sum and integral:

\[
\frac{1}{2} \left( \sum_{n=-\infty}^{\infty} - \int_{-\infty}^{\infty} dn \right) e^{-\alpha n^2} = \sum_{m=1}^{\infty} \sqrt{\frac{\pi}{\alpha}} e^{-4\pi^2 m^2/\alpha}. \tag{7.721}
\]

The left-hand side appears directly in (7.717), which therefore becomes

\[
\left( \sum_{k_z=k_0}^{'} - \int_0^\infty \frac{dk_zd}{\pi} \right) e^{-\tau k_z^2} = \frac{d}{\sqrt{\pi \tau}} \sum_{m=1}^{\infty} e^{-d^2 m^2/\tau}. \tag{7.722}
\]

To obtain the energy difference (7.716), this has to be multiplied by $\tau^{-3/2}$ and integrated over $d\tau/\tau$. In the dually transformed integral, convergence at small $\tau$ is automatic. The $\tau$-integral can be done yielding

$$
\int_0^\infty \frac{d\tau}{\tau} \frac{\tau^{-3/2}}{\sqrt{\pi \tau}} \sum_{m=1}^\infty e^{-d^2m^2/\tau} = \frac{d}{\sqrt{\pi}} \frac{\Gamma(2)}{d^4} \sum_{m=1}^\infty \frac{1}{m^4}.
$$

(7.723)

The sum over $m$ involves Riemann’s zeta function

$$
\zeta(z) \equiv \sum_{m=1}^\infty \frac{1}{m^z},
$$

(7.724)

encountered before in calculations of statistical mechanics [see (2.277)]. Here it is equal to [recall (2.317)]

$$
\zeta(4) = \frac{8\pi^4}{4!}|B_4| = \frac{8\pi^4}{4!} \frac{1}{30}.
$$

(7.725)

With this number, the energy difference (7.716) becomes, recalling the values of the Gamma function $\Gamma(2) = 1$ and $\Gamma(-1/2) = -2\sqrt{\pi}$,

$$
\Delta E = \hbar c L_x L_y \frac{1}{4\pi \Gamma(-\frac{1}{2})} \int_0^\infty \frac{d\tau}{\tau} \tau^{-3/2} \left( \sum_{k_z=k_y^2} \int_0^\infty \frac{dk_z d}{\pi} \right) e^{-\tau k_z^2} = \hbar c L_x L_y \frac{\Gamma(2)}{4\pi^{3/2} \Gamma(-\frac{1}{2}) d^3} = -L_x L_y \hbar c \frac{\pi^2}{720d^3}.
$$

(7.726)

This is the same result as in Eq. (7.699). To be satisfactory, these calculations should all be done for the general expression (7.713), followed by an analytic continuation of the power $z$ to $-1/2$ at the end.

While being formal, with the above justification via analytic continuation, we may be mathematically even more sloppy, and process the $\tau$-integral in the energy difference (7.716) as a shortcut via (7.717) and write

$$
\int_0^\infty \frac{d\tau}{\tau} \tau^{-3/2} \left( \sum_{k_z=k_y^2} \int_0^\infty \frac{dk_z d}{\pi} \right) e^{-\tau k_z^2} = \frac{1}{2} \left( \sum_{n=-\infty}^\infty - \int_0^\infty dn \right) \int_0^\infty \frac{d\tau}{\tau} \tau^{-3/2} e^{-\tau \pi^2 n^2/d^2} = \Gamma(-\frac{1}{2}) \left( \sum_{n=-\infty}^\infty - \int_0^\infty dn \right) n^2.
$$

(7.727)

The divergent sum-minus-integral over $n^3$ can formally be identified with the zeta function $\sum_{n=1}^\infty n^{-z}$ at the negative argument $z = -3$. With the help of formula (2.314), we find

$$
\zeta(-3) = -\frac{B_4}{4} = \frac{1}{120}.
$$

(7.728)

Inserting this together with $\Gamma(-\frac{1}{2}) = -\frac{\pi}{2} \Gamma(-\frac{1}{2})$ into (7.727), and this further into (31.15), we obtain

$$
\Delta E = -L_x L_y \hbar c \frac{\pi^2}{720d^3}.
$$

(7.729)
The correctness of this formal approach is ensured by the explicit duality transformation done before.

The formal procedure of evaluating apparently meaningless sum-minus-integrals over powers $n^\nu$ with $\nu > -1$, using zeta functions of negative arguments defined by analytic continuation, is known as the zeta function regularization method. We shall see later in Eq. (11.127), that by a similar analytic continuation the integral $\int_0^\infty dk k^\alpha$ vanishes for all $\alpha$. For this reason, the zeta-function regularization applies even to the pure sum, allowing us to replace $\sum_{n=1}^\infty n^\nu$ by $\zeta(\nu)$.

Let us rederive also the repulsive result (7.710) with the help of this regularization method. The energy difference has again the form (7.716), but with the discrete values of $k^n_z$ running through all $\pi n/d$ with half-integer values of $n = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$. Thus we must calculate (7.717) with the sum over all half-integer $n$. This can be done with a slight modification of the Poisson summation formula (1.205):

$$\sum_{m=-\infty}^\infty e^{2\pi i m (\frac{1}{2})^m} = \sum_{m=-\infty}^\infty e^{2\pi i (\mu - \frac{1}{2}) m} = \sum_{n=-\infty}^\infty \delta(\mu - n - \frac{1}{2}).$$

As a result we obtain a modified version of (7.723):

$$\int_0^\infty \frac{d\tau}{\tau^{3/2}} \frac{d}{\sqrt{\pi \tau}} \sum_{m=1}^\infty e^{-d^2 m^2/\tau} (-1)^m = \frac{d}{\sqrt{\pi}} \frac{\Gamma(2)}{d} \sum_{m=1}^\infty \frac{(-1)^m}{m^4}.$$ \hfill (7.731)

It contains the modified zeta function

$$\zeta(x) \equiv \sum_{m=1}^\infty \frac{(-1)^m}{m^x}.$$ \hfill (7.732)

It is now easy to derive the identity

$$\zeta(x) \equiv \sum_{m=1}^\infty \left[ -\frac{1}{m^x} + 2 \frac{1}{(2m)^x} \right] = -(1 - 2^{1-x})\zeta(x),$$

so that

$$\zeta(4) \equiv -\left(1 - \frac{1}{8}\right) \zeta(4) = -\frac{7}{8} \zeta(4).$$ \hfill (7.734)

As in the previous result (7.710), there is again a sign change and a factor $7/8$ with respect to the attractive result (7.726) that was obtained there from the Euler-MacLaurin summation formula.

### 7.14 Dimensional Regularization

The above results can also be obtained without the duality transformation (7.720) by a calculation in an arbitrary dimension $D$, and continuing everything to the physical dimension $D = 3$ at the end. In $D$ spatial dimensions, the energy difference (7.713) becomes

$$\Delta E = \hbar c V_{D-1} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \left( \sum_{k_z=k^n_z} + \int_0^\infty \frac{dk_z d}{\pi} \right) (k_x^2 + k_y^2 + k_z^2)^{\nu-\nu},$$ \hfill (7.735)
where $V_{D-1}$ is the $D-1$-dimensional volume of the plates, the dimensional extension of the area $L_xL_y$. After using the integral formula (7.714), we obtain the $D$-dimensional generalization of (7.715):

$$\Delta E = \hbar c V_{D-1} \frac{1}{\Gamma(-\frac{D}{2})} \int_0^\infty \frac{d\tau}{\tau} \tau^{-1/2} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \left( \sum_{k_z=k_z^*} \int_0^\infty \frac{dk_zd}{\pi} \right) e^{-\frac{\tau}{4} (k_x^2+k_y^2+k_z^2)}.$$  

(7.736)

Performing now the $D-1$ Gaussian momentum integrals yields

$$\Delta E = \hbar c V_{D-1} \frac{1}{\Gamma(-\frac{D}{2})} \frac{\Gamma(-\frac{D}{2})}{\pi^{D/2}} \left( \sum_{k_z=k_z^*} \int_0^\infty \frac{dk_zd}{\pi} \right) e^{-\frac{\tau}{4} k_z^2}.$$  

(7.737)

After integrating over all $\tau$, this becomes

$$\Delta E = \hbar c V_{D-1} \frac{\Gamma(-\frac{D}{2})}{\Gamma(-\frac{D}{2})} \left( \frac{1}{4\pi} \right)^{D/2} \left( \sum_{k_z=k_z^*} \int_0^\infty \frac{dk_zd}{\pi} \right) k_z^D.$$  

(7.738)

Inserting $k_z = k_z^* = \pi n/d$, the last factor is seen to be equal to

$$\left( \sum_{k_z=k_z^*} \int_0^\infty \frac{dk_zd}{\pi} \right) k_z^D = \left( \frac{\pi}{d} \right)^D \left( \sum_n \int_0^\infty \frac{dn}{\pi} \right) n^D.$$  

(7.739)

We now make use of the heuristic Veltman integral rule. This states that in any renormalizable quantum field theory (which a free-field theory trivially is), we may always set the following integrals equal to zero:

$$\int_0^\infty dk k^n = 0.$$  

(7.740)

The proof of this can be found in the textbook [7] around Eq. (8.33). With this rule we shall rewrite the sum-minus-integral expression in the energy difference (7.738), using a zeta function (7.724) as

$$\left( \sum_{k_z=k_z^*} \int_0^\infty \frac{dk_zd}{\pi} \right) k_z^D = \left( \frac{\pi}{d} \right)^D \zeta(-D),$$  

(7.741)

and the energy difference (7.738) becomes

$$\Delta E = \hbar c V_{D-1} \frac{\Gamma(-\frac{D}{2})}{\Gamma(-\frac{D}{2})} \left( \frac{1}{4\pi} \right)^{D/2} \left( \frac{\pi}{d} \right)^D \zeta(-D).$$  

(7.742)

For $D = 3$ dimensions, this is equal to

$$\Delta E = \hbar c V_2 \frac{\Gamma(-\frac{3}{2})}{\Gamma(-\frac{3}{2})} \left( \frac{1}{4\pi} \right)^{3/2} \zeta(-3).$$  

(7.743)
Inserting $\Gamma(-\frac{3}{2}) = 4\sqrt{\pi}/3$, $\Gamma(-1/2) = -2\sqrt{\pi}$, this becomes

$$\Delta E = -\hbar c V_2 \frac{\pi^2}{6d^3} \zeta(-3).$$ (7.744)

The value of $\zeta(-3)$ cannot be calculated from the definition (7.724) since the sum diverges. There exists, however, an integral representation for $\zeta(z)$ which agrees with (7.724) for $z > 1$ where the sum converges, but which can also be evaluated for $z \leq 1$. From this, one can derive the reflection formula

$$\Gamma(z/2)\pi^{-z/2}\zeta(z) = \Gamma((1-z)/2)\pi^{-(1-z)/2}\zeta(1-z).$$ (7.745)

This allows us to calculate

$$\zeta(-3) = \frac{3}{4\pi^4} \zeta(4) = \frac{1}{120},$$ (7.746)

so that

$$\Delta E = -\hbar c V_2 \frac{\pi^2}{6d^3} \frac{3}{4\pi^4} \zeta(4) = -V_2 \hbar c \frac{\pi^2}{720d^3},$$ (7.747)

in agreement with Eq. (7.729).

Note that this result via dimensional regularization agrees with what was obtained before with the help of the sloppy treatment in Eq. (7.716). The reason for this is the fact that the reflection formula (7.745) is a direct consequence of the duality transformation in Eq. (7.720), which was the basis for the previous treatment.

