Weak Interactions

The decay of nuclei proceeds via $\alpha$-, $\beta$-, $\gamma$-emission, or by fission. The first is governed by nuclear forces, the third by electromagnetic interactions. The $\beta$-emission has an entirely different origin. It is caused by weak interactions. Initially, such processes posed a serious puzzle: The observed emerging particles, electrons or positrons, did not come out with a definite energy equal to the mass difference between initial and final nuclei. Instead, they had an energy distribution with a maximal energy at the expected value. This led Nils Bohr to speculate that energy conservation could possibly be violated in quantum physics. In 1939, however, Wolfgang Pauli suggested in a historic letter to colleagues [1] meeting in Tübingen, that the missing energy was carried off by a neutral particle, now called neutrino. Its spin had to be $1/2$, to preserve angular momentum.

The simplest nuclear particle which shows $\beta$-decay is the neutron itself which decays, after about 900 seconds, as follows:

$$n \rightarrow p + e^- + \bar{\nu}_e,$$

where $\bar{\nu}_e$ denotes a right-handed version of Pauli’s particle, which is nowadays referred to as an antineutrino associated with the electron.

As time went on, many more particles were discovered that arose from similar interactions. All of them are now called leptons. Recall the classification in Section 24.1.

27.1 Fermi Theory

In 1934, Fermi made a first theoretical attempt to describe $\beta$-decay in lowest-order perturbation theory [2]. He suggested that the interaction is due to a local current-current interaction with a Lagrangian density of the form

$$\mathcal{L} = C \bar{p}(x) \gamma^\mu n(x) \bar{e}(x) \gamma_\mu \nu(x) + \text{c.c.}.$$

In accordance with the standard short notation for the fields, we have denoted $\psi_e(x)$ by $e(x)$ and $\psi_p(x)$ by $p(x)$, etc. This interaction looks similar to the electric
interaction in Eq. (12.85), except that the Coulomb potential describing the electric long-range interaction is replaced by a short-range $\delta$-function potential. Fermi also realized that Lorentz invariance and locality allow a more general Lagrangian density of the form

$$\mathcal{L} = C_S \bar{p}(x)n(x) \bar{\nu}(x) + C_P \bar{p}(x)i\gamma_5 n(x) \bar{\nu}(x) + C_V \bar{p}(x)\gamma^\mu n(x) \bar{\nu}(x) + C_A \bar{p}(x)\gamma^\mu\gamma_5 n(x) \bar{\nu}(x) + \frac{i}{2} C_T \bar{p}(x)\sigma^{\lambda\kappa} n(x) \bar{\nu}(x) + \text{c.c.} \quad (27.3)$$

Fermi postulated the validity of time-reversal invariance, which to our present knowledge is fulfilled to a high degree of accuracy. A tiny violation of this is observable only in very few particularly sensitive processes. Hence the coefficients $C_i$ in (27.3) have to be real, with only very small imaginary parts. If $\Gamma_i$ denotes the five Dirac matrices $1, i\gamma_5, \gamma^\mu, \gamma^\nu\gamma_5$, and $\sigma^{\mu\nu}/\sqrt{2}$, one may write the interaction (27.3) short as

$$\mathcal{L} = \sum_{i=S,P,V,A,T} C_i \bar{p}(x)\Gamma_i n(x) \bar{\nu}(x) + \text{c.c.} \quad (27.4)$$

The decay of an initial nucleus at rest $|\Psi_i\rangle$ into a final nucleus $|\Psi_f\rangle$ is governed by the matrix elements

$$\langle \Psi_f | \bar{p}(x)\Gamma_i n(x) | \Psi_i \rangle \quad (27.5)$$

Due to the small energy differences involved, the final state will be almost at rest (since the decaying nucleus is usually big enough), and one may use the nonrelativistic limit for the matrix elements between nucleons. Taking the Dirac matrices in Dirac’s original representations, one can use the direct-product expressions (27.6),

$$\gamma^0_D = \sigma^3 \times 1, \quad \gamma^i_D = i\sigma^2 \times \sigma, \quad \gamma^5_D = \sigma^1 \times 1, \quad (27.6)$$

to rewrite the matrix elements of $\bar{p}(x)\Gamma_i n(x)$ as follows:

$$S: \quad \bar{p}(x)n(x) \approx p^1(x)n(x),$$
$$P: \quad \bar{p}(x)i\gamma_5^0 n(x) \approx -p^1(x) \sigma^2 \times 1 n(x) \approx 0,$$
$$V: \quad \bar{p}(x)\gamma^0_D n(x) \approx p^1(x)n(x),$$
$$\bar{p}(x)\gamma^i_D n(x) \approx -p^1(x) \sigma^1 \times \sigma^i n(x) \approx 0,$$
$$A: \quad \bar{p}(x)\gamma^0_D\gamma^5_D n(x) \approx p^1(x) \sigma^1 \times 1 n(x) \approx 0,$$
$$\bar{p}(x)\gamma^i_D\gamma^5_D n(x) \approx p^1(x) 1 \times \sigma^i n(x),$$
$$T: \quad \bar{p}(x)\sigma^{0i} n(x) \approx -p^1(x) \sigma^2 \times \sigma^i n(x) \approx 0,$$
$$\bar{p}(x)\sigma^{ij} n(x) \approx \epsilon_{ijk} p^1(x) 1 \times \sigma^k n(x). \quad (27.7)$$

The lines containing mixtures of upper and lower components are very small for small momenta, and can be omitted. Thus we are left with only two types of nuclear matrix elements:

$$\langle \Psi_f | p^1(x)n(x) | \Psi_i \rangle \equiv \langle 1 \rangle,$$
$$\langle \Psi_f | p^1(x)\sigma n(x) | \Psi_i \rangle \equiv \langle \sigma \rangle, \quad (27.8)$$
where $\sigma$ in front of a Dirac spinor acts only upon the spin label.

The $S$-matrix elements are then approximately given by

$$S = 1 - 2\pi i \delta(E_t + E_e + E_\nu - E_i)T,$$

with the $T$-matrix [recall (9.293)]

$$T = \frac{\sqrt{2m_\nu}}{V} \left\{ \langle 1 | \left[ C_S \bar{u}(p_e)v(p_\nu) + C_V \bar{u}(p_e)\gamma^0 v(p_\nu) \right] 
+ \langle \sigma | \left[ \gamma^0 C_T \bar{u}(p_e)\sigma v(p_\nu) + C_A \bar{u}(p_e)\gamma^5 v(p_\nu) \right] \right\} . \quad (27.10)$$

The spin labels $s_3$ for the electron and antineutrino have been omitted, for brevity. We have also normalized the antineutrino states in the same way as photon states like

$$\langle p'_\nu | p_\nu \rangle = 2p'_\nu(\bar{\nu}(p'_\nu - p_\nu), \quad (27.11)$$

to allow a smooth limit as $m_\nu \to 0$.

When calculating decay rates, we may sum over the antineutrino polarizations which are hard to measure. The polarization sum yields

$$\sqrt{2m_\nu} \sum_{\text{spin pol}} \bar{v}(p_\nu)v(p_\nu) = \not{p}_\nu - m_\nu, \quad (27.12)$$

and $m_\nu$ is set equal to zero at the end. For unpolarized nuclei, we have

$$\langle \sigma^i \sigma^j \rangle = \frac{1}{3} |\sigma|^2, \quad \langle \sigma^i 1 \rangle = 0. \quad (27.13)$$

If also the electrons are unpolarized, we obtain

$$\sum_{\text{spin pol}} |T|^2 = \frac{1}{2m_e V^2} \left[ C_S^2 |\langle 1 |^2 |1|^2 \text{tr}(\not{p}_\nu \not{p}_e) + |C_V|^2 |\langle 1 |^2 |1|^2 \text{tr}(\gamma_0 \not{p}_\nu \gamma_0 \not{p}_e) 
+ \frac{1}{2} C_A \langle \sigma |^2 |\langle 1 |^2 |1|^2 \text{tr}(\gamma^i \gamma_5 \not{p}_\nu \gamma_5 \gamma^i \not{p}_e) 
+ 2C_S C_V |\langle 1 |^2 |m_e \text{tr}(\gamma_0 \not{p}_\nu) + \frac{1}{2} C_A |\langle \sigma |^2 |1|^2 \text{tr}(\sigma^i \gamma_5 \gamma_0 \gamma^i \not{p}_e) \right] . \quad (27.14)$$

or more explicitly

$$\sum_{\text{spin pol}} |T|^2 = \frac{4E_\nu E_\nu}{2m_e V^2} \left[ C_S^2 |\langle 1 |^2 |1|^2 \left( 1 - \frac{p_e P_\nu}{E_e E_\nu} \right) + |C_V|^2 |\langle 1 |^2 |1|^2 \left( 1 + \frac{p_e P_\nu}{E_e E_\nu} \right) 
+ \frac{1}{2} C_A \langle \sigma |^2 \left( 1 - \frac{1}{3} \frac{p_e P_\nu}{E_e E_\nu} \right) + \frac{1}{4} C_A^2 |\langle \sigma |^2 \left( 1 + \frac{1}{3} \frac{p_e P_\nu}{E_e E_\nu} \right) 
+ 2C_S C_V |\langle 1 |^2 |m_e \frac{2m_e}{E_e} - C_A C_T |\langle \sigma |^2 \frac{m_e}{E_e} \right] . \quad (27.15)$$

The interference terms $C_S C_V$ and $C_T C_A$ show a strong threshold-dependence proportional to $1/E_e$. Since this is not observed experimentally, one concludes that

$$C_S C_V \approx 0, \quad C_T C_A \approx 0. \quad (27.16)$$
One distinguishes now Gamow-Teller transitions, by for which $|\langle \sigma \rangle|^2 \neq 0$, and Fermi transitions, for which $|\langle 1 \rangle|^2 \neq 0$ (both without nuclear parity change). For transitions in which initial and final nuclear spins are the same, only $C_S^2$ and $C_V^2$ contribute. These are distinguishable by the electron-antineutrino correlation in momentum space. In the first case, electron and antineutrino come out preferably in opposite direction ($p_e p_\nu < 0$), in the second case in the same direction ($p_e p_\nu > 0$).

