Any intelligent fool can make things bigger, more complex, and more violent. It takes a touch of genius—and a lot of courage—to move in the opposite direction.

E. F. Schumacher (1911–1977)

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Nonlinear $\sigma$-Model

Another field theoretic model that is exactly solvable in a limit $N \to \infty$ and illustrates some typical properties of such theories is the so-called nonlinear $\sigma$-model. It has its origin in statistical mechanics where it arises as a limit of the classical Heisenberg spin model of ferromagnetism. There it is known under the name of spherical model. The nonlinear $\sigma$-model yields a complementary approach to the critical phenomena described previously by the $O(N)$-symmetric $\phi^4$-theory.

19.1 Definition of Classical Heisenberg Model

The model consists of a fluctuating field $n_a(x)$ of unit vectors with $N$ components

$$n_a(x) \quad (a = 1, \ldots, N), \quad n^2_a(x) = 1. \quad (19.1)$$

We shall discuss here the euclidean formulation in which the model is described by a partition function

$$Z = \prod_x \left[ \int \frac{d^{N-1}n_a(x)}{S_N} \right] \exp \left[ -\frac{1}{2g} \int d^Dx [\nabla n_a(x)]^2 \right]. \quad (19.2)$$

The constant $S_N$ is the surface of a sphere in $N$ dimensions covered by the directional integral $d^Dn^a$. This ensures a unit integral $\int d^{N-1}n_a/S_N = 1$. Explicitly, $S_N = 2\pi^{N/2}/\Gamma(N/2)$ [recall (11.126)]. This model can be thought of as arising from the euclidean action of a previous $O(N)$-symmetric $\phi^4$-theory in Chapter 18, where

$$A = \int d^Dx \left[ \frac{1}{2} (\nabla \phi_a)^2 + \frac{m^2}{2} \phi_a^2 + \frac{g}{4} (\phi_a^2)^2 \right] \quad (19.3)$$

by going to the limit of very large negative $m^2$. For negative $m^2$, the $O(N)$-symmetry is spontaneously broken, and the initially degenerate $N$ field fluctuations decompose into a massive fluctuation of the field size $\phi \equiv \sqrt{\phi_a^2}$, the so-called radial fluctuations, and $N - 1$ fluctuation modes around the direction of the field $\phi_a$, the so-called azimuthal fluctuations, described by the unit vectors

$$n_a \equiv \phi_a/\sqrt{\phi_a^2}. \quad (19.4)$$
These are the Goldstone bosons which arise whenever there is a spontaneous breakdown of a continuous symmetry (see Section 16.2). In the limit of \( m^2 \to \infty \), the size fluctuations are completely frozen out. The action can be written approximately as

\[
\mathcal{A} \to \int d^D x \left[ -\frac{(m^2)^2}{4g} + \frac{1}{2} |\phi|^2 (\nabla n_a)^2 \right].
\]  

(19.5)

Dropping the first term with the constant condensation energy, we arrive at the \( O(N) \) nonlinear \( \sigma \)-model (19.3) with a coupling constant

\[
g = \frac{1}{|\phi|^2}.
\]  

(19.6)

Readers familiar with models of statistical mechanics will recognize the intimate relation of the model (19.2) with the so-called classical \( O(N) \)-symmetric Heisenberg model of ferromagnetism. It is defined on a lattice, where its energy is written as a sum over nearest-neighbor interactions between local \( D \)-dimensional spin vectors \( S_i \), which are conventionally normalized to unit length:

\[
E = -\frac{J}{2} \sum_{\{i,j\}} S_i \cdot S_j \quad \text{with} \quad J > 0 \quad \text{and} \quad S_i^2 = 1.
\]  

(19.7)

The symbol \( \{i,j\} \) denotes neighboring index pairs. If the spin vectors \( S_i \) are not of unit length, the length fluctuates and \( S_i \) represents an order field. The expectation value \( M \equiv \langle S_i \rangle \) is often called magnetization. If \( M \) is nonzero, the system is said to exhibit a spontaneous magnetization. This happens only at low temperatures. Above a critical temperature \( T_{MF}^c \), such a system becomes usually demagnetized with \( M = 0 \). This is referred to as the normal state. Then \( T_{MF}^c \) is called the Curie temperature.

A special case of this model is the famous Ising model where the direction of the vector \( S_i \) is restricted to a single axis, pointing parallel or antiparallel to it. The phase transition of magnetic systems with strong anisotropy can be described by this model. Then \( S_i \) can be replaced by a scalar field with positive and negative signs, and the symmetry which is spontaneously broken in the low-temperature phase is the reflection symmetry \( S(x) \to -S(x) \).

Let us denote the lattice points by a vector \( x \) on a simple cubic lattice, i.e., we let \( x \) take the values

\[
x_{(m_1,m_1,\ldots,m_d)} \equiv \Sigma_{i=1}^D m_i \mathbf{i}_i,
\]  

(19.8)

where \( \mathbf{i} \) are the basis vectors of a \( D \)-dimensional hypercubic lattice, and \( m_i \) are integer numbers 0, \( \pm 1, \pm 2, \ldots \).

The partition function associated with the generalized Heisenberg model is

\[
Z = \sum_{\text{spin configurations}} e^{-E/k_B T},
\]  

(19.9)
where the sum runs over all possible spin configurations. The unit spin vectors $\mathbf{S}_i$ are now identified with the unit vector fields $\mathbf{n}(x)$ in (19.1). The sum over products $\sum_{\{i,j\}} \mathbf{n}_i \mathbf{n}_j$ in the energy (19.7) can then be rewritten as

$$
\sum_{\{i,j\}} \mathbf{n}_i \mathbf{n}_j = \sum_{\{i,j\}} [\mathbf{n}_i (\mathbf{n}_j - \mathbf{n}_i) + 1] = 2 \sum_x \sum_i [\mathbf{n}(x)[\mathbf{n}(x + i) - \mathbf{n}(x)]] + 1
$$

$$
= - \sum_x \left\{ \sum_i [\mathbf{n}(x + i) - \mathbf{n}(x)]^2 - 2D \right\}. \quad (19.10)
$$

Now we introduce lattice gradients:

$$
\nabla_i \mathbf{n}(x) \equiv \frac{1}{a} [\mathbf{n}(x + i) - \mathbf{n}(x)], \quad (19.11)
$$

$$
\nabla_i \mathbf{n}(x) \equiv \frac{1}{a} [\mathbf{n}(x) - \mathbf{n}(x - i)], \quad (19.12)
$$

where $a$ is the lattice spacing, and we can rewrite (19.10) as

$$
\sum_{\{i,j\}} \mathbf{n}_i \mathbf{n}_j = - \sum_x \left\{ a^2 \sum_i [\nabla_i \mathbf{n}(x)]^2 - 2D \right\}. \quad (19.13)
$$

With this, the partition function (19.9) takes the form

$$
Z = \prod_x \left[ \int \mathbf{n}(x) \exp \left\{ -\frac{J}{2k_B T} \sum_x \left( \sum_i a^2[\nabla_i \mathbf{n}(x)]^2 - 2D \right) \right\} \right]. \quad (19.14)
$$

Here we can use the lattice version of partial integration\(^1\)

$$
\sum_x \sum_i [\nabla_i f(x)] g(x) = - \sum_x \sum_i [f(x) \nabla_i g(x)], \quad (19.15)
$$

which is valid for all lattice functions with periodic boundary conditions, to express (19.13) in terms of a lattice version of the Laplace operator $\nabla_i \nabla_i$, where repeated lattice unit vectors are summed. Then

$$
\sum_x \left\{ \sum_i a^2[\nabla_i \mathbf{n}(x)]^2 - 2D \right\} = - \sum_x \left[ a^2 \mathbf{n}(x) \nabla_i \nabla_i \mathbf{n}(x) + 2D \right]. \quad (19.16)
$$

Ignoring the irrelevant constant term $\sum_x 2D$, we can then write the sum over all spin configurations in the partition function (19.9) as a product of integrals over a unit sphere at each lattice point

$$
Z = \prod_x \left[ \int \mathbf{n}(x) \exp \left\{ \frac{Ja^2}{2k_B T} \sum_x \mathbf{n}(x) \nabla_i \nabla_i \mathbf{n}(x) \right\} \right]. \quad (19.17)
$$

For small $a$, the sum in (19.16) has a continuum limit

$$a^{2-D} \int d^D x \left\{ [\nabla n_a(x)]^2 - \frac{2D}{a^2} \right\} = -a^{2-D} \int d^D x \left\{ n_a(x) \nabla^2 n_a(x) + \frac{2D}{a^2} \right\}. \quad (19.18)$$

After dropping the irrelevant $D/a^2$ term, the Boltzmann factor in the partition function (19.17) agrees with the exponential in the partition function (19.2) of the nonlinear $\sigma$-model with a coupling constant

$$g = \frac{k_B T}{J} a^{D-2}. \quad (19.19)$$

In these models, $g$ grows with temperature, and due to the significance of the model for statistical mechanics, we find it convenient to replace the prefactor $1/g$ in (19.2) by a reduced inverse temperature $1/T$, thus discussing the model as a function of a reduced temperature rather than of the coupling strength. As we see from (19.19), the so-defined reduced temperature corresponds to the true one if that is measured in units of $J/a^{D-2}$.

As mentioned in the beginning of this chapter, the limit $N \to \infty$ of this model is referred to as the spherical model.