The derivation of the factor $-7/8$ for the energy between the mixed electric and magnetic conductor plates is more direct in the present approach than before. Here the expression (7.748) becomes

$$\left( \sum_{k_z = k_z^D} - \int_0^\infty \frac{dk_z d}{\pi} \right) k_z^D = \left( \frac{\pi}{d} \right)^D \left( \sum_{n=1/2, 3/2, \ldots} - \int_0^\infty dn \right) n^D,$$ (7.748)

and we may express the right-hand side in terms of the Hurwitz zeta function defined by

$$\zeta(x, q) \equiv \sum_{n=0}^\infty \frac{1}{(q + n)^x}$$ (7.749)

as

$$\left( \sum_{k_z = k_z^D} - \int_0^\infty \frac{dk_z d}{\pi} \right) k_z^D = \left( \frac{\pi}{d} \right)^D \zeta(-D, 1/2).$$ (7.750)

Now we use the property

$$\zeta(x, q) = (2^x - 1)\zeta(x),$$ (7.751)

\[\text{References:}\]
\[24\] I.S. Gradshteyn and I.M. Ryzhik, op. cit., Formula 9.521
\[25\] ibid., Formula 9.535.1
to calculate the right-hand side of (7.750) as
\[ \zeta(-3, 1/2) = -\frac{7}{8} \zeta(-3). \]  
(7.752)

This exhibits immediately the desired factor $-7/8$ with respect to (7.747).

At a finite temperature, the sum over the vacuum energies $2\hbar \sum_k \omega_k/2$ is modified by the Bose occupation factor $n_\omega$ of Eq. (2.264),
\[ n_\omega = \frac{1}{2} + \frac{1}{e^{\omega/k_B T} - 1} = \frac{1}{2} \coth \frac{\hbar \omega}{2k_B T}. \]  
(7.753)

It becomes $2\hbar \sum_k \omega_k n_\omega$.

How does this modify the vacuum energy, which in Eq. (7.744) was shown to be equal to
\[ \Delta E = -\hbar c V_2 \frac{\pi^2}{6d^3} \zeta(-3)? \]  
(7.754)

For small $T$, the sum over $n^3$ in $\zeta(-3)$ is replaced by a sum over $2nDk_B T/n$. The subject is discussed in detail in Ref. [17], and the resulting force between a plate and a sphere of radius $R$ is
\[ F = \frac{\zeta(3)}{4} \frac{Rk_B T}{d^2}. \]  
(7.755)

This has apparently been observed recently.

### 7.15 Accelerated Frame and Unruh Temperature

The vacuum oscillations of a field show unusual properties when they are observed from an accelerated frame. Such a frame is obtained by applying to a rest frame an infinitesimal boost $e^{-i\kappa \mathbf{M}}$, successively after each proper time interval $d\tau$. The rapidity $\zeta$ increases as a function of the invariant time $\tau$ like $\zeta(\tau) = a\tau/c$, where $a$ is the invariant acceleration. Hence the velocity $v = c \arctanh \zeta$ increases as a function of $\tau$ like
\[ v(\tau) = c \arctan(at/c) \]  
(7.756)

If the acceleration points into the $z$-direction, the space and time coordinates evolve like
\[ z(\tau) = (c/a) \cosh(at/c), \quad t(\tau) = (c/a) \sinh(at/c). \]  
(7.757)

The velocity seen from the initial Minkowski frame increases therefore like
\[ dv(t)/dt = a(1 - v^2/c^2)^{3/2}. \]  
(7.758)

The energy of a particle transforms into
\[ p_0(\tau) = \cosh \zeta(\tau) p_0 + \sinh \zeta(\tau) p_3. \]  
(7.759)
A massless particle has \( p_3 = p_0 \), where this becomes simply
\[
p_0(\tau) = e^{\xi(\tau)}p_0 = e^{\alpha \tau/c}p_0.
\] (7.760)

The temporal oscillations of the wave function of the particle \( e^{i\omega \tau} \) with \( \omega = cp_0/\hbar \) is observed in an accelerated frame to have a frequency \( e^{-\omega(\tau)\tau} = e^{-\omega_0 \alpha \tau/c} \). As a consequence, a wave of a single momentum \( p_3 \) in the rest frame is seen to have an entire spectrum of frequencies. Their distribution can be calculated by a simple Fourier transformation [18]:
\[
f(\omega, \Omega) = \int_{-\infty}^{\infty} dt \, e^{-i\Omega t} e^{-i\omega c/c/a} = \left( \frac{\omega c}{a} \right)^{-i\Omega c/a} \Gamma \left( \frac{\Omega c}{a} \right),
\] (7.761)
or
\[
f(\omega, -\Omega) = e^{-\pi \Omega c/2a} \Gamma \left( -i \frac{\Omega c}{a} \right) e^{i\Omega c/a} \log(\omega c/a).
\] (7.763)

We see that the probability to find the frequency \( \Omega \) in \( f(\omega, \Omega) \) is
\[
|f(\omega, \Omega)|^2 = 2\pi c \frac{1}{\Omega a} e^{2\pi \Omega c/a} - 1 = 2\pi c \frac{\Omega a}{\Omega a} n_\Omega.
\] (7.765)

The factor \( n_\Omega \) is the thermal Bose-Einstein distribution of the frequencies \( \Omega \) [recall Eq. (2.423)] at a temperature \( T_U = \hbar a/2\pi ck_B \), the so-called Unruh temperature [19]. The particles in this heat bath can be detected by suitable particle reactions (for details see Ref. [20]).

Let us describe the situation in terms of the quantum field. For simplicity, we focus attention upon the relevant dimensions \( z \) and \( t \) and study the field only at the origin \( z = 0 \), where it has the expansion [compare (7.16)]
\[
\phi(0, t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \sqrt{\hbar} \left( e^{-i\omega_k t} a_k + e^{i\omega_k t} a_k^\dagger \right), \quad \omega_k \equiv c|k|.
\] (7.766)

The creation and annihilation operators satisfy the commutation rules
\[
\begin{align*}
[a_k, a_{k'}] &= [a_k^\dagger, a_{k'}^\dagger] = 0, \\
[a_k, a_{k'}^\dagger] &= 2k^0 2\pi \delta(k - k').
\end{align*}
\] (7.767)

They can be recovered from the fields by the Fourier transform [compare (7.16)] :
\[
\begin{align*}
\{ a_k \} &= 2\Omega \omega_k \int_{-\infty}^{\infty} dt \, e^{i\omega_k t} \phi(0, t).
\end{align*}
\] (7.768)
The accelerated field has the form
\[ \phi_a(z, t) = \int \frac{dk}{2\pi} \frac{1}{\omega_k} \left( e^{-ie^{\omega_k t} a_k} + e^{ie^{\omega_k t} a_k^\dagger} \right). \] (7.769)

Inverting the Fourier transformation (7.761), we can expand
\[ \phi_a(z, t) = c \int \frac{dK}{2\pi} \left( e^{-iK(ct-z)} A_K + e^{iK(ct-z)} A_K^\dagger \right), \] (7.770)
where
\[ A_K = \int \frac{dk}{2\pi}. \] (7.771)

In the following sections we shall study the quantum-statistical properties of this field in a thermal environment of temperature \( T \).

### 7.16 Photon Propagator in Dirac Quantization Scheme

Let us calculate the propagator of the \( A_\mu \)-field in the Dirac quantization scheme. We expand the vector field \( A_\mu \) as in (7.419) and calculate
\[ G^{\mu\nu}(x, x') = \langle 0_{\text{phys}} | T A^\mu(x) A^\nu(x') | 0_{\text{phys}} \rangle, \] (7.772)

obtaining for the time-ordered operators between the vacuum states the same terms as in (7.478)–(7.479). The transverse polarization terms contribute, as before, a sum
\[ e^\mu(k, 1)^* e^\nu(k, 1) + e^\mu(k, 2)^* e^\nu(k, 2). \] (7.773)

However, in contrast to the Gupta-Bleuler case, the creation operators \( a_k^\dagger \) annihilate the physical vacuum to their right, and \( a_k \) do the same thing to their left. Hence there are no contributions proportional to \( e^\mu(k, \bar{s})^* e^\nu(k, s) \) and \( e^\mu(k, s)^* e^\nu(k, \bar{s}) \). For the same reason, there are no contributions from (7.479). Thus we obtain, instead of (7.509),
\[ G^{\mu\nu}(x, x') = \langle 0_{\text{phys}} | T A^\mu(x) A^\nu(x') | 0_{\text{phys}} \rangle = \Theta(x_0 - x'_0) \frac{1}{V} \sum_k \frac{1}{2k^0} e^{-ik(x-x')} \sum_{\lambda=1}^2 e^\mu(k, \lambda)^* e^\nu(k, \lambda) \]
\[ + \Theta(x'_0 - x_0) \frac{1}{V} \sum_k \frac{1}{2k^0} e^{ik(x-x')} \sum_{\lambda=1}^2 e^\mu(k, \lambda)^* e^\nu(k\lambda). \] (7.774)

This is the same propagator as the one in Eq. (7.361), which was obtained in the noncovariant quantization scheme where only the physical degrees of freedom of the vector potential became operators. Thus, although the field operators have been quantized with covariant commutation relations (7.386), the selection procedure of the physical states has made the propagator noncovariant.
7.17 Free Green Functions of \( n \) Fields

For free fields of spin 0 and 1/2 quantized in the preceding sections we have shown that the vacuum expectation value of the time-ordered product of two field operators, i.e., the field propagator is equal to the Green function associated with the differential equation obeyed by the field. In the case of the real Klein-Gordon field we had, for example,

\[
G(x_1, x_2) = G(x_1 - x_2) \equiv \langle 0| \hat{T} \phi(x_1) \phi(x_2)|0 \rangle, \tag{7.775}
\]

satisfying

\[
(-\partial^2 - M^2)G(x, x_2) = i\delta^{(4)}(x - x_2). \tag{7.776}
\]

It is useful to define a generalization of the vacuum expectation values (7.775) to any number \( n \) of free fields:

\[
G^{(n)}(x_1, x_2, x_3, \ldots, x_n) \equiv \langle 0| \hat{T} \phi(x_1) \ldots \phi(x_n)|0 \rangle. \tag{7.777}
\]

This object will be referred to as the free \( n \)-point propagator, or \( n \)-point function.

For massive vector fields and other massive fields with spin larger than 1/2, one must watch out for the discrepancy between propagator and Green function by Schwinger terms. For scalar fields, there is no problem of this kind. For complex scalar fields, the appropriate definition is

\[
G^{(n,m)}(x_1, \ldots, x_n; x'_1, \ldots, x'_m) \equiv \langle 0| \hat{T} \phi(x_1) \ldots \phi(x_n) \phi^\dagger(x'_1) \ldots \phi^\dagger(x'_m)|0 \rangle. \tag{7.782}
\]

For free fields, the latter is nonzero only for \( n = m \). Corresponding expressions are used for fields of any spin. We shall see later that the set of all free \( n \)-point functions plays a crucial role in extracting the information contained in an arbitrary interacting field theory.

The \( n \)-point functions (7.778) may be considered as the relativistic generalizations of the set of all nonrelativistic many-particle Schrödinger wave functions. In the local \( \mathbf{x} \)-basis, these would consist of all scalar products

\[
\langle \mathbf{x}_1, \ldots, \mathbf{x}_n | \psi(t) \rangle = \langle 0 | \psi(\mathbf{x}_1, 0) \ldots \psi(\mathbf{x}_n, 0) | \psi(t) \rangle. \tag{7.779}
\]

Choosing for the wave vectors \( | \psi(t) \rangle \) any state in the local \( n \)-particle basis

\[
| \mathbf{x}_1', \ldots, \mathbf{x}_n'; t \rangle = \psi^\dagger(\mathbf{x}_1', t) \ldots \psi^\dagger(\mathbf{x}_n', t)|0 \rangle, \tag{7.780}
\]

we arrive at the scalar products

\[
\langle 0 | \psi(\mathbf{x}, 0) \ldots \psi(\mathbf{x}_n, 0) \psi^\dagger(\mathbf{x}_1', t) \ldots \psi^\dagger(\mathbf{x}_n', t)|0 \rangle. \tag{7.781}
\]

These, in turn, are a special class of propagators:

\[
G^{(n,m)}(\mathbf{x}_1, 0, \mathbf{x}_2, 0, \ldots, \mathbf{x}_n, 0; \mathbf{x}'_1, t, \ldots, \mathbf{x}'_m, t), \tag{7.782}
\]
where only two time arguments appear. When going to relativistic theories, the set of such equal-time objects is no longer invariant under Lorentz transformations. Two events which happen simultaneously in one frame do not do so in another frame. This is why the propagators $G^{(n,m)}(x, \ldots, x_n; x'_1, \ldots, x'_m)$ require arbitrary spacetime arguments to represent the relativistic generalization of the local wave functions. This argument does not yet justify the time ordering in the relativistic propagators in contrast to the relevance of the causal propagator in the nonrelativistic case. Its advantage will become apparent only later in the development of perturbation theory.