Experiments done for the decay
\[ \text{Ne}^{19} \rightarrow \text{F}^{19} + e + \bar{\nu}, \quad (27.17) \]
in which the nuclear spin is 1/2 before and after the decay show that the latter is true. Thus we can approximate $C_S \approx 0$.

Transitions which change the nuclear spin by one unit are sensitive to $C_A^2$ and $C_T^2$, with similar momentum correlations. The decay
\[ \text{He}^6 \rightarrow \text{Li}^6 + e + \bar{\nu} \quad (27.18) \]
involves a change from nuclear spin 0 to 1. The electron and the neutrino emerge antiparallel, from which one deduces that $C_T^2 \approx 0$. Thus only $C_V$ and $C_A$ are appreciable.

The antineutrinos which appear in these decays have been found to interact with matter extremely weakly. Only many years later, in 1953, was it possible to detect them directly via the inverse reaction
\[ \bar{\nu}_e + p \rightarrow n + e^+, \quad (27.19) \]
using a prolific antineutrino source of a large nuclear reactor filled with a high density of thermal neutrons.

After many precise measurements of angular correlations and electron and nuclear polarizations, the coefficients of the general interaction (27.3) have been quite well determined and led to
\[ \mathcal{L} = -\frac{G_W}{\sqrt{2}} \left[ \bar{p} \gamma^\lambda \left( 1 - \frac{g_A}{g_V} \gamma_5 \right) n \right] \left[ \bar{e} \gamma_\lambda (1 - \gamma_5) \nu \right] + \text{c.c.}, \quad (27.20) \]
with the weak-interaction coupling constant
\[ G_W = (1.14730 \pm 0.000641) \times 10^{-5}\text{GeV}^{-2}, \quad (27.21) \]
and the ratio
\[ g_A/g_V = 1.255 \pm 0.006. \quad (27.22) \]
The $\bar{e}\nu$-part of the interaction violates space reflection symmetry in the maximal possible way. Under a parity transformation, the left-helicity field $(1 - \gamma_0)\nu(x)$ that possesses nonzero components only in the first two entries is multiplied by
\[ \gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]
This moves the nonzero entries to the lower two components producing a right-helicity field \((1 + \gamma_5)\gamma_0\nu\). However, such a field is not present in the Lagrangian density (27.20). This maximal parity violation is responsible for the asymmetry of the decay electrons with respect to the neutron spin as observed in Madame Wu’s experiment [3].

The ratio \(g_A/g_V\) can be measured in neutron decay via the up-down asymmetry by analogy with Madame Wu’s experiment. The expectation \(\langle s_n \cdot p_e \rangle\) is proportional to

\[
|C_A|^2 + \text{Re}C_V C_A^* \propto (1 - g_A/g_V). \tag{27.23}
\]

The deviation of the ratio \(g_A/g_V\) from unity can be explained by strong interactions. This possibility was first suggested by the weak decay of the 1937 discovered muon (initially mistaken for a pion). The muon decays weakly as follows

\[
\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu. \tag{27.24}
\]

### 27.2 Lepton-Number Conservation

In 1962, a Brookhaven experiment showed that the antineutrino emerging in the decay had to be distinguished from that emerging in the \(\beta\)-decay of the muon. The two neutrinos are now referred to as muon neutrino \(\bar{\nu}_\mu\) and electron antineutrino \(\bar{\nu}_e\), respectively. With this assignment, the total number of electrons plus the associated electron neutrinos, and the total number of muons plus the associated muon neutrinos, are both separately conserved if antiparticles are counted negatively. These laws are called lepton-number conservation laws. The muon has no strong interaction and its decay is described by the Lagrangian density

\[
\mathcal{L}_\mu = \frac{G_\mu}{\sqrt{2}} \bar{\nu}_\mu \gamma^\lambda(1 - \gamma_5)\mu \left[ \bar{e}_\gamma \lambda(1 - \gamma_5)\nu_e(x) \right] + \text{c.c.}. \tag{27.25}
\]

The value of \(G_\mu\) is given by

\[
G_\mu = 1.16637(1) \times 10^{-5}\text{GeV}^{-2}. \tag{27.26}
\]

It is 2% larger than \(G_W\) of the \(\beta\)-decay in Eq. (27.21).

The parameter \(G_\mu\) is defined as the Fermi constant. There are two reasons for choosing it rather than \(G_W\). First, the experimental value of the lifetime of \(\mu^-\) is extremely well measured:

\[
\tau_\mu = 2.197035(40) \times 10^{-6}\text{s}^{-1}. \tag{27.27}
\]

Second, this lifetime determines \(G_\mu\) via a very reliable theoretical relation that follows from QED [4]:

\[
\tau_\mu^{-1} = \frac{G_\mu^2 m_\mu^5}{192\pi^3} \int \left( \frac{m_e^2}{m_\mu^2} \right) \left( 1 + \text{R.C.} \right) \left( 1 + \frac{3}{5} \frac{m_\mu^2}{m_W^2} + \ldots \right). \tag{27.28}
\]
The abbreviation R.C. denotes radiative corrections

\[
\text{R.C.} = \frac{\alpha}{2\pi} \left( \frac{25}{4} - \pi^2 \right) \left[ 1 + \frac{\alpha}{\pi} \left( \frac{2}{3} \log \frac{m_\mu}{m_e} - 3.7 \right) \right] + \left( \frac{\alpha}{\pi} \right)^2 \left( \frac{4}{9} \log^2 \frac{m_\mu}{m_e} - 2.0 \log \frac{m_\mu}{m_e} + C \right) + \ldots \right]. \quad (27.29)
\]

These can be calculated with great accuracy by solving the renormalization group equation

\[
(m_e \partial_{m_e} + \beta(\alpha) \partial_\alpha)\text{R.C.} = 0, \quad \beta(\alpha) = \frac{2}{3} \frac{\alpha^2}{\pi} + \frac{1}{2} \frac{\alpha^3}{\pi^2} + \ldots , \quad (27.30)
\]

where \(\beta(\alpha)\) is the beta-function that governs the change of the interaction strength upon a change in the scale of the theory. It was defined in (20.31) for a scalar field theory with \(\phi^4\)-interaction. For QED it was first introduced by Stückelberg and Petermann [5], and soon after that by Gell-Mann and Low [6].

### 27.3 Cabibbo Angle

In order to explain why \(G_W\) is smaller than \(G_\mu\), Nicola Cabibbo, in 1964, put forward the hypothesis that the reduction is due to a simple geometric property of the weak [7]. Studying the \(\beta\)-decay the strange particles

\[
\begin{align*}
\Lambda & \rightarrow p + e^- + \bar{\nu}_e, \\
\Sigma^+ & \rightarrow \Lambda + e^+ + \nu_e, 
\end{align*} \quad (27.31)
\]

he observed that their weak coupling constant \(G_W^{\text{strange}}\) was considerably smaller than that of neutrons. Moreover, he was able to fit the three decays (27.1), (27.24), and (27.31), quite well by assuming that the weak coupling constant defined as in (27.20) for the ordinary \(\beta\)-decay, and a similar \(G_W^{\text{strange}}\) for the decays (27.31) can be parametrized by setting

\[
\begin{align*}
G_W &= \cos \theta_c G_\mu, \quad (27.32) \\
G_W^{\text{strange}} &= \sin \theta_c G_\mu, \quad (27.33)
\end{align*}
\]

where \(G_\mu\) is the weak coupling constant (27.26) of the leptonic decay, if one chooses the value

\[
\theta_c = 0.21 \quad (27.34)
\]

for the so-called Cabibbo angle.