### 19.2 Spherical Model

The $N \to \infty$ -limit of the Heisenberg model was first solved by Berlin and Kac in 1952.\footnote{T.H. Berlin and M. Kac, Phys. Rev. 86, 82 (1952).} The solution of the model is quite simple. We liberate $n_a$ from its constraint $n_a^2 = 1$, allowing each component to fluctuate between $-\infty$ and $\infty$. The constraint is enforced by an extra functional integral over a Lagrange multiplier $\lambda(x)$. Thus we write the lattice partition function as

$$Z = \prod_x \left[ \int_{-\infty}^{\infty} \frac{d\lambda(x)}{4\pi i T} \int_{-\infty}^{\infty} \frac{d^N n_a(x)}{S_N} \right] \exp \left\{ -\frac{1}{2g} \int d^D x \left[ (\nabla n_a)^2 + \lambda (n_a^2 - 1) \right] \right\}. \quad (19.20)$$

In this form, the integrals over the components $n_a$ are Gaussian, so that they can be performed, leaving only a functional integral over $\lambda(x)$:

$$Z = \prod_x \left[ \int_{-\infty}^{\infty} \frac{d\lambda(x)}{4\pi i T} \right] \exp \left[ -\frac{N}{2} \text{Tr} \log \left[ -\nabla^2 + \lambda(x) \right] + \frac{1}{2T} \int d^D x \lambda(x) \right]. \quad (19.21)$$

In the limit $N \to \infty$, this is the partition function of the spherical model. In this limit, the first term in the exponent grows so large that the $\lambda$ fluctuations become frozen at the extremum of the action. The limit is non-trivial if we allow $g^{-1}$ to carry a factor $N$. Thus we set

$$g \equiv t/N. \quad (19.22)$$
Now we can go to large $N$, and find the partition function
\[
Z \xrightarrow{N \to \infty} e^{-F/t},
\] (19.23)
with a free energy per $O(N)$-degree of freedom and per temperature unit equal to
\[
\frac{F}{t} \xrightarrow{N \to \infty} \frac{1}{2} \left[ \text{Tr} \log \left[ -\nabla^2 + \lambda(x) \right] - \frac{1}{t} \int d^Dx \lambda(x) \right],
\] (19.24)
to be evaluated at the extremum where the functional derivative $\delta F/\delta \lambda(x)$ vanishes. Explicitly, this condition reads
\[
\frac{1}{t} = \left[ -\nabla^2 + \lambda(x) \right]^{-1}(x, x).
\] (19.25)
Each solution $\lambda_i(x)$ of this equation must be found, and the partition function is given by the sum over the Boltzmann factors (19.23) for all of them. In general, these solutions are hard to find. One solution, however, is straightforward: that with a constant $\lambda(x) \equiv \lambda$, for which the free energy (19.24) yields
\[
\frac{F}{tL^D} \xrightarrow{N \to \infty} \frac{1}{2} \frac{S_D}{(2\pi)^D} \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{D}{2} \right) \lambda^{D/2} - \frac{\lambda}{2t},
\] (19.28)
where $L^D$ is the spatial volume of the system. The extremality condition (19.25) turns into the so-called gap equation:
\[
\frac{1}{t} = \frac{S_D}{2(2\pi)^D} \frac{1}{\bar{S}_D} \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{D}{2} \right) \lambda^{D/2} - \frac{\lambda}{2t},
\] (19.29)
Its solution will now be discussed.

### 19.3 Free Energy and Gap Equation in $D > 2$ Dimensions

Using formulas (11.134) and (11.137), the integrals in (19.26) and (19.27) can be done right-away in $D$ dimensions with the result
\[
\frac{F}{tL^D} \xrightarrow{N \to \infty} \frac{1}{2} \frac{S_D}{D} \Gamma \left( \frac{D}{2} \right) \Gamma \left( 1 - \frac{D}{2} \right) \lambda^{D/2} - \frac{\lambda}{2t},
\] (19.28)
for the free energy and
\[
\frac{1}{t} = \frac{S_D}{2} \Gamma \left( \frac{D}{2} \right) \Gamma \left( 1 - \frac{D}{2} \right) \lambda^{D/2-1}
\] (19.29)
for the gap equation, where $S_D = S_D/(2\pi)^D = 2/(4\pi)^D/\Gamma(D/2)$ from Eq. (11.129). It is easy to check that the free energy (19.28) is extremal for $\lambda$ satisfying (19.29).
In order to discuss the physical consequences of these equations, consider first the case of two dimensions. Letting $\epsilon \equiv D - 2$ approach zero, the equations become

$$\frac{F}{tL^D} \rightarrow_{N \rightarrow \infty} -\frac{1}{2} S_D \frac{1}{\epsilon^{1 + \epsilon/2}} - \frac{\lambda}{2t} + \mathcal{O}(\epsilon), \quad (19.30)$$

$$\frac{1}{t} = -S_D \frac{1}{\epsilon} \lambda^{1/2} + \mathcal{O}(\epsilon). \quad (19.31)$$

Note that we use $\epsilon$ for the deviations from $D = 2$ dimensions, in contrast to $\varepsilon$ for the deviations from $D = 4$.

The gap equation (19.31) has a solution only for negative $t$. To avoid the infinities in the limit $\epsilon \rightarrow 0$, we introduce an arbitrary mass scale $\mu$ and a renormalized temperature $t_R$ via the defining equation for $t_R$:

$$\frac{1}{t} = \frac{\mu^\epsilon}{t_R} - S_D \frac{1}{\epsilon} \mu^\epsilon. \quad (19.32)$$

Then we can take the limit $\epsilon \rightarrow 0$ and find

$$\frac{F}{tL^D} \rightarrow_{N \rightarrow \infty} -S_D \frac{1}{4} \lambda \left( \log \frac{\lambda}{\mu^2} - 1 \right) - \frac{\lambda}{2t_R} \quad (19.33)$$

$$\frac{1}{t_R} = -S_D \frac{1}{2} \log \frac{\lambda}{\mu^2}. \quad (19.34)$$

The renormalized gap equation is solved for $\lambda = \bar{\lambda}$ satisfying the equation

$$\frac{\bar{\lambda}}{\mu^2} = e^{-2/t_R S_D}. \quad (19.35)$$

For this value, the free energy has a maximum. It corresponds to a minimum in the $\lambda$-integrals in (19.21) which runs along the imaginary direction. Only the immediate neighborhood of this minimum contributes to the integral (see Fig. 19.1).

![Figure 19.1 Free energy as a function of $\lambda$ for $D = 2$ (schematically). The gap equation (19.35) determines the value $\bar{\lambda}$ of the maximum.](image-url)
The value of $\lambda$ at the minimum is a parameter characterizing the solution, having the dimension of a square-mass. From the role played by $\lambda$ in Eq. (19.20) and in the trace of the logarithm of the fluctuations (19.24) we see that $\lambda$ is the square-mass of the $n_a$-fluctuations. This implies that even though we had started out with a model describing $N - 1$ massless Goldstone bosons, the interacting system is no longer massless, but all $N$ fields have acquired a nonzero mass $M = \sqrt{\lambda}$. What we witness here is the spontaneous restoration of the $O(N)$-symmetry due to the violence of the directional fluctuations in two dimensions.

It is instructive to consider also the case of more than two dimensions and treat it in a way which has a smooth limit for $\epsilon \equiv D - 2 \to 0$. If $\epsilon$ is arbitrary, the free energy in $D \equiv 2 + \epsilon$ dimensions becomes

$$F(\lambda)_{tL^D} = -\frac{c_\epsilon}{2} \frac{\lambda^{1+\epsilon/2}}{1+\epsilon/2} - \frac{\lambda}{2t},$$

(19.36)

where

$$c_\epsilon \equiv -\frac{\bar{S}_D}{2} \Gamma \left( \frac{D}{2} \right) \Gamma \left( 1 - \frac{D}{2} \right) = \frac{1}{\epsilon} S_D \frac{\pi \epsilon/2}{\sin (\pi \epsilon/2)}.$$  \hspace{1cm} (19.37)

For small $\epsilon$ this behaves like

$$c_\epsilon \approx \frac{1}{2\pi \epsilon} + O(\epsilon).$$ \hspace{1cm} (19.38)

In order to have a smooth limit $\epsilon \to 0$ of the free energy, we write

$$F(\lambda)_{tL^D} = \begin{cases} -\frac{1}{2} \frac{c_\epsilon}{1+\epsilon/2} \left[ \left( \frac{\lambda}{\mu^2} \right)^{\epsilon/2} - \left( \frac{\epsilon}{2} \right) \lambda - \frac{\lambda}{2t_R} \right] \mu^\epsilon, \end{cases}$$

(19.39)

where $t_R$ is the renormalized temperature defined by

$$\frac{1}{t} = \mu^\epsilon - c_\epsilon \mu^\epsilon.$$ \hspace{1cm} (19.40)

19.3.1 High-Temperature Phase

At high temperatures, where $t_R$ is large, the free energy (19.36) has a maximum. Its position $\lambda = \bar{\lambda}$ of the maximum is determined by the unrenormalized gap equation as

$$\frac{1}{t} = -c_\epsilon \bar{\lambda}^{\epsilon/2}.$$ \hspace{1cm} (19.41)

For $2 > \epsilon > 0$, i.e., between two and four dimensions, there is a solution for the bare parameter $t < 0$. Equation (19.40) implies that the renormalized temperature is, in this regime, larger than a certain critical value $t^c_R$ given by

$$t^c_R = c_\epsilon^{-1} > 0.$$ \hspace{1cm} (19.42)
In terms of $t_R$, Eq. (19.40) reads

$$\frac{1}{t} = \mu' \left( \frac{1}{t_R} - \frac{1}{t_{cR}} \right),$$

(19.43)

and Eq. (19.43) gives the gap $\bar{\lambda}$ as a function of $t_R > t_{cR}$:

$$\bar{\lambda} = \mu^2 \left( 1 - \frac{t_{cR}}{t_R} \right)^{2/\epsilon}.$$  

(19.44)

For $t_R < t_{cR}$, the maximum lies at $\lambda = \bar{\lambda} = 0$. For $t_R > t_{cR}$, it becomes positive (see Fig. 19.2).

