23

Exactly Solvable $O(N)$-Symmetric Four-Fermion Theory in $2 + \epsilon$ Dimensions

As in the $O(N)$-symmetric $\phi^4$-theory in Section 18.2, there exists also a class of fermionic field theories which can be solved exactly by introducing collective quantum fields of the type discussed in Section 18.1. In this chapter we shall investigate the properties of a model that contains a large number $N$ of identical fermion fields, coupled by an $O(N)$-symmetric four-field interaction.

23.1 Four-Fermion Self-Interaction

The Lagrange density reads at finite $N$:

$$\mathcal{L} = \bar{\psi}_a (i\gamma^\mu \partial_\mu - m_0) \psi_a + \frac{g_0}{2N} \left( \bar{\psi}_a \psi_a \right)^2,$$  \hspace{1cm} (23.1)

where the index $a$ runs from 1 to $N$. This model has first been studied in slightly different forms by Anselm, by Vaks and Larkin, and by Nambu and Jona-Lasinio to exhibit the phenomenon of spontaneous breakdown of chiral symmetry. In a so-called chiral version of the model, the latter authors were the first to point out the existence of Nambu-Goldston bosons as consequence of the spontaneous symmetry breakdown. Moreover, an extended version of the model proposed in 1976 [1] was capable of illustrating the properties of all low-energy phenomena in hadron physics.

However, some years later it was discovered that the approximations used in that model were too crude so that the approximate treatment used so far was not really reliable [2]. The chiral fluctuations are so strong that the spontaneously broken symmetry is restored. This will be discussed in Section 23.10.

The original inspiration for studying this model had come from the microscopic theory of superconductivity which had been found in 1957 by Bardeen, Cooper, and Schrieffer [3]. A satisfactory definition of the model is only possible in $D = 2 + \epsilon$ spacetime dimensions, where it is renormalizable and called the Gross-Neveu model [4]. At the mean-field level, the effective action becomes

$$\Gamma [\Psi, \overline{\Psi}] = A [\Psi, \overline{\Psi}] = \int d^Dx \left[ \overline{\Psi} (i\gamma^\mu \partial_\mu - m_0) \Psi + \frac{g_0}{2N} \left( \overline{\Psi}_a \Psi_a \right)^2 \right].$$  \hspace{1cm} (23.2)
It yields an equation of motion

\[
\left( i\partial - m_0 + \frac{g_0}{N} \Psi_a \right) \Psi_b(x) = 0. \tag{23.3}
\]

This equation differs in an important physical aspect from the Bose case. Unlike the Bose field expectation, the expectation value of a Fermi field \( \psi_a(x) \),

\[
\Psi_a \equiv \langle 0 | \psi_a | 0 \rangle, \tag{23.4}
\]

can never become non-zero, since \( \Psi_a \) is an anticommuting Grassmann field. Nevertheless, the model can exhibit a spontaneous symmetry breakdown. As before, we shall discuss the phenomenon in any dimension \( D \).

The generating functional contains linear terms in which the fields \( \psi_a(x) \) are coupled to fermionic, anticommuting sources \( \eta_a(x) \) and \( \bar{\eta}_a(x) \):

\[
Z[\eta, \bar{\eta}] = e^{iW[\eta_a, \bar{\eta}_a]} = \prod_{a=1}^{N} \int D\psi_a D\bar{\psi}_a e^{iA[\psi_a, \bar{\psi}_a] + \bar{\psi}_a \eta_a + c.c.} \tag{23.5}
\]

We introduce a collective field \( \sigma \), that fluctuates on the average around \( g \bar{\psi}_a \psi_a \), very similar to Eq. (18.4), and rewrite (23.5) as

\[
Z[\eta, \bar{\eta}] = \prod_{a=1}^{N} \int D\psi_a D\bar{\psi}_a D\sigma e^{i \int d^Dx \left[ \bar{\psi}_a (i\partial - m_0 - \sigma) \psi_a + (\bar{\psi}_a \eta_a + c.c.) \right] - \frac{N^2 g_0^2}{2} \sigma^2}. \tag{23.6}
\]

If we integrate out the \( \sigma \)-field, one recovers the original partition function (23.5). Instead, if we integrate out the fields \( \psi_a \) according to the rule (14.98), we obtain the generating functional in a form that contains only the collective field \( \sigma \):

\[
Z[\eta, \bar{\eta}] = \int D\sigma e^{iA_{\text{coll}}[\sigma] - \bar{\eta}_a G_\sigma \eta_a}, \tag{23.7}
\]

where \( A_{\text{coll}}[\sigma] \) is the collective action

\[
A_{\text{coll}}[\sigma] = N \left\{ -\frac{N}{2g_0} \int d^Dx \sigma^2 - iN \text{Tr} \log \left[ i\partial - m_0 - \sigma \right] \right\}, \tag{23.8}
\]

and

\[
G_\sigma(x, x') \equiv \left. \frac{i}{i\partial - m_0 - \sigma} \right|_{\sigma = 0}(x, x') \tag{23.9}
\]

is the propagator of the fermi field \( \psi_a(x) \) in the presence of the collective \( \sigma \)-field. This is analogous to Eq. (18.6).

In the limit \( N \to \infty \), the field \( \sigma \) is squeezed into the extremum of the action and we obtain the effective action [compare (18.24)]

\[
\frac{1}{N} \Gamma[\Sigma, \Psi_a, \bar{\Psi}_a] = -\frac{1}{2g_0} \int d^Dx \Sigma^2(x) - i\text{Tr} \log \left[ i\partial - m_0 - \Sigma(x) \right] + \frac{1}{N} \int d^Dx \Psi_a(x) \left[ i\partial - m_0 - \Sigma(x) \right] \Psi_a(x). \tag{23.10}
\]
23.1 Four-Fermion Self-Interaction

The extremum of $\Gamma[\Sigma, \Psi_a, \bar{\Psi}_a]$ yields the equations of motion,

$$\left[i\partial - m_0 - \Sigma(x)\right]\Psi_a(x) = 0,$$

(23.11)

and

$$\Sigma(x) = g_0\text{tr} \left( \frac{1}{i\partial - m_0 - \Sigma} \right) (x, x) - \frac{1}{N} g_0 \bar{\Psi}_a \Psi_a(x),$$

(23.12)

where $\text{tr}$ indicates a trace over the Dirac indices. As argued above, the field expectation $\Psi_a(x)$ must vanish, so that the only equation to be solved is

$$\Sigma(x) = g_0\text{tr} \left( \frac{1}{i\partial - m_0 - \Sigma} \right) (x, x).$$

(23.13)

It is called the gap equation. The name alludes to the first appearance of such an equation to explain the energy gap in the electron gas of superconductors.

Thus, as far as the extremum is concerned, we may study only the purely collective part of the exact action

$$\frac{1}{N} \Gamma[\Sigma] = -\frac{1}{2g_0} \int d^Dx \Sigma^2 - i\text{Tr} \log (i\partial - m_0 - \Sigma).$$

(23.14)

Let us seek for an extremal constant solution $\Sigma_0$. Then the gap equation reduces to

$$\Sigma_0 = ig_0 \text{tr} \int \frac{d^Dp}{(2\pi)^D} \frac{\hat{p} + m_0 + \Sigma_0}{p^2 - (m_0 + \Sigma_0)^2}$$

$$= \text{tr}(1) g_0 \int \frac{d^Dp_E}{(2\pi)^D} \frac{m_0 + \Sigma_0}{p_E^2 + (m_0 + \Sigma_0)^2}.$$  

(23.15)

The Dirac matrices have disappeared, except for the unit matrix inside the trace. This makes it possible to use this equation in any desired number dimensions. We only need to know the dimension of the Dirac matrices, which is $2^{D/2}$ for even $D$. In this form, Eq. (23.15) may be extrapolated analytically to any non-integer value of $D$ [10].

For a constant $\Sigma$, the effective action gives rise to an effective potential

$$\frac{1}{N} v(\Sigma) = -\frac{1}{NV} \Gamma[\Sigma] = -\frac{1}{2g_0} \Sigma^2 - \text{tr}(1) \frac{1}{2} \int \frac{d^Dp_E}{(2\pi)^D} \log \left[ p_E^2 + (m_0 + \Sigma)^2 \right],$$

(23.16)

where $V \equiv \int d^Dx$ is the total volume of the system. The last term is obtained from the $\text{Tr} \log$ in (23.10) by the following calculation

$$\int \frac{d^Dp}{(2\pi)^D} \log (\hat{p} - m_0 - \Sigma) = \frac{1}{2} \int \frac{d^Dp}{(2\pi)^D} \left[ \log (\hat{p} - m_0 - \Sigma) + \log (-\hat{p} - m_0 - \Sigma) \right]$$

$$= \frac{1}{2} \int \frac{d^Dp}{(2\pi)^D} \log \left[ p^2 + (m_0 + \Sigma)^2 \right]$$

$$= \frac{i}{2} \int \frac{d^Dp_E}{(2\pi)^D} \log \left[ p_E^2 + (m_0 + \Sigma)^2 \right].$$
The integral is performed with the help of formula (11.134), leading to
\[
\frac{1}{N} v(\Sigma) = \frac{1}{2g_0} \Sigma^2 - \frac{2D/2-1}{2} \mu^{D-2} \frac{1}{2} S_D \Gamma(D/2) \Gamma(1-D/2) \frac{2}{D} \left( \frac{m_0 + \Sigma}{\mu} \right)^D \mu^2. \quad (23.17)
\]

We have introduced, as usual, an arbitrary mass scale \( \mu \), which will enable us to study the theory in the limit \( m_0 = 0 \).

We now focus attention upon the spacetime dimensions in the neighborhood of \( D = 2 \), setting
\[
D = 2 + \epsilon, \quad \text{with} \quad \epsilon > 0. \quad (23.18)
\]

Then the potential (23.17) can be written as
\[
\frac{1}{N} v(\Sigma) = \mu^\epsilon \left[ \frac{\Sigma^2}{g_0 \mu^\epsilon} - b_\epsilon \left( \frac{m_0 + \Sigma}{\mu} \right)^{2+\epsilon} \right], \quad (23.19)
\]
where the constant \( b_\epsilon \) stands for
\[
b_\epsilon = \frac{2}{D} 2^{\epsilon/2} S_D \Gamma(D/2) \Gamma(1-D/2) = \frac{1}{D} 2^{\epsilon/2} \pi^{1-\epsilon/2} \Gamma(-\epsilon/2). \quad (23.20)
\]

For small \( \epsilon \), the constant \( b_\epsilon \) behaves like
\[
b_\epsilon \sim -\frac{1}{\epsilon \pi} \left[ 1 - \frac{\epsilon}{2} \log \left( 2\pi e^{-\gamma/2} \right) \right] + \mathcal{O}(\epsilon). \quad (23.21)
\]

Therefore, the bare parameters must somehow diverge if the theory is supposed to remain finite in the limit \( \epsilon \to 0 \).

For simplicity, consider first the case of a vanishing bare mass, i.e., \( m_0 = 0 \). Then a renormalized coupling constant can be defined via
\[
\frac{1}{g_0 \mu^\epsilon} - b_\epsilon = \frac{1}{g}. \quad (23.22)
\]

The limit \( \epsilon \to 0 \) can now be taken at a finite \( g \) and we obtain the renormalized potential
\[
\frac{1}{N} v(\Sigma) \to \frac{1}{2} \left[ \frac{\Sigma^2}{g} + \frac{\Sigma^2}{\pi} \log \left( \frac{\Sigma}{\mu} \right) \right] = \frac{\Sigma^2}{2\pi} \log \left( \frac{\Sigma}{\mu e^{-\pi/g}} \right). \quad (23.23)
\]

The minimum of this lies at \( \Sigma = \Sigma_0 \) which satisfies the equation
\[
\frac{\Sigma_0}{\pi} \left[ \log \left( \frac{\Sigma_0}{\mu e^{-\pi/g}} \right) + \frac{1}{2} \right] = \frac{\Sigma_0}{\pi} \left[ \log \left( \frac{\Sigma_0}{\mu e^{-\pi/g-1/2}} \right) \right] = 0. \quad (23.24)
\]
The nontrivial solution of this is

\[ \Sigma_0 = M \equiv \mu e^{-c}, \]  

(23.25)

where \( c \) is the constant \( c = \pi/g + 1/2 \).

Let us study the effect of a nonzero bare fermion mass \( m_0 \) by going through the same discussion once more starting from the potential (23.19). Setting \( m_0 + \Sigma \equiv \Sigma_1 \) the renormalized potential (23.23) becomes

\[ \frac{1}{N} v(\Sigma) \to \frac{1}{2} \left[ \frac{\Sigma^2}{g} + \frac{\Sigma_1^2}{\pi} \log \left( \frac{\Sigma_1}{\mu} \right) \right], \] 

(23.26)

which is minimized at the modified version of (23.24)

\[ -\frac{m_0}{g} + \frac{\Sigma_1}{\pi} \log \left( \frac{\Sigma_1}{\mu e^{-c}} \right) = \frac{\Sigma_1}{\pi} \log \left( \frac{\Sigma_1}{Me^{-m_0\pi/\Sigma_1 g}} \right) = 0. \] 

(23.27)

For small \( m_0 \), this is solved by

\[ \Sigma_1 = M \left( 1 - \frac{m_0 \pi}{M g} + \ldots \right). \] 

(23.28)

Thus a small bare mass lowers the physical mass by a small amount.

For arbitrary \( 0 < \epsilon < 2 \), we may define a renormalized potential \( v(\Sigma) \) as

\[ \frac{1}{N} v(\Sigma) = \frac{\mu^\epsilon}{2} \left\{ \frac{\Sigma^2}{g} + b_\epsilon \Sigma^2 \left[ 1 - \left( \frac{\Sigma}{\mu} \right)^\epsilon \right] \right\}, \] 

(23.29)

It reduces to the expression (23.23) for \( \epsilon \to 0 \).

Let us also see that the potential minimum at \( \Sigma = \Sigma_0 \) of Eq. (23.25) solves the gap equation (23.13) for fermions with a vanishing bare mass \( m_0 = 0 \). In the renormalized version, \( \Sigma_0 \) at minimum of (23.29) satisfies

\[ 1 = gb_\epsilon \left[ \left( 1 + \frac{\epsilon}{2} \right) \left( \frac{\Sigma_0}{\mu} \right)^\epsilon - 1 \right]. \] 

(23.30)

In the limit \( \epsilon \to 0 \), this becomes

\[ 1 = -\frac{1}{\pi g} \left( \frac{1}{2} + \log \frac{\Sigma_0}{\mu} \right), \] 

(23.31)

yielding once more the energy gap (23.25).

The question arises whether this non-trivial solution corresponds to the true ground state of the problem. For this, we differentiate \( v(\Sigma) \) once more, and find

\[ \frac{1}{N} v''(\Sigma) = \frac{1}{g_0} - b_\epsilon (D - 1) \frac{D}{2} \Sigma^\epsilon. \] 

(23.32)
Inserting (23.30), this becomes
\[
\frac{1}{N} v''(\Sigma) = -\frac{1}{g_0} (D - 2) = -e_b (1 + \epsilon/2) \Sigma_0',
\]
which is positive for \( D > 2 \) if
\[
g_0 < 0.
\] (23.34)

What does this condition mean for the renormalized coupling \( g \)? Using (23.22), we see that \( g_0 < 0 \) and \( g_0 > 0 \) amount to \( g > g^* \) and \( g < g^* \), respectively, with
\[
g^* \equiv -b^{-1}_r = \epsilon \pi.
\] (23.35)

This is a critical coupling constant above which the model has a phase with a non-vanishing field expectation \( \Sigma_0 \). If the renormalized coupling lies below \( g^* \), the bare coupling constant is positive and only the trivial solution \( \Sigma_0 = 0 \) has a positive \( v''(\Sigma_0) \) indicating stability.

What are the physical properties of the two solutions? Looking back at the effective action (23.110), we see that \( \Sigma_0 \) increased the fermion mass term to \( M = m_0 + \Sigma_0 \). In the present zero-\( m_0 \) case, we have
\[
M = \Sigma_0.
\] (23.36)

We therefore conclude that, for \( g < g^* \), the massless initial fermions remain massless. For \( g > g^* \), on the other hand, the massless fermions acquire a mass \( M = \Sigma_0 \) from the fluctuations. We observe a spontaneous generation of fermion mass. The result may also be phrased differently: In the weak-coupling phase with \( g < g^* \), the fermions keep their initial long-range correlations. In the strong-coupling phase with \( g > g^* \), however, the spontaneously generated mass limits the range of the correlation functions to \( 1/M \).

### 23.2 Spontaneous Symmetry Breakdown

The spontaneous mass generation is closely related to the fact that the model displays, for zero initial mass, the phenomenon of spontaneous symmetry breakdown (recall Chapters 18, and see the later Chapter 25). Indeed, for \( m_0 = 0 \), the Lagrangian (23.1) possesses an additional symmetry called \( \gamma_5 \)-invariance. In two dimensions, we may choose the following \( \gamma \)-matrices
\[
\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma^1, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma^2,
\] (23.37)

which satisfy
\[
\{ \gamma^\mu, \gamma^\nu \} = 2 g^{\mu\nu} = 2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{\mu\nu}.
\] (23.38)
The Hermitian $\gamma^5$-matrix is defined by analogy with the four-dimensional case in Eq. (4.544) as

$$\gamma^5 = \frac{1}{2!} \epsilon_{\mu\nu} \gamma^\mu \gamma^\nu = \gamma^0 \gamma^1,$$  \hspace{1cm} (23.39)

where $\epsilon_{\mu\nu}$ is the completely antisymmetric tensor, with $\epsilon_{01} = 1$. In the representation (23.37), $\gamma_5$ is equal to the Pauli matrix $\sigma_3$:

$$\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \hspace{1cm} (23.40)$$

We now introduce $\gamma_5$-transformations $T_{\gamma_5}$ as follows:

$$\psi \xrightarrow{T_{\gamma_5}} \gamma_5 \psi,$$ \hspace{1cm} (23.41)

which satisfies $T_{\gamma_5}^2 = 1$. Under $T_{\gamma_5}$, we have

$$\bar{\psi} \xrightarrow{T_{\gamma_5}} \psi \gamma_5 \gamma_0 = -\bar{\psi} \gamma_5.$$ \hspace{1cm} (23.42)

Hence:

$$\bar{\psi} \psi \xrightarrow{T_{\gamma_5}} -\bar{\psi} \psi,$$

$$\bar{\psi} \gamma^\mu \psi \xrightarrow{T_{\gamma_5}} \bar{\psi} \gamma^\mu \psi.$$ \hspace{1cm} (23.43)

If the bare mass $m_0$ in (23.1) is zero, the Lagrangian is invariant under $T_{\gamma_5}$.