Nowadays it is believed that weak interactions of all hadrons proceed via the decay of quarks inside their wave functions. The non-strange hadrons decay via the elementary transitions

\[
\begin{align*}
d & \rightarrow u + e^- + \nu_e, \\
d & \rightarrow u + \mu^- + \nu_\mu, \\
u & \rightarrow d + e^+ + \bar{\nu}_e, \\
u & \rightarrow d + \mu^+ + \bar{\nu}_\mu. 
\end{align*} \quad (27.35)
\]
The strange hadrons decay by similar transitions, with the down quarks \( d \) getting replaced by a strange quark \( s \). Cabibbo’s hypothesis amounts to the ansatz for the interaction Lagrangian density

\[
\mathcal{L} = -\frac{G_\mu}{\sqrt{2}} \bar{u} \gamma^\lambda (1 - \gamma_5) \left( \cos \theta_c \, d + \sin \theta_c \, s \right) \cdot \left( \bar{e} \gamma^\lambda (1 - \gamma_5) \nu \right) + \text{h.c.}
\]

The first expression implies that the weak interactions couple only to an isospin-rotated combination of down and strange quarks. Moreover, they appear only in a special combination of vector and axial vector currents, which we previously found to be conserved Noether currents of QCD (or partially conserved in the axial vector case). In fact we can write

\[
\begin{align*}
\bar{u} \gamma^\lambda d &= \sqrt{2} j_\pi^\lambda (1 + i\gamma_5) / \sqrt{2} = \sqrt{2} j_{\pi^+}^\lambda, \\
\bar{u} \gamma^\lambda \gamma_5 d &= \sqrt{2} j_5 (1 + i\gamma_5) / \sqrt{2} = \sqrt{2} j_{5\pi^+}^\lambda, \\
\bar{u} \gamma^\lambda s &= \sqrt{2} j_5 (1 + i\gamma_5) / \sqrt{2} = \sqrt{2} j_{5K^+}^\lambda,
\end{align*}
\]

(27.36)

where the subscripts \( \pi^+\), \( K^+ \) indicate the SU(3)-quantum numbers which can be created by these currents. In terms of these currents, we can rewrite (27.36) as

\[
\mathcal{L} = -G_\mu \left\{ \cos \theta_c \left[ j_{\pi^+}^\lambda - j_{5\pi^+}^\lambda \right] + \sin \theta_c \left[ j_{K^+}^\lambda - j_{5K^+}^\lambda \right] \right\} \left[ \bar{e} \gamma^\lambda (1 - \gamma_5) \nu \right] + \text{h.c.} \ .
\]

(27.37)

\section*{27.4 Cabibbo Mass Matrix}

The Cabibbo theory can be phrased in a different way. The combination of quarks \( d \theta \equiv \cos \theta_c \, d + \sin \theta_c \, s \) entering the interaction Lagrangian density (27.36) may be thought of as a result of a diagonalization of a nondiagonal mass matrix of down and strange quarks:

\[
(d', u') \mathcal{M} \begin{pmatrix} d' \\ u' \end{pmatrix} = \begin{pmatrix} m_d & m_{ds} \\ m_{ds} & m_s \end{pmatrix} \begin{pmatrix} d' \\ u' \end{pmatrix}.
\]

(27.38)

This matrix is diagonalized by the physical down and strange quarks

\[
\begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} \\ -V_{us} & V_{ud} \end{pmatrix} \begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} d' \\ s' \end{pmatrix},
\]

(27.39)

with a Cabibbo angle solving the equation

\[
\tan \theta = \frac{m_{ds}}{m_s - \bar{m}} = \frac{m_d - \bar{m}}{m_{ds}},
\]

(27.40)
where
\[ \bar{m} = \frac{1}{2} \left[ (m_d + m_s) + \sqrt{(m_d + m_s)^2 + 4m_{ds}^2} \right]. \] (27.42)

The weak interactions involve only the Cabibbo-rotated *bare quark* field \( d' = d_\theta \). This means that the weak quark current entering the Lagrangian density (27.36) has the form
\[ \bar{u}\gamma^\lambda(1 - \gamma^5)d_\theta, \] (27.43)
so that Lagrangian reads
\[ \mathcal{L} = -\frac{G_\mu}{\sqrt{2}} \bar{u}\gamma^\lambda(1 - \gamma^5)d_\theta \cdot \left[ \bar{e}\gamma^\nu(1 - \gamma^5)\nu \right] + \text{h.c.}. \] (27.44)

### 27.5 Heavy Vector Bosons

Historically, the fact that the coupling constant of weak interactions carries a dimension of an inverse square mass was a great obstacle in calculating higher-order corrections to weak processes: Up to that point, the only successful field theory of fundamental interactions, quantum electrodynamics, had only a three-particle interaction, with the photon mediating the interactions: This led to the hypothesis that weak interactions could be due to a similar coupling of a massive charged vector meson called \( W \). As we shall see later in the analysis of perturbation theory, this can generate a weak coupling of the same type as in (27.20), but with \( G_W \) replaced by
\[ G_W(q^2) = -\frac{e_W^2}{q^2 - M_W^2}, \] (27.45)
where \( q \) is the momentum carried off by the leptons in the decay, and \( e_W \) is the coupling strength of the heavy vector meson \( W \) to the leptons and quarks. If \( M_W \) is much larger than 1 BeV, the \( q \)-dependence can be ignored and one can approxiamate
\[ G_W(0) \approx \frac{e_W^2}{M_W^2}. \] (27.46)

Since \( e_W \) is dimensionless, it was suggestive to assume it to be equal to the coupling strength of the photons. Then weak interactions would carry only one additional parameter, the mass of the vector meson. Using
\[ e^2 = 4\pi\alpha, \] (27.47)
this requires as mass
\[ M_W = \sqrt{\frac{4\pi\alpha}{G}} \approx 90\text{GeV}. \] (27.48)
A charged meson which has the desired properties was found near this predicted mass in 1983 [8]. Its experimental value is

\[ M_W = 80.423 \pm 0.039 \text{ GeV}. \] (27.49)

With the usual weak decays carrying off a non-zero charge via a lepton, the question has been raised whether weak interactions could also lead to a decay into \( \nu_\mu \bar{\nu}_\mu, e \bar{e}, \) or \( \mu \bar{\mu} \). The first two could have easily escaped experimental detection, the last two would be hard to find since they would hide under the similar but much larger electromagnetic interaction. Such decays would come from couplings such as

\[ \mathcal{L} = G^0 \left[ \bar{u} \gamma^\mu (1 - \gamma_5) u \right] \left[ \left\{ e \gamma^\mu e + \bar{\nu}_e (1 - \gamma_5) \nu \right\} + \left\{ e \rightarrow \mu \right\} \right]. \] (27.50)

The field combinations \( \bar{u} \gamma^\mu (1 - \gamma_5) u, \bar{e} \gamma^\mu (1 - \gamma_5) e \) contained in (27.50) do not change the charge of the decaying particles and are called neutral currents. They appear in the same combination \( V^\mu - A^\mu \) of vector and axial vector currents as in the interaction (27.38).

Experimental evidence for such neutral currents has been observed in 1973 in a purely leptonic reaction

\[ \bar{\nu}_\mu + e^- \leftrightarrow \bar{\nu}_\mu + e^-, \] (27.51)

as well as in similar reactions involving quarks

\[ \nu_\mu + \left( \begin{array}{c} u \\ d \end{array} \right) \rightarrow \nu_\mu + \left( \begin{array}{c} u \\ d \end{array} \right), \]

\[ \bar{\nu}_\mu + \left( \begin{array}{c} u \\ d \end{array} \right) \rightarrow \bar{\nu}_\mu + \left( \begin{array}{c} u \\ d \end{array} \right). \] (27.52)

Also a heavy neutral vector meson has been found which, by analogy with \( W \), mediates the neutral weak interactions. It is called \( Z \)-boson and its mass is given by [11]

\[ M_Z = 92.9 \pm 1.6 \text{ GeV}. \] (27.53)

### 27.6 Standard Model of Electroweak Interactions

All these interactions, plus the electromagnetic interactions, were unified in one field theory by Glashow, Weinberg, and Salam [9] in the standard model of electromagnetic and weak interactions (the combination being also called electroweak interactions). We shall briefly review this theory here in a reduced version in which the only leptons involved are electrons and electron-neutrinos [10]. All terms involving these leptons will have to be extended by similar terms involving the other two families of leptons discovered so far:

\[ \left( \begin{array}{c} \nu_\mu \\ \mu \end{array} \right), \left( \begin{array}{c} \nu_\tau \\ \tau \end{array} \right). \] (27.54)
The charged lepton masses are well known:

\[ m_e = 0.511 \text{ MeV}, \quad m_\mu = 105.7 \text{ MeV}, \quad m_\tau = 1777 \text{ MeV}. \]  
(27.55)

The free lepton Lagrangian density for the electron part is

\[ \mathcal{L} = (\bar{\nu}_{eL}, \bar{e}_L) \mathcal{\bar{\phi}} \left( \begin{array}{c} \nu_{eL} \\ e_L \end{array} \right) + \bar{e}_R i \mathcal{\bar{\phi}} e_R. \]  
(27.56)

It is invariant under the rotation group SU\(_L\)(2) of the left-handed leptons, and a trivial group U\(_R\)(1) of the right-handed leptons. In the latter, the neutrino \( \nu_R \) is absent. This Lagrangian density is made locally SU\(_L\)(2) \( \times \) U\(_R\)(1)-invariant by means of two types of gauge fields. For the SU\(_L\)(2)-symmetry of the left-handed particles one uses a 2 \( \times \) 2-matrix gauge field

\[ W_\mu = W^a_\mu \frac{\tau_a}{2}, \]  
(27.57)

in which \( \tau_a \) acts on the two SU\(_L\)(2)-isospin indices. Since these are mathematically equivalent to isospin, one speaks of a weak isospin. For the U\(_R\)(1)-symmetry of the right-handed electrons one uses a single vector field \( B_\mu \). In the Lagrangian, the ordinary derivatives are replaced by the covariant derivatives

\[ D_\mu = \partial_\mu + igW_\mu, \quad D_\mu = \partial_\mu + ieB_\mu. \]  
(27.58)

After this, one adds dynamics to the gauge fields themselves via the Lagrangian density

\[ \mathcal{L}_{GF} = -\frac{1}{4} \text{tr} (W_\mu W^{\mu\nu}) - \frac{1}{4} B^{2}_{\mu\nu}, \]  
(27.59)

where

\[ W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + ig[W_\mu, W_\nu] \]  
(27.60)

is the covariant curl of the SU(2)\(_L\)-gauge field, and

\[ B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \]  
(27.61)

the usual abelian curl of the U(1)\(_R\)-gauge field.