Note that in order to make the functional integral of the theory (19.20) well-defined for $t_R > t_{cR}$ where $t < 0$, the $n^a$-integrations have to be taken along the imaginary field axis. We shall see later in Section 19.5 that this apparently unphysical condition is an artifact of the analytic continuation in the dimension $D$.

### 19.3.2 Low-Temperature Phase

In the low-temperature regime

$$t_R < t_{cR},$$

(19.45)

where the unrenormalized temperature is positive, the gap equation (19.41) has no solution. The reason for this is an incorrect treatment of the fluctuations at zero momentum. In order to see what went wrong in the above calculations we may proceed in two ways. One is inspired by the treatment in Subsection 2.15.3 of the ideal Bose gas at a fixed particle number $N$. There we observed that the equation for the particle number containing a momentum integral over the Bose distribution could be performed only down to a certain temperature, where the chemical potential
first hits zero (from the negative side). For lower temperatures, the state of zero
momentum accumulates a macroscopic number of states, the so-called \textit{condensate}. To
account for its presence, it was essential to treat the system in a finite volume, where the
momenta have to be summed up rather than being integrated over. Then, the zero-momentum
state can be isolated and treated separately. This is a typical solution of an infrared problem of a field system. Let us do the same thing here.

If the system is enclosed in a box of volume $L^D$, we calculate the free energy (19.24) as

$$
\frac{F}{L^D} = \frac{1}{2L^D} \log \lambda + \frac{1}{2L^D} \sum_{k \neq 0} \log \left( k^2 + \lambda \right) - \frac{\lambda}{2t}.
$$

The first term is responsible for generating a local maximum at a real $\lambda = \bar{\lambda}$ which
is a minimum for the fluctuations in $\delta \lambda \equiv \lambda - \bar{\lambda}$ along the imaginary $\lambda$-axis. The
maximum’s value $\bar{\lambda}$ is determined by the gap equation

$$
\frac{1}{L^D} \frac{1}{\lambda} + \frac{1}{L^D} \sum_{k \neq 0} \frac{1}{k^2 + \lambda} = \frac{1}{t}.
$$

If the system has more than two dimensions, the sum over $1/(k^2 + \lambda)$ with $k \neq 0$
can be replaced by a phase-space integral $\int d^Dk L^D/(2\pi)^D (k^2 + \lambda)$. The relative
error is only of order $1/\lambda L^2$, which can be calculated exactly using a $D$-dimensional
version of the \textit{Euler-Maclaurin approximation} to the sum in Eq. (7.693). Thus the
gap equation (19.41) reads, more precisely,

$$
\frac{1}{t} = \frac{1}{L^D \bar{\lambda}} - c_\epsilon \bar{\lambda}^{\epsilon/2}.
$$

For $t < 0$, and a large volume $L^D$, this equation has the same solution $\bar{\lambda}$ as before
in (19.41), since the first term $1/L^D \bar{\lambda}$ on the right-hand side can be ignored. For
$t > 0$, however, this term makes an essential difference. It gives rise to a nonzero
solution $\bar{\lambda}$, which is very small for a large volume $L^D$. In this regime, the second
term in (19.48) can be neglected with respect to the first term, so that for positive $t$:

$$
\bar{\lambda} \approx \frac{t}{L^D}.
$$

The general solution for both finite and infinite volumes $L^D$ is illustrated in Fig. 19.3. In
the limit of infinite volume, the solution is

$$
\bar{\lambda} = 0, \quad t \geq 0,
\bar{\lambda} > 0, \quad t < 0.
$$

There is a phase transition at $t = 0$. For $t > 0$, all modes become massive and the
$O(N)$-symmetry is restored. For $t < 0$, the original $N-1$ massless Goldstone modes
survive.
What does this imply for renormalized quantities? Using (19.42) and (19.43), we find from (19.48) the properly renormalized gap equation correcting the previous Eq. (19.44):

$$1 - \frac{t_R^c}{t_R} = -\frac{t_R^c}{\mu^2 L^D \lambda} + \left( \frac{\lambda}{\mu^2} \right)^{\epsilon/2}. \tag{19.51}$$

For $t_R \geq t_R^c$ and large volume $L^D$, we can neglect the first term on the right-hand side, and (19.51) coincides with (19.44). For $t_R < t_R^c$, on the other hand, we can neglect the second term on the right-hand side, to find for the spontaneously generated square-mass $\bar{\lambda}$ of $n_a$-fluctuations:

$$\bar{\lambda} \approx \frac{t_R}{(\mu L)^c L^2} \left( 1 - \frac{t_R^c}{t_R} \right)^{-1}. \tag{19.52}$$

This almost zero part of the gap in the low-temperature phase is pictured by the right-hand branch in Fig. 19.3. In the thermodynamic limit of infinite volume, $\bar{\lambda}$ goes to zero. In this limit, the system has a phase transition at the critical temperature $t_R^c > 0$, where the symmetry changes as described above. The statements for $t < 0$ and $t > 0$ hold now for $t_R > t_R^c$ and $t_R < t_R^c$, respectively. This is, of course, the same transition found earlier in the $O(N)$-symmetric $\phi^4$-theory, only in quite a different description.

In a finite volume, the square-mass $\bar{\lambda}$ is always nonzero, and there exists no phase in which the $n_a$-fluctuations are massless. The spontaneous breakdown of $O(N)$-symmetry is always restored by fluctuations.

## 19.4 Approaching the Critical Point

It is instructive to study the way in which the mass vanishes as $t_R$ approaches $t_R^c$ from above. From (19.44) we know that

$$\bar{\lambda} = \mu^2 \left( 1 - \frac{t_R^c}{t_R} \right)^{2/\epsilon}. \tag{19.53}$$
Since \( \bar{\lambda} \) is the square-mass of all \( n_a \)-fluctuations, this implies that the correlation functions of the \( n_a \) fields fall off exponentially with

\[
\langle n_a(x)n_a(0) \rangle \xrightarrow{x \to \infty} e^{-|x|/\xi},
\]

where \( \xi = \bar{\lambda}^{-1/2} \) is the correlation length. As \( t_R \) approaches the critical temperature, \( \xi \) goes to infinity, according to the power law

\[
\xi \sim (t_R - t^c_R)^{-\nu}
\]

with

\[
\nu = \frac{1}{\epsilon}.
\]

This number is called the critical exponent of the correlation length. The power behavior of \( \xi \) as a function of temperature is shown in Fig. 19.4. A similar power behavior will be obtained later in Chapter 20, for finite \( N \) as well, using renormalization-group techniques, although for finite \( N \) it will no longer be possible to find an exact result.

![Figure 19.4 Temperature behavior of the correlation length.](image)

### 19.5 Physical Properties of the Bare Temperature

The reader may rightfully wonder whether the entire treatment is consistent. We have observed above that the phase in which the symmetry has been restored possesses a negative bare temperature. This is not the sign of \( t \) for which the model was originally defined. Only for field integrations \( d\eta_a \) along the imaginary field direction
does the partition function make sense. Fortunately, this apparent inconsistency is not intrinsic to the theory, but merely a mathematical artifact of the dimensional regularization of the divergent integral. A more physical regularization would have proceeded via a cutoff $\Lambda$ in momentum space. Let us verify how this would change the range of the bare temperature if the phase transition is approached from positive $t$.

In order to be specific, consider first the cases of $D = 2$ and 3 dimensions. Using the integrals

\[
\int_{|k|<\Lambda} \frac{d^2 k}{(2\pi)^2} \log \left( k^2 + \lambda \right) = \frac{1}{4\pi} \left[ \left( \Lambda^2 + \lambda \right) \log \left( \Lambda^2 + \lambda \right) - \Lambda^2 - \lambda \log \lambda \right] - \frac{1}{4\pi} \left[ \Lambda^2 \left( \log \Lambda^2 - 1 \right) + \left( \log \Lambda^2 + 1 \right) \lambda - \lambda \log \lambda \right],
\]

\[
\int_{|k|<\Lambda} \frac{d^3 k}{(2\pi)^3} \log \left( k^2 + \lambda \right) = \frac{1}{6\pi^2} \left[ \Lambda^3 \log \left( \Lambda^2 + \lambda \right) - \frac{2}{3} \Lambda^3 + 3\Lambda \lambda - 2\lambda^2 \arctan \frac{\Lambda}{\sqrt{\lambda}} \right] - \frac{1}{6\pi^2} \left[ \Lambda^3 \left( \log \Lambda^2 - \frac{2}{3} \right) + 3\Lambda \lambda - \pi\lambda^{3/2} \right],
\]

the free energy (19.24) becomes

\[
\frac{F}{tL^2} = \frac{1}{2} \frac{1}{4\pi} \left[ \Lambda^2 \left( \log \Lambda^2 - 1 \right) + \left( \log \Lambda^2 + 1 \right) \lambda - \lambda \log \lambda \right] - \frac{1}{2t},
\]

\[
\frac{F}{tL^3} = \frac{1}{2} \frac{1}{6\pi^2} \left[ \Lambda^3 \left( \log \Lambda^2 - \frac{2}{3} \right) + 3\Lambda \lambda - \pi\lambda^{3/2} \right] - \frac{1}{2t}.
\]

In writing these expressions, we have assumed all momenta and masses to be dimensionless quantities. For mathematically proper expressions, all these quantities should be divided by some standard mass $\mu$, which we have assumed to be equal to one.