In addition, the actions in the exponents of (23.6) and (23.7) are invariant if we assign to $\sigma \sim g \bar{\psi} \psi$ the transformation

$$\sigma \xrightarrow{T_{\gamma_5}} -\sigma,$$ \hspace{1cm} (23.44)

in accordance with (23.39) and (23.41).

Thus the $m_0 = 0$ collective action (23.8) is symmetric in $\sigma$. It is precisely this $\gamma_5$-symmetry which is broken by the non-vanishing expectation value $\langle \sigma \rangle = \Sigma^0$. We may compare this result with the Bose discussion in Section 18.4. There we found that the continuous $O(N)$-symmetry could not be broken in less or equal to two dimensions, due to fluctuations. This would imply that such a state would contain massless Nambu-Goldstone bosons in two dimensions, whose correlation functions diverge at any distance, which is unphysical. This is in contrast to a discrete symmetry that can be broken in two dimensions.

### 23.3 Dimensionally Transmuted Coupling Constant

It is worth pointing out an important structural property of the final result: The characteristic parameter of the two-dimensional system is the fermion mass which
is determined from the coupling strength and the arbitrary mass parameter $\mu$ via (23.25), (23.36) as

$$M = \mu e^{-(1/2 + \pi/g)}.$$  \hspace{1cm} (23.45)

The original theory with $m_0 = 0$ had only a single free parameter, namely the coupling strength $g_0$. The auxiliary mass parameter $\mu$ was only introduced for the purpose of renormalizing the theory in the massless case. The renormalized coupling $g$ depends on the choice of $\mu$, and (23.22) should more explicitly be written as

$$\frac{1}{g_0 \mu^\epsilon} - b_\epsilon = \frac{1}{g(\mu)}.$$  \hspace{1cm} (23.46)

In this way, a system with a single parameter $g_0$ has been recharacterized by two parameters, $\mu$ and $g(\mu)$.

This increase of parameters is certainly an artifact. There must be a relation between $\mu$ and $g(\mu)$ such that different pairs $(\mu, g(\mu))$ correspond to the same set of Green’s functions, i.e., to the same physical theory. Indeed, such a relation follows from (23.22). At a fixed $g_0$, we can plot curves in the $(\mu, g)$ plane which correspond to one and the same theory. In the limit $\epsilon \to 0$, this relation between $\mu$ and $g$ has a subtlety. It goes over into the mass relation (23.45).

The fermion mass is a physically observable finite quantity. There are infinitely many pairs of parameters $\mu$ and couplings $g(\mu)$, which lead to the same fermion mass, and this mass is the most economic single parameter by which all properties of the theory can be expressed. Let us illustrate this by reexpressing the potential (23.23) in terms of $M$. For this we write the renormalized gap equation in the form (with $\Sigma_0 = M$)

$$\frac{\pi}{g} + \frac{1}{2} + \log \frac{M}{\mu} = 0.$$  \hspace{1cm} (23.47)

Multiplying this by $\Sigma^2/2$, and subtracting the result from the potential (23.23), we find

$$\frac{1}{N} v(\Sigma) = \frac{\Sigma^2}{2\pi} \log \frac{\Sigma}{M} - \frac{1}{2}.$$  \hspace{1cm} (23.48)

This has indeed the desired property that neither $\mu$ nor $g(\mu)$ appear but only the single parameter $M$. The same property can, of course, be verified in $D > 2$ dimensions. Here we may combine (23.30) with (23.29) and find

$$\frac{1}{N} v(\Sigma) = \frac{\mu^\epsilon}{2} \left[ \frac{\Sigma^2}{g_0 \mu^\epsilon} - b_\epsilon \left( \frac{\Sigma}{\mu} \right)^{2+\epsilon} \mu^2 \right]$$

$$= \frac{M^\epsilon}{2} b_\epsilon \Sigma^2 \left[ 1 + \frac{\epsilon}{2} - \left( \frac{\Sigma}{M} \right)^\epsilon \right].$$  \hspace{1cm} (23.49)

For $\epsilon \to 0$, this reduces to (23.48).
23.4 Scattering Amplitude for Fermions

We calculate the scattering amplitude for fermions. As we know from the discussion in the Bose case, this is given entirely by the exchange of $\Sigma$-propagators. These can be extracted from the effective action (23.10) with $m_0 = 0$ by forming the second functional derivative at $\Sigma_0$ [compare the discussion leading to (18.137) and (18.138)].

The quadratic piece in $\Sigma' \equiv \Sigma - \Sigma_0 = \Sigma - M$

\[ \delta^2 \Gamma = -\frac{N}{2} \left[ \frac{1}{g_0} \int d^Dx \Sigma'^2 + i \text{Tr} \left( \frac{i}{i\partial - M} \Sigma' \frac{i}{i\partial - M} \Sigma' \right) \right]. \tag{23.50} \]

From this we extract the propagator of the $\sigma'$-fluctuations in momentum space

\[ G_{\sigma'\sigma'}(q) = -\frac{1}{N} \frac{i}{g_0^{-1} + \Pi(q)}, \tag{23.51} \]

where the self-energy of the $\sigma'$-field is given by the integral

\[ \Pi(q) = i \text{tr} \int \frac{d^Dk}{(2\pi)^D} \frac{i}{k - M + i\eta (k - q) - M + i\eta} \]

\[ = -i \text{tr} \int \frac{d^Dk}{(2\pi)^D} \frac{[(k + M)(k - q) + M]}{(k^2 - M^2) [(k - q)^2 - M^2]}, \tag{23.52} \]

with the momenta illustrated in the Feynman diagram shown in Fig. 23.1.

\[ \Gamma \]

\[ k \]

\[ k - q \]

\[ q \]

**Figure 23.1** One-loop Feynman diagram in the inverse propagator of the $\sigma'$-field.

Forming the trace and going to euclidean momentum space yields

\[ \Pi(q) = -2^{D/2} \int \frac{d^Dk_E}{(2\pi)^D} \frac{k(k - q)_E - M^2}{(k^2_E + M^2) [(k - q)_E^2 + M^2]}, \tag{23.53} \]

In the last line we have gone to the euclidean form. The denominator can be treated with the help of the Feynman formula (11.157), and we may write [compare (11.160)]

\[ \Pi(q) = -2^{D/2} \int \frac{d^Dk_E}{(2\pi)^D} \int_0^1 dx \frac{k(k - q)_E - M^2}{(k^2_E - 2k_EQEX + q^2_EX + M^2)^2}. \tag{23.54} \]

The integrand can be rearranged to

\[ \frac{(k - qx)_E^2 + (k - qx)E(qE(2x - 1) - q^2_EX(1 - x) - M^2}{[(k - qx)_E^2 + q^2_EX(1 - x) + M^2]^2}, \tag{23.55} \]
Upon integration over the shifted momentum \( k' \equiv k - qx \), the second term in the numerator will vanish since it is odd in \( k' \). We therefore remain with the integral

\[
\Pi(q) = -2^{D/2} \int_0^1 dx \int \frac{d^Dk_E}{(2\pi)^D} \times \left\{ \frac{1}{k_E^2 + q_E^2 x(1-x) + M^2} - 2 \frac{M^2 + q_E^2 x(1-x)}{[k_E^2 + q_E^2 x(1-x) + M^2]^2} \right\}.
\]

This can be integrated using formulas (11.123), leading to

\[
\Pi(q) = -2^{D/2} \frac{1}{2} \frac{\Gamma(D/2)\Gamma(1-D/2)}{\Gamma(1)} - 2 \frac{1}{2} \frac{\Gamma(D/2)\Gamma(2-D/2)}{\Gamma(2)}
\]

\[
\times \int_0^1 dx \left[ q_E^2 x(1-x) + M^2 \right]^{D/2-1}
\]

\[
= -2^{D/2-1} S_D(D-1)\Gamma(D/2)\Gamma(1-D/2) M^\epsilon \int_0^1 dx \left[ \frac{q_E^2}{M^2} x(1-x) + 1 \right]^{-\epsilon/2},
\]

where we have introduced the parameter \( b_\epsilon \) of (23.20). Inserting this into (23.50), we obtain the propagator in terms of the renormalized coupling \( g \):

\[
G_{\sigma',\sigma} = -i \frac{N^\epsilon}{N^\mu} \left\{ \frac{1}{g_0^\mu \epsilon} - \frac{D(D-1)}{2} \frac{b_\epsilon (M/\mu)^\epsilon}{\Gamma(1)} \int_0^1 dx \left[ \frac{q_E^2}{M^2} x(1-x) + 1 \right]^{\epsilon/2} \right\}^{-1}
\]

\[
= -i \frac{N^\epsilon}{N^\mu} \left\{ \frac{1}{g} + b_\epsilon \left[ 1 - \frac{D(D-1)}{2} \left( \frac{M}{\mu} \right)^\epsilon \int_0^1 dx \left[ \frac{q_E^2}{M^2} x(1-x) + 1 \right]^{\epsilon/2} \right] \right\}^{-1}.
\]

The expression in the last curly brackets behaves for small \( \epsilon \) like

\[
1 - \left( 1 + \frac{\epsilon}{2} \right) (1+\epsilon) \left\{ 1 + \frac{\epsilon}{2} \int_0^1 dx \log \left[ \frac{q_E^2}{M^2} x(1-x) + 1 \right] + \epsilon \log \frac{M}{\mu} \right\}
\]

\[
= -\frac{\epsilon}{2} \left[ \int_0^1 dx \log \left( \frac{q_E^2}{M^2} x(1-x) + 1 \right) \right] + 3 + 2 \log \frac{M}{\mu},
\]

so that the denominator in (23.51) becomes, in \( D = 2 \) dimensions,

\[
\left( \frac{1}{g} + \frac{1}{2\pi} \left\{ \int_0^1 dx \log \left[ \frac{q_E^2}{M^2} x(1-x) + 1 \right] + 3 + 2 \log \frac{M}{\mu} \right\} \right).
\]

This result can be expressed completely in terms of \( M \). For this purpose, we subtract again the gap equation (23.31) and find

\[
\frac{1}{2\pi} \int_0^1 dx \left\{ \log \left[ \frac{q_E^2}{M^2} x(1-x) + 1 \right] + 2 \right\}.
\]
Then the $\sigma'$-propagator takes the form

$$G_{\sigma'\sigma'}(q) = -\frac{i}{N} \frac{1}{q_0^{-1} + \Pi(q)} = -\frac{i}{N} \frac{2\pi}{\int_0^1 dx \log \left[ \frac{q_E^2}{M^2 x(1-x) + 1} \right] + 2}.$$  \hspace{1cm} (23.63)

The integral in the denominator has been solved before in Eqs. (22.49)-(11.175), where we found, with $z \equiv \frac{q_E^2}{M^2}$,

$$J(z) = \int_0^1 dx \log [zx(1-x) + 1] = -2 + 2\theta \coth \theta,$$  \hspace{1cm} (23.64)

where

$$\theta = \tanh \sqrt{\frac{z}{z+4}} = \tanh \sqrt{\frac{q_E^2}{q_E^2 + 4M^2}}, \quad \sinh \theta = \sqrt{\frac{z}{4}} = \sqrt{\frac{q_E^2}{4M^2}}.$$  \hspace{1cm} (23.65)

The function $J(z)$ is monotonously increasing in $q_E^2$, with the minimum lying at the origin, as shown in Fig. 23.2. The propagator itself starts out, in momentum space, with the value

$$G_{\sigma'\sigma'}(q)|_{q_E^2=0} = -\frac{\pi}{N},$$  \hspace{1cm} (23.66)

and has a monotonously decreasing size for growing euclidean momentum $q_E$.

Figure 23.2 Function $J(z) + 2$ in the denominator of the $\sigma'$-propagator (23.63). For $z < -4$, the curve shows the real part, the dashed curve the imaginary part.

We may now ask whether there exists a scalar ground state in the fermion antifermion scattering amplitude, which is usually called $\sigma$-particle, by analogy with a resonance of $\pi^+\pi^-$ in the proton-proton scattering amplitude which is seen at roughly 700 MeV, and which was the origin of using the name $\sigma$ for the collective field $\sigma \sim \bar{\psi}\psi$ from the beginning. This particle would have to manifest itself in a pole in the propagator $G_{\sigma'\sigma'}(q^2_E)$ at timelike $q^2$, i.e., at a negative value of $q^2_E = -s = -M^2_\sigma$. Indeed, the denominator (23.64) is seen to vanish for

$$s = 4M^2,$$  \hspace{1cm} (23.67)
as can be verified by continuing (23.64) to $0 < s < 4M^2$ using

$$\theta \coth \theta = \bar{\theta} \coth \bar{\theta}. \quad (23.68)$$

In terms of $s$, the parameter $\theta$ is given by

$$\theta = \arctan \sqrt{\frac{s}{s - 4M^2}}, \quad \sin \theta = \sqrt{\frac{s}{4M^2}}. \quad (23.69)$$

Close to $4M^2$ the propagator behaves like

$$G_{\sigma'\sigma} = -\frac{2i}{N} \sqrt{\frac{s}{s - 4M^2}}. \quad (23.70)$$

Thus we see that there is no proper particle pole at $s = 4M^2$, but only a branch cut which runs from $s = 4M^2$ to infinity. Such a cut is present in every scattering amplitude. It is commonly referred to as the elastic cut. This is an artifact of the leading large-$N$ approximation investigated in the present discussion. For finite $N$, the Green function does have a proper bound-state pole before the cut starts.

As in the effective potential, there is only the fermion mass $M$ which characterizes the theory, rather than the pair of parameters $\mu$ and bare coupling $g_0$, thereby avoiding the problem that the latter becomes undefined for $\epsilon \to 0$. The fermion mass $M$ is independent of the particular renormalization procedure. It may come as a surprise that a dimensionless quantity $g_0$ has been replaced by a quantity with the dimension of a mass. This process is often referred to as dimensional transmutation. It was first observed in the microscopic theory of superconductivity. There are many superconductors with different coupling strengths $g$ and mass parameters $\mu$, that characterized the energy scale of the pair interactions caused by phonon exchange (see Chapter 17). But there is only one quantity that governs the superconductivity properties, namely the critical temperature $T_c$. Theories with the same $T_c$ are identical superconductors independent on what $g$ or $\mu$ they were derived from.

Note that in the fundamental Lagrangian (23.1), the mass parameter $\mu$ and the associated $g(\mu)$ are not detectable separately by any physical experiment. Only their combination $M$ is. If a superconductor is investigated at microscopic scales, both quantities are properties of the substructure. They can both be measured. The supercurrent properties, however, are completely described by the knowledge of the Lagrangian of type (23.1) and the parameters $\mu$ and $g(\mu)$.

This points at an important physical aspect of the renormalization procedure: Every theory that requires renormalization of the coupling constant has a redundancy in its parameterization with a mass parameter $\mu$ and a renormalized coupling constant $g(\mu)$. This redundancy cannot be resolved at the level of the theory itself. But there may be a more microscopic theory in which both parameters $\mu$ and $g(\mu)$ acquire a separate physical significance. Until now, theoretical physics has gone precisely this way. Every theory which was initially considered to be microscopic turned later out to be a phenomenological description of even more microscopic substructures.
23.4 Scattering Amplitude for Fermions

The reader may wonder how this description applies to the phase in which the model is at small coupling constants \( g < g^* \) or \( g^0 > 0 \), where the fermions remain massless and the correlations have a long range. The potential may still be parametrized in the form (23.19), (23.29), with \( m_0 = 0 \):

\[
\frac{1}{N} v(\Sigma) = \frac{\mu^\epsilon}{2} \left[ \frac{\Sigma^2}{g_0 \mu^\epsilon} - b_c \left( \frac{\Sigma}{\mu} \right)^{2+\epsilon} \right] - \frac{\mu^2}{2} \left\{ \frac{\Sigma^2}{g} - b_c \Sigma^2 \left[ \left( \frac{\Sigma}{\mu} \right)^\epsilon \mu^2 - 1 \right] \right\}.
\]  

(23.71)

It contains the arbitrary mass parameter \( \mu \), and the renormalized coupling \( g \) depending on the choice of \( \mu \). There is no fermion mass in terms of which the result can be expressed in a renormalization-independent fashion. Nevertheless, it is still possible to substitute the pair of parameters \( \mu, g(\mu) \) by a single parameter whose dimension is “mass”. For this we simply define \( M \) by the equation

\[
1 \equiv -g_0 \mu^\epsilon b_c D^2 \left( \frac{M}{\mu} \right)^\epsilon.
\]  

(23.72)

Then \( v(\Sigma) \) can be rewritten as

\[
\frac{1}{N} v(\Sigma) = -\frac{M^\epsilon}{b_c \Sigma^2} \left[ \frac{D^2}{2} + \left( \frac{\Sigma}{M} \right)^\epsilon \right].
\]  

(23.73)

Note that this potential exists only if \( \epsilon \) is truly larger than zero. For \( \epsilon \to 0 \), there is no finite limit. This is due to the fact that only for negative \( g_0 \), \( 1/g_0 \mu^\epsilon - b_c \) can be compensated to become a finite quantity \( 1/g \) in the limit \( \epsilon \to 0 \).