Instead of the 2\( \times \)2 matrix notation (27.60), one can use the vector notation of Eq. (27.57), and write the covariant curl in (27.60) as

\[ W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - g(W^a_\mu \times W_\nu)^a. \]  
(27.62)

The Lagrangian density (27.59) becomes

\[ \mathcal{L}_{GF} = -\frac{1}{4}(W^a_{\mu\nu})^2 - \frac{1}{4} B^{2}_{\mu\nu}. \]  
(27.63)
We now introduce the charged $W$-boson fields

$$W^\pm_\mu = \frac{1}{\sqrt{2}} \left( W^1_\mu \pm i W^2_\mu \right),$$

and rewrite the interactions as

$$L_{\text{int}} = -g (\bar{\nu}_e \gamma^\lambda e_L) \frac{\tau_\lambda}{2} \left( \begin{array}{c} \nu e_L \\ e_L \end{array} \right) - g' e_R \gamma^\lambda B_\lambda e_R$$

$$= -g (\bar{\nu}_e \gamma^\lambda e_L) \left( \begin{array}{cc} W^3_{\lambda}/2 & W^3_{-}/\sqrt{2} \\ W^3_{+}/\sqrt{2} & -W^3_{\lambda}/2 \end{array} \right) \left( \begin{array}{c} \nu e_L \\ e_L \end{array} \right) - g' e_R \gamma^\lambda B_\lambda e_R. \quad (27.65)$$

Actually, there is some freedom in choosing the coupling to the vector meson $B_\nu$, since we can always absorb an equal coupling to $(\bar{\nu}_e, e_L)$ into the term involving the field $W^3$. Hence we may choose, instead of $-g' e_R \gamma^\lambda B_\lambda e_R$, the interaction

$$g' (\bar{\nu}_e, e_R) \gamma^\lambda Y \left( \begin{array}{c} \nu e_L \\ e_L \end{array} \right),$$

where the $3 \times 3$-matrix $Y$ is called the weak hypercharge and has the form

$$Y = \left( \begin{array}{ccc} y_L & y_L & y_R \end{array} \right). \quad (27.67)$$

The parameter $y_L$ can be chosen at will since it can be absorbed into $g$, for instance

$$y_L = 1. \quad (27.68)$$

Then the interaction reads

$$L_{\text{int}} = -g \sqrt{2} \left( W^3_\lambda \bar{\nu}_e \gamma^\lambda e_L + \text{c.c.} \right)$$

$$- \frac{1}{2} \left( gW^3 - g' B_\lambda \right) \bar{\nu}_e \gamma^\lambda \nu e_L$$

$$+ \frac{1}{2} \left( gW^3 + g' B_\lambda \right) \bar{\nu}_e \gamma^\lambda e_L - \frac{g'}{2} y_R e_R \gamma^\lambda e_R. \quad (27.69)$$

Now we observe that the two fields $W^3_\lambda$ and $B_\lambda$ are initially both massless and allow for arbitrary linear mixing. One combination of them remains massless after the mixing. This may be identified as the physical photon field:

$$A_\mu = \cos \theta_W B_\mu + \sin \theta_W W^3_\mu. \quad (27.70)$$

The masslessness fixes the mixing angle $\theta_W$ to satisfy

$$\sin \theta_W = \frac{g'}{\sqrt{g^2 + g'^2}}, \quad \cos \theta_W = \frac{g}{\sqrt{g^2 + g'^2}}. \quad (27.71)$$
The mixing angle $\theta_W$ is also called weak angle or Weinberg angle [9]. The orthogonal linear combination

$$Z_\mu = \sin \theta_W B_\mu - \cos \theta_W W^\mu_3$$  \hspace{1cm} (27.72)

has a mass

$$M_Z = \frac{v}{2} \sqrt{g^2 + g'^2}. \hspace{1cm} (27.73)$$

The exchange of the $Z_\mu$-field gives rise to another contribution to the weak interactions with no charge transfer, a so-called $\Delta Q = 0$-weak interaction.

If we choose the other coupling in (27.68) to be $y_R = -1$ and require that the photon field couples to the electron via the usual electromagnetic current density, which consists of a sum of a right-handed and left-handed part:

$$j_{em}^\mu = \bar{e}_L \gamma^\mu e_L + \bar{e}_R \gamma^\mu e_R. \hspace{1cm} (27.74)$$

This is possible if we set

$$e = \frac{gg'}{\sqrt{g^2 + g'^2}}, \hspace{1cm} (27.75)$$

and the interaction density (27.69) becomes

$$\mathcal{L}_{int} = -\frac{g}{2\sqrt{2}} \left( \bar{\nu}_e L \gamma^\lambda e_L W^\lambda_\mu + \text{c.c.} \right) - e j_{em}^\lambda A_\lambda$$  \hspace{1cm} (27.76)

$$- \frac{e}{2} \left[ \tan \theta_W (2\bar{e}_R \gamma^\lambda e_R + \bar{e}_L \gamma^\lambda e_L + \bar{\nu}_e L \gamma^\lambda \nu_e L) - \cot \theta_W \left( \bar{e}_L \gamma^\lambda e_L - \bar{\nu}_e L \gamma^\lambda \nu_e L \right) \right] Z_\mu. \hspace{1cm} (27.76)$$

This can be decomposed into three terms: the desired contribution from the electromagnetic current (27.74), a contribution from the left-handed weak currents written as vectors in weak isospace

$$j_L^\pm = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \nu_e L \\ \bar{\nu}_e L \end{array} \right) e e \gamma^\mu \tau^{\pm}, \hspace{1cm} (27.77)$$

$$j_L^3 = \frac{1}{2} \left( \begin{array}{c} \nu_e L \\ \bar{\nu}_e L \end{array} \right) e e \gamma^\mu \nu_e L = \frac{1}{2} \left( \nu_e \gamma^\mu \nu_e L - \bar{\nu}_e \gamma^\mu \bar{\nu}_e L \right), \hspace{1cm} (27.78)$$

and a contribution from the weak neutral current:

$$j_{nc} = j_L^3 + \sin^2 \theta_W j_{em}. \hspace{1cm} (27.79)$$

The total interaction (27.76) may then be written as

$$\mathcal{L}_{int} = e \left[ j_{em}^\mu A_\mu - \frac{1}{\sin \theta_W} (j_L^+ W^\mu_\mu \text{+ c.c.}) + \frac{1}{\sin \theta_W} j_{nc}^\mu Z_\mu \right]. \hspace{1cm} (27.80)$$

The weak angle $\theta_W$ parametrizes the admixture of the ordinary electric current to the neutral weak isospin current.
Up to this point, the model had been constructed at the phenomenological level by Glashow in 1961. To explain the weakness of the interaction, he had added to the Lagrangian suitable mass terms for the $W$ and the $Z$ meson

$$\frac{M_W^2}{2} (W^+ + W^-) + \frac{M_Z^2}{2} Z^2,$$

choosing the value for $M_W$ to achieve the correct weak coupling constant. At that time he did not know what value he should choose for $M_Z$. In this way he obtained a model which parametrized the known weak interactions, while predicting unknown neutral currents.

### 27.7 Masses from Meissner-Ochsenfeld-Higgs Effect

The theory constructed so far had an important drawback: It did not permit the calculation of higher-order corrections to weak interaction processes. The masses of $W^\pm$- and $Z$-bosons in (27.81) gave rise to divergencies in the higher-order Feynman diagrams which could not be removed by a redefinition of couplings, masses, and field normalization. The theory was not renormalizable.

This defect was corrected by generating the masses $m_W$ and $m_Z$ by a mechanism that had been discovered in 1950 to explain the finite range of the magnetic field observed in a superconductor. It is known as the Meissner-Ochsenfeld effect discussed in Section 17.2. In contrast to the superconductor, however, the theory needed more degrees of freedom. The single complex field of the Ginzburg-Landau Lagrangian (17.9) was not sufficient. An extension was needed that contained at least a doublet of complex fields, one charged and one uncharged:

$$\varphi(x) = \left( \begin{array}{c} \varphi^+ \\ \varphi^0 \end{array} \right).$$

Their dynamics is governed by a generalization of the Ginzburg-Landau Lagrangian (17.9), the so-called Higgs model:

$$A_{\varphi, \varphi^*, A} = \int d^D x \left\{ \frac{1}{2} D_\mu \varphi^* D^\mu \varphi - \frac{m^2}{2} |\varphi|^2 - g \frac{1}{4} |\varphi|^4 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right\}. \quad (27.83)$$

The nongradient part involving the scalar field is now invariant under $SU_R(2) \times U_L(1)$-rotations of the two complex fields $\varphi^+$ and $\varphi^0$:

$$\varphi(x) \rightarrow e^{-i\alpha(x)/2} \times e^{-i\gamma(x)/2} \varphi(x).$$