The dimensionally regularized expression (19.36), on the other hand, reads for dimensions $D = 2 + \epsilon$ with small $\epsilon > 0$

\[
\frac{F}{tL^2N} = -\frac{1}{4\pi} \frac{1}{2} \frac{1}{2} \frac{2}{\epsilon} \lambda - \frac{1}{4\pi} \frac{1}{2} \frac{1}{2} \log \frac{\lambda}{\mu^2} - \frac{\lambda}{2t},
\]

and for $D = 3$:

\[
\frac{F}{tL^3N} = -\frac{1}{4\pi} \frac{1}{2} \frac{1}{2} \frac{2}{3} \frac{3}{\lambda^2} - \frac{\lambda}{2t}.
\]

For $D = 2 + \epsilon$, the positively divergent cutoff expression $(\log \Lambda^2 + 1) \lambda/8\pi$ in the second term of (19.60) is replaced by the negatively divergent $-\lambda/4\pi\epsilon$ in the first term of (19.61). For $D = 3$, the dimensionally regularized expression in (19.61) shows no trace of the infinite linearly divergent term $\Lambda \lambda/4\pi^2$ in (19.60). Thus the inverse temperature $1/t$ in the dimensionally regularized expressions must be thought of as being an unphysical version of the original inverse temperature $1/t$ in
the initial bare energy, from which the constants \([\log(\Lambda^2 + 1) + 2/\epsilon]/4\pi\) for \(D = 2\), or \(\Lambda/2\pi^2\) for \(D = 3\), have been subtracted. Since these are very large quantities, there is no wonder that the dimensionally regularized discussion yields the phase with restored symmetry at a negative unphysical \(t\).

The analytic continuation is taking advantage of the inobservability of the bare value of \(t\) in a continuous field theory. In any system in which there are well-defined physical mechanisms which regularize the short-wavelength fluctuations (for example lattice structures) and in which the bare temperature \(t\) is measurable by microscopic methods, the dimensional regularization cannot be used directly to properly represent the physical circumstances of the bare theory. This must be kept in mind when applying such a method, in particular, when dealing with the non-renormalizable \(O(N)\)-lattice models of statistical mechanics, to which the present nonlinear \(\sigma\)-model is related in the continuum limit. This issue is discussed in the following section.

### 19.6 Spherical Model on Lattice

In order to illustrate the dimensional regularization issue discussed in the previous section it is instructive to go through the treatment once more in the lattice formulation of the spherical model. Then the lattice spacing \(a\) provides us with a natural short-distance cutoff, so that there is no need for introducing a momentum space cutoff \(\Lambda\), nor dimensional regularization. On a simple hypercubic lattice, the partition function (19.17) reads

\[
Z = \prod_x \left( \int \frac{d\lambda (x)}{4\pi i} \right)^2 \prod_x \left( \int \frac{d^n x}{S_D} \right) \exp \left( -\frac{\beta a^2}{2} \sum_x \left\{ \left( (\nabla_i n)^2 + \lambda n^2 \right) - \lambda \right\} \right),
\]

where the gradient term has been brought to the lattice form (19.14) omitting the irrelevant term \(-2D\). The dimensionless parameter

\[
\beta \equiv \frac{1}{k_B T} = \frac{J}{k_B T}
\]

is called the stiffness of the directional fluctuations. The free energy per lattice site is given by

\[
\beta f = \frac{1}{2} \left( \int_{-\pi/a}^{\pi/a} N \frac{d^D k a^D}{(2\pi)^D} \log \left\{ \sum_{i=1}^D [2 - 2 \cos(k_i a)] + a^2 \lambda \right\} - \beta a^2 \lambda \right),
\]

where \(a^{-2} \sum_{i=1}^D [2 - 2 \cos (k_i a)]\) is the Fourier transform of the lattice Laplacian \(-\nabla_i \nabla_i\). Extremizing this with respect to \(\lambda\) yields the gap equation

\[
\beta = N v_{\alpha^2 \lambda}^D(0),
\]

where

\[
v_{\alpha^2 \lambda}^D(x) = \int_{-\pi/a}^{\pi/a} d^D k a^D \left( \frac{e^{i k x}}{\sum_{i=1}^D [2 - 2 \cos(k_i a)] + a^2 \lambda} \right).
\]
is the dimensionless lattice Green function of mass $m$ in $D$ dimensions. For small $\beta$, i.e., high temperatures, the gap equation has a solution $\lambda > 0$ and the fluctuations are massive. The solution exists up to a critical $\beta_c$ given by

$$\beta_c = N v_0^D(0).$$

(19.68)

For $D = 2$, the momentum integral diverges at the origin: $v_0^2(0) = \infty$. This implies $\beta_c = 0$, so that a nonzero $\lambda$ exists for all temperatures $T$ larger than zero. Here the approximation (19.68) becomes useless, and a special discussion is necessary [12].

The Green function (19.67) is the dimensionless lattice version of the Yukawa potential in $D$ dimensions:

$$V^D_{m^2}(x) = \int \frac{d^Dk}{(2\pi)^D} \frac{e^{ikx}}{k^2 + m^2}. \quad (19.69)$$

The lattice version has the Fourier representation

$$v_{a^2m^2}^D(x) = \prod_i \left[ \int_{-\pi/a}^{\pi/a} \frac{d(ak_i)}{2\pi} \frac{e^{ikx}}{\sum_{i=1}^{D} (2 - 2 \cos ak_i) + a^2m^2} \right]. \quad (19.70)$$

The denominator can be rewritten as $\int_0^\infty ds e^{-s[\sum_{i=1}^{D} (2 - 2 \cos ak_i) + a^2m^2]}$, leading to the multiple integral

$$v_{a^2m^2}^D(0) = \int_0^\infty ds e^{-s(a^2m^2 + 2D) \prod_{i=1}^{D} \left[ \int_{-\pi}^{\pi} \frac{dK_i}{2\pi} e^{2s \cos K_i} \right]} \quad (19.71)$$

The integrations over $k_i$ can now easily be performed, and we obtain the integral representation

$$v_{a^2m^2}^D(0) = \int_0^\infty ds e^{-s(a^2m^2 + 2D)[I_0(2s)]^D}, \quad (19.72)$$

where $I_0(2s)$ is the modified Bessel function. Integrating this numerically, we find for $D = 3, 4, \ldots$ the values shown in Table 19.1. A power series expansion of the

<table>
<thead>
<tr>
<th>$D$</th>
<th>$v_0^D(0)$</th>
<th>$v_1^D(0)$</th>
<th>$v_2^D(0)$</th>
<th>$v_3^D(0)$</th>
<th>$v_4^D(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.2527</td>
<td>0.1710</td>
<td>0.1410</td>
<td>0.1214</td>
<td>0.1071</td>
</tr>
<tr>
<td></td>
<td>0.2171</td>
<td>0.1691</td>
<td>0.1407</td>
<td>0.1214</td>
<td>0.1071</td>
</tr>
<tr>
<td>4</td>
<td>0.1549</td>
<td>0.1271</td>
<td>0.1105</td>
<td>0.0983</td>
<td>0.0888</td>
</tr>
<tr>
<td></td>
<td>0.1496</td>
<td>0.1265</td>
<td>0.1104</td>
<td>0.0983</td>
<td>0.0888</td>
</tr>
</tbody>
</table>

See also the Tables on pages 178 and 241 of the textbook [3].
$D$th power of the modified Bessel function in (19.72),

$$[I_0(2s)]^D = 1 + Ds^2 + D(2D - 1)s^4 + D(6D^2 - 9D + 4)s^6 + \ldots,$$

leads to the so-called hopping expansion for $v_{a^2\lambda}(0)$:

$$v_{a^2\lambda}(0) = \frac{1}{2D + a^2\lambda} + \frac{2D}{(2D + a^2\lambda)^3} + \frac{6D(2D - 1)}{(2D + a^2\lambda)^5} + \frac{20D(6D^2 - 9D + 4)}{(2D + a^2\lambda)^7} + O(D^{-9}).$$

It converges rapidly for large $D$, and yields for $D = 3, 4$ the approximate values shown in the lower entries of Table 19.1. They lie quite close to the exact values in the upper entries.

The lattice potential at the origin $v_{a^2\lambda}(0)$ in the gap equation (19.66) is always smaller than the massless potential $v_0^D(0)$. A nonzero value for $\lambda$ can therefore only be found for sufficiently small values of the stiffness $\beta$, i.e., for sufficiently high temperatures $T$ [recall (19.64)]. The temperature $T_c$, at which the gap equation (19.66) has the solution $\lambda = 0$, determines the Curie point. Thus we have

$$T_c = \frac{J}{\beta_c k_B},$$

where $\beta_c$ is the critical stiffness (19.68):

$$\beta_c = Nv_0^D(0).$$

This result, derived for large $N$, turns out to be amazingly accurate even for rather small $N$. As an important example, take $N = 2$ where the model consists of planar spins and is referred to as XY-model. For $D = 3$, it describes accurately the critical behavior of the superfluid transition in helium near the $\lambda$-transition. From (19.76) and the value $v_0^D(0) \approx 0.2527$ in Table 19.1, we estimate the critical value

$$\beta_c \approx 0.5054.$$ 

In Monte-Carlo simulations of this model one obtains, on the other hand,

$$\beta_c \approx 0.45.$$ 

Thus, in three dimensions, we can use the large-$N$ result (19.76) practically for all $N \geq 2$.