At the present level where we are dealing only with free quantum fields, the $n$-point functions have a special property. They can all be expanded into a sum of products of the simple propagators. Before we develop the main expansion formula it is useful to recall that normal products were introduced in Subsecs. 7.1.5 and 7.4.2 for Bose and Fermi fields, respectively. They were contrived as a simple trick to eliminate a bothersome infinite energy of the vacuum to the zero-point oscillations of free fields, that helped us to avoid a proper physical discussion. This product is useful in a purely mathematical discussion to evaluate vacuum expectation values of products of any number of field operators.

Given an arbitrary set of $n$ free field operators $\phi_1(x_1) \cdots \phi_n(x_n)$, each of them consists of a creation and an annihilation part:

$$\phi_i(x_i) = \phi^c_i(x_i) + \phi^a_i(x_i).$$

(7.783)

Some $\phi_i$ may be commuting Bose fields, some anticommuting Fermi fields. The normally ordered product or normal product of $n$ of these fields was introduced in (7.34) as $:\phi_1(x)\phi(x_2)\cdots\phi(x_n):$. From now on we shall prefer a notation that is more parallel to that of the time-ordered product: $\hat{N}(\phi_1(x)\phi(x_2)\cdots\phi(x_n))$. For the subsequent discussion, the function symbol $\hat{N}(\ldots)$ will be more convenient than the earlier double-dot notation. The normal product is a function of a product of field operators which has the following two properties:

i) **Linearity**: The normal product is a linear function of all its $n$ arguments, i.e., it satisfies

$$\hat{N}((\alpha\phi_1 + \beta\phi'_1)\phi_2\phi_3\cdots\phi_n) = \alpha\hat{N}(\phi_1\phi_2\phi_3\cdots\phi_n) + \beta\hat{N}(\phi'_1\phi_2\phi_3\cdots\phi_n).$$

(7.784)

If every $\phi_i$ is replaced by $\phi^c_i + \phi^a_i$, it can be expanded into a linear combination of terms which are all pure products of creation and annihilation operators.

ii) **Normal Reordering**: The normal product reorders all products arising from a complete linear expansion of all fields according to i) in such a way that all annihilators stand to the right of all creators. If the operators $\phi_i$ describe fermions, the definition requires a factor $-1$ to be inserted for every transmutation of the order of two operators.
For example, let \( \phi_1, \phi_2, \phi_3 \) be scalar fields, then normal ordering produces for two field operators

\[
\hat{N}(\phi_1^c \phi_2^c) = \phi_1^c \phi_2^c = \phi_2^c \phi_1^c, \\
\hat{N}(\phi_1^a \phi_2^a) = \phi_1^a \phi_2^a, \\
\hat{N}(\phi_1^a \phi_2^c) = \phi_2^c \phi_1^a, \\
\hat{N}(\phi_1^c \phi_2^a) = \phi_1^a \phi_2^a = \phi_2^a \phi_1^a, 
\]

and for three field operators

\[
\hat{N}(\phi_1^c \phi_2^c \phi_3^c) = \phi_1^c \phi_2^c \phi_3^c = \phi_3^c \phi_1^c \phi_2^c, \\
\hat{N}(\phi_1^c \phi_2^a \phi_3^c) = \phi_1^c \phi_3^c \phi_2^c = \phi_2^c \phi_3^c \phi_1^c, \\
\hat{N}(\phi_1^a \phi_2^a \phi_3^c) = \phi_2^c \phi_3^c \phi_1^a = \phi_3^c \phi_2^c \phi_1^a. 
\]

If the operators \( \phi_i \) are fermions, the effect is

\[
\hat{N}(\phi_1^c \phi_2^c) = \phi_1^c \phi_2^c = -\phi_2^c \phi_1^c, \\
\hat{N}(\phi_1^c \phi_2^a) = \phi_1^c \phi_2^a, \\
\hat{N}(\phi_1^a \phi_2^c) = -\phi_2^c \phi_1^a, \\
\hat{N}(\phi_1^a \phi_2^a) = \phi_1^a \phi_2^a = -\phi_2^a \phi_1^a, 
\]

and

\[
\hat{N}(\phi_1^c \phi_2^c \phi_3^c) = \phi_1^c \phi_2^c \phi_3^c = -\phi_3^c \phi_1^c \phi_2^c, \\
\hat{N}(\phi_1^c \phi_2^a \phi_3^c) = -\phi_1^c \phi_3^c \phi_2^c = \phi_2^c \phi_3^c \phi_1^c, \\
\hat{N}(\phi_1^a \phi_2^a \phi_3^c) = \phi_2^c \phi_3^c \phi_1^a = -\phi_3^c \phi_2^c \phi_1^a. 
\]

The normal product is uniquely defined. The remaining order of creation or annihilation parts among themselves is irrelevant, since these commute or anticommute with each other by virtue of the canonical free-field commutation rules. In the following, the fields \( \phi \) may be Bose or Fermi fields and the sign of the Fermi case will be recorded underneath the Bose sign.

The advantage of normal products is that their vacuum expectation values are zero. There is always an annihilator on the right-hand side or a creator on the left-hand side which produces 0 when matched between vacuum states. The method of calculating all \( n \)-point functions will consist in expanding all time-ordered products of \( n \) field operators completely into normal products. Then only the terms with no operators will survive between vacuum states. This will be the desired value of the \( n \)-point function.

Let us see how this works for the simplest case of a time-ordered product of two identical field operators

\[
\hat{T}(\phi(x_1)\phi(x_2)) \equiv \Theta(x_1^0 - x_2^0)\phi(x_1)\phi(x_2) \pm \Theta(x_2^0 - x_1^0)\phi(x_2)\phi(x_1). 
\]
The basic expansion formula is

\[ \hat{T}(\phi(x_1)\phi(x_2)) = \hat{N}(\phi(x_1)\phi(x_2)) + \langle 0|\hat{T}(\phi(x_1)\phi(x_2))|0 \rangle. \tag{7.790} \]

For brevity, we shall denote the propagator of two fields as follows:

\[ \langle 0|\hat{T}(\phi(x_1)\phi(x_2))|0 \rangle = \phi(x_1)\phi(x_2) = G(x_1 - x_2). \tag{7.791} \]

The hook which connects the two fields is referred to as a *contraction* of the fields.

We shall prove the basic expansion formula (7.790) by considering it separately for the creation and annihilation parts \( \phi^c \) and \( \phi^a \). This will be sufficient since the time-ordered product is linear in each field just as the normal product. Now, in both cases \( x_1^0 < x_2^0 \) we have

\[
\hat{T}(\phi^c(x_1)\phi^c(x_2)) = \left\{ \phi^c(x_1)\phi^c(x_2), \pm \phi^c(x_2)\phi^c(x_1) \right\} = \phi^c(x_1)\phi^c(x_2) + \langle 0|\left\{ \pm \phi^c(x_2)\phi^c(x_1) \right\}|0 \rangle, \tag{7.792}\]

which is true since \( \phi^c(x_1)\phi^c(x_2) \) commute or anticommute with each other, and annihilate the vacuum state \( |0 \rangle \). The same equation holds for \( \phi^a(x_1)\phi^a(x_2) \). The only nontrivial cases are those with a time-ordered product of \( \phi^c(x_1)\phi^a(x_2) \) and \( \phi^a(x_1)\phi^c(x_2) \). The first becomes for \( x_1^0 < x_2^0 \):

\[
\hat{T}(\phi^c(x_1)\phi^a(x_2)) = \left\{ \phi^c(x_1)\phi^a(x_2), \pm \phi^a(x_2)\phi^c(x_1) \right\} = \phi^c(x_1)\phi^a(x_2) + \langle 0|\left\{ \pm \phi^a(x_2)\phi^c(x_1) \right\}|0 \rangle. \tag{7.793}\]

For \( x_1^0 > x_2^0 \), this equation is obviously true. For \( x_1^0 < x_2^0 \), the normal ordering produces an additional term

\[
\pm(\phi^a(x_2)\phi^c(x_1) \mp \phi^c(x_1)\phi^a(x_2)) = \pm[\phi^a(x_2), \phi^c(x_1)]_+. \tag{7.794}\]

As the commutator or anticommutator of free fields is a \( c \)-number, they may equally well be evaluated between vacuum states, so that we may replace (7.794) by

\[
\pm \langle 0| [\phi^a(x_2), \phi^c(x_1)]_+ |0 \rangle. \tag{7.795}\]

Moreover, since \( \phi^a \) annihilates the vacuum, this reduces to

\[
\pm \langle 0|\phi^a(x_2), \phi^c(x_1)|0 \rangle. \tag{7.796}\]

The oppositely ordered operators \( \phi^a(x_1)\phi^c(x_2) \) can be processed by complete analogy.
We shall now generalize this basic result to an arbitrary number of field operators. In order to abbreviate the expressions let us define the concept of a contraction inside a normal product

\[ \hat{N} \left( \phi_1 \cdots \phi_{i-1} \phi_i \phi_{i+1} \cdots \phi_{j-1} \phi_j \phi_{j+1} \cdots \phi_n \right) \]

\[ \equiv \eta \phi_i \phi_j \hat{N} \left( \phi_1 \cdots \phi_{i-1} \phi_{i+1} \cdots \phi_{j-1} \phi_{j+1} \cdots \phi_n \right). \] (7.797)

The phase factor \( \eta \) is equal to 1 for bosons and \((-1)^{j-i-1}\) for fermions accounting for the number of fermion transmutations necessary to reach the final order. A normal product with several contractions is defined by the successive execution of each of them. If only one field is left inside the normal ordering symbol, it is automatically normally ordered so that

\[ \hat{N}(\phi) = \phi. \] (7.798)

Similarly, if all fields inside a normal product are contracted, the result is no longer an operator and the symbol \( \hat{N} \) may be dropped using linearity and the trivial property

\[ \hat{N}(1) \equiv 1. \] (7.799)

The fully contracted normal product will be the relevant one in determining the \( n \)-particle propagator. With these preliminaries we are now ready to prove Wick’s theorem for the expansion of a time-ordered product in terms of normally ordered products.\(^{26}\)

### 7.17.1 Wick’s Theorem

The theorem may be stated as follows. An arbitrary time ordered product of free fields can be expanded into a sum of normal products, one for each different contraction between pairs of operators:

\[ \hat{T}(\phi_1 \cdots \phi_n) = \hat{N}(\phi_1 \cdots \phi_n) \] (no contraction)

\[ + \hat{N}(\phi_1 \phi_2 \phi_3 \phi_4 \cdots \phi_n) + \hat{N}(\phi_1 \phi_2 \phi_3 \phi_4 \cdots \phi_n) + \ldots + \hat{N}(\phi_1 \phi_2 \phi_3 \phi_4 \cdots \phi_n) \] (one contraction)

\[ + \hat{N}(\phi_1 \phi_2 \phi_3 \phi_4 \phi_5 \cdots \phi_n) + \hat{N}(\phi_1 \phi_2 \phi_3 \phi_4 \phi_5 \cdots \phi_n) + \hat{N}(\phi_1 \phi_2 \phi_3 \phi_4 \phi_5 \cdots \phi_n) + \cdots \] (two contractions)

\[ + \cdots + \hat{N}(\phi_1 \phi_2 \phi_3 \phi_4 \phi_5 \phi_6 \cdots \phi_{n-1} \phi_n). \] (remaining contractions)

\(^{26}\)G.C. Wick, Phys. Rev. 80, 268 (1950); F. Dyson, Phys. Rev. 82, 428 (1951).
In this particular expression we have assumed \( n \) to be even so that it is possible to contract all operators. Otherwise each term in the last row would have contained one uncontracted field. In either case the expansion on the right-hand side will be abbreviated as
\[
\sum_{\text{allpaircontractions}} \hat{N}(\phi_1 \cdots \phi_n), \quad (7.801)
\]
i.e., we shall state Wick’s theorem in the form
\[
\hat{T}(\phi_1 \cdots \phi_n) = \sum_{\text{allpaircontractions}} \hat{N}(\phi_1 \cdots \phi_n). \quad (7.802)
\]
Using Wick’s theorem, it is a trivial matter to calculate an arbitrary \( n \)-point function of free fields. Since the vacuum expectation values of all normal products are zero except for the fully contracted ones, we may immediately write for an even number of real fields:
\[
G^{(n)}(x_1 \cdots x_n) = \langle 0| \hat{T}(\phi(x_1) \cdots \phi(x_n)) |0\rangle = \sum_{\text{allfullpaircontractions}} \hat{N}(\phi(x_1) \cdots \phi(x_n)). \quad (7.803)
\]
A simple combinatorical analysis shows that the right-hand side consists of
\[
n!/(n/2)!2^{n/2} = 1 \cdot 3 \cdot 5 \cdots (n-1) \equiv (n-1)!!
\]
pair terms.