The gradient term $|\partial_\mu \varphi|^2$ is not invariant under (27.84), but picks up extra gradient terms of $\alpha(x)$ and $\gamma(x)$. These can be compensated by gauge fields of $SU_L(2) \times U_R(1)$-rotations: a singlet vector field $B^\mu(x)$ for the $U_R(1)$ transformations $e^{-i\gamma(x)/2}$, and an $SU_L(2)$-triplet vector field $W^\mu(x)$ for the $SU_L(2)$ transformations $e^{-i\alpha(x)/2}$. For this purpose one defines a covariant derivative

$$D_\mu \varphi(x) = \partial_\mu \varphi(x) + i \frac{g}{2} B_\mu \varphi(x) + i \frac{g'}{2} W_\mu \cdot \varphi(x).$$

$$\quad (27.85)$$
The extra gradients can be absorbed by the gauge transformations

\[ B_\mu(x) \rightarrow B_\mu(x) + \frac{1}{ig'} e^{-i\gamma(x)/2} \partial_\mu e^{i\gamma(x)/2} = B_\mu(x) + \frac{1}{2g'} \partial_\mu \gamma(x), \]  

(27.86)

\[ W_\mu(x) \rightarrow e^{-i\alpha(x)\cdot x/2} W_\mu e^{i\alpha(x)\cdot x/2} + \frac{1}{ig'} e^{-i\alpha(x)\cdot x/2} \partial_\mu e^{i\alpha(x)\cdot x/2}. \]  

(27.87)

The term \( F_{\mu\nu} F^{\mu\nu} \) is a gauge-invariant Lagrangian density of the gauge fields themselves. It contains the field strength of the \( B_\mu \)-field and the field strength of the \( W_\mu \)-field

\[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} \text{tr}(W_{\mu\nu} W^{\mu\nu}), \]  

(27.88)

where \( B_{\mu\nu} B^{\mu\nu} \) is the square of the ordinary curl of the vector potential \( B_\mu(x) \):

\[ B_{\mu\nu}(x) = \partial_\mu B_\nu(x) - \partial_\nu B_\mu(x), \]  

(27.89)

and \( \text{tr}(W_{\mu\nu} W^{\mu\nu}) \) the nonabelian generalization of it associated with \( W^a_\mu(x) \) [recall (27.62)]

\[ W^a_{\mu\nu}(x) = \partial_\mu W^a_\nu(x) - \partial_\nu W^a_\mu(x) - g(W_\mu(x) \times W_\nu(x))^a. \]  

(27.90)

Just as the finite penetration depth in a superconductor, nonzero masses of the vector bosons are now assumed to be the results of a nonzero expectation value of the \( \varphi \)-field:

\[ \langle \varphi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}. \]  

(27.91)

Allowing for a spacetime dependence of the neutral components of the \( \varphi \)-field by setting

\[ \varphi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + \rho(x) \end{pmatrix}, \]  

(27.92)

the Lagrangian density of the scalar field in (27.83) takes then the form

\[ \mathcal{L} = \frac{1}{2} [\partial_\mu \rho(x)]^2 + \frac{1}{8} [v + \rho(x)]^2 [g'B_\mu - gW^3] + g^2 (W^1_\mu W^{1\mu} + W^2_\mu W^{2\mu}) - m^2 [v + \rho(x)]^2 - \frac{g}{8} [v + \rho(x)]^4. \]  

(27.93)

By diagonalizing the quadratic terms in the vector fields \( W^1_\mu \) and \( W^2_\mu \), we observe the generation of a charged field

\[ W^\pm = \frac{1}{\sqrt{2}} (W^1_\mu \pm W^2_\mu). \]  

(27.94)
The mass of the $\rho$-field is determined by

$$m_\rho^2 = \frac{-2m^2}{2}.$$ (27.99)

This is the relation anticipated in Eq. (17.37).

### 27.8 Lepton Masses

Thus we have found an explanation for the masses of the vector bosons by the spontaneous breakdown of a continuous symmetry. The nonzero ground-state expectation value of a scalar field $v$ sets the mass scale of $m_W$ in Eq. (27.95) and of $m_Z$ in Eq. (27.73). It is now suggestive that also the fermion masses are due a similar mechanism. Thus we assume the leptons to start life as massless fermi fields whose mass is generated by the simplest possible coupling to the scalar field via a Lagrangian density

$$\mathcal{L} = -G_\varepsilon \bar{\varphi}^0 (\bar{e}_R e_L + \bar{e}_L e_R) + (e \rightarrow \mu).$$ (27.100)

Inserting the scalar field (27.92), we see that the spontaneous breakdown gives to the initially massless electrons and muons the new masses

$$m_e = G_e v / \sqrt{2}, \quad m_\mu = G_\mu v / \sqrt{2}.$$ (27.101)

### 27.9 More Leptons

Apart from electrons and muons, more leptons have been discovered experimentally. They are listed in Table 27.1. Their weak interactions proceed in the same way as those of electrons and their neutrinos by the coupling to the vector bosons $W^\pm$ and $Z^0$. 
Table 27.1 List of leptons and their properties taken from the Particle Data Group [22].

<table>
<thead>
<tr>
<th>Lepton</th>
<th>Charge</th>
<th>Mass (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu_e$</td>
<td>0</td>
<td>$&lt; 2 \cdot 10^{-6}$</td>
</tr>
<tr>
<td>$\nu_\mu$</td>
<td>0</td>
<td>$&lt; 0.19$</td>
</tr>
<tr>
<td>$\nu_\tau$</td>
<td>0</td>
<td>$&lt; 18.2$</td>
</tr>
<tr>
<td>$e$</td>
<td>-1</td>
<td>0.510998928(11)</td>
</tr>
<tr>
<td>$\mu$</td>
<td>-1</td>
<td>105.6583715(35)</td>
</tr>
<tr>
<td>$\tau$</td>
<td>-1</td>
<td>1776.82 $\pm$ 0.16</td>
</tr>
</tbody>
</table>

27.10 Weak Interaction of Hadrons

In order to accommodate also the weak interactions of hadrons, the above Lagrangian density (27.80) is extended by a term involving the quarks

$$\mathcal{L}_{\text{int}} = \frac{g}{2\sqrt{2}} \left( W_{\mu}^+ j_{h}^\mu + \text{c.c.} \right) + e A_{\mu}^{em} j_{\mu}^{em} - \sqrt{g^2 + g'^2} Z^\mu \left( \bar{q}_L \frac{\tau_3}{2} \gamma_\mu q_L - \sin^2 \theta_W A_{\mu}^{em} \right),$$

(27.102)

where

$$j_{h}^{\mu+} = \bar{u}_L \gamma_\mu d_L,$$

(27.103)

$$j_{\mu h}^{em} = \frac{2}{3} \bar{u}_\mu u - \frac{1}{3} \left( \bar{d}_L \gamma_\mu d + \bar{s}_L \gamma_\mu s \right).$$

(27.104)

with

$$d_\theta = d \cos \theta_c + s \sin \theta_c.$$  

(27.105)

When the theory arrived at this stage it was realized that it contained an unpleasant contradiction with experiments. The neutral vector meson $Z$ was coupled to a quark current

$$\bar{q}_L \gamma^\mu \tau_3 q_L = \bar{u}_L \gamma^\mu u_L - \cos^2 \theta_c \bar{d}_L \gamma^\mu d_L,$$

$$-i \sin \theta_c \cos \theta_c \left( \bar{d}_L \gamma^\mu s_L + \bar{s}_L \gamma^\mu d_L \right) - i \sin^2 \theta_c \bar{q}_L \gamma^\mu c_L.$$ 

(27.106)

The third term changed the strangeness by one unit. However, processes with a strangeness-changing neutral current were known to be strongly suppressed in nature. Otherwise one would have observed particle decays

$$K^\pm \rightarrow \pi^\pm \nu \bar{\nu},$$

$$K^0 \rightarrow \mu^+ \mu^-,$$

(27.107)

with a much larger branching ratio than the experimentally observed $6 \times 10^{-7}, 10^{-8}$, respectively (see Fig. 27.1).
In order to achieve the desired suppression, Glashow, Iliopoulos, and Maiani [12] postulated the existence of a further quark, called charmed quark, denoted by $c$, which was supposed to form a weak isodoublet with $s$ in the same way as $u$ does with $d$. Then they considered the two doublets

$$\begin{pmatrix} u \\ d_\theta \end{pmatrix}_L, \begin{pmatrix} c \\ s_\theta \end{pmatrix}_L$$

with

$$s_\theta = -u \sin \theta_c + s \cos \theta_c$$

as two families of the same sort, both with weak hypercharge 1/3, and wrote the left-handed neutral current as

$$\left( \bar{u}, \bar{d}_\theta \right) \frac{\tau_3}{2} \gamma^\mu \left( u, d_\theta \right)_L + \left( \bar{c}, \bar{s}_\theta \right) \frac{\tau_2}{2} \gamma^\mu \left( c, s_\theta \right)_L - i \sin^2 \theta W j_{\mu m}^e.$$  \hspace{1cm} (27.110)

After this, the strangeness changing currents cancel exactly. The cancellation in the $K^0$ decays goes as follows. Take, for instance the decay $K^0 \rightarrow \mu^+\mu^-$. The compensating diagrams are shown in Fig. 27.2.

$$\Delta j_{\mu h} = \frac{2}{3} \bar{c}\gamma^\mu c.$$ \hspace{1cm} (27.111)
The cancellation of the strangeness-changing neutral currents in weak interactions if names after the authors [12] as GIM mechanism. GIM mechanism needed in the space of all leptons to suppress the strangeness changing neutral currents is discussed in Ref. [23].