For $D > 2$, where $\beta_c$ has a finite positive value, the solution for $\beta > \beta_c$ requires special consideration. The situation is similar to what we encountered earlier in the free Bose liquid. One observes the formation of a condensate consisting of Goldstone bosons of zero momentum. For its theoretical description, we enclose the system

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4See pages 390 and 391 in the textbook [3].
in a finite box of volume $V$ and replace the integral over $k$ by the corresponding momentum sum

$$\int \frac{d^D k a^D}{(2\pi)^D} \to \frac{a^D}{V} \sum_k.$$  \hfill (19.79)

Since the integrand in (19.67) is singular at $k = 0$, this mode has to be treated separately. We split the right-hand side of the gap equation (19.68) for $\beta > \beta_c$ as follows

$$1 = N \frac{a^D}{V} \left\{ \frac{1}{\beta \lambda a^2} + \frac{1}{\beta} \sum_{k \neq 0} \frac{1}{\sum_{i=1}^{D} \left[ 2 - 2 \cos (k_i a) \right] + \lambda a^2} \right\}. \hfill (19.80)$$

This equation can now be solved for $\beta > \beta_c$. For $\beta \gg \beta_c$, the very small prefactor $a^D/V$ guarantees a solution with a very small $\lambda a^2$, that is of the order of $a^D/V \beta = a^D/L^D \beta$, where $L$ is the linear size of the system. This follows from the observation that the first nonzero contribution in the sum is of the order

$$\frac{1}{(2\pi a/L)^2 + \lambda a^2} \approx \frac{1}{(2\pi a/L)^2 + a^D/L^D}. \hfill (19.81)$$

For $D > 2$ and large $L$ one has $a^D/L^D \ll a^2/L^2$, such that the second term in the denominator can be ignored, and the first term in (19.94) is much larger than the second term. For $\beta > \beta_c$, the gap equation in a large volume shows that

$$1 \approx N \frac{a^D}{\sqrt{\beta \lambda a^2}}. \hfill (19.82)$$

What happens if $\beta$ is only slightly larger than $\beta_c$? Since $D > 2$, the sum on the right-hand side can again be replaced by an integral and we find the gap equation, valid for very large $V$, but arbitrary $\beta > \beta_c$:

$$1 = \frac{a^D}{V} \frac{1}{\beta \lambda a^2} + \frac{N}{\beta} v_0(0) = \frac{a^D}{V} \frac{N}{\beta} \frac{N}{\beta} \lambda a^2 + \frac{\beta_c}{\beta}. \hfill (19.83)$$

Hence the square-mass $\lambda$ of directional fluctuations is found to be

$$\lambda = \frac{1}{N} \frac{a^D}{V} \frac{1}{\beta - \beta_c}. \hfill (19.84)$$

This is positive and infinitely small for a large volume $V$. As a function of temperature $t = 1/\beta$, this looks very similar to the plot in (19.3).

Let us look in detail at the difference between the treatment of the gap equation on the lattice and that of the dimensionally regularized gap equation in the continuum field theory. This will allow us to understand the reason for the negative sign
of $t$ in the ordered phase in the continuum model. To be specific, we assume $D = 3$. The integral appearing in the gap equation is, in dimensional regularization,

$$
\int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2 + m^2} = S_D \frac{1}{2} \Gamma \left( \frac{D}{2} \right) \Gamma \left( \frac{1 - D}{2} \right) m^{D-2}\bigg|_{D=3}
$$

$$
= S_3 \frac{1}{2} \Gamma \left( \frac{3}{2} \right) \Gamma \left( -\frac{1}{2} \right) m
$$

$$
= -\frac{1}{4\pi} m.
$$

(19.85)

The corresponding three-dimensional lattice expression is

$$
\frac{1}{a^3} \int_{\pi/a}^{\pi/a} \frac{d^3ka}{(2\pi)^3} \frac{1}{a^{-2} \Sigma_i \left[ 2 - 2 \cos (k_i a) \right] + m^2} = \frac{1}{a} v_{m^2}^3(0)
$$

(19.86)

where, for small $m^2$,

$$
\frac{1}{a} v_{0}(0) \equiv \frac{1}{4\pi} m + \mathcal{O}(m).
$$

(19.87)

In the lattice gap equation (19.68), the first term on the right-hand side of (19.94) goes to infinity in the continuum limit $a \to 0$. In the analytic continuation (19.67), on the other hand, it is absent. It is this term which, in the lattice model, is responsible for the fact that $\beta$ is positive in both phases, with a positive $\beta_c > 0$ separating the two phases. In contrast to this, the two phases in the continuum theory exist for opposite signs of the bare quantity $\beta$. This curious situation, caused entirely by the mathematics of analytic regularization, is removed after renormalizing the theory. Along the renormalized $1/t_R$ -axis, the situation is again analogous to the more physical one in the lattice model, where $\beta$ needs no renormalization.

**19.7 Background Field Treatment of Cold Phase**

There is another more elegant way of dealing with the low temperature phase. It does not require the delicate treatment of the $k = 0$ -mode, and yields a consistent gap equation for all temperatures without the intermediate consideration of a finite volume. It follows from minimizing the effective action of the model, if that is calculated after expanding $n^a$ around a background field. Let us denote this background field by $N^a(x)$, and the fluctuations around it by $\delta n^a(x) = n^a(x) - N^a(x)$. Then the one-loop effective action becomes

$$
\Gamma[N^a] = \frac{N}{2t} \int d^Dx \left\{ (\nabla N_a)^2 + \lambda(x) N_a^2(x) \right\} + \frac{N}{2} \text{Tr} \log \left[ -\nabla^2 - \lambda(x) \right]
$$

$$
+ \frac{N}{2t} \int d^Dx \lambda.
$$

(19.88)

This has to be evaluated at the minimum with respect to $N^a(x)$ and the maximum with respect to $\lambda$. In the limit $N \to \infty$, the one-loop effective action $(1/N) \Gamma [N^a]$ is the exact one.
The new gap equation reads
\[-\frac{N_a}{t}^2 + \frac{1}{t} = \int \frac{d^Dk}{(2\pi)^D} \frac{1}{k^2 + \lambda},\]  
(19.89)
and the background field \(N_a\) satisfies the equation of motion
\[\left[-\nabla^2 + \bar{\lambda}(x)\right] N_a(x) = 0.\]  
(19.90)
This equation shows that \(\sqrt{\lambda}^{-1}\) determines the correlation length of the \(N_a\) field. The ground state is obtained for a constant \(N_a\) field, which satisfies
\[\bar{\lambda}N_a = 0.\]  
(19.91)
This has two solutions, \(N_a = 0\) or \(\bar{\lambda} = 0\). In the first case, the gap equation reduces to the previous one, which in dimensional regularization had a solution only for \(t > 0\). In the other phase with \(t > 0\), the gap equation turns into an equation for the size of \(N_a\)
\[-\frac{N_a^2}{t} + \frac{1}{t} = \int \frac{d^Dk}{(2\pi)^D} \frac{1}{k^2}.\]  
(19.92)
The direction of \(N_a\) is arbitrary and will be chosen by the system statistically. This shows that for \(t > 0\), the ground state displays a spontaneous breakdown of \(O(N)\)-symmetry. It displays a spontaneous magnetization.

Comparison with Eq. (19.47) shows that the gap equation (19.92) coincides precisely with the previous gap equation on the lattice (19.94) for the low temperature phase, with \(N_a^2/t\) playing the role of \(1/L^D\lambda\). Given the previous solution for \(\bar{\lambda}\), we can therefore identify, for \(t > 0\),
\[N_a^2 = \frac{t}{\lambda L^D}.\]  
(19.93)
The quantity \(N_a^2\) can be considered as an order parameter of the low temperature phase. It specifies the magnetization \(M \equiv \sqrt{N_a^2}\).

In the framework of dimensional regularization, the integral on the right-hand side of the gap equation vanishes and the order parameter for \(t_R < t^*\) has the value
\[N_a^2 = 1.\]  
(19.94)
The background vector has the unit length of the original fluctuating field \(n_a(x)\). The fact that \(N_a\) is of unit length for all \(t > 0\) or \(t_R < t^*\), is once more an artifact of dimensional regularization. On a lattice, the magnetization \(M\) goes to zero smoothly, with a power law in \(\beta - \beta_c\):
\[M \propto (\beta - \beta_c)^\beta.\]  
(19.95)
The power is conventionally denoted by $\beta$, which should, when it appears in this context, not be confused with the inverse temperature $\beta$. On a lattice, the gap equation (19.92) reads, for $\beta > \beta_c$,

$$1 - N_a^2 = \frac{N}{\beta} \int_{-\pi/a}^{\pi/a} d^D k a^D \frac{1}{\sum_{i=1}^D [2 - 2 \cos (k_i a)]},$$

such that

$$1 - N_a^2 = \frac{N v_D(0)}{\beta}.$$  \hspace{1cm} (19.97)

From this we extract the power behavior

$$M = \sqrt{N_a^2} = \left(1 - \frac{\beta_c}{\beta}\right)^{1/2},$$ \hspace{1cm} (19.98)

showing that the critical exponent $\beta$ has the value 1/2.

19.8 Quantum Statistics at Nonzero Temperature

Let us also study quantum fluctuations in the nonlinear $\sigma$-model. For this we assume that one of the spatial coordinates ($x_1, \ldots, x_D$), for instance $x_D$, is an imaginary time variable $\tau$ restricted to the interval $\tau \in (0, \hbar \beta)$ with $\beta = 1/k_B T$. If the fields are periodic in $\tau$ with period $\hbar \beta$, we may think of this model as a nonlinear $\sigma$-model on an infinitely long spatial strip with periodic boundary conditions, whose width along the $x_D$-axis is $\beta$. In this context, we shall write the $D - 1$-dimensional purely spatial vectors as bold letters, whereas the $D$-dimensional spacetime vectors are denoted by light letters.

Since temperature enters the theory now via the period $\hbar \beta = 1/k_B T$, we shall return to the original notation $g$ for the coupling constant, as in the partition function (19.2) [which had been traded for the reduced temperature $t$ in (19.22) and (19.19)].