In the case of complex fields, \( G^{(n,m)} \) can only be nonzero for \( n = m \) and all contractions \( \phi \phi \) and \( \phi^\dagger \phi^\dagger \) vanish. Thus there are altogether \( n! \) different contractions.

The proof of (7.802) goes by induction: For \( n = 2 \) the theorem was proved in Eq. (7.790). Let it be true for a product of \( n \) fields \( \phi_1 \cdots \phi_n \), and allow for an additional field \( \phi \). We may assume that it is earlier in time than the other fields \( \phi_1, \ldots, \phi_n \). For if it were not, we could always choose the earliest of the \( n \) fields, move it completely to the right in the time ordered product, with a well-defined phase factor due to Fermi permutations, and proceed. Once the rightmost field is the earliest, it may be removed trivially from the time ordered product. The resulting product can be expanded according to the Wick’s theorem for a product of \( n \) field operators as
\[
\hat{T}(\phi_1 \cdots \phi_n \phi) = \hat{T}(\phi_1 \cdots \phi_n)\phi = \sum_{\text{allpaircontractions}} \hat{N}(\phi_1 \cdots \phi_n)\phi. \quad (7.804)
\]
We now incorporate the extra field \( \phi \) on the right-hand side into each of the normal products in the expansion. When doing so we obtain, from each term, precisely the contractions required by Wick’s theorem for \( n + 1 \) field operators. The crucial formula which yields these is
\[
\hat{N}(\phi_1 \cdots \phi_n)\phi = \hat{N}(\phi_1 \cdots \phi_n)\phi + \hat{N}(\phi_1 \phi_2 \cdots \phi_n)\phi + \hat{N}(\phi_1 \phi_2 \cdots \phi_n\phi) + \hat{N}(\phi_1 \cdots \phi_{n-1}\phi_n)\phi + \hat{N}(\phi_1 \cdots \phi_n), \quad (7.805)
\]
7 Quantization of Relativistic Free Fields

which is valid as long as $\phi$ is earlier than the others. This formula is proved by splitting $\phi$ into creation and annihilation parts and considering each part separately, which is allowed because of the linearity of the normal products. For $\phi = \phi^a$, the formula is trivially true since all contractions vanish. Indeed, for a field $\phi^a$ with an earlier time argument than that of all others, we find:

$$\hat{\phi}_i \phi^a \equiv \langle \hat{T}(\phi(x_i))\phi^a(x)|0\rangle = \langle \hat{\phi}(x_i)\phi^a(x)|0\rangle = 0.$$  

(7.806)

For $\phi = \phi^c$, let us for a moment assume that all other $\phi^i$’s are annihilating parts $\phi^c_i$. If one or more creation parts $\phi^c_i$ are present, they will appear as left-hand factors in each normally ordered term of (7.805), and participate only as spectators in all further operations. Thus we shall prove that

$$\hat{N}(\phi^a_1 \cdots \phi^a_n)\phi^c = \hat{N}(\phi^c_1 \cdots \phi^c_n) + \hat{N}(\phi^a_1 \cdots \phi^c_n) + \ldots + \hat{N}(\phi^a_1 \cdots \phi^a_n)\phi^c,$$  

(7.807)

for any creation part of a field $\phi^c$ that lies earlier than the annihilation parts $\phi^a_i$’s. The proof proceeds by induction.

To start, we show that (7.807) is true for $n = 2$:

$$\hat{N}(\phi^a)\phi^c = \hat{N}(\phi^a\phi^c) + \hat{N}(\phi^a\phi^c).$$  

(7.808)

Using (7.798), the left-hand side is equal to $\phi^a \phi^c$, which can be rewritten as

$$\phi^a \phi^c = \pm \phi^c \phi^a [\phi^a, \phi^c]_+. $$  

(7.809)

Since the commutator or anticommutator is a $c$-number, it can be replaced as in (7.795) by its vacuum expectation value, and thus by a contraction (7.806):

$$[\phi^a, \phi^c]_+ = \langle 0|[[\phi^a, \phi^c]_+|0\rangle = \phi^a \phi^c. $$  

(7.810)

Rewriting $\pm \phi^c \phi^a$ as $\hat{N}(\phi^c \phi^a)$ and trivially as $\hat{\phi}^a \hat{\phi}^c$, we see that (7.808) is true.

Now suppose (7.807) to be true for $n$ factors and consider one more annihilation field $\phi^c_0$ inside a normal product. This field can immediately be taken outside:

$$\hat{N}(\phi^a_0 \phi^a_1 \cdots \phi^a_n)\phi^c = \phi^a_0 \hat{N}(\phi^a_1 \cdots \phi^a_n)\phi^c.$$  

(7.811)

The remainder can be expanded via (7.807):

$$\hat{N}(\phi^a_0 \phi^a_1 \phi^a_2 \cdots \phi^a_n)\phi^c \equiv \phi^a_0 [N(\phi^a_1 \phi^a_2 \cdots \phi^a_n)\phi^c] + \hat{N}(\phi^a_1 \phi^a_2 \cdots \phi^a_n)\phi^c)$$

+ \hat{N}(\phi^a_1 \phi^a_2 \cdots \phi^a_n)\phi^c) + \ldots + \hat{N}(\phi^a_1 \phi^a_2 \cdots \phi^a_n)\phi^c). $$  

(7.812)

The additional $\phi^a_0$ can now be taken into all terms where the creation operator $\phi^c$ appears in a contraction:

$$\hat{N}(\phi^a_0 \phi^a_1 \phi^a_2 \cdots \phi^a_n)\phi^c \equiv \phi^a_0 + N(\phi^a_0 \phi^a_1 \phi^a_2 \cdots \phi^a_n)\phi^c) + \hat{N}(\phi^a_0 \phi^a_1 \phi^a_2 \cdots \phi^a_n)\phi^c)$$

+ \hat{N}(\phi^a_0 \phi^a_1 \phi^a_2 \cdots \phi^a_n)\phi^c) + \ldots + \hat{N}(\phi^a_0 \phi^a_1 \phi^a_2 \cdots \phi^a_n)\phi^c). $$  

(7.813)
Only in the first term work is needed to reorder the uncontracted creation operator. Writing out the normal product explicitly, this term reads

\[ \eta \phi_0^c \phi_1^a \cdots \phi_n^a, \quad (7.814) \]

where \( \eta \) denotes the number of fermion transmutations necessary to bring \( \phi^c \) from the right-hand side to its normal position. This product can now be rewritten in normal form by using one additional commutator or anticommutator:

\[ \pm \eta \phi^c \phi_0^a \phi_1^a \cdots \phi_n^a + \eta [\phi_0^c, \phi^c] \phi_1^a \cdots \phi_n^a. \quad (7.815) \]

But the commutator or anticommutator is again a \( c \)-number, and since \( \phi_0^c \) is earlier than \( \phi_1^a \), it is equal to the contraction \( \phi_0^c \phi_1^a \). The two terms in (7.815) can therefore be rewritten as normal products

\[ \hat{N}(\phi_0^c \cdots \phi_n^a) + \hat{N}(\phi_0^c \cdots \phi_n^a \phi_1^a \phi_2^a \cdots \phi_n^a). \quad (7.816) \]

In both terms we have brought the field \( \phi^c \) back to its original position, thereby canceling the sign factors \( \eta \) and \( \pm \eta \) in (7.815). These two are just the missing terms in (7.812) to verify the expansion (7.807) for \( n+1 \) operators, and thus Wick’s theorem (7.802).

Examples are

\[ T(\phi_1 \phi_2 \phi_3) = \hat{N}(\phi_1 \phi_2 \phi_3) + \phi_1 \phi_2 N(\phi_3) \pm \phi_1 \phi_3 N(\phi_2) + \phi_2 \phi_3 N(\phi_1), \quad (7.817) \]

and

\[ \hat{T}(\phi_1 \phi_2 \phi_3 \phi_4) = \hat{N}(\phi_1 \phi_2 \phi_3 \phi_4) \]
\[ + \phi_1 \phi_2 N(\phi_3 \phi_4) \pm \phi_1 \phi_3 N(\phi_2 \phi_4) + \phi_1 \phi_4 N(\phi_2 \phi_3) \]
\[ + \phi_2 \phi_3 N(\phi_1 \phi_4) \pm \phi_2 \phi_4 N(\phi_1 \phi_3) + \phi_3 \phi_4 N(\phi_1 \phi_2) \]
\[ = \phi_1 \phi_2 \phi_3 \phi_4 \pm \phi_1 \phi_3 \phi_2 \phi_4 + \phi_1 \phi_4 \phi_2 \phi_3. \quad (7.818) \]

Each expansion contains \((n-1)!!\) fully contracted terms.

There is a useful lemma to Wick’s theorem which helps in evaluating vacuum expectation values of ordinary products of field operators rather than time-ordered ones. An ordinary product can be expanded in a sum of normal products with all possible pair contractions just as in (7.801), except that contractions stand for commutators of the creation and annihilation parts of the fields, which are \( c \)-numbers:

\[ \phi(x_1) \phi(x_2) \equiv [\phi^a(x_1), \phi^c(x_2)]. \quad (7.819) \]

The proof proceeds by induction in the same way as before, except that one does not have to make the assumption of the additional field \( \phi \) in the induction being the earliest of all field operators. For the calculation of scattering amplitudes we shall need both expansions.
7.18 Functional Form of Wick’s Theorem

Although Wick’s rule for expanding the time ordered product of \( n \) fields in terms of normal products is quite transparent, it is somewhat cumbersome to state them explicitly for any \( n \), as we can see in formula (7.801). It would be useful to find an algorithm which specifies the expansion compactly in mathematical terms. This would greatly simplify further manipulations of operator products. Such an algorithm can easily be found. Let \( \frac{\delta}{\delta \phi} \) denote the functional differentiations for functions of space and time, i.e.,

\[
\frac{\delta \phi(x)}{\delta \phi(x')} = \delta^{(4)}(x - x').
\]

Then we see immediately that the basic formula (7.790) can be rewritten as

\[
\hat{T}(\phi(x_1)\phi(x_2)) = \left( 1 \pm \frac{1}{2} \int d^4y_1 d^4y_2 \frac{\delta}{\delta \phi(y_1)} G(y_1, y_2) \frac{\delta}{\delta \phi(y_2)} \right) \hat{N}(\phi(x_1)\phi(x_2)),
\]

where functional differentiation treats field variables as \( c \)-numbers. The differentiation produces the single possible contraction which by (7.791) is equal to the free propagator \( G(x_1, x_2) \). For three operators, the Wick expansion

\[
\hat{T}(\phi(x_1)\phi(x_2)\phi(x_3)) = \hat{N}(\phi(x_1)\phi(x_2)\phi(x_3)) + \hat{N}(\dot{\phi}(x_1)\phi(x_2)\phi(x_3)) + \hat{N}(\phi(x_1)\phi(x_2)\dot{\phi}(x_3)) + \hat{N}(\phi(x_1)\phi(x_2)\phi(x_3))
\]

can once more be obtained via the same operation

\[
\left( 1 \pm \frac{1}{2} \int d^4y_1 d^4y_2 \frac{\delta}{\delta \phi(y_1)} G(y_1, y_2) \frac{\delta}{\delta \phi(y_2)} \right) \hat{N}(\phi(x_1)\phi(x_2)\phi(x_3)).
\]

If we want to recover the correct signs for fermions, we have to imagine the field variables \( \phi(x) \) to be anticommuting objects satisfying \( \phi(x)\phi(y) = -\phi(y)\phi(x) \), which results in the antisymmetry \( G(x_1, x_2) = -G(x_2, x_1) \).