27.11 Quantum Oscillations

The standard model of weak interactions explains some interesting novel phenomena in particle physics. The nature of a fundamental particle is sometimes not fixed but can oscillate as a function of time. The first particles, where this phenomenon was observed, was the system of neutral kaons, where the $K$-mesons $K^0$ and $\bar{K}^0$ oscillate between each other.

27.11.1 Oscillations between Neutral Kaons

In our discussion of the SU(3)-symmetry of strong interactions in Section 24.4, we mentioned that the mesons $K^0$ and $\bar{K}^0$ have strangeness $-1$ and $+1$, respectively. It is believed that strong interactions strictly conserve these quantum numbers. The weak interactions described by the standard model, however, conserve the symmetry $CP$. If we want to describe the weak decay of the strongly produced $K^0$ and $\bar{K}^0$ mesons, we must decompose them into eigenstates of $CP$. These are the states

$$|K^0_S\rangle \equiv \frac{1}{\sqrt{2}} \left( |K^0\rangle + |\bar{K}^0\rangle \right), \quad |K^0_L\rangle \equiv \frac{1}{\sqrt{2}} \left( |K^0\rangle - |\bar{K}^0\rangle \right).$$

They have different decay channels. At rest, the first state is even under $CP$ (recall Subsections 7.1.6 and 7.1.6), the second is odd. Hence the first state can decay into two pions, while the second cannot. The lifetime of $K^0_S$ is $0.8922 \pm 0.0020 \times 10^{-10}$ sec, producing almost exclusively $\pi^+ \pi^-$ and $\pi^0 \pi^0$ at a ratio 2:1.

The $CP$-odd state $K^0_L$ must at least decay into three pions and has therefore a much longer lifetime. This is why $K^0_S$ and $K^0_L$ are also called $K^0_S$ and $K^0_L$.

Let $m_S$, $m_L$ be their masses and $\Gamma_S$, $\Gamma_L$ their decay rates. A beam of $K^0$ produced in a strong-interaction process will then evolve in its rest frame as an oscillating superposition of $K_S$ and $K_L$:

$$|K^0\rangle = \frac{1}{\sqrt{2}} \left[ e^{-(imS+\Gamma S/2)r} |K_S\rangle + e^{-(imL+\Gamma L/2)r} |K_L\rangle \right],$$

where $\tau$ is the proper time. Since the two components decay at a different rate, the beam will contain oscillating admixtures of $\bar{K}^0$. These can be detected if the beam is directed towards a strongly interacting target, say a slab of copper. In it the forward scattering amplitude for $K^0$ is smaller than for $\bar{K}^0$. Thus, if one changes the thickness of the slab, the outcoming beam will have different admixtures of the short-lived component $K^0_S$. In this way, one is able to regenerate $K^0_S$ from a beam which had turned almost completely into $K^0_L$ by the decay into $\pi^+ \pi^-$ or $\pi^0 \pi^0$. 
The process was analyzed in 1957 [14], and the experiment showed clearly the regeneration effect [15, 16].

When experimentalists tried to set an upper limit to the decay of $K^0_L$ into two pions they found that this decay was not completely forbidden [17]. This implied that the weak decay of $K^0$-mesons contains a small term violating CP-symmetry. Thus the correct mixtures of short- and long-lived $K^0$ mesons should really be slightly different from (27.112):

\begin{align}
|K^0_S\rangle & \equiv \frac{1}{\sqrt{2(1+\epsilon^2)}} \left[ (1+\epsilon)|K^0\rangle + (1-\epsilon)|\bar{K}^0\rangle \right], \quad (27.114) \\
|K^0_L\rangle & \equiv \frac{1}{\sqrt{2(1+\epsilon^2)}} \left[ (1-\epsilon)|K^0\rangle - (1+\epsilon)|\bar{K}^0\rangle \right], \quad (27.115)
\end{align}

where $\epsilon$ is of the order of $10^{-3}$.

The parameter $\epsilon$ can be measured by studying a beam of $K^0$ as a function of time. Due to the decay of the $K^0_S$ content, the beam is expected to perform a damped oscillation from $K^0 = d\bar{s}$ to $\bar{K}^0 = s\bar{d}$, and back. The content of the two components can be studied by looking at the decay rates $K^0 \to \pi^\pm e^\pm \nu_e$ and $\bar{K}^0 \to \pi^\pm e^\pm \bar{\nu}_e$, arising from the decay of the quarks $\bar{s}$ and $s$, respectively, in accordance with the $\Delta Q = \Delta S$ rule. This permits measuring $\epsilon$ directly from the asymmetry

\begin{equation}
\frac{N_{K^0} - N_{\bar{K}^0}}{N_{K^0} + N_{\bar{K}^0}} = \frac{N_{\pi^- - \pi^+}}{N_{\pi^+ + \pi^-}} = \frac{N_{e^-} - N_{e^+}}{N_{e^+} + N_{e^-}} = \frac{2\text{Re} \epsilon}{1 + |\epsilon|^2} \equiv (3.320 \pm 0.074) \times 10^{-3}. \quad (27.116)
\end{equation}

Thanks to CP-violation, not only $K^0_S$ but also $K^0_L$ decays into two pions, and this makes it possible to observe oscillations in the decay of a $K^0$-beam into $\pi^+ \pi^-$. The ratio of the decay rates changes as a function of the proper time $\tau$ as follows:

\begin{equation}
I_{\pi^+\pi^-} = \frac{|\langle \pi^+\pi^- | K^0(\tau) \rangle|}{|\langle \pi^+\pi^- | \bar{K}^0_S(\tau) \rangle|} = e^{-\Gamma_S \tau} + 2|\eta_+|e^{-\Gamma_L \tau/2} \cos(\Delta m \tau - \phi_+) + |\eta_-|^2 e^{-\Gamma_L \tau}, \quad (27.117)
\end{equation}

where

\begin{equation}
\eta_+ \equiv |\eta_+|e^{i\phi_+} = \frac{|\langle \pi^+\pi^- | K^0(\tau) \rangle|}{|\langle \pi^+\pi^- | \bar{K}^0_S(\tau) \rangle|}, \quad \Delta m \equiv m_L - m_S. \quad (27.118)
\end{equation}

This damped oscillation was indeed observed experimentally [18], as shown in Fig. 27.3.

A best fit to the data yields the parameters

\begin{align}
\tau_S &= 1/\Gamma_S = (0.8926 \pm 0.0012) \times 10^{-8} \text{ sec}, \quad (27.119) \\
\tau_L &= 1/\Gamma_L = (5.17 \pm 0.04) \times 10^{-8} \text{ sec}, \quad (27.120) \\
\Delta m &= (0.5333 \times 0.0027) \times 10^{10} \text{ h/sec}, \quad (27.121) \\
|\eta_+| &= (2.269 \pm 0.023) \times 10^{-3} \approx |\epsilon| \quad (27.122) \\
\phi_+ &= (44.3 \pm 0.8)^\circ. \quad (27.123)
\end{align}
Figure 27.3 Left-hand: Asymmetry of the number of \( K^0 = s\bar{d} \) mesons with respect to the \( K^0 = \bar{s}d \) mesons as a function of time [19]. The asymmetry is measured by the ratio of the decays \( K^0 \to \pi^+e^\pm\nu_e \) and \( K^0 \to \pi^+\bar{e}^\pm\bar{\nu}_e \). After a long time, only \( K^0 \) survives which does not have an equal content of \( K^0 \) and \( \bar{K}^0 \) as a signal of \( CP \)-violation. Right-hand: Oscillation in the decay rate into \( \pi^+\pi^- \) of the \( K^0 \)-beam. Curve (a) shows the histogram of the raw data. Curve (b) shows the theoretical decay curve without oscillations, and the insert shows the best fit to the oscillations if the nonoscillating background is subtracted (figure from [18]).

Figure 27.4 Asymmetry of the number of \( B^0_d = d\bar{b} \) mesons with respect to the \( \bar{B}^0_d = b\bar{d} \) mesons as a function of time [19]. Here the long- and short-lived combinations have almost the same lifetime.
27.11 Quantum Oscillations

27.11.2 Mesons containing the Bottom Quark

By analogy with the strange mesons $K^0$ and $\bar{K}^0$ there exist neutral mesons $B^0 = d\bar{b}$ and $\bar{B}^0 = b\bar{d}$. The masses of their short-lived combination is $m_s = 5721$ MeV which oscillated with the long-lived combination of mass $m_s = 5747$ MeV. For these mesons, the asymmetry measurement yields (see Fig. 27.4) [20]:

\[
\frac{N_{B^0_d} - N_{\bar{B}^0_d}}{N_{B^0_d} + N_{\bar{B}^0_d}} = \frac{N_{e^+} - N_{e^-}}{N_{e^+} + N_{e^-}} = \frac{2\text{Re} \epsilon_{B_d}}{1 + |\epsilon_{B_d}|^2} = (0.0035 \pm 0.0103 \pm 0.0015).
\]  

(27.124)

Similar asymmetries can be observed for $B^0_s = s\bar{b}$ and $\bar{B}^0_s = b\bar{s}$.