In the limit $N \to \infty$, we can study the effects of temperature exactly. The partition function is (19.26), and the gap equation has the same form as in (19.27), except that the momentum integral $\int d p_D/2\pi$ is replaced by a sum $T \sum_{p_D=\omega_m}$ over the Matsubara frequencies

$$\omega_m = 2\pi T m, \quad m = 0, \pm 1, \pm 2, \ldots,$$

(19.99)

guaranteeing the periodicity of the fields on the interval $x_D \in (0, \beta)$. The gap equation (19.27) at finite temperature is therefore

$$\frac{1}{g} = (-\partial^2 + \lambda)^{-1} = \int \frac{d^{D-1} p}{(2\pi)^{D-1}} T \sum_m \frac{1}{\omega_m^2 + p^2 + \lambda}.$$ \hspace{1cm} (19.100)
We have gone to natural units with $\hbar = 1$, $k_B = 1$. This equation can be renormalized with the same renormalized coupling constant $g_R(\mu)$ as in the infinite system. We add and subtract, on the right-hand side, the $T = 0$-limit of the right-hand side and obtain

$$
\frac{1}{g} - \int \frac{d^Dp}{(2\pi)^D} \frac{1}{p^2 + \lambda} = \int \frac{d^{D-1}p^{\epsilon}}{(2\pi)^{D-1}} \left( T \sum_m \frac{1}{\omega_m^2 + p^2 + \lambda} - \int_\infty^{\Omega} \frac{d\omega}{2\pi} \frac{1}{\omega^2 + p^2 + \lambda} \right). \tag{19.101}
$$

The left-hand side is the zero-temperature gap equation (19.27).

### 19.8.1 Two-Dimensional Model

For the subsequent discussion we focus attention on the two-dimensional case. Then the zero-temperature gap equation can be renormalized in $D = 2 + \epsilon$ dimensions using Eq. (19.32). This brings the left-hand side in (19.101) to the form

$$
\frac{1}{g} + S_D \frac{1}{\epsilon} \mu^\epsilon + S_D \frac{\mu^\epsilon}{\epsilon} \left[ \left( \frac{\lambda}{\mu^2} \right)^{\epsilon/2} - 1 \right] \approx \frac{1}{g_R(\mu)} + \frac{1}{2} S_2 \log \frac{\lambda}{\mu^2}. \tag{19.102}
$$

The sum over $m$ on the right-hand side of Eq. (19.177) can be performed using the sum formula

$$
T \sum_{m=-\infty}^{\infty} \frac{1}{\omega_m^2 + \Omega^2} = \frac{1}{2\Omega} \coth \frac{\Omega}{2T}, \tag{19.103}
$$

with $\Omega \equiv \sqrt{p^2 + \lambda}$. In the limit $T \to 0$, this becomes

$$
\int_\infty^{-\infty} \frac{d\omega}{2\pi} \frac{1}{\omega^2 + \Omega^2} = \frac{1}{2\Omega}. \tag{19.104}
$$

Subtracting (19.104) from (19.103), and using (19.102), we obtain the temperature-dependent gap equation in two dimensions

$$
\frac{1}{g_R(\mu)} + \frac{1}{4\pi} \log \frac{\lambda}{\mu^2} = \int_\infty^{-\infty} \frac{d\omega}{2\pi} \frac{1}{2\Omega} \left( \coth \frac{\Omega}{2T} - 1 \right). \tag{19.105}
$$

Eliminating the arbitrary mass scale $\mu$ in favor of the solution $\lambda = \tilde{\lambda}$ of this equation for $T = 0$, which is

$$
\tilde{\lambda} = \mu^2 e^{4\pi/g_R(\mu^2)}, \tag{19.106}
$$

then Eq. (19.105) can be rewritten in a renormalization group invariant way:

$$
\frac{1}{4\pi} \log \frac{\lambda}{\tilde{\lambda}} = \int_\infty^{-\infty} \frac{d\omega}{2\pi} \frac{1}{2\Omega} \left( \coth \frac{\Omega}{2T} - 1 \right). \tag{19.107}
$$

The right-hand side is a function of the dimensionless variable $\lambda_T$

$$
\lambda_T \equiv \lambda/(2\pi T)^2 \tag{19.108}
$$
and will be denoted by
\[
\int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{1}{2\Omega} \left( \coth \frac{\Omega}{2T} - 1 \right) \equiv \frac{1}{2\pi} S_1(\lambda_T). \tag{19.109}
\]
In terms of this function, the gap equation reads simply [compare with the forthcoming fermionic gap function (23.225)]
\[
\log \frac{\lambda}{\bar{\lambda}} = 2 S_1(\lambda_T). \tag{19.110}
\]
The functions $S_1(\lambda_T)$ can also be rewritten as follows
\[
S_1(\lambda_T) = \int_{0}^{\infty} \frac{dp}{\Omega} \left( \coth \left( \frac{\Omega}{2T} \right) - 1 \right) = -2 \int_{0}^{\infty} \frac{dp}{\Omega} \left( 1 - e^{\Omega/T} \right)^{-1}. \tag{19.111}
\]
There exists no critical temperature $T_c$ where $\lambda$ vanishes.

The gap equation (19.110) is fast convergent only at low temperature. For high temperatures, it is better to keep the original sum over $\omega_m$ in (19.103). Inserting this sum into (19.111), and performing the integral over $p$ gives
\[
S_1(\lambda_T) = \frac{\pi}{2} \left( T \sum_{m=-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d\omega_m}{2\pi} \right) \frac{1}{\sqrt{\omega_m^2 + \lambda}}
= \frac{1}{2} \left( \sum_{m=-\infty}^{\infty} - \int_{-\infty}^{\infty} dm \right) \frac{1}{\sqrt{m^2 + \lambda_T}}. \tag{19.112}
\]
Treating the $m = 0$ term in the sum separately, the right-hand side can be rearranged as follows:
\[
S_1(\lambda_T) = \frac{1}{2\sqrt{\lambda_T}} + \sum_{m=1}^{\infty} \left( \frac{1}{\sqrt{m^2 + \lambda_T}} - \frac{1}{m} \right) + \sum_{m=1}^{\infty} \frac{1}{m} - \int_{0}^{\infty} \frac{dm}{\sqrt{m^2 + \lambda_T}}. \tag{19.113}
\]
The last two terms diverge. The two divergences cancel each other. To see this we truncate the sum and the integral at some finite large integer value $m = M$. Then the two terms have the limits
\[
\sum_{m=1}^{M} \frac{1}{m} \to \log M + \gamma + O \left( \frac{1}{M} \right), \tag{19.114}
\]
and
\[
\arcsin \frac{M}{\sqrt{\lambda_T}} \to \log \frac{2M}{\sqrt{\lambda_T}} + O \left( \frac{1}{M} \right), \tag{19.115}
\]
respectively. Combining them, we obtain the alternative expression for $S_1(\lambda_T)$:
\[
S_1(\lambda_T) = \tilde{S}_1(\lambda_T) + \frac{1}{2\sqrt{\lambda_T}} + \frac{1}{2} \log \frac{\lambda_T e^{2\gamma}}{4}, \tag{19.116}
\]
where $\tilde{S}_1(\lambda T)$ denotes the convergent sum:

$$
\tilde{S}_1(\lambda T) = \sum_{m=1}^{\infty} \left( \frac{1}{\sqrt{m^2 + \lambda T}} - \frac{1}{m} \right).
$$

(19.117)

Inserting this into (19.116), we arrive at the desired alternative form for the gap equation which converges fast at high temperatures:

$$
\log \frac{T}{\bar{T}} = \tilde{S}_1(\lambda T) + \frac{1}{2\sqrt{\lambda T}},
$$

(19.118)

where $\bar{T} \equiv \sqrt{\lambda e^\gamma/4\pi}$.

It is straightforward to calculate $\tilde{S}_1(\lambda T)$ as a function of $\lambda T$, and find from this $T(\lambda T)$ as well as $\lambda = \lambda_T 4\pi^2 T$. Plotting $\lambda(T)$ we see that $\lambda$ starts out with $\bar{\lambda}$ at $T = 0$, and grows rapidly to infinity.

The gap equation is obtained by extremizing the effective potential as a function of $\lambda$. The effective potential is given by

$$
\frac{1}{N}v(\lambda) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dp}{2\pi} T \sum_m \log(\omega_m^2 + p^2 + \lambda) - \frac{\lambda}{2g}.
$$

(19.119)

We split this again into the zero-temperature equation

$$
\frac{1}{N}v_0(\lambda) = \frac{1}{2} \int \frac{dp}{(2\pi)^2} \log(p^2 + \lambda) - \frac{\lambda}{2g} = -\frac{\lambda}{8\pi} \left( \log \frac{\lambda}{\bar{\lambda}} - 1 \right),
$$

(19.120)

and a finite temperature correction

$$
\frac{1}{N}\Delta_T v(\lambda) = T \int_{-\infty}^{\infty} \frac{dp}{2\pi} \log \left( 1 - e^{-\Omega/T} \right) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \left( T \sum_m - \int_{-\infty}^{\infty} \frac{d\omega_m}{2\pi} \right) \log(\omega_m^2 + p^2 + \lambda).
$$

(19.121)

The right-hand side depends only on $\lambda_T$, and will be denoted by $(\lambda/2\pi)S_0(\lambda_T)$, i.e.,

$$
\frac{1}{N} \Delta_T v(\lambda) = \frac{\lambda}{2\pi} S_0(\lambda_T).
$$

(19.122)

The integral over $p$ in (19.120) can be performed after analytic regularization. For $D = 2$, this amounts to the sequence of steps

$$
\int \frac{dp}{2\pi} \log(\omega_m^2 + p^2 + \lambda) = -\int_{0}^{\infty} \frac{d\tau}{\tau} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \exp[-\tau(\omega_m^2 + p^2 + \lambda)]
$$

$$
= -\int_{0}^{\infty} \frac{d\tau}{2\sqrt{\pi} \tau^{3/2}} \exp[-\tau(\omega_m^2 + \lambda)]
$$

$$
= -\frac{\Gamma(-1/2)}{2\sqrt{\pi}} \sqrt{\omega_m^2 + \lambda}.
$$

(19.123)
Hence
\[ S_0(\lambda T) = \frac{\pi}{\lambda} \left( T \sum_{m=-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d\omega_m}{2\pi} \right) \sqrt{\omega_m^2 + \lambda} \]
\[ = \frac{1}{2\lambda T} \left( \sum_{m=-\infty}^{\infty} - \int_{-\infty}^{\infty} dm \right) \sqrt{m^2 + \lambda T}. \quad (19.124) \]

Note that \( S_0(\lambda T) \) has the derivative
\[ \frac{d}{d\lambda T} \lambda T S_0(\lambda T) = \frac{1}{2} S_1(\lambda T), \quad (19.125) \]
so that a differentiation of the potential
\[ \frac{1}{N} v(\lambda) = -\frac{\lambda}{8\pi} \left( \log \frac{\lambda}{\bar{T}} - 1 \right) + \frac{\lambda}{2\pi} S_0(\lambda T) \quad (19.126) \]
leads properly to the gap equation (19.110).