A time-ordered product of four field operators has a Wick expansion

\[
T(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)) = \hat{N}(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)) + \sum_{\text{onepaircontractions}} \hat{N}(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)) + \sum_{\text{twopaircontractions}} \hat{N}(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)).
\]

Here we can obtain the same result by applying the functional differentiation

\[
\left\{ 1 \pm \frac{1}{2} \int d^4y_1 d^4y_2 \frac{\delta}{\delta \phi(y_1)} G(y_1, y_2) \frac{\delta}{\delta \phi(y_2)}
\right. \\
\left. + \frac{1}{8} \left( \int d^4y_1 d^4y_2 \frac{\delta}{\delta \phi(y_1)} G(y_1, y_2) \frac{\delta}{\delta \phi(y_2)} \right)^2 \right\}.
\]
This looks like the first three pieces of a Taylor expansion of the exponential

\[ \exp \left\{ \pm \frac{1}{2} \int d^4y_1d^4y_2 \frac{\delta}{\delta \phi(y_1)} G(y_1, y_2) \frac{\delta}{\delta \phi(y_2)} \right\} . \]

Indeed, we shall now prove that the general Wick expansion is given by the functional formula

\[ \hat{T}(\phi(x_1) \cdots \phi(x_n)) = e^{\pm \frac{1}{2} \int d^4y_1 \int d^4y_2 \frac{\delta}{\delta \phi(y_1)} G(y_1, y_2) \frac{\delta}{\delta \phi(y_2)} \hat{N}(\phi(x_1) \cdots \phi(x_n))} . \] (7.826)

To prepare the tools for deriving this result we introduce the following generating functionals of ordinary, time-ordered, and normal-ordered operator products, respectively:

\[ O[j] \equiv e^{\int d^4x j(x) \phi(x)} , \]
\[ T[j] \equiv \hat{T} e^{\int d^4x j(x) \phi(x)} , \]
\[ N[j] \equiv \hat{N} e^{\int d^4x j(x) \phi(x)} . \] (7.827)

Here \( j(x) \) is an arbitrary \( c \)-number local source term, usually referred to as an external current. For Fermi fields \( \phi(x) \), the currents have to be anticommuting Grassmann variables which also anticommute with the fields. Under complex conjugation, products of Grassmann variables interchange their order. The product \( j(x)\phi(x) \) has therefore the complex-conjugation property

\[ [j(x)\phi(x)]^* = -\phi(x) j(x) . \] (7.828)

The expressions (7.827) are operators. They have the obvious property that their \( n \)th functional derivatives with respect to \( j(x) \), evaluated at \( j(x) \equiv 0 \), give the corresponding operator products of \( n \) fields:

\[ \phi(x_1) \cdots \phi(x_n) = (-i)^n \left. \frac{\delta}{\delta j(x_1)} \cdots \frac{\delta}{\delta j(x_n)} O[j] \right|_{j(x)\equiv 0} , \] (7.829)
\[ \hat{T}(\phi(x_1) \cdots \phi(x_n)) = (-i)^n \left. \frac{\delta}{\delta j(x_1)} \cdots \frac{\delta}{\delta j(x_n)} T[j] \right|_{j(x)\equiv 0} , \] (7.830)
\[ \hat{N}(\phi(x_1) \cdots \phi(x_n)) = (-i)^n \left. \frac{\delta}{\delta j(x_1)} \cdots \frac{\delta}{\delta j(x_n)} N[j] \right|_{j(x)\equiv 0} . \] (7.831)

We can now easily prove Wick’s lemma for ordinary products of field operators. We simply separate the field in the generating functional \( O[j] \) into creation and annihilation parts:

\[ O[j] \equiv e^{\int d^4x j(x) [\phi^{\dagger}(x) + \phi(x)]} , \] (7.832)
and apply the Baker-Hausdorff formula (4.74) to this expression, with \( A \equiv i \int d^4x \, j(x) \phi^a(x) \) and \( B \equiv i \int d^4x \, j(x) \phi^c(x) \). Because of the anticommutativity of \( \phi(x) \) and \( j(x) \) for fermions, the commutator \([A, B]\) is

\[
[A, B] = - \int d^4x_1 \int d^4x_2 \left[ \phi^a(x_1), \phi^c(x_2) \right] = j(x_1) j(x_2),
\]

(7.833)

which is a \( c \)-number. Thus the expansion on the right-hand side of (4.74) terminates, and we can use the formula as

\[
e^{A+B} = e^{-\frac{1}{2}[A,B]} e^A e^B.
\]

(7.834)

This yields the simple result

\[
O[j] = \exp \left\{ \frac{1}{2} \int d^4x_1 \int d^4x_2 \left[ \phi^a(x_1), \phi^c(x_2) \right] j(x_1) j(x_2) \right\}
\]

\[
\times e^{i \int d^4x j(x) \phi^c(x)} e^{i \int d^4x j(x) \phi^a(x)}.
\]

(7.835)

The operator on the right-hand side of the curly bracket is recognized to be precisely the generating functional of the normal products:

\[
N[j] = e^{i \int d^4x j(x) \phi^c(x)} e^{i \int d^4x j(x) \phi^a(x)}.
\]

(7.836)

Now we observe that the currents \( j(x) \) in front of this operator are equivalent to a functional differentiation with respect to the fields, \( j(x) = -i \delta / \delta \phi(x) \). Wick’s lemma can therefore be rewritten in the following concise functional form

\[
O[j] = \exp \left\{ \mp \frac{1}{2} \int d^4x_1 \int d^4x_2 \left[ \phi^a(x_1), \phi^c(x_2) \right] j(x_1) j(x_2) \right\} \delta \phi(x_1) \delta \phi(x_2)} N[j].
\]

The \( \pm \)-sign appears when commuting or anticommuting \( \delta / \delta \phi(x) \) with \( j(x) \).

A Taylor expansion of the exponential prefactor renders, via the functional derivatives, precisely all pair contractions required by Wick’s lemma, where a contraction represents a commutator or anticommutator of the fields.

From the earlier proof in Section (7.17.1) it is now clear that exactly the same set of pair contractions appears in the Wick expansion of the time-ordered products of field operators, except that the pair contractions become free-field propagators

\[
G(x_1, x_2) \equiv \langle 0 | T \left( \phi(x_1) \phi(x_2) \right) | 0 \rangle.
\]

(7.837)

Thus we may immediately conclude that Wick’s theorem has the functional form

\[
T[j] = \exp \left\{ \pm \frac{1}{2} \int d^4x_1 \int d^4x_2 G(x_1, x_2) \frac{\delta}{\delta \phi(x_1)} \frac{\delta}{\delta \phi(x_2)} \right\} N[j],
\]

(7.838)

which becomes, after reexpressing the functional derivatives in terms of the currents,

\[
T[j] = e^{-\frac{1}{2} \int d^4y_1 d^4y_2 j(y_1) G(y_1, y_2) j(y_2) N[j].
\]

(7.839)
Expanding both sides in powers of $j(x)$, a comparison of the coefficients yields Wick’s expansion for products of any number of time-ordered operators.

The functional $T[j]$ has the important property that its vacuum expectation value

$$Z[j] \equiv \langle 0 | T[j] | 0 \rangle$$

(7.840)

collects all informations on $n$-point functions. Indeed, forming the $n$th functional derivative as in (7.830) we see that

$$G^{(n)}(x_1, \ldots, x_n) = (-i)^n \frac{\delta}{\delta j(x_1)} \cdots \frac{\delta}{\delta j(x_n)} Z[j].$$

(7.841)

For this reason, the vacuum expectation value $Z[j]$ is called the generating functional of all $n$-point functions. We can take advantage of Wick’s expansion formula and write down a compact formula for all $n$-point functions of free fields. We only have to observe that the generating functional of normal products $\hat{N}[j]$ has a trivial vacuum expectation value

$$\langle 0 | \hat{N}[j] | 0 \rangle \equiv 1.$$  

(7.842)

Then (7.839) leads directly to the simple expression

$$Z[j] = e^{-\frac{1}{2} \int d^4y_1 d^4y_2 j(y_1) G(y_1, y_2) j(y_2)}.$$  

(7.843)

Expanding the right-hand side in powers of $j(x)$, each term consists of a sum of combinations of propagators $G(x_i, x_{i+1})$. These are precisely the fully pair contracted terms appearing in the Wick expansions (7.803) for the $n$-point functions.

There exists another way of proving Eq. (7.843), by making use of the equation of motion for the field $\phi(x)$. For simplicity, we shall consider a Klein-Gordon field satisfying the equation of motion

$$(-\partial^2 - M^2) \phi(x) = 0.$$  

(7.844)

This equation can be applied to the first functional derivative of $Z[j]$: 

$$\frac{\delta Z[j]}{\delta j(x)} = \langle 0 | T \phi(x) e^{i \int d^4y \phi(y)} | 0 \rangle,$$  

(7.845)

to find the differential equation

$$(-\partial^2 - M^2) \frac{\delta Z[j]}{\delta j(x)} = -ij(x) Z[j].$$  

(7.846)

The correctness of this equation is verified by integrating it functionally with the initial condition

$$Z[0] = 1,$$  

(7.847)

and by recovering the known equation (7.843) with $G(x, y) = i/(-\partial^2 - M^2)$. A direct proof of the differential equation (7.846) proceeds by expanding both sides in
a Taylor series in \( j(x) \) and processing every expansion term as in (7.43)–(7.45). If we were able to pass \((-\partial^2 - m^2)\) through the time-ordering operation, we would obtain zero. During the passage, however, \( \partial^2_0 \) encounters the Heaviside function \( \Theta(x_0 - x_0') \) of the time ordering which generates additional terms. These are collected by the left-hand side a term \( nT\phi(x)[\int dz j(z)\phi(z)]^{n-1}/n! \). Applying \((-\partial^2 - m^2)\) and using the steps (7.43)–(7.45), we obtain \( n \) times a canonical commutator between \( \phi(x) \) and \( \phi(z) \) which can be written as \(-j(x)T[\int dz j(z)\phi(z)]^{n-2}/(n-2)! \). Summing these up gives \(-jZ[j]\). These properties will find extensive use in Chapter 13.