All such neutral particle mixings can simply be understood by assuming that the Lagrangian density for these particles contains a nondiagonal mass matrix

\[
\mathcal{M} = \begin{pmatrix}
M_{11} - \frac{i}{2} \Gamma_{11} & M_{12} - \frac{i}{2} \Gamma_{12} \\
M_{21} - \frac{i}{2} \Gamma_{21} & M_{22} - \frac{i}{2} \Gamma_{22}
\end{pmatrix}.
\]  

(27.125)

This matrix must be Hermitian, implying that

\[
M_{21} = M_{12}^*, \quad \Gamma_{21} = \Gamma_{12}^*.
\]  

(27.126)

Moreover, as a consequence of CPT-invariance, the diagonal elements must satisfy

\[
M_{11} = M_{22} = M, \quad \Gamma_{11} = \Gamma_{22} = \Gamma.
\]  

(27.127)

The most general mass matrix is therefore

\[
\mathcal{M} = \begin{pmatrix}
M - \frac{i}{2} \Gamma & M_{12} - \frac{i}{2} \Gamma_{12} \\
M_{12}^* - \frac{i}{2} \Gamma_{12}^* & M - \frac{i}{2} \Gamma
\end{pmatrix}.
\]  

(27.128)

27.11.3 General Flavor Mixing

The Cabibbo theory was set up at a time when only the three quarks $u, d, s$ were known. After the discovery of the heavier three quarks it turned out that all quarks can be grouped into three families of charge $2/3$ and $-1/3$, respectively, and that the weak current (27.43) has the generalization

\[
(u, c, t) \gamma^\lambda(1 - \gamma_5) \begin{pmatrix}
V_{ud} & V_{us} & V_{ub} \\
V_{cd} & V_{cs} & V_{cb} \\
V_{td} & V_{ts} & V_{tb}
\end{pmatrix} \begin{pmatrix}
d \\ s \\ b
\end{pmatrix}.
\]  

(27.129)

The $3 \times 3$-matrix $V$ is called CKM-matrix, after the initials of the authors Cabibbo, Kobayashi, and Maskawa [7, 13]. It is often expressed in terms of four angles $\theta_{12}, \theta_{13}, \theta_{23}$, and $\delta_{13}$ as

\[
V = \begin{pmatrix}
c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta_{13}} \\
-s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta_{13}} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta_{13}} & s_{23} c_{13} \\
s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta_{13}} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i\delta_{13}} & c_{23} c_{13}
\end{pmatrix},
\]  

(27.130)
where \( s_{12} \equiv \sin \theta_{12} \), \( c_{12} \equiv \cos \theta_{12} \). The angle \( \delta_{13} \) serves to explain the small \( CP \)-violation observed in the decay of \( K_L^0 \) discussed in the previous subsection. The mixing matrix is permanently updated whenever new experimental data are available. The presently used numbers are [21]:

\[
V = \begin{pmatrix}
0.974 \text{ to } 0.9756 & 0.219 \text{ to } 0.226 & 0.0025 \text{ to } 0.0048 \\
0.219 \text{ to } 0.226 & 0.9732 \text{ to } 0.9748 & 0.038 \text{ to } 0.044 \\
0.004 \text{ to } 0.014 & 0.037 \text{ to } 0.044 & 0.9990 \text{ to } 0.9993
\end{pmatrix}.
\] (27.131)

The most precise constraints on the size of the elements of the CKM-matrix are extracted from the low-energy \( s \to u \) and \( d \to u \) semileptonic transitions. The determination is a specialized part of phenomenological particle physics, on which excellent reviews are available from the Particle Data Group [13].

### 27.12 Neutrino Mixing

By analogy with the flavor mixing matrix \( V \) for quarks in Eq. (27.129), there exists a mixing matrix for the three neutrinos \( \nu_e, \nu_\mu, \nu_\tau \). It is known as the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix which transforms three bare neutrinos \( \nu_1, \nu_2, \nu_2 \) into the physical particles:

\[
\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau
\end{pmatrix} = U
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3
\end{pmatrix} = \begin{pmatrix}
U_{e1} & U_{e2} & U_{e3} \\
U_{\mu1} & U_{\mu2} & U_{\mu3} \\
U_{\tau1} & U_{\tau2} & U_{\tau3}
\end{pmatrix}
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3
\end{pmatrix}.
\] (27.132)

The matrix \( U \) is parametrized in terms of angles in the same way as the matrix \( V \) in (27.130), and the best determination at present yields [24]

\[
U = \begin{pmatrix}
0.82 \pm 0.01 & 0.54 \pm 0.02 & 0.15 \pm 0.03 \\
0.35 \pm 0.06 & 0.70 \pm 0.06 & 0.62 \pm 0.06 \\
0.44 \pm 0.06 & 0.45 \pm 0.06 & 0.77 \pm 0.06
\end{pmatrix}.
\] (27.133)

Neutrino flavor mixing is a cause for the notorious detection problem of solar neutrinos [25]. It is also the source of the so-called atmospheric neutrino anomaly [26].

The neutrino masses are too small to be experimentally measurable. What is measurable is the mass difference between the neutrinos associated with the electron and the muon. This is done by counting the number of times by which electron neutrinos are transformed into muon neutrinos and back on their way from the sun to us.

An alternative possibility to measure small mass differences comes from recent observations of the time modulation of two-body weak decays of heavy ions. This reveals the mass content of the electron neutrinos via interference patterns in the recoiling ion wave function. From the modulation period one derives the difference of the square masses \( \Delta m^2 \approx 22.5 \times 10^{-5}\text{eV}^2 \), which is about 2.8 times larger than
that derived from a combined analysis of KamLAND and solar neutrino oscillation experiments. It is, however, compatible with a data regime to which the KamLAND analysis attributes a smaller probability.

At the GSI in Darmstadt, Germany, the experimental storage ring ESR permits observing completely ionized heavy atoms \( I \) or hydrogen-like heavy ions \( I^+ \) over a long time [27, 28] and thus to measure the time dependence of their weak two-body decays \( I^+ \to I + \nu_e \) or \( I \to I^+ + \bar{\nu}_e \). The first decay is due to the well-known electron-capture (EC) process. In the second an electron comes out of the nucleus together with an antineutrino, and the electron remains in an orbit. The virtue of such experiments is that the properties of the neutrino or antineutrino can be deduced from the time dependence of the transition, by observing only the initial and final ions. The special efficiency of these experiments becomes clear in the Dirac sea interpretation of the second process, where the initial ion simply absorbs a negative-energy antineutrino in the vacuum. Since the vacuum has all negative-energy states filled, the vacuum is a source of negative-energy neutrinos of maximally possible current density, i.e., the best possible neutrino source in the universe. This is why the ESR experiments yield information on neutrino properties with great precision even if the targets and exposure times are quite small, in particular much smaller than the \( 2.44 \times 10^{32} \) proton-yrs (2881 ton-yrs) in the famous KamLAND experiments [29], which are only sensitive to the much less abundant positive-energy neutrinos produced by nuclear reactors.

### 27.13 Simple Theory of Two-Neutrino Mixing

To illustrate this we consider here the small mass difference of the two lightest neutrinos. According to Pontecorvo [30, 31], the Dirac fields of the physical electron and muon-neutrinos \( \nu_f = (\nu_e, \nu_\mu) \), the so-called flavor fields, are superpositions of neutrino fields \( \nu_i(x) = (\nu_1(x), \nu_2(x)) \) of masses \( m_1 \) and \( m_2 \):

\[
\nu_e(x) = \nu_1(x) \cos \theta + \nu_2(x) \sin \theta, \quad \nu_\mu(x) = -\nu_1(x) \sin \theta + \nu_2(x) \cos \theta.
\]  

(27.134)

Here \( \theta \) is a mixing angle. The free Dirac action has the form

\[
\mathcal{A} = \sum_f \int d^4x \bar{\nu}_f(x) (i\gamma^\mu \partial_\mu - \mathcal{M}) \nu_f(x),
\]  

(27.135)

where \( \gamma^\mu \) are the Dirac matrices, and \( \mathcal{M} \) is a mass matrix, whose diagonal and off-diagonal elements are \( m_f = (m_e, m_\mu) \) and \( m_{e\mu} = m_{\mu e} \), respectively. The eigenvalues \( m_i = (m_1, m_2) \) are related to \( m_f \) by [30, 31, 32, 33],

\[
m_e = m_1 \cos^2 \theta + m_2 \sin^2 \theta, \quad m_\mu = m_1 \sin^2 \theta + m_2 \cos^2 \theta, \quad m_{e\mu} = m_{\mu e} = (m_2 - m_1) \sin \theta \cos \theta.
\]  

(27.136)

The weak transition between the electron \( e \) and its neutrino \( \nu_e \) is governed by the interaction

\[
\mathcal{A}_{\text{int}} = \frac{g}{\sqrt{2}} \int d^4x W^-_\mu(x) J^{+\mu}(x) + \text{h.c.} = \frac{g}{\sqrt{2}} \int d^4x W^-_\mu(x) \bar{e}(x) \gamma^\mu (1 - \gamma_5) \nu_e(x) + \text{h.c.},
\]  

(27.137)
where $\gamma_5$ is the product of Dirac matrices $i\gamma^0\gamma^1\gamma^2\gamma^3$.