If we calculate (23.61) for a vanishing fermion mass, we obtain

\[
G_{\sigma'\sigma}(q) = -\frac{i}{N \mu} \left\{ \frac{1}{g_0 \mu^\epsilon} - \frac{D(D-1)}{2} b_c \left( \frac{q_E^2}{\mu^2} \right)^{\epsilon/2} \int_0^1 dx \ [x(1-x)]^{\epsilon/2} \right\}^{-1}
\]  

\[
= -\frac{i}{N \mu^\epsilon} \left\{ \frac{1}{g_0 \mu^\epsilon} - \frac{D(D-1)}{2} b_c \frac{\Gamma^2(1+\epsilon/2)}{\Gamma(2+\epsilon)} \left( \frac{q_E^2}{\mu^2} \right)^{\epsilon/2} \right\}^{-1}.
\]  

(23.74)

This may be expressed in terms of the auxiliary mass parameter (23.72) as

\[
G_{\sigma'\sigma}(q) = \frac{i}{N} \left( \frac{D}{2} b_c M^\epsilon \right)^{-1} \left[ 1 + \frac{\Gamma^2(1+\epsilon/2)}{\Gamma(1+\epsilon)} \left( \frac{q_E^2}{M^2} \right)^{\epsilon/2} \right]^{-1}.
\]  

(23.75)

The following should be pointed out here. If we had calculated the propagator \( G_{\sigma'\sigma}(q) \) by expanding the action around the wrong ground state solution, say \( \Sigma_0 = 0 \) for \( g_0 < 0 \), \( g > g^* \), the resulting propagator would show the mistake of doing this.
The above Eq. (23.74) is an example for this. It is singular in euclidean momentum space by having an unphysical tachyon pole \[q^2_E = \left[ \frac{1}{g_0 b_0} \frac{\Gamma(2+\epsilon)}{\Gamma^2(1+\epsilon/2) D(D-1)} \right]^{2/\epsilon}. \tag{23.76}\]

This may also be expressed in terms of renormalized quantities as

\[
\frac{q^2_E}{\mu^2} = \left[ \left( 1 + \frac{1}{g_0 b_0} \right) \frac{\Gamma(2+\epsilon)}{\Gamma(1+\epsilon/2) D(D-1)} \right]^{2/\epsilon}. \tag{23.77}\]

When going to Minkowski space, the propagator (23.75) has a particle pole at

\[q^2 = -q^2_E.\]

The situation is similar to that in the scalar \(\phi^4\)-theory in four dimensions. Also there we found such a particle with an imaginary mass, which in Minkowski spacetime travels faster than light and is therefore unphysical. There it appeared for very large \(q^2\), here for very small \(q^2\). Since a tachyon can have states with arbitrary negative energy, there must be another ground state for the theory which lies lower than the zero-field configuration. Indeed, the potential \(v(\Sigma)\) in Eq. (23.73) shows that a phase, in which there might possibly exist a tachyon, cannot really exist in nature. If it did, the energy could be lowered by going into another phase where the energy is lower. An unstable house will collapse until the ruins are stable. See the similar situation in \(\phi^4\)-theory discussed on p. 1134.

It can be argued that, for finite \(N\), positive couplings \(g_0\) correspond to another interesting physical phase for which the collective field \(\sigma \sim (g_0/N)\bar{\psi}_a \psi_a\) is no longer appropriate. Instead, a collective field that is proportional to \(\psi_a \psi_a\) would lead to a more appropriate description. This will become clearer after the next section.

There is one more observation that should be made for the massive phase. One may express the potential (23.29) also in terms of \(g^*\), and find the form

\[
\frac{1}{N} v(\Sigma) = \frac{\mu^2}{2} \frac{1}{g} \left[ \left( 1 - \frac{g}{g^*} \right) \Sigma^2 + \frac{g}{g^*} \left( \frac{\Sigma}{\mu} \right)^\epsilon \right] \epsilon^2. \tag{23.78}\]

This form exhibits very nicely the unstable origin for \(g > g^*\) and the stabilization due to the term \(\Sigma^{2+\epsilon}\). The potential looks very similar to the previously discussed \(\phi^4\)-theory. In fact, for \(\epsilon \to 2\) \((D \to 4)\) it takes exactly this form. The minimum lies at \(\Sigma = \Sigma_0 = M\), where \(M\) is the fermion mass

\[
\frac{M}{\mu} = \left[ \frac{2}{D} \frac{g^*}{g} \left( \frac{g}{g^*} - 1 \right) \right]^{1/\epsilon}, \tag{23.79}\]

in terms of which the potential may be written in a natural parametrization

\[
\frac{1}{N} v(\Sigma) = \frac{\mu^2}{2} M^2 \frac{1}{g} \left( \frac{g}{g^*} - 1 \right) \left[ - \left( \frac{\Sigma}{M} \right)^2 + \frac{2}{D} \left( \frac{\Sigma}{M} \right)^D \right]. \tag{23.80}\]
The mass scale $\nu$ can be replaced by the physical mass at infinite $g$:

$$M_\infty = \left( \frac{2}{D} \right)^{1/\epsilon}. \quad (23.81)$$

Then (23.79) reads

$$\frac{M}{M_\infty} = \left( 1 - \frac{g^*}{g} - 1 \right)^{1/\epsilon}. \quad (23.82)$$

We argued before that $g, g^*$ play the role of temperature $T$ and critical temperature $T_c$ in surface layers. There the mass goes with

$$\frac{M}{\mu} \propto \left( \frac{T}{T_c} - 1 \right)^{1/\epsilon} \approx \left( \frac{g}{g^*} - 1 \right)^{1/\epsilon}. \quad (23.83)$$

It vanishes at the critical point in which case $v$ takes on a pure power behavior

$$\frac{1}{N} v(\Sigma) \Sigma \to 0 \sim \Sigma^D \left( \frac{\Sigma}{\mu} \right)^D. \quad (23.84)$$

This power can also be seen at arbitrary $T$, if $\Sigma$ is increased to be much larger than the mass scale $M$ [ultraviolet (UV) limit of the theory].

Note that in the opposite limit of small $\Sigma$ [infrared (IR) limit], the power behavior is

$$\frac{1}{N} v(\Sigma) \Sigma \to 0 \sim \Sigma^2, \quad (23.85)$$

which corresponds to $g \to 0$, i.e., the free-field limit of the theory. One says that the theory behaves IR-free. Such UV- and IR-power behaviors are typical at a critical point. They have been the subject of extended experimental and theoretical investigation.

### 23.5 Nonzero Bare Fermion Mass

Before we come to that, let us briefly indicate what happens to the Gross-Neveu model if there is a fermion mass from the beginning, a possibility that we discarded so far for the sake of simplicity. We may assume $m_0$ to be positive, since otherwise its sign can be changed by a simple $\gamma_5$ transformation under which $m_0 \bar{\psi} \psi \to -m_0 \bar{\psi} \psi$.

Differentiating (23.19), we obtain the gap equation

$$\frac{\Sigma_0}{g_0 \mu^\epsilon} = \left( 1 + \frac{\epsilon}{2} \right) b_\epsilon \left( \frac{m_0 + \Sigma_0}{\mu} \right)^{1+\epsilon} \mu \quad (23.86)$$
which has a solution \( \Sigma_0 > 0 \) for \( g_0 < 0 \), or a solution \(-m_0 < \Sigma_0 < 0\) for \( g_0 > 0 \). In other words, for a repulsive interaction the mass becomes larger and for an attractive interaction it becomes smaller. The second derivative is

\[
\frac{1}{N} v''(\Sigma) = \mu^e \left[ \frac{1}{g_0 \mu^e} - \frac{D}{2} (D - 1) \left( \frac{m_0 + \Sigma}{\mu} \right)^{D-2} \right],
\]

and at the solution \( \Sigma_0 \) of the gap equation (23.86):

\[
\frac{1}{N} v''(\Sigma_0) = \mu^e \frac{1}{g_0 \mu^e} \left[ 1 - \left(1 + \epsilon\right) \frac{\Sigma_0}{m_0 + \Sigma} \right]
= - \frac{1}{g_0} \frac{\epsilon (\Sigma + m_0) - (1 + \epsilon)m_0}{m_0 + \Sigma}.
\]

From this we deduce the stability regions for negative and positive \( g_0 \).

Let us renormalize the effective potential (23.19). Introducing the sum \( \Sigma_{\text{tot}} \equiv m_0 + \Sigma \), whose equilibrium value \( \Sigma_{\text{tot}}^0 = m_0 + \Sigma_0 \) is the total finite fermion mass \( M \), we obtain

\[
\frac{1}{N} v(\Sigma) = \frac{\mu^e}{2} \left[ \frac{\Sigma_{\text{tot}}^2 - 2\Sigma_{\text{tot}}^0 m_0 + m_0^2}{g_0 \mu^e} - b_e \left( \frac{\Sigma_{\text{tot}}}{\mu} \right)^{2+\epsilon} \right].
\]

Here we renormalize the coupling again by using Eq. (23.22). The term \( 2\Sigma_{\text{tot}}^0 m_0 / g_0 \mu^e \) is made finite by defining the renormalized mass as

\[
\frac{m_0}{g_0 \mu^e} = \frac{m}{g},
\]

i.e.,

\[
\frac{m_0}{m} = \frac{g_0 \mu^e}{g} = 1 - g_0 \mu^e b_e = (1 + gb_e)^{-1}.
\]

The term \( m_0^2 / g_0 \mu^e \) in (23.89) is not finite for \( \epsilon \to 0 \). It needs a trivial additive renormalization of the vacuum energy. Assuming that this has been supplied, we find the renormalized potential [compare (23.29)]

\[
\frac{1}{N} v(\Sigma) = \frac{\mu^e}{2} \left\{ \frac{(\Sigma_{\text{tot}} - m)^2}{g} - b_e \Sigma_{\text{tot}} \left( \frac{\Sigma_{\text{tot}}}{\mu} \right)^{\epsilon} \right\},
\]

in which \( g \) and \( m \) depend on \( \mu \). The renormalized gap equation becomes

\[
1 = gb_e \left[ \left( \frac{1 + \epsilon}{2} \right) \left( \frac{M}{\mu} \right)^{\epsilon} - 1 \right] \frac{M}{M - m},
\]

where \( \Sigma_{\text{tot}}^0 \) has been replaced by the fermion mass \( M \). At \( \Sigma_{\text{tot}}^0 = M \), the effective potential has the value

\[
\frac{1}{N} v(\Sigma) = \frac{\mu^e}{2} b_e \left[ \frac{m(m - \Sigma_{\text{tot}})}{g} + b_e \Sigma_{\text{tot}} \left( \frac{\epsilon}{2 + \epsilon} \right) \right].
\]
Note that the potentials (23.29) and (23.92) for the massless and the massive models differ by a term
\[ \Delta v(\Sigma) = \frac{\mu^2}{2} \frac{(\Sigma^{\text{tot}} - m)^2}{g}. \] (23.95)

### 23.6 Pairing Model and Dynamically Generated Goldstone Bosons

The model discussed in the last section is somewhat uninteresting, since the symmetry which is broken is discrete. It is instructive to consider a slightly modified situation in which there is a spontaneous breakdown of a continuous symmetry. From the Nambu-Goldstone theorem we then expect the appearance of a massless particle. For this purpose consider once more a theory of \( N \) fields \( \psi_a \) in \( D = 2 + \epsilon \) dimensions, but now we take the Lagrangian to be
\[ L = \bar{\psi}_a (i\partial - m_0) \psi_a + \frac{g_0}{2N} \left( \bar{\psi}_a C \bar{\psi}^T_a \right) \left( \psi^T_b C \psi_b \right). \] (23.96)

Here \( C \) is the matrix of charge conjugation which is defined by [recall (4.602)]
\[ C\gamma^\mu C^{-1} = -\gamma^{\mu T}. \] (23.97)

In two dimensions, where the \( \gamma \)-matrices have the explicit form (23.37), we may use \( C = \gamma^1 \):
\[ C = \gamma^1 = -i\sigma^2. \] (23.98)

It is the same matrix which was introduced in the four-dimensional discussion in Eq. (26.69) as the \( 2 \times 2 \)-submatrix \( c \) of the \( 4 \times 4 \) charge conjugation matrix \( C \).

Due to the antisymmetry of \( C \), we have
\[ \left( \bar{\psi}_a C \bar{\psi}^T_a \right)^\dagger = \bar{\psi}^T_a C \psi_a. \] (23.99)

As a consequence, the interaction potential in (23.96) is negative for \( g_0 < 0 \), amounting to an attractive potential.

Now we introduce a collective field by adding to \( L \) the term
\[ \frac{N}{2g_0} \left| \Delta - \frac{g_0}{N} \bar{\psi}^T_a C \psi_a \right|^2, \]
leading to the partition function
\[ Z[\eta, \bar{\eta}] = \prod_{a=1}^N \int D\psi_a D\bar{\psi}_a D\Delta DD\Delta^\dagger \exp \left\{ i \int d^Dx \left[ \bar{\psi}_a (i\partial - m_0) \psi_a + \frac{1}{2} \left( \Delta^\dagger \bar{\psi}^T_a C \psi_a + \text{c.c.} \right) + \bar{\psi}_a \eta_a + \bar{\eta}_a \psi_a - \frac{N}{2g_0} |\Delta|^2 \right] \right\}. \] (23.100)
In order to integrate out the Fermi fields, we rewrite the free part of the Lagrangian in the matrix form

\[
\frac{1}{2} \left( \psi^T C, \bar{\psi} \right) \left( \begin{array}{cc} 0 & i\hat{\theta} - m_0 \\ i\hat{\theta} - m_0 & 0 \end{array} \right) \left( \begin{array}{c} \psi \\ C\bar{\psi}^T \end{array} \right),
\]

which is the same as \( \bar{\psi} (i\hat{\theta} - m_0) \psi \), since

\[
\psi^T C C \bar{\psi}^T = -\psi^T \psi^T = \bar{\psi} \psi,
\]
\[
\psi^T C \hat{\theta} C \bar{\psi}^T = \psi^T \hat{\theta} \psi^T = \bar{\psi} \hat{\theta} \psi.
\] (23.101)

But then the interaction with \( \Delta \) can be combined with (23.84) in the form

\[
\frac{1}{2} \phi^T iG^{-1}_\Delta \phi,
\]

where

\[
\phi \equiv \left( \begin{array}{c} \psi \\ C\bar{\psi}^T \end{array} \right), \quad \phi^T \equiv \left( \begin{array}{c} \psi^T, \bar{\psi}C^{-1} \end{array} \right)
\] (23.102)

denotes the doubled fermion fields. The matrix \( G^{-1}_\Delta \) is the inverse propagator of the doubled fermion fields:

\[
iG^{-1}_\Delta = \left( \begin{array}{cc} C & 0 \\ 0 & C \end{array} \right) \left( \begin{array}{cc} \Delta^\dagger & i\hat{\theta} - m_0 \\ i\hat{\theta} - m_0 & \Delta \end{array} \right)
\]

in the presence of the external field \( \Delta \). Observe that \( \phi \) is a quasi-real field, since \( \phi^* \) is related to \( \phi \) by the similarity transformation

\[
\phi^* = \left( \begin{array}{c} \psi^* \\ C\bar{\psi}^{*T} \end{array} \right) = \left( \begin{array}{cc} 0 & C\gamma^0 \\ C\gamma^0 & 0 \end{array} \right) \left( \begin{array}{c} \psi \\ C\bar{\psi}^T \end{array} \right) = \left( \begin{array}{cc} 0 & C\gamma^0 \\ C\gamma^0 & 0 \end{array} \right) \phi.
\] (23.103)

For a quasi-real field, \( G^{-1}_\Delta \) must be an antisymmetric matrix in the combined spinor- and functional space, as can easily be verified:

\[
\left( \begin{array}{cc} C\Delta^\dagger & C(\hat{\theta} - m_0) \\ C(\hat{\theta} - m_0) & C\Delta \end{array} \right)^T = \left( \begin{array}{cc} C\Delta^\dagger & (\hat{\theta}^T - m_0)C^T \\ (\hat{\theta}^T - m_0)C^T & C\Delta \end{array} \right) = -\frac{1}{2} \left( \begin{array}{cc} C\Delta^\dagger & C(\hat{\theta} - m_0) \\ C(\hat{\theta} - m_0) & C\Delta \end{array} \right),
\]

using the identity \( C\hat{\theta}^T C^{-1} = \gamma^\mu \hat{\theta}_\mu^T = -\hat{\theta} \). The transposition applies to the combined space, and this is the origin of the negative sign in the relation \( C\gamma^\mu T C^{-1} = -\gamma^\mu \) of Eq. (23.97). In spinor space, partial integration makes the derivative \( \hat{\theta} \) equivalent to the manifestly antisymmetric functional matrix \( \frac{1}{2}(\hat{\theta} - \hat{\theta}) \). In momentum space, the kinetic part of \( iG^{-1}_\Delta(p', p) = \delta^{(2)}(p' + p) iG^{-1}_\Delta(p) \) is antidiagonal. This ensures
that \( G^T(p, p') = -G_\Delta(p, p') \), since \( \delta^{(D)}(p' + p)p \) is an antisymmetric functional matrix. The minus sign is necessary to have a nonzero kinetic part in the fermionic Lagrangian, which reads in terms of the quasi-real field \( \phi(p) \):

\[
\int d^Dp' d^Dp \, \phi(p') \, iG_\Delta^{-1}(p', p)\phi(p) = \int d^Dp \, \phi(-p) \, iG_\Delta^{-1}(p)\phi(p).
\]

(23.108)

We can now perform the functional integral over the fermion fields, according to the rule (14.97), leading to

\[
Z[j] = \int D\Delta D\Delta^\dagger e^{iN\mathcal{A}[\Delta] + \frac{i}{2} j^T G\Delta j},
\]

(23.109)

where \( \mathcal{A}[\Delta] \) is the collective action

\[
\mathcal{A}[\Delta] = -\frac{1}{2g_0} \int d^Dx |\Delta|^2 - \frac{i}{2} \text{Tr} \log iG_\Delta^{-1},
\]

(23.110)

and \( j_a \) is the doubled version of the external source, by analogy with (23.89),

\[
j_a = \left( \frac{\bar{\eta}_a^T}{C^{-1} \eta_a} \right).
\]

(23.111)

This is chosen to ensure that

\[
\bar{\psi}\eta + \bar{\eta}\psi = \frac{1}{2} \left( j^T \phi - \phi^T j \right),
\]

(23.112)

where we have omitted the subscript \( a \), for brevity. A quadratic completion gives

\[
\frac{1}{2} \phi^T iG_\Delta^{-1} \phi + \frac{1}{2} (j^T \phi - \phi^T j) = \frac{1}{2} (\phi^T + j^T iG_\Delta^T) iG_\Delta^{-1} (\phi + iG_\Delta j) - \frac{i}{2} j^T G_\Delta j.
\]

(23.113)

Note the sign change in front of \( \frac{1}{2} j^T G_\Delta j \) in Eq. (23.109), with respect to the Bose case, in accordance with the negative relative sign of the source term \( \frac{1}{2} (j^T \phi - \phi^T j) \).