At high temperatures, it is useful to do the same manipulations with \( S_0(\lambda T) \) as with \( S_1(\lambda T) \), rewriting it as follows
\[ S_0(\lambda T) = \tilde{S}_0(\lambda T) + \frac{1}{2} \left( \log \frac{\lambda T e^{2\gamma}}{4} - 1 \right), \quad (19.127) \]
where
\[ \tilde{S}_0(\lambda T) = \frac{2\pi}{\lambda} \sum_{m=1}^{\infty} \left( \sqrt{\omega_m^2 + \lambda - \omega_m - \frac{\lambda}{2\omega_m}} \right) \]
\[ = \frac{1}{\lambda T} \sum_{m=1}^{\infty} \left( \sqrt{m^2 + \lambda T - m - \frac{\lambda T}{2m}} \right). \quad (19.128) \]

Then the total potential becomes
\[ \frac{1}{N} v(\lambda) = -\frac{\lambda}{4\pi} \log \frac{T}{\bar{T}} + \frac{\lambda}{4\pi \sqrt{\lambda T}} + \frac{\lambda}{2\pi} \tilde{S}_0(\lambda T). \quad (19.129) \]

In the close neighborhood of two dimensions, we set \( D = 2 + \epsilon > 2 \), and split the zero-temperature potential \( v_0(\lambda) \) conveniently as follows:
\[ \frac{1}{N} v_0(\lambda) = -\frac{\lambda}{2g} - \frac{\lambda}{2} c_\epsilon \left( \frac{1}{1 + \epsilon/2} \right)^{\lambda/2} = -\frac{\lambda}{2g} - \frac{\lambda \mu^\epsilon}{2} c_\epsilon - \frac{\lambda}{2} c_\epsilon \left[ \frac{1}{1 + \epsilon/2} \right]^{\lambda/2 - \mu^\epsilon}, \quad (19.130) \]
with \( c_\epsilon \) of Eq. (19.37) with the small-\( \epsilon \) behavior (19.38).

The finite-temperature correction can be written as follows:
\[ \frac{1}{N} \Delta_T v_T(\lambda T) = \frac{\lambda^{D/2}}{2\pi} a_{D} S_0(\lambda T), \quad (19.131) \]
with
\[ S_0(\lambda_T) = a_D^{-1} \frac{2\pi}{\lambda^{D/2}} \frac{1}{2} \int \frac{d^{D-1}p}{(2\pi)^{D-1}} \left( T \sum_{m=-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d\omega_m}{2\pi} \right) \log(\omega_m^2 + p^2 + \lambda). \] (19.132)

Here the constant \( a_D \) is chosen so that \( S_0(\lambda_T) \) has a convenient form for all \( D \):
\[ a_D = -\Gamma \left( \frac{1}{2} - \frac{D}{2} \right) \frac{1}{(4\pi)^{(D-1)/2}}. \] (19.133)

Special values are \( a_2 = 1 \) and \( a_4 = -1/6\pi \) in \( D = 2 \) and \( D = 4 \) dimensions. By rewriting \( \log a \) as the analytically regularized integral
\[ -\int_0^\infty \left( \frac{d\tau}{\tau} \right) e^{-\tau a}, \]
Eq. (19.132) becomes
\[ S_0(\lambda_T) = -a_D^{-1} \frac{\pi}{\lambda^{D/2}} \left( T \sum_{m=-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d\omega_m}{2\pi} \right) \]
\[ \times \int_0^\infty \frac{d\tau}{\tau} \int \frac{d^{D-1}p}{(2\pi)^{D-1}} \exp(-\tau p^2) \exp \left[ -\tau (\omega_m^2 + \lambda) \right]. \] (19.134)

The integral over \( p \) can now be performed, with the result
\[ \int \frac{d^{D-1}p}{(2\pi)^{D-1}} \exp(-\tau p^2) = \frac{1}{(4\pi \tau)^{(D-1)/2}}. \] (19.135)

Using here the integral formula
\[ \int_0^\infty \frac{d\tau}{\tau} \tau^{-(D-1)/2} e^{-\tau E} = \Gamma \left( \frac{1-D}{2} \right) E^{(D-1)/2}, \] (19.136)
we find
\[ S_0(\lambda_T) = \frac{2\pi}{\lambda^{D/2}} \frac{1}{2} \left( T \sum_{m=-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d\omega_m}{2\pi} \right) \omega_m^2 + \lambda \right)^{(D-1)/2}
\[ = \frac{1}{2\lambda_T^{D/2}} \left( \sum_{m=-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{dm}{m} \right) (m^2 + \lambda_T)^{(D-1)/2}. \] (19.137)

For \( D = 2 \), this reduces to the previous sum (19.124).

In going from (19.132) to (19.137), we have of course rederived formula (11.134) in the form
\[ \int \frac{d^{D-1}q}{(2\pi)^{D-1}} \log(\omega_m^2 + q^2) = \frac{1}{(4\pi)^{(D-1)/2}} \Gamma \left( \frac{1}{2} - \frac{D}{2} \right) \sqrt{\omega_m^{2D-1}}. \] (19.138)

### 19.8.2 Four-Dimensional Model

For \( D = 4 \), we obtain
\[ S_0(\lambda_T) = \frac{\pi}{\lambda^2} \left( T \sum_{m=-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d\omega_m}{2\pi} \right) \sqrt{\omega_m^2 + \lambda^3}
\[ = \frac{1}{2\lambda_T^2} \left( \sum_{m=-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{dm}{m} \right) \sqrt{m^2 + \lambda_T^3}. \] (19.139)
By analogy with the two-dimensional expression (19.127), this can be processed further to a convergent sum
\[
\tilde{S}_0(\lambda_T) = \frac{1}{\lambda_T^2} \sum_{m=1}^{\infty} \left\{ \sqrt{m^2 + \lambda_T^3} - m^3 - \frac{3}{2} \lambda_T m - \frac{3 \lambda_T^2}{8} \right\}. \tag{19.140}
\]

### 19.8.3 Temperature Behavior in Any Dimension

In any dimension \(D\), the expression \(a^D S_0(\lambda_D)\) in the temperature correction (19.131) is derived as follows: First we perform the integral over \(\omega_m\) in (19.137), and rewrite \(a^D S_0(\lambda_T)\) as
\[
a^D S_0(\lambda_T) = a_D \frac{1}{\lambda_T D/2} \left[ \lambda_T^{(D-1)/2} + 2 \sum_{m=1}^{\infty} (m^2 + \lambda_T)^{(D-1)/2} \right] + \pi c. \tag{19.141}
\]

Then we expand the second term in powers of \(\lambda_T\) as
\[
a^D S_0(\lambda_T) = a^D \frac{1}{\lambda_T D/2} \left[ \lambda_T^{(D-1)/2} + 2 \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} \left( \frac{(D-1)/2}{k} \right) m^{D-1-2k} \lambda_T^k \right] + \pi c. \tag{19.142}
\]

Performing the sum over \(m\) in the curly brackets gives
\[
\left[ \lambda_T^{(D-1)/2} + 2 \sum_{k=0}^{\infty} \left( \frac{(D-1)/2}{k} \right) \zeta(2k+1-D) \lambda_T^k \right]. \tag{19.143}
\]

For an even dimension \(D = \bar{D}\), the term with \(k = \bar{D}/2\) has a singularity for \(D\) close to \(2, 4, \ldots\). This singularity cancels a corresponding one in \(c\).

Let \(D\) be close to the even dimension \(\bar{D}\), say \(D = \bar{D} + \epsilon\). Then there are two singular terms in \(a^D S_0(\lambda_T)\):
\[
a^D S_0(\lambda_T) \bigg|_{\text{sing}} = -\Gamma \left( \frac{1}{2} - \frac{D}{2} \right) \frac{1}{\sqrt{4\pi}^{D-1}} \frac{1}{\lambda_T^{D/2}} \frac{\Gamma \left( \frac{1}{2} + \frac{D}{2} \right)}{\Gamma \left( \frac{D}{2} + 1 \right)} \zeta(1 - \epsilon) \lambda_T^{D/2} - \frac{1}{2\sqrt{4\pi}^{D-2}} \frac{\Gamma \left( 1 - \frac{\bar{D}}{2} - \frac{\epsilon}{2} \right)}{\bar{D} + \epsilon}. \tag{19.144}
\]