Since the interaction (27.137) involves only two flavor fields in (27.134), the states of masses $m_i$ will always be produced as coherent superpositions. The weakness of the interaction allows us to calculate the shape of the mixed wave packet from perturbation theory. Consider the decay $I \to I_H + \bar{\nu}_e$ which is a superposition of the states of masses $m_1$ and $m_2$. Let us write down formulas which are applicable for electron capture and, after the interchange of $M_H$ with the mass $M$ of the bare ion, treats the case of outgoing neutrinos. In the center-of-mass (CM) frame of the initial bare ion of mass $M$, the final $H$-like ion has the same momentum as the antineutrino $\bar{\nu}_i$ ($i = 1, 2$), whose energy is $\omega_i \equiv \omega_{k,i} = \sqrt{k^2 + m^2_i}$ determined by

$$M \equiv M_H + Q = \omega_i + \sqrt{M^2_H + k^2_i} = \omega_i + \sqrt{M^2_H + \omega^2_i - m^2_i}, \quad i = 1, 2,$$  

(27.138)

so that $\omega_i = [(2M_H + Q)Q + m^2_i]/2(M_H + Q)$. Subtracting $\omega_2$ and $\omega_1$ from each other we find the energy difference

$$\Delta \omega \equiv \omega_2 - \omega_1 = \frac{m^2_2 - m^2_1}{2M} \equiv \frac{\Delta m^2}{2M}. \quad (27.139)$$

The denominator $M$ is of the order of 100 GeV and much larger than $\Delta m^2$, so that $\Delta \omega$ is extremely small. It is the difference of the recoil energies transferred to the outcoming ion by the antineutrinos of masses $m_1$ and $m_2$. Without recoil, we would have found an energy difference at the same momentum which is four orders of magnitude larger than that, namely

$$\Delta \omega_k = \omega_{k,2} - \omega_{k,1} = (\Delta m^2 + \omega^2_{k,1})^{1/2} - \omega_{k,1} \approx \Delta m^2/2\omega_{k,1} \approx \Delta m^2/2Q. \quad (27.140)$$

This is the frequency with which the incoming negative-energy neutrino current of momentum $k$ oscillates in the vacuum. Note that although $\Delta \omega$ is small, the momentum difference $\Delta k \equiv k_2 - k_1$ associated with the energies $\omega_1,2$ is as large as $\Delta \omega_k$, but has the opposite sign.

### 27.14 Experiments

The best experimental results are available for the EC-processes reported in Ref. [27], where an electron is captured from the K-shell and converted into an electron-neutrino which runs off to infinity. On the average, the decay is exponential with a rate expected from a standard-model calculation. In addition, however, the rate shows oscillations with a frequency $\Delta \omega$. The experimental results are [34]

$$\begin{align*}
^{140}\text{Pr}^{58+} & \to ^{140}\text{Ce}^{58+} : \quad \Delta \omega \approx 0.890(11) \text{ sec}^{-1} \quad (Q = 3.386 \text{ keV}), \quad (27.141) \\
^{142}\text{Pm}^{60+} & \to ^{142}\text{Nd}^{60+} : \quad \Delta \omega \approx 0.885(31) \text{ sec}^{-1} \quad (Q = 4.470 \text{ keV}). \quad (27.142)
\end{align*}$$

In both cases the period of oscillations is roughly 7 sec, and it scales with $M$ (see Fig. 27.5).
We expect these oscillations to be associated with the frequency $\Delta \omega$ determined by Eq. (27.139), and thus to give information on $\Delta m^2$. Inserting the experimental numbers for $\Delta \omega$ into Eq. (27.139) and taking into account that the particles in the storage ring run around with 71% of the light velocity with a Lorentz factor $\gamma \approx 1.43$, we find from the two processes [35]

$$\Delta m^2 \approx 22.5 \times 10^{-5} \text{eV}^2. \quad (27.143)$$

This is by a factor $\approx 2.8$ larger than the result $\Delta m^2 \approx 7.58^{+0.3}_{-0.3} \times 10^{-5} \text{eV}^2$ favored by the KamLAND experiment [29, 37, 38], but it lies close to their less favored result [36], which the authors excluded by $2.2\sigma$ in 2005, now by $6\sigma$ [29] (see Fig. 27.6).
So far, the origin of this discrepancy is unclear. One explanation has been attempted in Ref. [39] where the authors investigate the influence of the strong Coulomb field around the ion upon the process. We refer the reader to recent discussions of this question [41].

### 27.15 Entangled Wavefunction

It is easy to describe theoretically the oscillations in Fig. 27.5. Consider the decay of the initial ion $I$ into the ion $I_H$ plus an electron-antineutrino $\bar{\nu}_e$. Ignoring all spins and the finite size of the ions, the effective interaction for this process is

$$A_{\text{int}} = \int d^4x I_H^\dagger(x)\bar{\nu}^\dagger_e(x)I(x) = \int d^4x \left[ \cos \theta I_H^\dagger(x)\bar{\nu}^\dagger_1(x)I(x) + \sin \theta I_H^\dagger(x)\bar{\nu}^\dagger_2(x)I(x) \right],$$

(27.144)

where $I(x)$, $\bar{\nu}_e(x)$, and $I_H(x)$ are the field operators of the involved particles. In the CM frame, the initial ion is at rest, while the final moves nonrelativistically. The outgoing wave is spherical. The role of the antineutrinos $\bar{\nu}_1$ and $\bar{\nu}_2$ is simply to create a coherent superposition of two such waves with the two different radial momenta $k$ and associated frequencies $\omega$ whose difference was calculated in (27.143):

$$\langle x|\psi^{(+)}; t \rangle \equiv -\frac{g}{r} \left[ \cos \theta e^{i(k_1 r - \omega_1 t)} + \sin \theta e^{i(k_2 r - \omega_2 t)} \right].$$

(27.145)

The resulting wave carries a radial current density of ions $I_H$:

$$j_r = \frac{g^2}{M_H r^2} \left[ \cos^2 \theta k_1 + \sin^2 \theta k_2 + \sin \theta \cos \theta (k_1 + k_2) \cos(\Delta k r - \Delta \omega t) \right].$$

(27.146)

In order to find the decay rate we integrate this over a sphere of radius $R$ surrounding the initial ion. We can choose $R$ any size $\ll 1/\Delta k \approx 10^4$ m. From this surface we find the outgoing probability current density

$$\dot{P} = 4\pi g^2 \frac{k}{M} \left[ 1 + \sin(2\theta) \cos(\Delta \omega t) \right],$$

(27.147)

where we have approximated $k_1$ and $k_2$ by their average $\bar{k}$. This $\dot{P}$ explains directly the observed oscillating decay rate of the initial ions.

A comment is in place on several recent publications [42, 43, 44, 45, 46, 47]. These deny a relation between neutrino oscillations and the nonexponential decay seen in the GSI experiment for various reasons. In Ref. [42], the basic argument is that the antineutrino oscillations set in after their emission, so that they cannot be observed in the GSI experiment. The present discussion shows that this argument is true, although it fails to give the correct explanation of the data [48]. Indeed, the GSI data do not care about the neutrino oscillations while they propagate away from the decay center. The neutrinos merely serve to give the ion a coherent kick of two different momenta and energies. What is measured are the resulting oscillations of the ion wave function caused by this kick.
It is noteworthy that this analysis, in which we extract the properties of the unobserved antineutrino from the behavior of the ion, corresponds precisely to the usual entanglement analysis of decay processes, such as $\pi^0 \to \gamma + \gamma$. There the measurement of the polarization of one photon tells us immediately the polarization properties of the other, unobserved photon.

Let us finally remind the reader of the similarities of these neutrino oscillations with similar oscillation phenomena in associate production processes of particles, where they appear together with an oscillating partner. These have been proposed and controversially discussed before by many authors. See Subsection 27.11.1, where $\Lambda$-hyperons are produced together with neutral Kaons [49]. Or the mixing arising in the production of neutral mesons containing a bottom quark discussed in Subsection 27.11.2. Or the production of $\mu^-$-particles together with antineutrinos in the decay $\pi^- \to \mu^- + \bar{\nu}_e$ [50]. The presently-discussed data provide us with the most definite experimental confirmation of such a phenomenon.

We refer the reader to the many interesting ideas presented in other publications [44, 45, 46, 47].

Notes and References

[1] See the www address (http://www.library.ethz.ch/exhibit/pauli/neutrino_e.html).
M. Kobayashi and T. Maskawa, Prog. Theor. Phys. 49, 652 (1973);
S. Weinberg, Phys. Rev. Lett. 19, 1264 (1967);
A. Salam, in Elementary Particle Physics, Relativistic Groups, and Analyticity, Eighth Nobel Symposium, ed. by N. Svartholm, Almqvist and Wiksell, Stockholm, 1968, p. 367.


[29] S. Abe et al., Phys. Rev. Lett. 100, 221803 (2008); See also:


[34] The right-hand superscript indicates the ionization degree of the associated atom.
[35] The isotope masses of the final ions are $M \approx 130.319 \text{ GeV}$ and $M \approx 132.186 \text{ GeV}$, for $^{140}_{58}\text{Ce}$ and $^{142}_{60}\text{Nd}$, respectively. Also recall that a frequency $1/\text{sec}$ corresponds to $\approx 6.6 \times 10^{-16} \text{ eV}$.
[36] See the upper allowed region in Fig. 4 of the KamLAND data in T. Araki et al., Phys. Rev. Lett. 94, 081801 (2005).
[48] The author of Refs. [42] and [43] suggest the existence of two closely lying nuclear states whose beats are observed.