In the limit \( N \to \infty \) we obtain, from (23.110), the effective action

\[
\frac{1}{N} \Gamma[\Delta, \Psi] = \frac{1}{2g_0} |\Delta|^2 - \frac{i}{2} \text{Tr} \log iG_\Delta^{-1} + \frac{1}{2N} \Phi_a^T iG_\Delta^{-1} \Phi_a,
\]

(23.114)

in the same way as in the last chapter for the simpler model with a real \( \sigma \)-field.

The ground state has \( \Phi = 0 \), whereas \( \Delta = \Delta_0 \) satisfies the gap equation

\[
\frac{\Delta_0}{g_0} = -\frac{1}{2} \text{tr} G \Delta_0 \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix},
\]

(23.115)

where we may assume \( \Delta_0 \) to be real, as we shall show later.

As before, we shall consider first the case of zero initial mass \( m_0 \). Then the Green’s function is inverted as follows

\[
G_{\Delta_0}(x, y) = \int \frac{d^Dp}{(2\pi)^D} e^{-ip(x-y)} \frac{i}{p^2 - \Delta_0} \begin{pmatrix} -\Delta_0 & \phi \\ \phi & -\Delta_0 \end{pmatrix} \begin{pmatrix} C^{-1} & 0 \\ 0 & C^{-1} \end{pmatrix}.
\]

(23.116)
We can verify this by multiplying it with (23.105). Thus the gap equation (23.115) becomes simply

\[
\frac{1}{g_0} = 2^{D/2} \int \frac{d^Dp}{(2\pi)^D} \frac{1}{p^2 + M^2},
\]  

(23.117)

where we have introduced the notation

\[M \equiv \Delta_0\]

(23.118)

to indicate the significance of \(\Delta_0\) as a spontaneously generated fermion mass. We have also taken the trace in Dirac space to be \(2^{D/2}\) in \(D\) dimensions.

The integral in (23.116) can be performed just as before and we find

\[
\frac{1}{g_0 \mu^\epsilon} = -b_\epsilon \frac{D}{\epsilon} \left( \frac{M}{\mu} \right)^\epsilon.
\]  

(23.119)

For \(\Phi_a = 0\), the effective potential is now

\[
\frac{1}{N} v(\Delta) = \frac{1}{2g_0} |\Delta|^2 + i \int \frac{d^Dp}{(2\pi)^D} \log \left( \frac{\Delta^\dagger}{\phi} \frac{\phi}{\Delta} \right),
\]  

(23.120)

such that we obtain, after a Wick rotation,

\[
\frac{1}{N} v(\Delta) = \frac{1}{2g_0} |\Delta|^2 - \frac{1}{2} 2^{D/2} \int \frac{d^Dp_E}{(2\pi)^D} \log \left( p_E^2 + |\Delta|^2 \right).
\]

Performing the integral gives

\[
\frac{1}{N} v(\Delta) = \frac{1}{2g_0} |\Delta|^2 - \frac{1}{2} 2^{D/2-1} S_D \Gamma(D/2) \Gamma \left( 1 - \frac{D}{2} \right) \frac{2}{D} |\Delta|^D \mu^\epsilon
\]

\[
= \frac{\mu^\epsilon}{2} \left[ \frac{|\Delta|^2}{g_0 \mu^\epsilon} - b_\epsilon \left( \frac{|\Delta|}{\mu} \right)^{2+\epsilon} \mu^2 \right],
\]

(23.121)

from which the gap equation (23.119) can again be recovered by differentiation. Stability is insured for \(g_0 < 0\), i.e., for attractive interactions. For \(\Delta_0 = 0\), only the trivial solution \(\Delta_0 = 0\) is stable.

For \(\Delta_0 \neq 0\), we may use (23.119) and express the potential in terms of \(M\) rather than the bare coupling constant \(g_0\) as

\[
\frac{1}{N} v(\Delta) = \frac{M^\epsilon}{2} \frac{b_\epsilon}{g_0} |\Delta|^2 \left[ \frac{D}{2} - \left( \frac{|\Delta|}{M} \right)^\epsilon \right].
\]  

(23.122)

As before in (23.22), \(v(\Delta)\) can be expressed in terms of the renormalized coupling:

\[
\frac{1}{g_0 \mu^\epsilon} - b_\epsilon = \frac{1}{g},
\]

(23.123)
which brings it to the alternative form

$$\frac{1}{N} v(\Delta) = \frac{\mu \epsilon}{\mu} \left\{ \frac{|\Delta|^2}{g} + b \left[ 1 - \left( \frac{|\Delta|}{\mu} \right)^{D-2} \right] |\Delta|^2 \right\}. \quad (23.124)$$

From either expression, we find in the limit of $\epsilon \to 0$:

$$\frac{1}{N} v(\Delta) = \frac{1}{2\pi} \left( \log \frac{|\Delta|}{\mu} - \frac{1}{2} \right) |\Delta|^2$$

$$= \frac{1}{2} \left[ \frac{|\Delta|^2}{g} + \frac{1}{\pi} |\Delta|^2 \log \frac{|\Delta|}{\mu} \right], \quad (23.125)$$

by analogy with (23.23) and (23.48).

Let us now study the propagator of the complete $\Delta$-field. For small deviations $\Delta' \equiv \Delta - \Delta_0$ from the ground state value of $\Delta$ we find, from (23.114), the quadratic term

$$\frac{1}{N} \delta^2 \Gamma = -\frac{1}{2} \left\{ \int d^D x \frac{|\Delta|^2}{g_0} + i \frac{1}{2} \text{Tr} \left[ \begin{array}{cc} \Delta' & \Delta' \end{array} \right] G_M \begin{pmatrix} \Delta' & \Delta' \end{pmatrix} G_M \right\}. \quad (23.126)$$

In momentum space, the trace term may be written more explicitly as

$$\frac{i}{2} \left\{ M^2 [\Delta'(q)\Delta'(-q) + \Delta'^*(q)\Delta'^*(-q)] 2^{D/2} \int \frac{d^D k}{(2\pi)^D k^2 - M^2 (k - q)^2 - M^2} + [\Delta'(q)\Delta'^*(-q) + \Delta'(-q)\Delta'^*(q)] \int \frac{d^D k}{(2\pi)^D k^2 - M^2 (k - q)^2 - M^2} \text{tr}[ \Delta' \Delta' \Delta' \Delta' ] \right\}. \quad (23.127)$$

In a Wick-rotated form, this becomes

$$\frac{1}{2} \left\{ M^2 [\Delta'(q)\Delta'(-q) + \Delta'^*(q)\Delta'^*(-q)] \tilde{\Pi}(q_E^2/M^2) + [\Delta'(q)\Delta'^*(-q) + \Delta'(-q)\Delta'^*(q)] \left[ \Pi(q_E^2/M^2) - M^2 \tilde{\Pi}(q_E^2/M^2) \right] \right\}, \quad (23.127)$$

where $\Pi(q_E^2/M^2)$ is the previous self-energy (23.52) calculated in (23.58). The slightly simpler quantity $\Pi(q_E^2/M^2)$ stands for

$$\tilde{\Pi} \left( q_E^2/M^2 \right) = i 2^{D/2} \int \frac{d^D k}{(2\pi)^D k^2 - M^2 (k - q)^2 - M^2}.$$

It is calculated as follows:

$$\tilde{\Pi} \left( q_E^2/M^2 \right) = 2^{D/2} \int \frac{d^D k}{(2\pi)^D} \int_0^1 dx \frac{1}{k_E^2 + q_E^2 x (1 - x) + M^2}$$

$$= 2^{D/2} S_D \Gamma(D/2) \Gamma(2 - D/2) \int_0^1 dx \left[ q_E^2 x (1 - x) + M^2 \right]^{D/2 - 2}$$

$$= -\frac{D}{2} b \left( 1 - D/2 \right) \int_0^1 dx \left[ q_E^2 x (1 - x) + M^2 \right]^{D/2 - 2}.$$
As a result, the action for the quadratic deviations from $\Delta_0$ can be written as

$$\frac{1}{N} \delta^2 \Gamma = -\frac{V}{4} \int \frac{d^D q}{(2\pi)^D} \left\{ \left( \frac{1}{g_0} + A \right) \left[ \Delta'(q) \Delta'^*(-q) + \Delta'(-q) \Delta'^*(q) \right] + B \left[ \Delta'(q) \Delta'(-q) + \Delta'^*(q) \Delta'^*(-q) \right] \right\},$$

with the coefficients

$$A = \Pi(q_E^2/M^2) - \tilde{\Pi}(q_E^2/M^2),$$

$$B = \tilde{\Pi}(q_E^2/M^2),$$

and the integrals

$$J_1^r(z) = \int_0^1 dx \left[ zx(1-x) + 1 \right]^{D/2-1}, \quad J_2^r(z) = \int_0^1 dx \left[ zx(1-x) + 1 \right]^{D/2-2}.$$ 

Thus the propagators of real and imaginary parts of the field $\Delta'$ are

$$G_{\Delta'^r \Delta'^r} = -\frac{i}{N} \frac{1}{g_0^{-1} + A + B},$$

$$G_{\Delta'^i \Delta'^i} = -\frac{i}{N} \frac{1}{g_0^{-1} + A - B},$$

and for the complex fields $\Delta'$, $\Delta'^\dagger$:

$$G_{\Delta'^\dagger \Delta'^\dagger} = -2i \frac{1}{N} \frac{1}{(g_0^{-1} + A)^2 - B^2} (-B),$$

$$G_{\Delta'^\dagger \Delta'^\dagger} = -2i \frac{1}{N} \frac{1}{(g_0^{-1} + A)^2 - B^2} (g_0^{-1} + A).$$

The expressions (23.132)–(23.135) can be made finite by using the gap equation (23.119). The term involving $1/g_0$,

$$\frac{1}{g_0} + A = \frac{D}{2} b_e M^e \left\{ \left[ 1 - (D-1)J_1^r(q_E^2/M^2) \right] - (1 - D/2) J_2^r(q_E^2/M^2) \right\},$$

depends only on the parameter $M$. It remains finite for $\epsilon \to 0$, where it becomes

$$\frac{1}{g_0} + A \to \frac{1}{2\pi} \left\{ J(q_E^2/M^2) + 2 - J_2^0(q_E^2/M^2) \right\},$$
with \( J(z) = 2dJ_2(z)/d\epsilon \big|_{\epsilon=0} \) being the function (23.64). The second function, \( B \), in (23.134) and (23.135) needs no renormalization. It has the \( \epsilon \to 0 \) -limit

\[
B \to \frac{1}{2\pi} J_2^0 (q_E^2/M^2).
\]

(23.138)

We now observe that there is a zero mass excitation in the imaginary part of the \( \Delta' \)-field, that describes the component of \( \Delta \) pointing orthogonally to the real ground state value \( \Delta_0 = M \) in complex field space. To show this we consider the denominator of the propagator (23.133):

\[
\frac{1}{g_0} + A - B = \frac{D}{2} b_\epsilon M^\epsilon \left\{ \left[ 1 - (D - 1) J_1^\epsilon (q_E^2/M^2) \right] - (2 - D) J_2^\epsilon (q_E^2/M^2) \right\}.
\]

(23.139)

By expanding in powers of \( z \equiv q_E^2/M^2 \),

\[
J_1^\epsilon \sim 1 + \frac{D-2}{12} z + \mathcal{O}(z^2),
\]

\[
J_2^\epsilon \sim 1 + \frac{D-4}{12} z + \mathcal{O}(z^2),
\]

(23.140)

we find, for small momenta,

\[
\frac{1}{g_0} + A - B = \frac{D}{2} b_\epsilon M^\epsilon \left\{ \left[ 1 - (D - 1) \left( 1 + \frac{D-2}{12} z \right) \right] - (2 - D) \left( 1 + \frac{D-4}{12} z \right) \right\} + \mathcal{O}(z^2)
\]

\[
= -\frac{D}{2} b_\epsilon M^\epsilon \frac{D-2}{4} z + \mathcal{O}(z^2),
\]

(23.141)

such that the propagator of (23.133) becomes, expressed in terms of the Minkowski square momentum \( q^2 = -q_E^2 \),

\[
G_{\Delta_{im} \Delta'_{im}} = -\frac{1}{N} \frac{2}{D(D-2)b_\epsilon} 4M^{2-\epsilon} \frac{i}{q^2} + \text{regular part at } q^2 = 0.
\]

(23.142)

Since \( b_\epsilon < 0 \) for \( D > 2 \), the residue is positive

\[
\text{Res } G_{\Delta_{im} \Delta'_{im}} \approx -\frac{1}{N} \frac{2}{D(D-2)b_\epsilon} 4M^{2-\epsilon} \to \frac{4\pi}{N} M^2,
\]

(23.143)

such that the propagator exhibits a proper particle pole at \( q^2 = 0 \). The positive sign is necessary for a positive norm of the corresponding particle state in the Hilbert space.

In the limit \( \epsilon \to 0 \), expression (23.139) becomes

\[
\frac{1}{g_0} + A - B \to \frac{1}{2\pi} [J(z) + 2 - 2J_2^0 (z)].
\]

(23.144)
The integral $J(z)$ was calculated in Eq. (23.64). For the integral $J_0^2(z)$ we find, by a similar calculation,

$$J_0^2(z) = \frac{2\theta}{\sinh 2\theta} = \frac{\theta \coth \theta}{\cosh^2 \theta} = \frac{2}{z + 4} [J(z) + 2],$$

such that we obtain, in Minkowski space, using the notation with $z \equiv q^2/E^2 = -q^2/M^2$:

$$G_{\Delta^\prime_m \Delta^\prime_m} = \frac{i}{N} \frac{2\pi}{2\theta \tanh \theta} \frac{1}{J(-q^2/M^2) + 2}. \quad (23.145)$$

The real part of the fluctuating field $\Delta^\prime = \Delta - \Delta_0$ has, for $\epsilon \to 0$, the propagator

$$G_{\Delta^\prime_\re \Delta^\prime_\re} = -\frac{i}{N} \frac{2\pi}{J(-q^2/M^2) + 2}. \quad (23.146)$$

### 23.7 Spontaneously Broken Symmetry

We have identified the pole at $q^2 = 0$ as a Nambu-Goldstone boson of a spontaneously broken symmetry. To justify this identification, we have to exhibit the continuous symmetry which has been spontaneously broken by the ground state solution. Looking back at the original Lagrangian (23.96), we see that it is invariant under global gauge transformations

$$\psi \to e^{i\alpha} \psi, \quad \alpha = \text{const.},$$

$$\bar{\psi} \to e^{-i\alpha} \bar{\psi}. \quad (23.148)$$

By the same token, the collective action (23.100) remains invariant, if the collective pair field $\Delta$, that is on the average equal to two $\psi$-fields, is transformed with twice the phase angle:

$$\Delta \to e^{2i\alpha} \Delta. \quad (23.149)$$

This invariance has been used before when we chose a real ground state expectation $\Delta_0$. Any other phase would have given the same physical result. Of course, once this phase is chosen, the invariance (23.148) is destroyed. Thus the zero-mass particle is indeed a Nambu-Goldstone particle. It corresponds to an excitation whose long-wavelength limit reduces to a pure global gauge transformation.

Strictly speaking, this zero mass boson can only exist in dimensions $D > 2$, as follows from a very general theorem of Mermin, Wagner, and Coleman (see Ref. 5 in Chapter 18). Indeed, we have seen before, in the Bose case, that fluctuations prevent the spontaneous breakdown of a continuous symmetry, which might be present at the mean-field level. Thus we may conclude that, if fluctuations are included in the
collective field, the theory will also exhibit this general feature in two space-time dimensions. In the limit $N \to \infty$ there are no fluctuations in $\Delta$. Thus Coleman's theorem should be satisfied after including all $1/N$-corrections. In two dimensions, however, things are more subtle. There is a critical coupling strength where a quasi-ordered state does exist. This will be discussed in Subsection 23.8.

The physical interpretation of the field $\Delta$ is the following: Due to the attraction for $g_0 < 0$, the fermions form bound-states of a particle and an antiparticle, the analogs of the Cooper pairs in a superconductor. These are bosons which can form a condensate, just as before the bosonic $\phi$-fields in $\phi^4$-theory (recall in Chapter 18). In fact, the effective potential for the $\Delta$-field looks qualitatively very similar to that of the bosonic potential $v(\Delta)$ of the O($N$)-symmetric theory for negative $m^2$. The origin of this potential is unstable and there is a new minimum at $\Delta_0 \neq 0$ with an arbitrary phase [see (23.103) with $m_0 = 0$]. Just as in the previous model with an interaction $(g_0/2N)\left(\bar{\psi}_a \psi_a\right)^2$, the opposite sign $g_0 > 0$ does not lead to a spontaneous symmetry breakdown, and massless fermions remain massless.