The formulas can be generalized to complex fields by using the generating functionals

\[
\begin{align*}
O[j,j^\dagger] &= e^{\int d^4x [j^\dagger(x)\phi(x) + j(x)\phi^\dagger(x)]}, \\
T[j,j^\dagger] &= \tilde{T}e^{\int d^4x [j^\dagger(x)\phi(x) + j(x)\phi^\dagger(x)]}, \\
N[j,j^\dagger] &= \tilde{N}e^{\int d^4x [j^\dagger(x)\phi(x) + j(x)\phi^\dagger(x)]},
\end{align*}
\]

Note that for fermions, the interaction \( j^\dagger(x)\phi(x) \) is odd under complex conjugation, just as for real anticommuting fields in (7.827).

Now the propagators of \( n \) fields \( \phi \) and \( m \) fields \( \phi^\dagger \) are obtained from the expectation value

\[
Z[j,j^\dagger] = \langle 0|\tilde{T}[j,j^\dagger]|0 \rangle 
\]

as the functional derivatives

\[
G^{(n,m)}(x_1,\ldots,x_n; y_1\ldots y_m) = (-i)^{m+n} \delta \frac{\delta}{\delta j^\dagger(x_1)} \cdots \delta \frac{\delta}{\delta j^\dagger(x_n)} \delta \frac{\delta}{\delta j(y_1)} \cdots \delta \frac{\delta}{\delta j(y_m)} Z[j,j^\dagger] \bigg|_{j(x)\equiv 0}.
\]

The explicit form of the generating functional \( Z[j,j^\dagger] \) is

\[
Z[j,j^\dagger] = e^{-\int d^4x d^4y j^\dagger(x)G(x,y)j(y)}. 
\]

The latter form is valid for fermions if one uses anticommuting currents \( j(x), j^\dagger(x) \).

### 7.18.1 Thermodynamic Version of Wick’s Theorem

By analogy with the generalization of the one-particle Green function (7.775) to the thermodynamic \( n \)-point functions \( G^{(n)} \) of (7.777) it is useful to generalize also the thermal propagator (2.391). We shall treat here explicitly only the nonrelativistic case. The generalization to relativistic fields is straightforward.

For this we introduce the thermal average of an imaginary-time-ordered product of \( n + n' \) fields:

\[
G^{(n,n')}(x_1, \tau_1, \ldots, x_n, \tau_n; x'_1, \tau'_1, \ldots, x'_{n'}, \tau'_{n'}) = \frac{\text{Tr} \left[ e^{-H_G/k_BT} T_{\tau} \left( \psi(x_1, \tau_1) \cdots \psi(x_n, \tau_n) \psi^\dagger(x'_1, \tau'_1) \cdots \psi^\dagger(x'_{n'}, \tau'_{n'}) \right) \right]}{\text{Tr} \left[ e^{-H_G/k_BT} \right]},
\]

(7.852)
where the free fields are expanded as in (2.405):

\[
\psi(x, \tau) = \int d^3 p \, f_p(x, \tau) a_p, \quad (7.853)
\]

\[
\psi^\dagger(x, \tau) = \int d^3 p \, f_p^\ast(x, \tau) a_p^\dagger, \quad (7.854)
\]

with the wave functions

\[
f_p(x, \tau) \equiv e^{i [p x - \xi(p) \tau] / \hbar} \equiv f_p^\ast(x, -\tau), \quad \xi(p) = \varepsilon(p) - \mu, \quad (7.855)
\]

where \( \mu \) is the chemical potential. We shall write all subsequent equations with an explicit \( \hbar \). An explicit evaluation of the thermal propagators (7.852) is much more involved than it was for vacuum expectation values. It proceeds using Wick’s theorem, expanding the thermal expectation values in \( G(n,n') \) into a sum of products of two-particle thermal propagators, the sum running over all possible pair contractions, just as in Eq. (7.803) for the \( n \)-point function in the vacuum. All thermal information is therefore contained in the one-particle propagators which have the Fourier decomposition (2.420) with (2.419). The detailed form is here irrelevant.

To prove now Wick’s expansion, we first observe that the free Hamiltonian conserves particle number, so that \( G^{(n,n')} \) vanishes unless \( n = n' \). As a next step we order the product of field operators according to their imaginary time arguments \( \tau_n \), thereby picking up some sign factor \( \pm 1 \) depending on the number of transpositions of Fermi fields. On the right-hand side of Wick’s expansion the corresponding permutations lead to the same sign. Thus we may assume all field operators to be time-ordered. The ordering in \( \tau \) yields a product in which the fields \( \psi \) and \( \psi^\dagger \) appear in a mixed fashion. We expand each field as \( \int d^3 p \, f_p \alpha_p \), using the same notation for \( \alpha_p = a_p \) and \( \alpha_p = a_p^\dagger \), with the tacit understanding that, in the latter case, the wave function is taken as \( f_p^\ast \) rather than \( f_p \). Expanding the product of field operators in this way, we have

\[
G^{(n,n')} = \int d^3 p_1 \cdots d^3 p_{2n} \ f_1 \cdots f_{2n} \\
= \text{Tr} \left( e^{-H_G/k_B T} \alpha_1 \cdots \alpha_{2n} \right) / \text{Tr} \left( e^{-H_G/k_B T} \right), \quad (7.856)
\]

where we have omitted the momentum labels on the operators \( \alpha \), using only their numeric subscripts. There are always equally many creators as annihilators among the \( \alpha_i \)'s. In order to reduce the traces to a pure number, let us commute or anti-commute \( \alpha_1 \) successively to the right via

\[
\alpha_1 \alpha_2 \alpha_3 \cdots \alpha_{2n} = [\alpha_1, \alpha_2] \pm \alpha_3 \cdots \alpha_{2n} \\
\pm \alpha_2 [\alpha_1, \alpha_3] \pm \alpha_4 \cdots \alpha_{2n} \\
+ \cdots \\
+ \alpha_2 \alpha_3 \cdots [\alpha_1, \alpha_{2n}] \mp \\
\pm \alpha_2 \alpha_3 \cdots \alpha_{2n} \alpha_1. \quad (7.857)
\]
The commutators or anticommutators among $\alpha_1$ and $\alpha_i$ will give 0 if both are creators or annihilators, 1 if $\alpha_1$ is an annihilator and $\alpha_i$ a creator, and $\mp 1$ in the opposite case. At any rate, they are $c$-numbers, so that they may be taken out of the trace leading to

$$\text{Tr} \left( e^{-H_G/k_BT} \alpha_1 \alpha_2 \cdots \alpha_{2n} \right) \mp \text{Tr} \left( e^{-H_G/k_BT} \alpha_2 \cdots \alpha_{2n} \alpha_1 \right) = [\alpha_1, \alpha_2] \mp \text{Tr} \left( e^{-H_G/k_BT} \alpha_3 \alpha_4 \cdots \alpha_{2n} \right) \mp \cdots \mp [\alpha_1, \alpha_{2n}] \mp \text{Tr} \left( e^{-H_G/k_BT} \alpha_2 \alpha_3 \cdots \alpha_{2n-1} \right).$$

(7.858)

Now we make use of the cyclic property of the trace and observe that

$$\alpha_1 e^{-H_G/k_BT} = e^{-H_G/k_BT} \alpha_1 e^{\eta \xi(p)/kB_T},$$

(7.859)

where the sign factor $\eta = \pm 1$ depends on whether $\alpha_1$ is a creator or an annihilator, respectively. Then we can rewrite the left-hand side in (7.858) as

$$\left( 1 \mp e^{\eta \xi(p)/kB_T} \right) \text{Tr} \left( e^{-H_G/k_BT} \alpha_2 \cdots \alpha_{2n} \alpha_1 \right).$$

(7.860)

Dividing the prefactor out, we obtain

$$\text{Tr} \left( e^{-H_G/k_BT} \alpha_1 \cdots \alpha_{2n} \right) = \text{Tr} \left( e^{-H_G/k_BT} \dot{\alpha}_1 \dot{\alpha}_2 \alpha_3 \cdots \alpha_{2n} \right) + \text{Tr} \left( e^{H_G/k_BT} \dot{\alpha}_1 \alpha_2 \dot{\alpha}_3 \cdots \alpha_{2n} \right) + \cdots + \text{Tr} \left( e^{-H_G/k_BT} \dot{\alpha}_1 \alpha_2 \alpha_3 \cdots \dot{\alpha}_{2n} \right),$$

(7.861)

where a thermal contraction is defined as

$$\dot{\alpha}_1 \dot{\alpha}_i = \frac{[a_p, a^\dagger_{p'}]}{1 \mp e^{\eta \xi(p)/kB_T}},$$

(7.862)

with the same rules of taking these $c$-numbers outside the trace as in the ordinary Wick contractions in Subsec. 7.17.1.

These contractions are just the Fourier transforms of the one-particle thermal propagators as given in (2.409) and (2.410). For if $\alpha_1$ and $\alpha_i$ are both creators or annihilators, this is trivially true: both expressions vanish identically. If $\alpha_1$ is an annihilator and $\alpha_i$ a creator, we see that [recalling (2.408) and (2.411)]

$$\dot{\alpha}_1 \dot{\alpha}_i = \frac{[a_p, a^\dagger_{p'}]}{1 \mp e^{\eta \xi(p)/kB_T}} = \delta^{(3)}(\pmb{p} - \pmb{p'}) (1 \pm n_{\xi(p)}).$$

(7.863)

In the opposite case

$$\dot{\alpha}_1 \dot{\alpha}_i = \frac{[a^\dagger_{p'}, a_p]}{1 \mp e^{\eta \xi(p)/kB_T}} = \delta^{(3)}(\pmb{p} - \pmb{p'}) n_{\xi(p)}.$$

(7.864)
7.18 Functional Form of Wick’s Theorem

Multiplying either of these expressions by the product of wave functions \( f_p(x) f'_p(x) \), and integrating everything over the entire momentum space, we find that these contractions are precisely the propagators \( G(x, \tau; x', \tau') = G(x - x', \tau - \tau') \):

\[
\hat{\psi}(x, \tau)\hat{\psi}^\dagger(x', \tau') = \pm \hat{\psi}^\dagger(x', \tau')\hat{\psi}(x, \tau) = G(x - x', \tau - \tau').
\]  

(7.865)

In (7.861), the \( n \)-particle thermal Green function \( G^{(n,n)} \) has been reduced to a sum of \( (n-1) \)-particle propagators, each multiplied by an ordinary propagator. Continuing this procedure iteratively, we arrive at Wick’s expansion for thermal Green functions.

As before, this result can be expressed most concisely in a functional form. For this we introduce

\[
Z[j, j^\dagger] = \operatorname{Tr} \left( e^{-\frac{H_G}{k_B T} T_\tau c_l \int d^4x dr [j(x, \tau)\hat{\psi}^\dagger(x, \tau) + \hat{\psi}(x, \tau)j^\dagger(x, \tau)]} \right) / \operatorname{Tr} \left( e^{-\frac{H_G}{k_B T}} \right).
\]  

(7.866)

as the generating functional of thermal Green functions. Then Wick’s expansion amounts to the statement

\[
Z[i, j^\dagger] = e^{-\int d^4x d\tau d^4x' d\tau' j^\dagger(x, \tau) G(x - x', \tau - \tau') j(x, \tau)}.
\]  

(7.867)

As before, this can be derived from the equation of motion [compare (2.399) and (2.400)]

\[
\left( -\partial_\tau + \frac{\nabla^2}{2M} + \mu \right) \frac{\delta}{\delta j^\dagger(x, \tau)} Z[j, j^\dagger] = -Z[j, j^\dagger] j(x, \tau),
\]

\[
Z[j, j^\dagger] \frac{\delta}{\delta j(x, \tau)} \left( -\partial_\tau + \frac{\nabla^2}{2M} + \mu \right) = -j^\dagger(x, \tau) Z[j, j^\dagger],
\]  

(7.868)

following the procedure to prove Eq. (7.846). Here the proof is simpler since only one time derivative needs to be passed through the time-ordering operation \( T_\tau \) producing a canonical equal-time commutator for every pair of fields, as in (2.237). As a check we integrate again (7.868) with the initial condition \( Z[0, 0] = 1 \), and recover (7.867).

It is worth deriving also a related theorem concerning the thermal expectations of normally ordered field operators. According to Wick’s theorem in the operator form (7.802), any product of time ordered operators can be expanded into normal products, so that

\[
Z[j, j^\dagger] = e^{-\int d^4x d^4x' j^\dagger(x) G(x - x') j(x)} \times \frac{\operatorname{Tr} \left( e^{-\frac{H_G}{k_B T} N c_l \int d^4x [j^\dagger(x)\hat{\psi}(x) + \hat{\psi}^\dagger(x)j(x)]} \right)}{\operatorname{Tr} \left( e^{-\frac{H_G}{k_B T}} \right)},
\]  

(7.869)

where we have used the four-vector notation \( x = (x_1, \tau) \) and \( d^4x = d^3x d\tau \), for brevity. We can show that the expectation of the normal product can be expanded once more in a Wick-like way. For this we introduce the expectation of two normally ordered operators

\[
G_N(x_1, x_2) \equiv \frac{\operatorname{Tr} \left[ e^{-\frac{H_G}{k_B T} N \int \hat{\psi}^\dagger(x_1)\hat{\psi}(x_2)} \right]}{\operatorname{Tr} \left( e^{-\frac{H_G}{k_B T}} \right)},
\]  

(7.870)
This quantity can be calculated by inserting (7.853) and (7.854), as well as intermediate states, with the result

\[ G^\hat{N}(x_1, x_2) = \int d^3p e^{i[p(x_2-x_1)+i\xi(p)\tau_1]/\hbar} n_\xi(p), \]  

(7.871)

where \( n_\xi \) is the particle distribution function for bosons and fermions, respectively:

\[ n_\xi = \frac{1}{e^{\xi/k_BT} \pm 1}. \]  

(7.872)

It can now be shown that the thermal expectation value of an arbitrary normal product can be expanded into expectation values of the one-particle normal products (7.870). This happens in the same way as the decomposition of the \( \tau \)-ordered products into the expectations of one-particle \( \tau \)-ordered propagators. The proof is most easily given functionally. We consider the generating functional

\[ \hat{N}[j, j^\dagger] \equiv \text{Tr} \left[ e^{-H_G/k_BT} N e^{i \int d^4x (\psi^\dagger(x) j(x) + j^\dagger(x) \psi(x))} \right] / \text{Tr} \left[ e^{-H_G/k_BT} \right] \]  

(7.873)

and differentiate this with respect to \( j^\dagger(x) \):

\[ \frac{\delta N[j, j^\dagger]}{\delta j^\dagger(x)} = \text{Tr} \left[ e^{-H_G/k_BT} N \left( \psi(x) e^{i \int d^4x (\psi^\dagger(x) j(x) + j^\dagger(x) \psi(x))} \right) \right] / \text{Tr} \left[ e^{-H_G/k_BT} \right]. \]  

(7.874)

Applying to this the field equation

\[ \left( -\partial_\tau + \frac{\nabla^2}{2M} + \mu \right) \psi(x) = 0, \]  

(7.875)

we see that \( \hat{N}[j, j^\dagger] \) satisfies

\[ \left( -\partial_\tau + \frac{\nabla^2}{2M} + \mu \right) \frac{\delta}{\delta j^\dagger} \hat{N}[j, j^\dagger] = 0. \]  

(7.876)

This is solved by

\[ \hat{N}[j, j^\dagger] = e^{-\int d^4x_1 d^4x_2 j^\dagger(x_1) G_N(x_1, x_2) j(x_2)}, \]  

(7.877)

where \( G_N(x_1, x_2) \) is a solution of the homogeneous differential equation. To obtain the correct result for the ordinary imaginary-time propagator, we expand both sides in \( j, j^\dagger \) and see that \( G_N(x_1, x_2) \) coincides with (7.870), (7.871).

**Appendix 7A Euler-Maclaurin Formula**

The Euler-Maclaurin formula serves to calculate discrete sums such as those in Eqs. (7.693), (7.702), and (7.693). It can be derived from a fundamental summation formula due to Bernoulli. Starting point is the observation that a power \( n^p \) may be integrated over a real variable \( n \) to

\[ \int_N^M d_n n^p = \frac{b_{p+1}(M) - b_{p+1}(N)}{p+1}, \]  

(7A.1)
where \( b_{p+1}(x) \equiv x^{p+1} \). The corresponding summation formula produces Bernoulli polynomials \( B_p(x) \) as follows:

\[
\sum_{n=N}^{M} n^p = \frac{B_{p+1}(M + 1) - B_{p+1}(N)}{p + 1}.
\]

Here \( B_p(x) \) denote the Bernoulli polynomials:

\[
\begin{align*}
B_0(x) &\equiv 1, \\
B_1(x) &= -\frac{1}{2} + x, \\
B_2(x) &= \frac{1}{6} - x + x^2, \\
B_3(x) &= \frac{1}{2}x - \frac{3}{2}x^2 + x^3, \\
B_4(x) &= -\frac{1}{30} + x^2 - 2x^3 + x^4, \\
B_5(x) &= -\frac{1}{6}x + \frac{5}{3}x^3 - \frac{5}{2}x^4 + x^5, \\
&\vdots
\end{align*}
\]

The more general integral

\[
\int_{N}^{M} dn (n + x)^p = \frac{b_{p+1}(M + x) - b_{p+1}(N + x)}{p + 1}
\]

goes into the sum:

\[
\sum_{n=N}^{M} (n + x)^p = \frac{B_{p+1}(M + 1 + x) - B_{p+1}(N + 1 + x)}{p + 1}.
\]

The values of the polynomials \( B_p(x) \) at \( x = 0 \) are called Bernoulli numbers \( B_p \):

\[
B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad B_5 = 0, \quad B_6 = \frac{1}{42}, \quad \ldots
\]

Except for \( B_1 \), all odd Bernoulli numbers vanish.

The close correspondence between integrals and sums has its counterpart in differential versus difference equations: By differentiating \( b_p(x) = x^p \), we find:

\[
b'_p(x) = px^{p-1},
\]

whose discrete counterpart is the difference equation

\[
B_p(x + 1) - B_p(x) = px^{p-1},
\]

and the initial condition

\[
B_0(x) \equiv 1.
\]

The differential relation

\[
b'_p(x) = pb_{p-1}(x)
\]

is completely shared by \( B_p(x) \):

\[
B'_p(x) = pB_{p-1}(x).
\]

There is another important property of the function \( b_p(x) = x^p \). The binomial expansion of \( (x + h)^p \),

\[
b_p(x + h) = \sum_{q=0}^{p} \binom{p}{q} b_q(x)h^{p-q},
\]

where
This follows from expanding the left-hand side into powers of \(h\)

\[
B_p(x + h) = \sum_{n=0}^{p} \binom{p}{n} B_n(x) h^{p-n}.
\]

(7A.13)

A useful property of the Bernoulli polynomials is caused by the discrete nature of the sum and reads

\[
B_p(1 - x) = (-1)^p B_p(x).
\]

(7A.14)

There exists a simple generating function for the Bernoulli polynomials which ensures the fundamental property (7A.8):

\[
\frac{te^{xt}}{e^t - 1} = \sum_{p=0}^{\infty} B_p(x) \frac{t^p}{p!}
\]

(7A.15)

The property (7A.5) can directly be used to calculate the difference between an integral and a discrete sum over a function \(F(t)\). If an interval \(t \in (a, b)\) is divided into \(m\) slices of width \(h = (b - a)/N\), we obviously have

\[
h \sum_{n=0}^{N-1} F(a + (n + x)h) - \int_a^b dt F(t) = \sum_{p=1}^{\infty} \frac{h^p}{p!} [B_p(N + x) - B_p(x)] F^{(p-1)}(a).
\]

(7A.16)

This follows from expanding the left-hand side into powers of \(h\). After this, the binomial expansion (7A.28) leads to

\[
\sum_{p=1}^{\infty} \frac{h^p}{p!} B_p(x + N + 1) F^{(p-1)}(a) = \sum_{p=1}^{\infty} \frac{h^p}{p!} \binom{p}{l} B_l(x) N^{p-l} F^{(p-1)}(a)
\]

\[
= \sum_{l=0}^{p} \frac{h^l}{l!} B_l(x) \sum_{p=1}^{\infty} \frac{1}{(p-l)!} (hN)^{p-l} F^{(p-1)}(a)
\]

\[
= \sum_{l=0}^{p} \frac{h^l}{l!} B_l(x) F^{(l-1)}(b),
\]

(7A.17)

which brings (7A.16) to Euler’s formula

\[
h \sum_{n=0}^{N-1} F(a + (n + x)h) - \int_a^b dt F(t) = \sum_{p=1}^{\infty} \frac{h^p}{p!} B_p(x) F^{(p-1)}(b) - F^{(p-1)}(a).
\]

(7A.18)

Since all odd Bernoulli numbers vanish except for \(B_1 = -1/2\), it is useful to remove the \(p = 1\) term from the right-hand side and rewrite (7A.25) as

\[
h \sum_{n=0}^{N-1} F(a + (n + x)h) - \int_a^b dt F(t) = (-1/2 + x) [F(b) - F(a)]
\]

\[
+ \sum_{p=1}^{\infty} \frac{h^{2p}}{(2p)!} B_{2p}(x) [F^{(2p-1)}(b) - F^{(2p-1)}(a)].
\]

(7A.19)

For \(x = 0\) we may extend the sum on the left-hand side to \(n = N\) and obtain the Euler-Maclaurin formula:27

\[
h \sum_{n=0}^{N} F(a + nh) - \int_a^b dt F(t) = \frac{1}{2} [F(b) + F(a)]
\]

\[
+ \sum_{p=1}^{\infty} \frac{h^{2p}}{(2p)!} B_{2p}(x) [F^{(2p-1)}(b) - F^{(2p-1)}(a)].
\]

(7A.20)

\[27\text{See M. Abramowitz and I. Stegun, Handbook of Mathematical Functions, Dover, New York, 1965, Eqs. 23.1.30 and 23.1.32.}\]
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If the sums (7A.5) are carried to infinity, they diverge. These infinities may, however, be removed by considering such sums as analytic continuations of convergent sums

\[ \zeta(z, x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^z}, \quad \text{Re} \, z > 1, \quad (7A.21) \]

known as Riemann’s zeta functions. A continuation is possible by an appropriate deformation of the contour in the integral representation

\[ \zeta(z, x) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1} e^{-xt}}{1 - e^{-t}}. \quad (7A.22) \]

This yields the relation:

\[ \zeta(-p, x) = \frac{-B_{p+1}(x)}{p+1}. \quad (7A.23) \]

Thus the finite sum (7A.5) can be understood as the difference between the regularized infinite sums

\[ \sum_{n=0}^{\infty} (n+x)^p = \zeta(-p, x) = \frac{-B_{p+1}(x)}{p+1}, \quad (7A.24) \]

and

\[ \sum_{n=m+1}^{\infty} (n+x)^p = \zeta(-p, m+x) = \frac{-B_{p+1}(m+1+x)}{p+1}. \]

With the help of these regularized infinite sums we can derive Euler’s formula (7A.25) by considering the left-hand side as a difference between the infinite sum

\[ \left[ h \sum_{n=0}^{\infty} F(a + (n+x)h) - \int_a^\infty dt \, F(t) \right] \quad (7A.25) \]

and a corresponding sum for \( b \) instead of \( a \). By expanding each sum in powers of \( h \) and using the formula (7A.24), we find directly (7A.25).