Finally we must justify why we have called the vacuum expectation $\Delta_0 = M$ the spontaneously generated fermion mass. Looking back at the collective effective action (23.114), we see that $M0$ appears in the combinations

$$\bar{\Psi}i\partial/\partial M\Psi = 1/2 \left(\Psi^T C \Psi + \bar{\Psi} C \Psi^T\right) = 1/2 \left(\Psi^T C, \Psi\right) \left(\begin{array}{c} M \\ i\phi/\partial M \end{array}\right) \left(\begin{array}{c} \Psi \\ C \Psi^T \end{array}\right).$$

There is a simple transformation which brings this to the canonical Dirac form. With the two-dimensional $\gamma^5$-matrix (23.39), we see that

$$\Psi' = \frac{1 - \gamma_5}{2} \Psi + \frac{1 + \gamma_5}{2} C \Psi^T, \quad \bar{\Psi}' = \bar{\Psi} \frac{1 + \gamma_5}{2} + \Psi^T C \frac{1 - \gamma_5}{2},$$

and hence

$$\bar{\Psi}' \Psi' = \frac{1}{2} \left(\Psi^T C \Psi + \bar{\Psi} C \Psi^T\right) - \frac{1}{2} \left(\Psi^T \gamma_5 \Psi - \bar{\Psi} \gamma_5 C \Psi^T\right).$$

We have used the projection property

$$P^2_+ = P_+$$

of the chiral projection matrix

$$P_+ \equiv \frac{1 + \gamma_5}{2},$$

and the fact that

$$\gamma_\mu P_+ = P_+ \gamma_\mu,$$

which implies that

$$P_+ \gamma_0 P_+ = 0, \quad P_+ \gamma_0 P_+ = \gamma_0.$$
The $\gamma_5$-terms in (23.152) vanish due to the relations
\[ C\gamma_5^T C^{-1} = C \left( \gamma_0^0 \gamma_1^1 \right)^T C^{-1} = -\gamma_0^0 \gamma_1^1 = -\gamma_5. \] (23.157)

This brings the mass term in Eq. (23.150) to the simple form $M\overline{\Psi}'\Psi'$.

The gradient term in (23.150) is invariant under the transformation (23.151):
\[ \overline{\Psi}'i\slash{\partial}\overline{\Psi}' = \overline{\Psi}i\slash{\partial}\overline{\Psi}. \] (23.158)

This follows again from (23.155), which here implies that
\[ P_\pm\gamma_0^\gamma_\mu P_\pm = \gamma_0^\gamma_\mu, \quad P_\pm\gamma_0^\gamma_\mu P_\mp = 0. \] (23.159)

In this context it should be mentioned that the entire model could have been written from the outset using $\Psi'$-fields, which are defined in terms of $\Psi$-fields as in (23.151). If we supplement the relation (23.152) by
\[ \overline{\Psi}'\gamma_5^5\Psi' = \frac{1}{2} \left( \Psi'^T C\Psi - \Psi C\Psi'^T \right) + \frac{1}{2} \left( \Psi'^T C\gamma_5^5\Psi + \Psi\gamma_5^5 C\Psi'^T \right), \] (23.160)

where the second parenthesis is again zero, we see that the fermion field part in the effective action (23.100) can be written for zero sources and mass as
\[ \overline{\Psi}_a i\slash{\partial}\Psi_a + \frac{1}{2}\Delta_{re} \left( \Psi'^a_a C\Psi_a + \Psi_a C\Psi'^a_a \right) - \frac{i}{2}\Delta_{im} \left( \Psi'^a_a C\Psi_a - \Psi_a C\Psi'^a_a \right) = \overline{\Psi}'_a (i\slash{\partial} - \sigma - i\pi^\gamma_5^5) \Psi'_a - \frac{N}{2g_0}(\sigma^2 + \pi^2), \] (23.161)

where we have identified
\[ \sigma \equiv -\Delta_{re}, \quad \pi = -\Delta_{im}. \] (23.162)

The invariance under global gauge transformation (23.148) becomes, in terms of $\Psi'$-fields, an invariance under the transformation
\[ \Psi' \equiv \begin{pmatrix} -\Psi'^*_1 \\ \Psi'^*_2 \end{pmatrix} \rightarrow \begin{pmatrix} -e^{ix}\Psi'^*_1 \\ e^{-ix}\Psi'^*_2 \end{pmatrix}, \] (23.163)
\[ = \begin{pmatrix} e^{ix} & 0 \\ 0 & e^{-ix} \end{pmatrix} \Psi' = e^{i\alpha^5^5}\Psi'. \] (23.164)

Such transformations involving $\gamma_5$ are referred to as chiral transformations. They play an important role in theories of weak interactions (see Chapter 27). Under the chiral transformation, $\overline{\Psi}'$ and $\overline{\Psi}'i\gamma_5^5\Psi'$ behave like a vector in a plane
\[ \overline{\Psi}' \rightarrow \overline{\Psi}' e^{2i\alpha^5^5}\Psi' = \cos 2\alpha\overline{\Psi}' \Psi' + \sin 2\alpha\overline{\Psi}' i\gamma_5^5\Psi', \]
\[ \overline{\Psi}'i\gamma_5^5\Psi' \rightarrow \overline{\Psi}' e^{i\alpha^5^5}\gamma_5^5 e^{i\alpha^5^5}\Psi' = -\sin 2\alpha\overline{\Psi}' \Psi' + \cos 2\alpha\overline{\Psi}' i\gamma_5^5\Psi'. \] (23.165)
Thus the transformation (23.149) becomes, with (23.152) and (23.160),

\[
\begin{pmatrix} \sigma \\ \pi \end{pmatrix} \rightarrow \begin{pmatrix} \sigma' \\ \pi' \end{pmatrix} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{pmatrix} \begin{pmatrix} \sigma \\ \pi \end{pmatrix}.
\] (23.166)

This leaves the transformed effective action (23.161) chirally invariant.

The ground state breaks chiral invariance since \( \sigma \) acquires an expectation value \( \sigma_0 = M \). The Nambu-Goldstone boson generated by this phase transition is the massless field \( \pi \). It is for this reason that chiral invariance is believed to be an important principle of strong interactions among elementary particles. There is a particle in nature, the pion, whose electrically neutral version has a mass roughly equal to 135 MeV/c\(^2\), and lies much lower than any other strongly interacting particle. One therefore interprets the pion as an almost Nambu-Goldstone particle of the underlying Lagrangian. It was in this context that Nambu initiated the study of chiral symmetry in particle physics.

Finally, let us remark that the inclusion of an initial fermion mass \( m_0 \neq 0 \) is possible but will not be done here, since it merely makes the discussion more involved but add little to the understanding of the model.

In \( D = 2 + \epsilon \) dimensions, the model with the chirally invariant Lagrangian

\[
\mathcal{L} = \bar{\psi}_a (i\partial - m_0 - \sigma - i\pi\gamma_5) \psi_a + \frac{g_0}{2N} \left[ (\bar{\psi}_a \psi_a)^2 + (\bar{\psi}_a i\gamma_5 \psi_a)^2 \right]
\] (23.167)
can be treated by introducing the collective fields \( \sigma \) and \( \pi \) as in Eq. (23.6), which leads to a Lagrangian

\[
\mathcal{L} = \bar{\psi}_a (i\partial - m_0 - \sigma - i\pi\gamma_5) \psi_a + \frac{N}{2g_0} (\sigma^2 + \pi^2).
\] (23.168)

This is called the chiral Gross-Neveu model [4]. At the mean-field level, it has an effective action (23.161).

### 23.8 Relation between Pairing and Gross-Neveu Model

Both versions of the fermionic O(N)-model discussed so far in this chapter, the Gross-Neveu-model with the Lagrangian (23.1) and the pairing model with the Lagrangian (23.96), showed a spontaneous mass generation in the limit of \( N \to \infty \). This happened only for one sign of the bare coupling constant. In the Gross-Neveu model with the interaction

\[
\frac{g_0}{N} (\bar{\psi}_a \psi_a)^2, \quad \text{for} \quad g_0 < 0,
\] (23.169)

and in the pairing model (23.100) with the interaction

\[
\frac{g_0}{N} \bar{\psi}_a^T C \psi_a \bar{\psi}_b C \bar{\psi}_b^T, \quad \text{for} \quad g_0 < 0.
\] (23.170)
The forces between the fermions described by the fields are in the latter case attractive, in the former repulsive. In either case, the opposite sign of $g_0$ leaves the fermions massless for $N \to \infty$.

These exactly soluble models may provide us with idealized approximations of two phases of the not exactly solvable $N = 1$ model for $D > 2$ [5]:

$$\mathcal{L} = \bar{\psi} i \gamma \psi + \frac{g_0}{2} (\bar{\psi} \psi)^2. \quad (23.171)$$

As before, we can certainly introduce a collective field $\sigma$ as in the partition function (23.6). For $N = 1$, the field $\sigma$ will fluctuate so violently, that the phase properties of the model cannot be derived in this way. Suppose now that fluctuations do not completely destroy the fact that, for $g_0 < 0$, there is a solution in which a mass is generated spontaneously, i.e., we suppose for a moment that the $N \to \infty$ limit gives at least a qualitatively correct description of the system for $g_0 < 0$. Then we might tend to believe that also the $N = 1$ version of the pairing model

$$\mathcal{L} = \bar{\psi} i \gamma \psi + \frac{g_0}{2} \psi^T C \psi \bar{\psi} C \psi^T \quad (23.172)$$

should have a solution for $g_0 < 0$ which resembles that of the $N \to \infty$ limit, i.e., a phase with massive fermions. The associated Cooper pair fields would then describe bound states which carry massless Nambu-Goldstone bosons.

The interesting observation is now that these solutions of the two large-$N$ models are two different phases of one and the same system. The point is that the interactions in the Lagrangians (23.171) and (23.172) are identical, apart from an opposite sign of the coupling constant, and a factor two. This follows directly by rewriting the interactions in terms of spin up and spin down components of the field $\psi = \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right)$. For the interaction in (23.171), we obtain

$$\left( \bar{\psi} \psi \right)^2 = \left( \psi_2^* \psi_1 + \psi_1^* \psi_2 \right)^2 = 2 \psi_2^* \psi_1 \psi_1^* \psi_2. \quad (23.173)$$

We have omitted terms containing squares of the fields $\psi_1^2 = \psi_2^2$ and their conjugates, since these vanish for Grassmann variables. Proceeding similarly with the interaction in (23.172), we obtain

$$|\psi^T C \psi|^2 = | - \psi_1 \psi_2 + \psi_2 \psi_1 |^2 = |2 \psi_2 \psi_1|^2$$

$$= 4 (\psi_2 \psi_1)^* \psi_2 \psi_1 = 4 \psi_1^* \psi_2^* \psi_2 \psi_1 = -4 \psi_2^* \psi_1 \psi_1^* \psi_2. \quad (23.174)$$

such that, indeed,

$$\left( \bar{\psi} \psi \right)^2 = -\frac{1}{2} |\psi^T C \psi|^2. \quad (23.175)$$

Thus we expect the following two phases for the $N = 1$ -model: for $g_0 < 0$ or $g > g^*$, a phase with massive fermions and a spontaneously broken $\gamma_5$-invariance. Here the system is symmetric under global gauge transformations. The other phase
for $g_0 > 0, g < g^*$ has again massive fermions but, in addition, massless Nambu-Goldstone modes due to a spontaneously broken global gauge symmetry. Physically, the second phase is distinguished from the first by strong long-range fluctuations which do not exist in the first phase.

For $N \to \infty$, either of the two phases becomes an exact solution of one of the two models. They differ by the arrangement of the indices over the four fermion fields in (23.169) and (23.170).

We pointed out before the analogy between the coupling constant in the model and the temperature in the euclidean formulation of the model. With this interpretation of $g$, the behavior of the model looks very similar to that found experimentally in thin films of $^4$He. It has a phase transition at a certain temperature $T_c$. Above $T_c$, there are only short-range correlations, the system is normal. Below $T_c$, there are long-range correlations, the system is super-fluid and there are Goldstone excitations of the condensate. By the Mermin-Wagner theorem discussed in Ref. [5] of Chapter 18), there can be no Nambu-Goldstone bosons in exactly two dimensions, but there do exist quasi-long-range fluctuations with power-like correlation functions of the type (18.99).

### 23.9 Comparison with the $O(N)$-Symmetric $\phi^4$-Theory

After having observed the possibility of spontaneously generating a mass in a massless theory via fluctuations, we may look once more back to the scalar $\phi^4$-version of the $O(N)$ model in $D = 4 - \varepsilon$ dimensions, that was discussed in Chapter 18. Note the different notation of $\varepsilon$ in $D = 4 - \varepsilon$, in contrast to $D = 2 + \varepsilon$ in the previous discussions. In the massless case the potential is

$$v(\Phi, \lambda) = \frac{1}{2} \lambda \Phi^2 - \frac{N}{4g_0} \lambda^2 + \frac{1}{2} N S_2 \frac{1}{2} \Gamma(D/2) \Gamma(1 - D/2) \frac{2}{D} \lambda^{D/2},$$

(23.176)

which may be written as

$$\frac{1}{N} v(\Phi, \lambda) = \frac{1}{2N} \lambda \Phi^2 - \frac{1}{4g_0} \lambda^2 - \frac{b'_\varepsilon}{4} \lambda^{D/2}$$

(23.177)

with

$$b'_\varepsilon \equiv -\frac{2}{D} S_2 \Gamma(D/2) \Gamma(1 - D/2) = \frac{4}{D} S_2 \frac{1}{\varepsilon} \approx \frac{1}{8\pi^2} \frac{1}{\varepsilon}. \quad (23.178)$$

Note the opposite sign of the last term (23.176), in comparison with the fermionic equation (23.17), and the absence in $b'_\varepsilon$ of a factor $2^{D/2-1}$ with respect to the fermionic $b$ in Eq. (23.20). The latter is caused by the Dirac trace in $D$ dimensions, and a factor $1/2$ caused by the linearity of the Dirac operator in $p$. The bosonic $b'_\varepsilon$ is related by $b'_\varepsilon = 4c/\varepsilon$ to the bosonic constant introduced in the discussion of
the nonlinear $\sigma$-model in Eq. (19.37) by $b'_\varepsilon = 4c_\varepsilon/D = b'_\varepsilon$. Extremization of (23.176) yields the two gap equations:

$$\lambda \Phi_a = 0, \quad \text{(23.179)}$$

to be solved with $\lambda = 0$ or $\Phi_a = 0$, and

$$\frac{1}{N} \Phi_a^2 - \frac{1}{g_0} \lambda - b'_\varepsilon \frac{D}{4} \lambda^{D/2-1} = 0. \quad \text{(23.180)}$$

A renormalized coupling constant may be introduced by setting

$$\frac{1}{g_0} \mu^{-\varepsilon} + b'_\varepsilon = \frac{1}{g}. \quad \text{(23.181)}$$

Then $v(\Phi, \lambda)$ becomes

$$\frac{1}{N} v(\Phi, \lambda) = \frac{1}{2N} \lambda \Phi_a^2 - \frac{\mu^{-\varepsilon}}{4} \left\{ \frac{\lambda^2}{g} + b'_\varepsilon \left[ \left( \frac{\lambda}{\mu^2} \right)^{-\varepsilon/2} - 1 \right] \lambda^2 \right\}. \quad \text{(23.182)}$$

In the renormalized form, the different signs of the bare coupling $g_0 > 0$ or $g_0 < 0$ correspond to $g < g^*$ or $g > g^*$ with $g^* = b'_\varepsilon^{-1} \approx 8\pi^2\varepsilon$.

There is a $O(N)$-symmetric phase with $\Phi_a = 0$ and $\lambda_0 \neq 0$ for $g_0 < 0$, where the bosons acquire a mass. For $g_0 < 0$ there is only the solution $\Phi_a = 0$, $\lambda_0 = 0$, which is again a symmetric phase. But contrary to the previous one, this phase is massless.

Both phases are stable, since the determinant of the matrix of second-derivatives $v_{ab}(\Phi_a, \lambda(\Phi_a^2))$ is nonnegative, whereas the subscripts $ab$ abbreviate the derivatives $\partial^2/\partial \Phi_a \partial \Phi_b$, and $\lambda(\Phi_a^2)$ is the solution of the gap equation (23.180).

Consider now the excitations, setting $\Sigma' \equiv \Sigma - \Sigma_0$. In the massless phase with $g < g^*$, we calculate the propagator $G_{\Sigma' \Sigma'}$ of $\Sigma'$-fluctuations from the quadratic variation

$$\delta^2 \Gamma[\Phi, \Sigma] = \delta^2 \Gamma[\Phi, \Sigma_0 + \Sigma'] = \frac{1}{2} \Sigma' \Gamma^{(2)} \Sigma', \quad \text{(23.183)}$$

where in euclidean space

$$\Gamma^{(2)} = -\frac{N}{2} \left[ \frac{1}{g_0} + I(q) \right] \quad \text{(23.184)}$$

with [see (11.155), (11.164), and (11.168)]

$$I(q) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k_E^2(k + q)^2} = - \left( 1 - \frac{D}{2} \right) \frac{D}{4} b'_\varepsilon \mu^{-\varepsilon} \left( \frac{q_E^2}{\mu^2} \right)^{-\varepsilon/2} \times \frac{\Gamma^2(1 - \varepsilon/2)}{\Gamma(2 - \varepsilon)} = c_\varepsilon \mu^{-\varepsilon} \left( \frac{q_E^2}{\mu^2} \right)^{\varepsilon/2}. \quad \text{(23.185)}$$
Here we have replaced $b'_\varepsilon$ by its small-$\varepsilon$ limit
\[ c_\varepsilon \equiv - \left( 1 - \frac{D}{2} \right) \frac{D}{4} b'_\varepsilon \approx \frac{1}{8\pi^2\varepsilon}. \] (23.186)

In this way, we obtain a propagator for the collective field $\Sigma'$:
\[ G_{\Sigma\Sigma'} = -\frac{2i}{N} \mu^\varepsilon \frac{1}{1/g_0\mu^{-\varepsilon} + c_\varepsilon (q_E^2/\mu^2)^{-\varepsilon/2}}. \] (23.187)

For $g_0 > 0$ with $g < g^*$, this is a physically acceptable quantity. For $g_0 < 0$ with $g > g^*$, however, there is a tachyon pole at
\[ \frac{q_E^2}{\mu^2} = \left( \frac{-1}{g_0\mu^{-\varepsilon}c_\varepsilon} \right)^{-2/\varepsilon}, \] (23.188)
indicating that we have expanded around the wrong vacuum value $\lambda_0 = 0$. This has led to an unphysical solution.