For completeness, let us mention that alternating sums are governed quite similarly by Euler polynomials\(^{28}\)

\[ \sum_{n=N}^{M} (-1)^{m-n} (n+x)^n = \frac{E_{n+1}(M+1+x) + (-1)^m E_{n+1}(p+x)}{2}. \quad (7A.26) \]

The Euler polynomials satisfy

\[ E_p(x+1) + E_p(x) = 2x^{p-2}, \quad E'_p(x) = pE_{p-1}(x), \quad E_p(1-x) = (-1)^p E_p(x), \quad (7A.27) \]

and

\[ E_p(x+h) = \sum_{n=0}^{p} \binom{p}{n} E_n(x) h^{p-n}. \quad (7A.28) \]

The generating function of the Euler polynomials is\(^{29}\)

\[ \frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \quad (7A.29) \]

\(^{28}\)Note the difference with respect to the definition of the Bernoulli numbers (7A.6) which are defined by the values of \( B_n(x) \) at \( x = 0 \).

\(^{29}\)Further properties of these polynomials can be found in M. Abramowitz and I. Stegun, Handbook of Mathematical Functions, Dover, New York, 1965, Chapter 23.
The values $E_p(0)$ are related to the Bernoulli numbers by
\[
E_p(0) = -2(2^{p+1} - 1)B_{p+1}/(p + 1). \tag{7A.30}
\]

The quantities $E_n \equiv 2^n E_n(1/2)$ are called Euler numbers. Their values are
\[
E_0 = 1, \ E_1 = 0, \ E_2 = -1, \ E_3 = 0, \ E_4 = 5, \ E_5 = 0, \ E_6 = -61, \ldots \tag{7A.31}
\]

Note that all odd Euler numbers vanish.

Appendix 7B Liénard-Wiechert Potential

Let us calculate the field around a point source with a time-dependent position $\bar{x}(t)$, whose current is
\[
j(t', \mathbf{x'}) = \int d\tau \delta^{(4)}(x' - \bar{x}(\tau)) = \frac{1}{d\bar{x}_0/d\tau} \delta^{(3)}(x' - \bar{x}(t')), \tag{7B.1}
\]
where $\tau$ is the proper time, and $d\bar{x}_0/d\tau = \sqrt{1 - \bar{v}^2(t)} = \gamma(t)$, with $\bar{v}(t)$ being the velocity along the trajectory $\bar{x}(t)$:
\[
\bar{v}(t) \equiv \dot{\bar{x}}(t). \tag{7B.2}
\]

After performing the spatial integral in (7.191), we obtain
\[
\phi(\mathbf{x}, t) = \int dt' \Theta(t - t') \frac{1}{4\pi R} \delta(t - t' - R(t')) \tag{7B.3}
\]
where $R(t)$ is the distance from the moving source position at the time $t$:
\[
R(t') = |\mathbf{x} - \bar{x}(t')|. \tag{7B.4}
\]

The integral over $t'$ gives a contribution only for $t' = t_R$ determined from $t$ by the retardation condition
\[
t - t_R = R(t_R) = 0. \tag{7B.5}
\]

At that point, the $\delta$-function can be rewritten as
\[
\delta(t - t' - R(t')) = \frac{1}{[d[t' + R(t')]/dt']_{t'=t_R}} \delta(t' - t_R) = \frac{1}{1 - \mathbf{n}(t_R) \cdot \bar{v}(t_R)} \delta(t' - t_R), \tag{7B.6}
\]
where $\mathbf{n}(t)$ is the direction of the distance vector from the charge.
\[
\mathbf{n}(t) = \frac{\mathbf{R}(t)}{|\mathbf{R}(t)|}. \tag{7B.7}
\]

Thus we find the Liénard-Wiechert-like potential:
\[
\phi(\mathbf{x}, t) = - \left[ \frac{1}{\gamma(1 - \mathbf{n} \cdot \bar{v})} \right]_{\text{ret}}, \tag{7B.8}
\]
and the retarded distance from the point source
\[
R_{\text{ret}}(t) = |\mathbf{x} - \bar{x}(t_R)|. \tag{7B.9}
\]
Appendix 7C  Equal-Time Commutator from Time-Ordered Products

As an exercise in handling the functions in Section 7.2, consider the relationship between the commutator at equal-times and the time-ordered product. The first can be obtained from the latter by forming the difference

\[ [\phi(x), \phi(x')]_{x^0 = x'^0} = \dot{T}(\phi(x)\phi(x'))|_{x^0 = x'^0} - \dot{T}(\phi(x)\phi(x'))|_{x^0 = x'^0 - \epsilon}, \quad (7C.1) \]

where \( \epsilon \) is an infinitesimal positive time. Thus, the Fourier representation of the equal-time commutator is obtained from that of the Feynman propagator by replacing the exponential \( e^{ip^0(x^0 - x'^0)} \) in the \( p^0 \)-integral of (7.66) by \( e^{-ip^0\epsilon} - e^{ip^0\epsilon} \). Thus we should get, for the commutator function at equal times, the energy-momentum integral:

\[ C(x - x')|_{x^0 = x'^0} = \int \frac{dp}{(2\pi)^3} e^{ip(x-x')} \int \frac{dp}{2\pi} \frac{e^{-ip^0\epsilon} - e^{ip^0\epsilon}}{p^2 - M^2 + i\eta} \frac{i}{p^2 - M^2 + i\eta}, \quad (7C.2) \]

The \( p^0 \)-integral can now be performed by closing the contour in the first term by a semicircle in the lower half-plane, in the second term by one in the upper half-plane. To do this, we express the Feynman propagator via the first of the rules (7.212) [see also (7.199)] as

\[ \frac{i}{p^2 - M^2 + i\eta} = \pi\delta_-(p^2 - M^2) = \pi\delta_-(p^0 - \omega_p^2), \quad (7C.3) \]

and use the simple integrals

\[ \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-ip\epsilon} \pi\delta_-(p^0 - \omega_p) = 1, \quad \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-ip\epsilon} \pi\delta_+(p^0 + \omega_p) = 0, \quad (7C.4) \]

\[ \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip\epsilon} \pi\delta_-(p^0 - \omega_p) = 0, \quad \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip\epsilon} \pi\delta_+(p^0 + \omega_p) = 1, \quad (7C.5) \]

to find a vanishing commutator function.

For its time derivative, these imply that exponentials \( e^{ip\epsilon} \) together with \( \delta_-(p^0 - \omega_p) \) and \( \delta_+(p^0 - \omega_p) \) have the same effect as Heaviside functions \( \Theta(\pm p^0) \) accompanied by \( 2\pi\delta_\pm(p^0 - \omega_p) \) and \( 2\pi\delta_\pm(p^0 + \omega_p) \), respectively. We can therefore make the following replacements in the \( p^0 \)-integrals of the Feynman representation (7C.2) of the commutator function:

\[ \int \frac{dp}{2\pi} \frac{e^{-ip\epsilon} - e^{ip\epsilon}}{p^2 - M^2 + i\eta} \rightarrow \int \frac{dp}{2\pi} \left[ \Theta(p^0) - \Theta(-p^0) \right] 2\pi\delta(p^2 - M^2). \quad (7C.6) \]

The result is the same as the \( p^0 \)-integral in Eq. (7.206) for \( C(x - x') \) at equal times, which vanishes.

For the time derivative of the commutator function \( C(x - x') \), the integrand carries an extra factor \(-p^0\), and (7C.6) becomes

\[ \int \frac{dp}{2\pi} \frac{(-ip^0)(e^{-ip\epsilon} - e^{ip\epsilon})}{p^2 - M^2 + i\eta} \rightarrow -i \int \frac{dp}{2\pi} \frac{2\pi\delta(p^2 - M^2)}{k^2 + i\eta} = -i, \quad (7C.7) \]

so that we find the correct result \( \dot{C}(x - x')|_{x^0 = x'^0} = -i\delta^{(3)}(x - x') \).

With the same formalism one may also check the consistency of the photon propagator (7.539) with the commutation rules (7.542)–(7.545). As in (7C.2), we write in the \( p^0 \)-integral of (7.66):

\[ [A^\mu(x), A^\nu(x')]|_{x^0 = x'^0} = \int \frac{dk}{(2\pi)^3} \left( e^{-ik^0\epsilon} - e^{ik^0\epsilon} \right) e^{ik(x-x')} \frac{i}{k^2 + i\eta} \left[ -g^{\mu\nu} + (1 - \alpha) \frac{k^\mu k^\nu}{k^2 + i\eta} \right]. \quad (7C.8) \]

The contribution of \(-p^\mu\nu\) in the brackets yields the commutator function \( C(x - x') \) of the scalar field. The second contribution proportional to \( k^\mu k^\nu \) is nontrivial. First we insert

\[ \frac{1}{k^2 + i\eta} = \delta_+(k^2) = \frac{1}{2\omega_k} \left[ \delta_+(k^0 - \omega_k) + \delta_+(k^0 + \omega_k) \right], \]
with \( \omega_k = |k| \). By forming the derivative \( \partial_\nu \) and taking \( \nu = 0 \), the first term gives

\[
i\delta^{(3)}(x - x').
\]

The second term is equal to

\[
-i(1 - \alpha) \int \frac{d^4k}{(2\pi)^4} (e^{-ik^0\epsilon} - e^{ik^0\epsilon}) e^{ik(x - x')} \frac{1}{2\omega_k} \left[ \delta_+ (k^0 - \omega_k) + \delta_+ (k^0 + \omega_k) \right] k^0.
\]

(7C.9)

Since \( k^0 \) changes the relative sign of the two pole contributions, the \( k \)-integration gives

\[
-i(1 - \alpha)\delta^{(3)}(x - x'),
\]

leading to the equal-time commutator

\[
[\partial_\mu A^\mu(x), A^0(x')]_{x_0 = x_0'} = i\alpha\delta^{(3)}(x - x')
\]

(7C.10)
in agreement with (7.542), provided that

\[
[\partial_\mu A^\mu(x), A^0(x')]_{x_0 = x_0'} = 0.
\]

(7C.11)

This commutator has a spectral decomposition coming from the second term in (7C.8): It is equal to

\[
-i(1 - \alpha) \int \frac{d^4k}{(2\pi)^4} (e^{-ik^0\epsilon} - e^{ik^0\epsilon}) e^{ik(x - x')} \frac{1}{2k^0} \left[ \delta_+ (k^0 - \omega_k) + \delta_+ (k^0 + \omega_k) \right] \times k^2 k^0 \frac{1}{2\omega_k} \left[ \delta_+ (k^0 - \omega_k) + \delta_+ (k^0 + \omega_k) \right].
\]

(7C.12)

By writing

\[
k^0 \frac{1}{2\omega_k} \left[ \delta_+ (k^0 - \omega_k) + \delta_+ (k^0 + \omega_k) \right] = k^0 \frac{1}{2\omega_k} \left[ \delta_+ (k^0 - \omega_k) - \delta_+ (k^0 + \omega_k) \right],
\]

we see that the \( \delta_\pm \)-functions appear in the combination \( [\delta_+ (k^0 - \omega_k)]^2 - [\delta_+ (k^0 + \omega_k)]^2 \). When closing the integration contours in the upper or lower \( k^0 \)-plane, the double poles give no contribution. Hence

\[
[\partial_\mu A^\mu(x), A^0(x')]_{x_0 = x_0'} = 0.
\]

(7C.13)

For the same reason, the commutators \( [\dot{A}^\mu(x), A^0(x')]_{x_0 = x_0'} \) receive only a contribution from the first term and yield \( -i\alpha \dot{\delta}^{(3)}(x - x') \). Other commutators are

\[
[D(x), \dot{A}^\mu(x')]_{x_0 = x_0'} = -[D(x), \partial_\mu A^0(x')]_{x_0 = x_0'}
\]

\[
= \int \frac{d^4k}{(2\pi)^4} (e^{-ik^0\epsilon} - e^{ik^0\epsilon}) e^{ik(x - x')} \frac{i}{k^2 + i\eta} \alpha k^0 k^4,
\]

(7C.14)
yielding

\[
\alpha i\dot{\delta}^{(3)}(x - x'),
\]

which shows that \( D(x) \) commutes with \( E^\mu(x) \) at equal times. Together with the previous result we recover the third of the canonical commutators (7.542)–(7.545).
Notes and References


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