We must insert the vacuum expectation for the field that minimizes the energy. Then there will be no tachyon since any unstable system will undergo a phase transition until the collapsed field configuration is stable. Recall the discussion on pages 1134 and 1294.

Alternatively, we can see that in the gap equation near four dimensions
\[ \frac{1}{g_0\mu^{-\varepsilon}} = \frac{1}{g} - S_4 \frac{1}{\varepsilon}, \] (23.189)
a finite renormalized coupling $g$ can only be achieved in the limit $\varepsilon \to 0$ for negative $g_0 < 0$, in which case the $\phi^4$-potential turns the wrong way around. In this case the only consistent solution for $\varepsilon \to 0$ is the free one with $g = 0$. We now realize the difference with the $N \to \infty$ Fermi case. There were two possible consistent phases. One had a mass that was spontaneously generated. That happened for $g > g^*, g_0 < 0$. In the other phase with $g < g^*$, the fermions remained massless. Only the latter phase is physically acceptable.

The four-dimensional theory has no consistent ground state with $\lambda = 0$, except for $g = 0$. In contrast, the three-dimensional theory does have one for $g < g^* \sim 8\pi^2\varepsilon$.

In the Gross-Neveu model we pointed out the existence of certain power laws in the massive phase. One deals with the physical mass as a function of $g - g^*$ or $T - T_c$ [see (23.80)]. The other is a power law for $v(\Sigma) \sim \Sigma$ at the critical point $g = g^*$, i.e., at $T = T_c$. For $T \neq T_c$, this power was valid up to the UV limit $\Sigma \gg M$. The opposite limit $\Sigma \ll M$, on the other hand, was shown to follow the IR-free power law $v(\Sigma) \sim \Sigma^2$.

We now demonstrate that the $O(N)$-symmetric $\phi^4$-theory displays quite a similar power behavior which, however, is opposite as far as IR and UV limits are concerned. For this the mass parameter $m_0^2$ has to be set proportional to $T/T_c - 1$, and the
critical theory will be obtained in the limit \( m_0 \to 0 \). In order to see this, consider the potential (23.176):

\[
\frac{1}{N} v(\Phi, \lambda(\Phi^2_a)) = \frac{1}{2N} \lambda \Phi^2_a + \frac{m_0^2}{2g_0} \lambda - \frac{1}{4g_0} \lambda^2 - \frac{b'}{4} \lambda D^{1/2} - \frac{m_0^4}{4g_0},
\]

(23.190)

where \( \lambda(\Phi^2_a) \) is the function of \( \Phi^2_a \), for which \( v(\Phi, \lambda) = 0 \):

\[
\frac{1}{N} \Phi^2_a = \frac{1}{g_0} \lambda - \frac{m_0^2}{g_0} + \frac{b'}{2} \frac{D}{2} \lambda^{1-\epsilon/2}.
\]

(23.191)

Suppose we are in the normal phase with \( \Phi_a = 0 \) and \( \lambda_0 \neq 0 \). Consider the critical regime with very small \( m_0^2 \). From (23.170) we see that \( \lambda \) behaves as a function of \( m_0^2 \) as

\[
\lambda \sim (m_0^2)^{1/(1-\epsilon/2)}.
\]

(23.192)

Inserting this into (23.190) we obtain, for the minimal value of \( v(\Phi, \lambda(\Phi^2_a)) \), the power behavior

\[
v_{\text{min}} \sim (m_0^2)^{1+1/(1-\epsilon/2)}.
\]

(23.193)

At the critical point with \( m_0 = 0 \), we have

\[
\frac{1}{N} \Phi^2_a = \frac{1}{g_0} \lambda + \frac{b'}{2} \frac{D}{2} \lambda^{1-\epsilon/2}.
\]

(23.194)

Contrary to the Gross-Neveu model there is no pure power behavior. Only if also \( g_0 = 0, g = 0 \) (free theory) or \( g_0 = \infty \) (\( g = g^* \)), a pure power behavior is obtained:

\[
\Phi^2_a \sim \lambda, \quad g_0 = 0 \quad g = 0 \quad \text{(or } \lambda \to \infty),
\]

\[
\Phi^2_a \sim \lambda^{1-\epsilon/2}, \quad g_0 = \infty \quad g = g^* \quad \text{(or } \lambda \to 0).
\]

(23.195)

The same behavior is found at any coupling strength for \( \lambda \to 0 \) (ultraviolet limit) or \( \lambda \to \infty \) (infrared limit), respectively. In the renormalized form of (23.195)

\[
\frac{1}{N} \Phi^2_a = \mu^\epsilon \left\{ \frac{1}{g} - b'_\epsilon \left[ 1 - \frac{D}{4} \left( \frac{\lambda}{\mu^2} \right)^{-\epsilon/2} \right] \right\} \lambda,
\]

(23.196)

the two limits are separated by the scale parameter \( \mu \). For small \( \lambda \), the potential itself behaves like

\[
v(\Phi) \sim (\Phi^2)^{1+1/(1-\epsilon/2)},
\]

(23.197)

as determined by the first and last term in (23.176). The small-\( \lambda \) behavior can be collected in the single formula valid for \( m_0^2 \)

\[
v(\Phi) \sim \left[ m_0^2 \left( \frac{\Phi^2}{m_0^2} + \frac{N}{g_0} \right) \right]^{1+1/(1-\epsilon/2)},
\]

(23.198)
which follows from writing (23.191) as

\[ \frac{1}{N} \Phi^2_a + \frac{m^2_0}{g_0} \sim b' \frac{D}{4} \lambda^{1-\varepsilon/2}, \]  

valid for small \( \lambda \), and reinserting this into (23.190).

If \( \Phi \) is interpreted as magnetization \( M \), and \( m^2_0 \approx (T/T_c - 1) \) is the relative distance of the temperature from the critical value, this corresponds to a general power law

\[ v(M) \sim M^{\delta+1} f \left( \frac{T/T_c - 1}{M^{1/\beta}} \right), \]  

which was first observed experimentally by Widom in magnetic systems. The present model has

\[ \beta = \frac{1}{2}, \quad \frac{\delta + 1}{2} = \frac{2 - \varepsilon/2}{1 - \varepsilon/2}. \]  

For large \( \lambda \) (UV-limit), there is again a free field behavior with

\[ \frac{1}{N} \Phi^2_a + \frac{m^2_0}{g_0} \sim \lambda \]  

and

\[ v \sim \left( \frac{\Phi^2_a}{N} + \frac{m^2_0}{g} \right)^2. \]

Note that there is great similarity with the Gross-Neveu model, as far as power behaviors are concerned. But the scaling limits are opposite to each other. In the Gross-Neveu model, there is only one scale, the fermion mass \( M \). Its size depends on the relative distance of \( g \) from \( g^* \) (or of \( T \) from \( T_c \)). If \( g \) hits \( g^* \) (or \( T \) hits \( T_c \)) then \( M \) vanishes. At such a critical point, the behavior is power-like in the fields. Near such a critical point, the powers are different in the UV- and IR-limits. Moreover, the energy is extremal at a field strength that varies like a power of the relative distance of \( g \) from \( g^* \) (or of \( T \) from \( T_c \)).

23.10 Two Phase Transitions in the Chiral Gross-Neveu Model

We shall now demonstrate that the chiral Gross-Neveu model in \( 2 + \epsilon \) dimensions has, for a small number \( N \) of fermions, two phase transitions, one corresponding to pair formation, the other to pair condensation [6]. In a first transition, fermions and antifermions acquire a mass spontaneously and form bound states in a chirally symmetric state, consisting of pairs which behave like a Bose liquid. In a second transition, the Bose liquid condenses into a coherent state that breaks chiral symmetry. This suggests the possibility that, in particle physics, the generation of quark masses may also happen separately from the breakdown of chiral symmetry [2].
Starting point of our discussion is Eq. (23.142) for the propagator of the massless Goldstone modes of the pair field in $D = 2 + \epsilon$ dimensions:

$$
G_{\Delta_m \Delta'_m} = \frac{1}{N} \frac{4}{\epsilon} \left( \frac{1}{g^*} - \frac{1}{g} \right)^{-1} M^{2-\epsilon} \frac{i}{q^2} + \text{regular part at } q^2 = 0. \quad (23.204)
$$

The sign of the pole term guarantees a positive norm of the corresponding particle state in the Hilbert space.

The residue of the pole term will allow us to conclude that the chiral model has two phase transitions. Consider first the case $\epsilon = 0$ where the collective field theory consists of a complex field $\Delta$ with the $O(2)$-symmetry $\Delta \to e^{i\phi}\Delta$. From the discussion of the Kosterlitz-Thouless transition [7, 8] we know that a complex field system possesses macroscopic excitations of the form of vortices and antivortices. These attract each other by a logarithmic Coulomb potential, just like a gas of electrons and positrons in two dimensions. At low temperatures, the vortices and antivortices form bound pairs. The grand-canonical ensemble of pairs exhibits quasi-long-range correlations. At some temperature $T_c$, the vortex pairs break up, and the correlations become short-range. The phase transition is of infinite order.

We have shown in the textbook on *Gauge Fields in Condensed Matter* [7] that this transition is most easily understood in a model field theory involving a pure phase field $\theta(x)$, with a Lagrange density

$$
\mathcal{L} = \frac{\beta}{2} [\partial \theta(x)]^2, \quad (23.205)
$$

where $\beta$ is the stiffness of the $\theta$-fluctuations. The important feature of the phase field $\theta$ is that it is a cyclic field with $\theta = \theta + 2\pi$. In order to ensure that such jumps by $2\pi$ carry no energy, the gradient in the Lagrange density needs a modification which allows the existence of vortices and antivortices. After including vortices and antivortices at positions $x_i, x_j$, their partition function can be written as

$$
Z = \sum_{\text{gas}} \exp \left\{ 4\pi^2 \beta \sum_{i<j} q_i q_j \frac{1}{2\pi} \log(|x_i - x_j|/r_0) \right\}, \quad (23.206)
$$

where $r_0$ characterizes the size of the vortices. For a single vortex-antivortex pair, the average square distance $r^2$ diverges as the stiffness falls below the critical value

$$
\beta_{\text{KT}} = 1/T_{\text{KT}} = 2/\pi \approx 0.63662. \quad (23.207)
$$

The large-stiffness state with bound vortex pairs has a coherent phase field $\theta(x)$, the low-stiffness state with separated vortex pairs exhibits incoherent phase fluctuations. The same situation is found in three dimensions, only that the excitations are vortex lines. These become infinitely long and prolific in a continuous phase transition at a critical point $\beta_c \approx 0.33$.

\[^1\text{Recall Section 11.10 in the textbook [7].}\]
The result (23.204) for $\epsilon = 0$ can now be used to estimate a critical value of the number of field components $N = N_c$, below which the phase fluctuations of the complex field $\Delta'$ become so violent that the system has a phase transition. For this we write $\Delta'_{im} = M\theta$ and find from (23.204) a propagator of the $\theta$-field:

$$G_{\theta\theta} \approx \frac{i}{N} \frac{4\pi}{q^2} + \text{regular terms.} \quad (23.208)$$

Comparing this with the propagator for the model Lagrange density (23.205),

$$G_{\theta\theta} = \frac{1}{\beta} \frac{i}{q^2}, \quad (23.209)$$

we identify the stiffness $\beta = N/4\pi$. The pair version of the chiral Gross-Neveu model has therefore a vortex-antivortex pair breaking transition, if $N$ falls below the critical value $N_c = 8$.

Consider now the model in $2 + \epsilon$ dimensions where pairs form at $g = g^* \approx \pi\epsilon$. A comparison between the propagator (23.204) and (23.209) yields a stiffness of phase fluctuations

$$\beta = \frac{N}{4\pi} M^s \left(1 - \frac{g^*}{g}\right). \quad (23.210)$$

The linear vanishing of the stiffness as a function of the distance of the coupling constant $g$ from the critical value $g^*$ is in agreement with a general scaling relation, according to which the critical exponent of bending rigidity should be equal to $(D - 2)\nu$.

The propagator for the real part of the pair field has, by (23.141), a correlation length

$$\xi = \left(\frac{D-1}{12M^2}\right)^{1/2}. \quad (23.211)$$

Inserting the $g$-dependence of $M$ from (23.82), we see that

$$\xi = \frac{1}{M_\infty} \left(\frac{D-1}{12}\right)^{1/2} \left(1 - \frac{g^*}{g}\right)^{-1/\epsilon}, \quad (23.212)$$

so that the coherence length diverges for $g \to g^*$ with a critical exponent $\nu = 1/\epsilon$. The stiffness (23.210) implies the existence of a phase transition in the neighborhood of two and three dimensions at roughly

$$N_c \approx 8 \left(1 - \frac{g^*}{g}\right)^{-1}, \quad D \approx 2, \quad N_c \approx 4.19 \left(1 - \frac{g^*}{g}\right)^{-1}, \quad D = 3. \quad (23.213)$$

As $N$ is lowered below these critical values, the phase fluctuations of the pair field $\Delta$ become incoherent and the pair condensate dissolves. The different phases are indicated in Fig. 23.3. In the chiral formulation of the same model, the intermediate phase has chiral symmetry in spite of a nonzero spontaneously generated “quark
mass" $M \neq 0$. The reason why this is possible is that the “quark mass” depends only on $|\Delta_0|$, thus allowing for arbitrary phase fluctuations preserving chiral symmetry.

The skeptical reader may wonder whether the solid hyperbola in Fig. 23.3 is not simply the proper (albeit approximate) continuation of the vertical line for smaller $N$. There are two simple counterarguments. One is formal: For infinitesimal $\epsilon$ the first transition lies precisely at $g = g^* = \pi \epsilon$ for all $N$, so that the horizontal transition line is clearly distinguished from it. The other argument is physical. If $N$ is lowered at some very large $g$, the binding energy of the pairs increases with $1/N$. Observe that in two dimensions, the exact binding energy is $4M \sin^2[\pi/2(N - 1)]$. It is then impossible that the phase fluctuations on the horizontal branch of the transition line, which are low-energy excitations, unbind the strongly bound pairs. This will only happen in the limit $N \to \infty$ where the binding energy becomes zero and the two transition curves merge into a single curve. This is the situation in the theory of superconductivity, where Cooper pair binding and pair condensation coincide.

In the ordinary Gross-Neveu model, the analog of the phase disordering transition is an Ising transition, in which the vacuum expectation value of $\sigma$ jumps between $-\Sigma_0$ and $\Sigma_0$ in a disorderly fashion. In two dimensions, this occurs at some critical value $N_c$. In $2 + \epsilon$ dimensions, this transition should again exist independently of the transition at which the system enters into a state of nonzero $\Sigma_0$. It will be interesting to see these two transitions confirmed by Monte-Carlo simulations.

There exists a four-dimensional version of this discussion in which it is shown that, very probably, a pion condensate cannot form in the Nambu-Jona-Lasinio model, due to directional fluctuations of the pion field [2].

### 23.11 Finite-Temperature Properties

It is useful to study also the behavior of the Gross-Neveu model at a finite temperature. The thermal properties of this model will closely resemble those of a
superconductor. For this we confine the imaginary-time variable \( \tau \) to the interval \( \tau \in (0, \bar{\hbar} \beta) \) with \( \beta = 1/k_B T \), and take the fields to be antiperiodic under \( \tau \to \tau + \bar{\hbar} \beta \). Equivalently, we may think of this model as a nonlinear \( \sigma \)-model on an infinitely long spatial strip with antiperiodic boundary conditions, whose width along the \( \tau \)-axis is \( \beta \). In the limit \( N \to \infty \), we can study the effects of temperature exactly. For simplicity, we consider the model only for a vanishing initial bare mass \( m_0 \). The corresponding effective potential of the \( \Sigma \)-field in Eq. (23.16),

\[
\frac{1}{N} v(\Sigma) = -\frac{1}{N} \Gamma[\Sigma] = \frac{1}{2g_0} \Sigma^2 - \text{tr}(1) \frac{1}{2} \int \frac{d^D p_E}{(2\pi)^D} \log (p_E^2 + \Sigma^2),
\]

is generalized to finite temperatures \( T \) by exchanging the momentum integral by a sum over Matsubara frequencies \( \omega_m = \pi (2m + 1) T/\bar{\hbar}, \ m = 0, \pm 1, \pm 2, \ldots \):

\[
\int \frac{d^D p_E}{(2\pi)^D} \to \frac{d^{D-1} p_E}{(2\pi)^{D-1}} \frac{1}{\bar{\hbar} \beta} \sum_{\omega_m = -\infty}^{\infty},
\]

thus becoming (in natural units with \( k_B = 1 \) and \( \bar{\hbar} = 1 \))

\[
\frac{1}{N} v(\Sigma) = \frac{1}{2g_0} \Sigma^2 - 2^{D/2} \frac{1}{2} \int \frac{d^{D-1} p_E}{(2\pi)^{D-1}} \frac{T}{\omega_m} \sum_{m = -\infty}^{\infty} \log (\omega_m^2 + p_E^2 + \Sigma^2). \tag{23.216}
\]

The gap equation is obtained by minimizing this potential. It reads [compare (23.15)]

\[
\frac{1}{g_0} = 2^{D/2} \int \frac{d^{D-1} p_E}{(2\pi)^{D-1}} \frac{T}{\omega_m + \Sigma^2} \sum_{m = -\infty}^{\infty} \frac{1}{\omega_m^2 + p_E^2 + \Sigma^2}. \tag{23.217}
\]

Using the well-known summation formula

\[
T \sum_{m = -\infty}^{\infty} \frac{1}{\omega_m^2 + \Omega^2} = \frac{1}{2\Omega} \tanh \left( \frac{\Omega}{2T} \right), \tag{23.218}
\]

the gap equation becomes an integral

\[
\frac{1}{g_0} = 2^{D/2} \int \frac{d^{D-1} p_E}{(2\pi)^{D-1}} \frac{T}{2\Omega} \tanh \left( \frac{\Omega}{2T} \right). \tag{23.219}
\]

We shall renormalize this by adding and subtracting, on the right-hand side, its zero-temperature limit:

\[
\frac{1}{g_0} = 2^{D/2} \int \frac{d^{D-1} p_E}{(2\pi)^{D-1}} \frac{1}{p_E^2 + \Sigma^2} + 2^{D/2} \int \frac{d^D p_E}{(2\pi)^D} \frac{1}{2\Omega} \left( \tanh \left( \frac{\Omega}{2T} \right) - 1 \right). \tag{23.220}
\]

Near two dimensions, the first integral in this expression can be written as

\[
2^{D/2-1} S_D \Gamma(D/2) \Gamma(1 - D/2)(\Sigma^2)^{D/2-1} = b_\epsilon \frac{D}{2} \Sigma^\epsilon \approx b_\epsilon \mu^\epsilon - \frac{\mu^\epsilon}{\pi} \left( \log \frac{\Sigma}{\mu} + \frac{1}{2} \right) + O(\epsilon), \tag{23.221}
\]
where we have used Eqs. (23.17) and (23.20). This leads to the renormalized gap equation at $\epsilon = 0$:

$$
\frac{1}{g_R(\mu^2)} = \frac{1}{\pi} \log \frac{\Sigma}{\mu} + \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{dp}{\Omega} \left( \tanh \frac{\Omega}{2T} - 1 \right).
$$

(23.222)

It is convenient to express the zero-temperature part of this equation without the arbitrary scale parameter $\mu$, using the renormalization group invariant mass

$$
M = \mu \exp \left[ -\frac{\pi}{g_R(\mu)} - \frac{1}{2} \right],
$$

(23.223)

which is the solution of the zero-temperature gap equation. Then we arrive at the finite-temperature equation

$$
\log \frac{\Sigma}{\Sigma_0} = S_1 \left( \frac{\Sigma}{2\pi T} \right),
$$

(23.224)

with the function [compare with the bosonic version of this in Eq. (19.112)]

$$
S_1 \left( \frac{\Sigma}{2\pi T} \right) = \int_0^\infty \frac{dp}{\Omega} \left( \tanh \frac{\Omega}{2T} - 1 \right) = -2 \int_0^\infty \frac{dp}{\Omega} \left( e^{\Omega/T} + 1 \right)^{-1}.
$$

(23.225)

This determines the dimensionless ratio $\Sigma/2\pi T$ as a function of $T$. The dimensionless ratio $\Sigma/2\pi T$ will in the following be denoted by $\Sigma$ for later convenience, i.e.,

$$
\hat{\Sigma} = \frac{\Sigma}{2\pi T}.
$$

(23.226)

The solution $\Sigma(T) \equiv M(T)$ of (23.224) is now the temperature-dependent fermion mass. For $T = 0$, the function $S_1(\Sigma)$ vanishes and $\Sigma(0) = \Sigma_0 = M$. As the temperature rises, the fermion mass $M(T)$ decreases, until it vanishes at a certain critical temperature $T_c$. The value of $T_c$ is found by assuming $\Sigma(T)$ to be small, and by approximating the right-hand side of (23.225) as

$$
2 \int_0^\infty \frac{dp}{2\pi} \left[ \frac{1}{p} \tanh \left( \frac{p}{2T} \right) - \frac{1}{\sqrt{p^2 + \Sigma^2}} \right].
$$

(23.227)

Integrating the first term by parts gives

$$
\frac{1}{\pi} \left\{ \log \left( \frac{p}{2T} \right) \tanh \left( \frac{p}{2T} \right) \right\}_{0}^{\infty} - \int_0^\infty dx \log x \cosh^{-2} x \right\}.
$$

(23.228)

The integral is convergent and equal to $-\log (4e^{\gamma}/\pi)$, where $\gamma = 0.577\ldots$ is the Euler number. This follows at once from the formula

$$
\int_0^\infty dx \ x^{\mu-1} \cosh^{-2} ax = \frac{4}{(2a)^{\mu}} (1 - 2^{2-\mu}) \frac{\Gamma(\mu)\zeta(\mu - 1)}{\mu}.
$$

(23.229)
in the limit \( \mu \to 1 \), where

\[
\zeta(z) \equiv \sum_{k=0}^{\infty} \frac{1}{k^z}
\]  

(23.230)
is Riemann’s zeta function. Using the property \( \zeta(0) = -1/2 \) and \( \zeta'(0) = -\frac{1}{2} \log(2\pi)\Gamma'(1) = -\gamma \), the second term in (23.227) can be integrated directly with the result

\[
\text{asinh} \left( \frac{p}{\Sigma} \right) = \log \left[ \frac{p}{\Sigma} + \sqrt{\frac{p^2}{\Sigma^2} + 1} \right].
\]

Hence (23.227) becomes

\[
\frac{1}{\pi} \left\{ \log \left( \frac{p}{2T} \right) \tanh \left( \frac{p}{2T} \right) - \log \left( \frac{p}{\Sigma} + \sqrt{\frac{p^2}{\Sigma^2} + 1} \right) \right\}_{0}^{\infty} + \log \left( \frac{4e^\gamma}{\pi} \right) = \frac{1}{\pi} \log \left( \frac{2e^\gamma}{\pi T} \right),
\]  

(23.231)

and (23.224) determines the critical temperature by the equation

\[
\log \left( \frac{\Sigma}{\Sigma_0} \right) = \log \left( \frac{\Sigma e^\gamma}{\pi T_c} \right),
\]  

(23.232)
or

\[
T_c = \Sigma_0 \frac{e^\gamma}{\pi} = M \frac{e^\gamma}{\pi}.
\]  

(23.233)

At this temperature, the fermion mass \( M(T) \) vanishes.

In order to study the full temperature behavior of \( M(T) \), the right-hand side of the gap equation (23.224) has to be evaluated numerically. For this purpose, we shall derive a more useful form of the gap equation. Let us once more go back to the original form (23.217), and rewrite it for \( \epsilon \sim 0 \) as [compare (23.47)]

\[
\frac{1}{g_0} - 2^{D/2} \int \frac{d^Dp}{(2\pi)^D} \frac{1}{p^2 + \Sigma^2} = \frac{1}{g_0} - b_\epsilon \mu^\epsilon + \frac{1}{2\pi} \left( \log \frac{\Sigma^2}{\mu^2} + 1 \right)
\]

\[
= \frac{1}{\pi} S_1(\hat{\Sigma}) = T \sum_{m=0}^{\infty} \frac{1}{\sqrt{\omega_m^2 + \Sigma^2}} - \int_{0}^{\infty} \frac{d\omega_m}{2\pi} \frac{1}{\sqrt{\omega_m^2 + \Sigma^2}}.
\]  

(23.234)

This can again be written as in (23.224):

\[
\log \frac{\Sigma}{\Sigma_0} = S_1(\hat{\Sigma}),
\]  

(23.235)

where \( S_1 \) is given by the sum-minus-integral expression

\[
S_1(\hat{\Sigma}) = \left( \sum_{m=0}^{\infty} - \int_{-1/2}^{\infty} dm \right) \frac{1}{\sqrt{(m + \frac{1}{2})^2 + \Sigma^2}}.
\]  

(23.236)

It is useful to reorganize the sum as follows:

\[
S_1(\hat{\Sigma}) = \sum_{m=0}^{\infty} \frac{1}{\sqrt{(m + \frac{1}{2})^2 + \Sigma^2}} - \frac{1}{m + \frac{1}{2}} \left( \sum_{m=0}^{\infty} \frac{1}{m + \frac{1}{2}} - \int_{-1/2}^{\infty} dm \frac{1}{\sqrt{(m + \frac{1}{2})^2 + \Sigma^2}} \right).
\]  

(23.237)
The integral up to some large number $m = \bar{m}$ gives

$$\text{asinh}(\bar{m}/\Sigma) \rightarrow \log(2\bar{m}/\Sigma).$$

The sum over $1/(m + 1/2)$ is equal to

$$\gamma + \log 4 + \psi(\bar{m} + 3/2) = \gamma + \log 4 + \psi(\bar{m} + 1/2) + 1/(\bar{m} + 1/2).$$

Using the large-$z$ behavior

$$\psi(z) = \log z - 1/(2z) - 1/(12z^2) + \ldots,$$

we find for $S_1(\hat{\Sigma})$ the convergent sum

$$S_1(\hat{\Sigma}) = \sum_{m=0}^{\infty} \left[ \frac{1}{\left(m + \frac{1}{2}\right)^2 + \hat{\Sigma}^2} - \frac{1}{m + \frac{1}{2}} \right] + \log \left(2e^\gamma \hat{\Sigma}\right)$$

$$\equiv \tilde{S}_1(\hat{\Sigma}) + \log(2e^\gamma \hat{\Sigma}). \quad (23.238)$$

The logarithm of $\hat{\Sigma}$ cancels a similar term on the left-hand side of the gap equation (23.224). Using now the relation (23.233) between $\Sigma_0$ and the critical temperature, we express $\Sigma_0$ in terms of $M$, and derive the gap equation in a form most suitable for a numerical evaluation:

$$\log \frac{T}{T_c} = \tilde{S}_1(\hat{\Sigma}). \quad (23.239)$$

This can be used to calculate $T/T_c$ as a function of $\hat{\Sigma}$, and from this $M(T) = \Sigma = 2\pi T\hat{\Sigma}$. The resulting function $M(T)$ is plotted in Fig. 23.4.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig23.4.png}
\caption{Solution of the temperature-dependent gap equation, showing the decrease of the fermion mass $M(T) = \Sigma(T)$ with increasing temperature $T/T_c$.}
\end{figure}

It is quite easy to calculate the way in which $M(T)$ vanishes as $T$ approaches the critical temperature $T_c$. We simply expand

$$\tilde{S}_1(\hat{\Sigma}) = \sum_{k=1}^{\infty} \left(-\frac{1}{2}\right)^k \hat{\Sigma}^{2k} \sum_{m=0}^{\infty} \left(\frac{1}{m + \frac{1}{2}}\right)^{2k+1}$$

$$= -\frac{1}{2} \hat{\Sigma}^2 \sum_{m=0}^{\infty} \left(\frac{1}{m + \frac{1}{2}}\right)^3 + \frac{3}{8} \hat{\Sigma}^4 \sum_{m=0}^{\infty} \left(\frac{1}{m + \frac{1}{2}}\right)^5 - \ldots. \quad (23.240)$$

Expanding the logarithm near $T_c$ as

$$\log \left(\frac{T}{T_c}\right) \sim T/T_c - 1,$$

we find in a first approximation

$$M(T) = \Sigma \approx \pi T_c \sqrt{\frac{8}{7\zeta(3)}} \sqrt{1 - \frac{T}{T_c}}. \quad (23.241)$$
In the opposite limit of low temperatures, the series (23.240) converges very slowly. It is, however, easy to find out how $\Sigma$ behaves near $T = 0$ by expanding in (23.225)

$$\tanh \frac{\Omega}{2T} - 1 = 2 \sum_{\tilde{m}=1}^{\infty} (-)^\tilde{m} e^{-\tilde{m}\Omega/T}. \quad (23.242)$$

Using the integral

$$\int_{-\infty}^{\infty} dp \frac{e^{-\tilde{m}\sqrt{p^2 + \Sigma^2}/T}}{\sqrt{p^2 + \Sigma^2}} = 2K_0 (\tilde{m}\Sigma/T), \quad (23.243)$$

where $K_0(z)$ is the associated Bessel function, we find the alternative expression

$$\log \frac{\Sigma}{\Sigma_0} = S_1 (\hat{\Sigma}) = 2 \sum_{\tilde{m}=1}^{0} (-)^\tilde{m} K_0 \left(2\pi\tilde{m}\hat{\Sigma}\right). \quad (23.244)$$

For small $T$, $K_0 (\tilde{m}\Sigma/T)$ has the asymptotic behavior

$$K_0 \left(\frac{\tilde{m}\Sigma}{T}\right) \sim \sqrt{\frac{\pi}{2\tilde{m}\Sigma/T}} e^{-\hat{\Sigma}/T}, \quad (23.245)$$

so that we can expand

$$\Sigma = \Sigma_0 \left(1 - \sqrt{\frac{2\pi}{\Sigma_0/T}} e^{-\Sigma/T}\right) + O \left(e^{-2\Sigma/T}\right) \quad (23.246)$$

and see that $\Sigma$ approaches its $T = 0$ -value $\Sigma_0$ exponentially fast from below (see Fig. 23.4).

Note that the properties of the gap equation (23.224) are quite similar to those of the corresponding equation (19.110) in the nonlinear $\sigma$-model.

It is instructive to go through the same discussion once more in $D$ dimensions. For this it is convenient to rewrite the gap equation (23.234) in accordance with the general procedure of dimensional regularization in Section 11.5 as

$$\frac{1}{g_0} - 2^{D/2} \int_0^\infty dt \int \frac{d^Dp}{(2\pi)^D} e^{-\tau(p^2 + \Sigma^2)} = \frac{\Sigma^{D-2}}{\pi} 2^{1-D/2} \pi^{(1-D)/2} \Gamma(3/2 - D/2) S_1 (\hat{\Sigma}). \quad (23.247)$$

The left-hand side corresponds to the zero-temperature gap equation and is integrated directly to

$$\frac{1}{g_0} - b_\epsilon \frac{D}{2} \Sigma^\epsilon = \frac{\mu^\epsilon}{g(\mu)} - b_\epsilon \left(\frac{D}{2} \Sigma^\epsilon - \mu^\epsilon\right), \quad (23.248)$$

while

$$S_1 (\hat{\Sigma}) = \pi \Sigma^{D-2} 2^{D-1} \pi^{(D-1)/2} \left[\Gamma \left(\frac{3}{2} - \frac{D}{2}\right)\right]^{-1} \quad (23.249)$$

$$\times \int_0^\infty dt \int \frac{d^{D-1}p}{(2\pi)^{D-1}} \left(T \Sigma \sum_{\tilde{m}=-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d\omega_\tilde{m}}{2\pi} \right) \exp \left[-\tau \left(\omega_\tilde{m}^2 + p^2 + \Sigma^2\right)\right].$$
The momentum integrations can now be done, with the result

\[
\frac{1}{g_0} - 2^{D/2} \frac{1}{2D} \pi^{-D/2} \int_0^\infty \frac{d\tau}{\tau^{D/2}} e^{-\tau \Sigma^2} = \Sigma^{D-2} 2^{1-D/2} \pi^{-1/2} \frac{\Gamma \left( \frac{3}{2} - \frac{D}{2} \right)}{\Gamma \left( \frac{3}{2} \right)} S_1(\hat{\Sigma}), \tag{23.250}
\]

where

\[
S_1(\hat{\Sigma}) = 2\pi \Sigma^{2-D} \left[ \Gamma \left( \frac{3}{2} - \frac{D}{2} \right) \right]^{-1} \int_0^\infty \frac{d\tau}{\tau^{D/2-1}} \left( T \sum_{m=0}^\infty -\int_0^\infty \frac{d\omega_m}{2\pi} \right) e^{-\tau (\omega_m + \Sigma^2)}. \tag{23.251}
\]

By performing the \(\tau\)-integral on the right-hand side, we find

\[
S_1(\hat{\Sigma}) = 2\pi \Sigma^{2-D} \left( T \sum_{m=0}^\infty -\int_0^\infty \frac{d\omega_m}{2\pi} \right) (\omega_m^2 + \Sigma^2)^{(D-3)/2}
= \Sigma^{2-D} \left( \sum_{m=0}^\infty -\int_{-1/2}^\infty dm \right) \left[ \left( m + \frac{1}{2} \right)^2 + \Sigma^2 \right]^{(D-3)/2}. \tag{23.252}
\]

Let us expand the sum over \(m\) formally in a power series of \(\Sigma\). Its contributions to \(S_1(\hat{\Sigma})\) are

\[
S_1(\hat{\Sigma}) \bigg|_{\text{sum part}} = \Sigma^{2-D} \sum_{k=0}^\infty \binom{(D-3)/2}{k} \hat{\Sigma}^k \zeta (2k + 3 - D, 1/2), \tag{23.253}
\]

where

\[
\zeta(z, a) \equiv \sum_{k=0}^\infty \frac{1}{(k + a)^z} \tag{23.254}
\]

is Riemann’s zeta function, with \(\zeta(z, 1) = \zeta(z)\) of Eq. (23.230). The integral over \(\omega_m\) in (23.252) subtracts from this the \(T = 0\)-limit of (23.253):

\[
2^{D/2-1} \pi^{(D-1)/2} \left[ \Gamma \left( \frac{3}{2} - \frac{D}{2} \right) \right]^{-1} b_0 \frac{D}{2} \pi = \sqrt{\pi} \left[ \Gamma \left( \frac{3}{2} - \frac{D}{2} \right) \right]^{-1} \frac{1}{2} \Gamma \left( 1 - \frac{D}{2} \right). \tag{23.255}
\]

For \(D\) near an even dimension \(\bar{D}\), say \(D = \bar{D} + \epsilon\), the \(k\)th term with \(k = \bar{D}/2 - 1\) has a \(1/\epsilon\)-singularity coming from the zeta function

\[
\zeta \left( 1 + \bar{D} - D, 1/2 \right) \sim \frac{1}{\epsilon} [1 + \epsilon \psi(1/2)]. \tag{23.256}
\]

This is canceled by a singularity of opposite sign in \(b_\epsilon\). We now expand up to first order in \(\epsilon\):

\[
\Sigma^{2-D} \left( \binom{D-3/2}{k} \right)_{k=\bar{D}/2} \hat{\Sigma}^{\bar{D}-2} \zeta (1 - \epsilon, 1/2) = \frac{\Gamma((D-1)/2)}{\Gamma(D/2) \Gamma(1/2 + \epsilon/2)} \hat{\Sigma}^{-\epsilon} \zeta (1 - \epsilon, 1/2)
\approx \frac{1}{\Gamma(3/2 - D/2) \Gamma(D/2) \Gamma(1/2)} \pi^{-1/2} \cos \pi \left( \bar{D}/2 - 1 \right) \frac{1}{\epsilon} \left[ 1 + \frac{\epsilon}{2} \left( -2 \log \hat{\Sigma} + \psi(1/2) \right) \right]
\approx -\frac{(-1)^D/2}{\Gamma(3/2 - D/2) \Gamma(D/2)} \sqrt{\pi} \epsilon \left[ 1 + \frac{\epsilon}{2} \left( -2 \log \hat{\Sigma} + \psi(1/2) + \psi \left( 3/2 - D/2 \right) \right) \right].
\]
Here \( \psi(x) \) is the Digamma function \( \Gamma'(z)/\Gamma(z) \). Similarly we expand the \( T = 0 \) -term (23.255) as:

\[
\frac{\sqrt{\pi}}{2} \left[ \Gamma \left( \frac{3}{2} - \frac{D}{2} \right) \right]^{-1} \Gamma \left( 1 - \frac{D}{2} \right) = \frac{(-1)^{D/2}}{\Gamma(D/2)\Gamma(3/2 - D/2)} \frac{\sqrt{\pi}}{\epsilon} \left\{ 1 - \frac{\epsilon}{2} \left[ \psi(\frac{D}{2}) - \psi \left( \frac{3}{2} - \frac{D}{2} \right) \right] \right\}. \tag{23.257}
\]

Subtracting the two terms from each other yields

\[
- \frac{(-1)^{D/2}}{\Gamma(D/2)\Gamma(3/2 - D/2)} \frac{\sqrt{\pi}}{\epsilon} \left[ 2 \log \hat{\Sigma} - \psi(1/2) - \psi \left( \frac{D}{2} \right) \right] - \frac{(-1)^{D/2}}{\Gamma(D/2)\Gamma(3/2 - D/2)} \frac{\sqrt{\pi}}{\epsilon} \left[ 2 \log(2e^\gamma \hat{\Sigma}) - \psi \left( \frac{D}{2} \right) - \gamma \right]. \tag{23.258}
\]

On the right-hand side we may replace the Digamma function \( \psi(\frac{D}{2}) \) for \( \hat{D} = 2 \) by \( \psi(\frac{D}{2}) = -\gamma \), and for \( \hat{D} > 2 \), \( \psi(\frac{D}{2}) = -\gamma + 2/\hat{D} + \ldots + 2/(\hat{D} - 2) \). Going now to the limit \( \epsilon \to 0 \), we obtain for even \( D = \hat{D} \):

\[
S_1(\hat{\Sigma}) = - \frac{(-1)^{D/2}}{\Gamma(D/2)\Gamma(3/2 - D/2)} \frac{\sqrt{\pi}}{2} \left[ 2 \log(2e^\gamma \hat{\Sigma}) - \psi(D/2) - \gamma \right] + \sum_{k=0, k \neq D/2-1}^{\infty} \left( \frac{(D - 3)/2}{k} \right) \hat{\Sigma}^{2k+2-D} \zeta(2k + 3 - D, 1/2). \tag{23.259}
\]

By taking the negative powers of \( \hat{\Sigma}^2 \) out of the sum, we split

\[
S_1(\hat{\Sigma}) = - \frac{(-1)^{D/2}}{\Gamma(D/2)\Gamma(9/2 - D/2)} \frac{\sqrt{\pi}}{2} \left[ 2 \log \left( 2e^\gamma \hat{\Sigma} \right) - \psi(D/2) - \gamma \right] + \sum_{k}^{D/2-2} \left( \frac{(D - 3)/2}{k} \right) \hat{\Sigma}^{2k+2-D} \zeta(2k + 3 - D, 1/2) + \hat{\Sigma}_1(\hat{\Sigma}) \tag{23.260}
\]

where

\[
\hat{\Sigma}_1(\hat{\Sigma}) \equiv \sum_{k=D/2}^{\infty} \left( \frac{(D - 3)/2}{k} \right) \hat{\Sigma}^{2k+2-D} \zeta(2k + 3 - D, 1/2). \tag{23.261}
\]

The sum \( \hat{\Sigma}_1(\hat{\Sigma}) \) can also be calculated from the sum part in (23.252) by performing \( D/2 \) subtractions:

\[
\hat{\Sigma}_1(\hat{\Sigma}) = \hat{\Sigma}^{2-D} \sum_{m=0}^{\infty} \left\{ \left[ \left( m + \frac{1}{2} \right)^2 + \hat{\Sigma}^2 \right]^{(D-3)/2} - \left( m + \frac{1}{2} \right)^{D-3} \frac{D-3}{2} \hat{\Sigma}^2 \left( m + \frac{1}{2} \right)^{D-5} + \ldots \right\}. \tag{23.262}
\]
This sum is convergent and starts out with \( \tilde{S}_1(0) = 0 \). It is the generalization of \( \tilde{S}_1(\hat{\Sigma}) \) in Eq. (23.238) to arbitrary even \( D \).

Let us also generalize to arbitrary \( D \) the other expressions for \( S_1(\hat{\Sigma}) \) in Eqs. (23.225) and (23.244). It gives the finite-temperature correction to the gap equation

\[
\Sigma^{-1}i\Delta T \text{Tr} \frac{1}{i\hat{\phi} - \Sigma} = \left[ \frac{2^{D/2}}{\pi} \int \frac{d^{D-1}p_E}{(2\pi)^{D-1}} T \sum_{m=\infty}^{\infty} \frac{1}{\omega_m^2 + p_E^2 + \Sigma^2} \right] - \left[ \ldots \right]_{T \to 0}
\]

\[
= \frac{\Sigma^{D-2}}{\pi} 2^{1-D/2}\pi^{(1-D)/2} \Gamma\left(\frac{3}{2} - \frac{D}{2}\right) S_1(\hat{\Sigma}).
\]  

(23.263)

The symbol

\[
\Delta_T X(T) \equiv X(T) - X(0)
\]

(23.264)
denotes the finite-temperature correction to any quantity \( X(T) \). By comparison with (23.249), we can identify

\[
S_1(\hat{\Sigma}) = \frac{2^{D-2}\pi^{(D+1)/2}}{\pi^{\Sigma^{2-D}}} \left[ \Gamma\left(\frac{3}{2} - \frac{D}{2}\right) \right]^{-1} \int \frac{d^{D-1}p}{(2\pi)^{D-1}} \frac{1}{\Omega} \left( \tanh \frac{\Omega}{2T} - 1 \right)
\]

\[
\Sigma^{2-D} \int_0^{\infty} dp p^{D-2} \frac{1}{\Omega} \left( \tanh \frac{\Omega}{2T} - 1 \right).
\]  

(23.265)

Writing \( \tanh (\Omega/2T) - 1 \) as \(-2(1 + e^{\Omega/T})^{-1}\), and expanding this in powers of \( e^{-\Omega/T} \), we have the alternative expression

\[
S_1(\hat{\Sigma}) = \frac{2\pi}{\Gamma\left(\frac{1}{2}\right)} \sum_{\hat{m}} \left( -\right)^{\hat{m}} \int_0^{\infty} dp p^{D-2} \Omega^{-1} e^{-\hat{m}\Omega/T}.
\]  

(23.266)

Using the integral representation for the modified Bessel function

\[
K_\nu(z) = \left(\frac{z}{2}\right)^\nu \frac{\Gamma(1/2)}{\Gamma(\nu + 1/2)} \int_0^{\infty} ds s^{2\nu} (s^2 + 1)^{-1/2} e^{-z\sqrt{s^2 + 1}},
\]  

(23.267)

we arrive at the expansion

\[
S_1(\hat{\Sigma}) = \pi \Gamma\left(\frac{3}{2} - \frac{D}{2}\right) \left( \frac{1}{2} \right) \]

\[
2^{D/2} \sum_{\hat{m}} \left( -\right)^{\hat{m}} \left( 2\pi \hat{m} \hat{\Sigma} \right)^{1-D/2} K_{D/2-1}(2\pi \hat{m} \hat{\Sigma}),
\]  

(23.268)

which, for \( D = 2 \), reduces properly to (23.244).

Let us also calculate the finite-temperature correction to the free energy for all dimensions \( D \). By integrating the gap equation (23.247) in \( \Sigma \), using (23.248) and (23.249), we obtain

\[
\frac{1}{Nv}(\Sigma) = \frac{\Sigma^2}{2g} + i\text{Tr} \log(i\hat{\phi} - \Sigma) + i\Delta_T \log(i\hat{\phi} - \Sigma),
\]  

(23.269)
with
\[ i \Delta_T \text{Tr} \log(i\vartheta - \Sigma) = -2^{D/2} \int \frac{d^{D-1}p_E}{(2\pi)^D} \frac{1}{2} \left( T \sum_{m=-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d\omega_m}{2\pi} \right) \log \left( \omega_m^2 - \Omega^2 \right) \]
\[ = -2^{D/2} T \int \frac{d^{D-1}p_E}{(2\pi)^{D-1}} \log \left( 1 + e^{-\Omega/T} \right). \] (23.270)

We now introduce a function \( S_0(\hat{\Sigma}) \), so that
\[ i \Delta_T \text{Tr} \log(i\vartheta - \Sigma) \equiv \frac{1}{N} \Delta_T v(\Sigma) = -\Sigma \frac{D}{\pi} 2^{-D/2} \pi^{1/2-D/2} \Gamma \left( \frac{1}{2} - \frac{D}{2} \right) S_0(\hat{\Sigma}), \] (23.271)
and find for \( S_0(\hat{\Sigma}) \) the following expression
\[ S_0(\hat{\Sigma}) = 4\pi \Sigma^{-D} \left[ \Gamma \left( \frac{1}{2} - \frac{D}{2} \right) \Gamma \left( \frac{D}{2} - \frac{1}{2} \right) \right]^{-1} T \int_{0}^{\infty} dp \ p^{D-2} \log \left( 1 + e^{-\Omega/T} \right), \] (23.272)
or, alternatively,
\[ S_0(\hat{\Sigma}) = \hat{\Sigma}^{-D} \left( \sum_{m=0}^{\infty} - \int_{-1/2}^{\infty} dm \right) \left[ \left( m + \frac{1}{2} \right)^2 + \hat{\Sigma}^2 \right]^{(D-1)/2}. \] (23.273)
The new function \( S_0(\hat{\Sigma}) \) is related to \( S_1(\hat{\Sigma}) \) by an integration:
\[ S_0(\hat{\Sigma}) = \frac{D}{2} - 1 \hat{\Sigma}^{-D} \int_{0}^{\hat{\Sigma}^2} d\Sigma^2 \left[ \Sigma^2 S_1(\hat{\Sigma}) \right]. \] (23.274)

We can now integrate (23.259)-(23.262), and see that the latter expression is separated into a convergent sum
\[ \tilde{S}_0(\hat{\Sigma}) = \hat{\Sigma}^{-D} \sum_{m=0}^{\infty} \left[ \left( m + \frac{1}{2} \right)^2 + \Sigma_T^2 \right]^{(D-1)/2} \]
\[ - \left( m + \frac{1}{2} \right)^{(D-1)/2} - \frac{D-1}{2} \hat{\Sigma}^2 \left( m + \frac{1}{2} \right)^{D-3} \ldots \right], \] (23.275)
with \( D/2 + 1 \) subtractions, an additional logarithmic term
\[ \frac{(-)^{D/2}}{\Gamma(D/2) \Gamma(1/2 - D/2)} \sqrt{\pi} \left[ 2 \log(2\gamma \hat{\Sigma}) - \psi(D/2 + 1/2) - \gamma \right], \] (23.276)
and a sum over negative powers of \( \hat{\Sigma} \) from the integral over the second term in (23.260).

Alternatively, we find from the integral (23.274) over the expansion (23.268):
\[ S_0(\Sigma) = 2\pi 2^{D/2} \left[ \Gamma \left( \frac{1}{2} - \frac{D}{2} \right) \Gamma \left( \frac{1}{2} \right) \right]^{-1} \sum_{m=1}^{\infty} (-)^m \left( 2\pi \bar{m} \hat{\Sigma} \right)^{D/2} K_{D/2}(2\pi \bar{m} \hat{\Sigma}). \] (23.277)
According to Eq. (23.274), this has to satisfy
\[
\frac{d \hat{\Sigma}^D S_0}{d \hat{\Sigma}^2} = -\frac{1 - D}{2} \hat{\Sigma}^{D-2} S_1, \tag{23.278}
\]
and a comparison with (23.268) shows that it does, since
\[
\left[ z^{D/2} K_{D/2}(z) \right]' = -z^{D/2} K_{D/2-1}(z). \tag{23.279}
\]
In the high-temperature limit, we use the small-z behavior
\[
\left[ \frac{z}{D/2} K_{D/2-1}(z) \right]' = -\frac{z}{D/2} K_{D/2} - \frac{1}{2} \frac{\Gamma(D/2)}{\Gamma(1/2 - D/2)} \sum_{\tilde{m}=1}^{\infty} (-1)^{\tilde{m}} (\tilde{m})^{-D} \tag{23.279}
\]
for small\( z \), and
\[
\sum_{\tilde{m}=1}^{\infty} (-1)^{\tilde{m}} (\tilde{m})^{-D} = -\left(1 - 2^{1-D}\right) \zeta(D). \tag{23.282}
\]
For \( D = 2 \), we thus obtain
\[
S_0(\hat{\Sigma}) \approx \frac{\pi}{2} \hat{\Sigma} - \frac{4}{\pi} T^2 \zeta(2). \tag{23.283}
\]
Recalling (23.271), this yields the low-temperature behavior of the effective potential in two dimensions:
\[
\frac{1}{N} \Delta_T v(\Sigma) \to \frac{1}{\pi} T^2 \zeta(2) = \frac{T^2 \pi}{6}. \tag{23.284}
\]
This is the well-known free energy of a hot or massless Fermi Gas in two dimensions. It could have been obtained directly by dimensional regularization from the \( \Sigma = 0 \)-expression
\[
\frac{1}{N} \Delta_T v(\Sigma) = -\int \frac{dp}{2\pi} \left( T \sum_{m=-\infty}^{\infty} - \int \frac{d\omega_m}{2\pi} \right) \log(\omega_m^2 + p^2)
\]
\[
= -T \sum_{m=-\infty}^{\infty} \sqrt{\omega_m^2} = -4\pi T^2 \zeta(-1, 1/2) = -T^2 \frac{\pi}{6}, \tag{23.285}
\]
the last equation following from \( \zeta(-1, 1/2) = (2^{-1} - 1)\zeta(-1) = 1/24 \).
In \( D = 4 \) dimensions, the result is
\[
S_0(\hat{\Sigma}) = \frac{1}{\pi^3} \hat{\Sigma}^{-4} \left[ \Gamma(-3/2) \sqrt{\pi} \right]^{-1} \sum_{\tilde{m}=1}^{\infty} (-1)^{\tilde{m}} \tilde{m}^{-4}
\]
\[
= \Sigma^{-4} 2^4 \pi T^4 \left[ \Gamma(-3/2) \sqrt{\pi} \right]^{-1} \frac{7}{8} \zeta(4), \tag{23.286}
\]
and for the potential:
\[
\frac{1}{N} \Delta_T v(\Sigma) \rightarrow -4T^4 \frac{7}{8} \frac{\pi^2}{g_D}. \tag{23.287}
\]

The latter follows directly from
\[
\frac{1}{N} \Delta_T v(\Sigma) = \frac{2}{2\pi^3} \int d^3p \int \frac{d\omega}{2\pi} T \sum_{m=-\infty}^{\infty} e^{-\tau(\omega^2_m+p^2)}
= (4\pi^{3/2})^{-1} \Gamma (-\frac{3}{2}) T \sum_{m=-\infty}^{\infty} (\omega^2_m)^{3/2} = T^4 4\pi^{3/2} \Gamma (-\frac{3}{2}) \zeta (-3, \frac{1}{2}) . \tag{23.288}
\]

It is the fermion equivalent to the *Stefan-Boltzmann law* for the free energy density \( f \) of hot or massless fermions. Compare the discussion for the bosonic model in the paragraph below Eq. (19.169), where we show that, for a pure power behavior of \( T^n \) of \( f \), the entropy density \( s = -\partial f/\partial T \) is related to \( f \) by a factor \(-4/T\). On the other hand, the internal energy density \( u \equiv E/V = f + TS \) and the specific heat at constant volume \( c_V = \partial u/\partial T \big|_V \) carry the relative factors \( 1-n \) and \( (1-n)n/T \) with respect to \( f \), respectively.

Thus, Eq. (23.288) yields for these three thermodynamic quantities
\[
f = -\frac{1}{3}u = -\frac{T}{4}s = -\frac{T}{12}c_V = -4 \frac{2}{3} \frac{\pi^2}{60} \frac{7}{8} T^4 . \tag{23.289}
\]

In proper physical units, this reads
\[
f = -\frac{1}{3}u = -\frac{T}{4}s = -\frac{T}{12}c_V = -4 \frac{2}{3} \frac{\sigma}{c} \frac{7}{8} T^4 , \tag{23.290}
\]

where \( \sigma \) is the Stefan-Boltzmann constant (19.172). The prefactor 4 accounts for the two polarization degrees of freedom of both particles and antiparticles, while the factor 7/8 is typical for fermions.

Recall that the black-body radiation of photons has a free energy
\[
f = -\frac{1}{3}u = -\frac{T}{4}s = -\frac{T}{12}c_V = -2 \frac{\pi^2}{90} T^4 = -2 \frac{2}{3} \frac{\sigma}{c} T^4 , \tag{23.291}
\]

where the prefactor 2 on the right-hand side accounts now for the two polarization degrees of photons. In contrast to the fermion case, particles and antiparticles coincide.

### Notes and References

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P. Langacker, Phys. Rev. Letters 34, 1592 (1975);
The Dirac algebra in arbitrary dimensions $D$ is discussed in

For calculations beyond the leading $1/N$ approximation see

The particular citations in this chapter refer to:
[1] A comprehensive explanation of the low-energy properties of hadron physics was given on the basis of this model by
[4] The model in $D = 2 + \epsilon$ dimensions is named after
[7] See the textbook
[8] See the lecture
[9] Singularities of this type were first discussed by Landau in quantum electrodynamics (QED) and are named after him. See

[10] For details, we refer to Bernard de Wit’s lecture notes on this subject in
http://www.phys.uu.nl/~bdewit/ftip/AppendixE.pdf.