Expanding \(\zeta(1 - \epsilon) = -\left( 1 - \frac{\epsilon}{\gamma} \right) / \epsilon + \ldots\), the \(1/\epsilon\)-singularity in the first term is given by
\[
\frac{1}{\sqrt{4\pi}^{D-1}} \frac{\Gamma \left( \frac{1}{2} - \frac{D}{2} \right)}{\Gamma \left( \frac{D}{2} + 1 \right)} \frac{\Gamma \left( \frac{1}{2} + \frac{D}{2} \right)}{\Gamma \left( \frac{D}{2} \right)} \frac{1}{\epsilon} = \frac{1}{\sqrt{4\pi}^{D-1}} \frac{(-)^{D/2}}{\Gamma \left( \frac{D}{2} \right) \frac{D}{2} \epsilon}. \tag{19.145}
\]
In the second term, we write
\[ \Gamma \left( \frac{1}{2} - \frac{\bar{D}}{2} - \frac{\epsilon}{2} \right) = (-1)^{\bar{D}/2} \frac{\Gamma (1 + \epsilon/2)}{\Gamma (\bar{D}/2 + \epsilon/2)} \frac{2^{\epsilon}}{\epsilon} \Gamma (1 - \epsilon/2) = \frac{(-1)^{\bar{D}/2 \pi}}{\Gamma (\frac{\bar{D}}{2} + \frac{\epsilon}{2})} \sin \frac{\pi \epsilon}{2}, \]
and see that the $1/\epsilon$-singularities cancel each other. The finite-$\epsilon$ independent contribution is obtained by expanding
\[ \Gamma (k + \epsilon) = \Gamma (k) [1 + \epsilon \psi (k)], \]
such that the first $1/\epsilon$ -singularity is accompanied by
\[ \left\{ 1 + \epsilon \left[ -\log \lambda_T - \psi \left( \frac{1}{2} - \frac{\bar{D}}{2} \right) + \psi \left( \frac{1}{2} + \frac{\bar{D}}{2} \right) - \psi \left( \frac{1}{2} \right) - 2\gamma - \log 4\pi \right] \right\}. \]
The second singularity has a residue
\[ \left\{ 1 + \epsilon \left[ -\log 4\pi - 2\frac{\bar{D}}{D} - \psi \left( \frac{\bar{D}}{2} \right) \right] \right\}. \]
Since
\[ \psi \left( \frac{1}{2} + \frac{\bar{D}}{2} \right) - \psi \left( \frac{1}{2} - \frac{\bar{D}}{2} \right) = \pi \cot \frac{\pi \bar{D}}{2} = 0, \]
the difference is
\[ -\epsilon \left( \log \frac{\lambda_T e^{2\gamma}}{4} - \sum_{k=1}^{D/2} \frac{1}{k} \right). \]
Thus, altogether, we obtain for an even number of dimension $D = 2, 4, 6, \ldots$ the finite sums
\[ S_0 (\lambda_T) = \frac{1}{2 \lambda_T^{D/2}} \left[ \lambda_T^{(D-1)/2} + 2 \sum_{k=0, \neq D/2}^{\infty} \left( \frac{(D-1)/2}{k} \right) \zeta (2k + 1 - D, \lambda_T^k) \right] + S_L^0 (\lambda_T), \]
where
\[ S_L^0 (\lambda_T) \equiv - \alpha_D^{-1} (-1)^{D/2} \frac{1}{4 \Gamma (1 + \frac{D}{2}) \sqrt{4e^{2\gamma}}} \left[ \log \left( \frac{\lambda_T}{4e^{-2\gamma}} \right) - \sum_{k=1}^{D/2} \frac{1}{k} \right]. \]
It is now easy to split (19.151) into
\[ S_0 (\lambda_T) = \tilde{S}_0 (\lambda_T) + \tilde{S}_0 (\lambda_T) + \tilde{S}_0 (\lambda_T) + \tilde{S}_0 (\lambda_T) + S_L^0 (\lambda_T). \]
with the convergent sum

\[ \tilde{S}_0(\lambda_T) = \frac{1}{\lambda_T^{D/2}} \sum_{k=D/2+1}^{\infty} \left( \frac{(D-1)/2}{k} \right) \zeta(2k + 1 - D) \lambda_T^k, \]  
(19.154)

and the finite sum

\[ \tilde{\tilde{S}}_0(\lambda_T) = \frac{1}{\lambda_T^{D/2}} \sum_{k=0}^{D/2-1} \left( \frac{(D-1)/2}{k} \right) \zeta(2k + 1 - D) \lambda_T^k. \]  
(19.155)

Inserting the divergent representation \( \sum_{m=1}^{\infty} m^{-z} \) for the \( \zeta \)-function at negative \( z \), this is formally equal to the divergent sum

\[ \tilde{\tilde{S}}_0(\lambda_T) = \frac{1}{\lambda_T^{D/2}} \sum_{m=0}^{\infty} \left( m^{D-1} + \frac{D-1}{2} \lambda_T m^{D-3} + \ldots \right), \]  
(19.156)

where the last omitted term in parentheses is proportional to \( \lambda_T^{D/2-1}/m \). The convergent sum (19.154) is obviously equal to a convergent power series expansion of the convergent infinite series

\[ \tilde{S}_0(\lambda_T) = \frac{1}{2\lambda_T^{D/2}} \sum_{m=1}^{\infty} \left( \sqrt{m^2 + \lambda_T^{D-1}} - m^{D-1} - \frac{D-1}{2} \lambda_T m^{D-3} + \ldots \right), \]  
(19.157)

where as many powers of \( \lambda_T \) are subtracted as needed for convergence. These are the generalizations of the two- and four-dimensional expressions (19.128) and (19.140), respectively. The expression (19.153) written in this way converges fast for high temperatures.

Let us now derive an expression which converges fast at low temperature \( T \). For this it is most convenient to use the \( D \)-dimensional generalization of Eq. (19.121):

\[ \frac{1}{N} \Delta_T v(\lambda) = \frac{1}{2} \int \frac{d^{D-1}p}{(2\pi)^{D-1}} \left( T \sum_m - \int_{-\infty}^{\infty} \frac{d\omega_m}{2\pi} \right) \log(\omega_m^2 + p^2 + \lambda) \]

\[ = T \int \frac{d^{D-1}p}{(2\pi)^{D-1}} \log \left( 1 - e^{-\Omega/T} \right), \]  
(19.158)

and expand the logarithm in powers of \( e^{-\Omega/T} \). This gives

\[ \frac{\lambda^{D/2}}{2\pi} a_D S_0(\lambda_T) = -T \sum_{\tilde{m}=1}^{\infty} \frac{1}{\tilde{m}} \int \frac{dp^{D-1}}{(2\pi)^{D-1}} e^{-\tilde{m}\sqrt{p^2 + \lambda}/T} \]  
(19.159)

\[ = -\frac{1}{2\pi} S_{D-1} \lambda^{D/2} \sqrt{\lambda_T} \sum_{\tilde{m}=1}^{\infty} \frac{1}{\tilde{m}} \int_0^\infty ds s^{D-2} e^{-\tilde{m}\sqrt{s^2 + 1}/T}. \]

We now use the integral formula

\[ K_{\nu}(z) = \left( \frac{z}{2} \right)^{\nu} \frac{1}{\Gamma \left( \nu + \frac{3}{2} \right)} \int_0^\infty ds s^{2\nu} (s^2 + 1)^{-1/2} e^{-z\sqrt{s^2 + 1}}, \]  
(19.160)
\[ S_0(\lambda T) = 2\pi \frac{2^{D/2}}{\Gamma \left( \frac{1}{2} - \frac{D}{2} \right) \Gamma \left( \frac{1}{2} \right)} \sum_{\tilde{m}=1}^{\infty} \frac{1}{\left( 2\pi \tilde{m} \sqrt{\lambda T} \right)^{D/2}} K_{D/2} \left( 2\pi \tilde{m} \sqrt{\lambda T} \right). \] (19.161)

The same expression will be found once more in an exactly solvable fermionic model in Eq. (23.277), except for a fermionic alternating sign \((-\tilde{m})^{\tilde{m}-1} \). For \( D = 2 \), the sum (19.161) reduces to

\[ S_0(\lambda T) = -2 \sum_{\tilde{m}=1}^{\infty} \frac{1}{2\pi^2 \lambda T} \tilde{m} = -\frac{1}{2\pi^2 \lambda T} \zeta(2) = -\frac{1}{12\lambda T}. \] (19.162)

This yields the free energy of black-body radiation in two dimensions

\[ \frac{1}{N} \Delta_T v(\lambda) \xrightarrow{T \to \infty} -T^2 \frac{\pi}{6}. \] (19.164)

The same result could have been obtained from the divergent expression (19.119) using the generalized Euler-Maclaurin formula based on analytic continuation as in the formal calculation of the Casimir effect in Eq. (7.727):

\[
\frac{1}{N} \Delta_T v(\lambda) = \frac{1}{2} \int \frac{d\omega}{2\pi} T \left[ \sum_{\tilde{m}=\infty}^{\infty} \frac{1}{\omega_m^2} - \int_{-\infty}^{\infty} \frac{d\omega_m}{2\pi} \right] \log(\omega_m^2 + p^2) = \frac{1}{2} T \sum_{\tilde{m}=\infty}^{\infty} \sqrt{\omega_m^2} = 2\pi T^2 \zeta(-1) = -T^2 \frac{\pi}{6}. \] (19.165)

In the second step we have used the integral formula (11.134) in the form

\[ \int \frac{d^{D-1}q}{(2\pi)^{D-1}} \log(\omega_m^2 + q^2) = \frac{1}{(4\pi)^{(D-1)/2}} \Gamma \left( \frac{1}{2} - \frac{D}{2} \right) \sqrt{\omega_m^{D-1}}. \] (19.166)

In four dimensions, we have

\[ S_0(\lambda T) = 6 \sum_{\tilde{m}=1}^{\infty} \frac{1}{\left( 2\pi \tilde{m} \sqrt{\lambda T} \right)^2} K_2 \left( 2\pi \tilde{m} \sqrt{\lambda T} \right) \] (19.167)

which becomes for large \( T \) [using \( K_2(z) \to 2z^2 \) at \( z \to 0 \)]

\[ S_0(\lambda T) \xrightarrow{T \to \infty} 12 \sum_{\tilde{m}=1}^{\infty} \frac{1}{\left( 2\pi \tilde{m} \sqrt{\lambda T} \right)^4} = 12 \frac{T^4}{\lambda^2} \zeta(4) = 12 \frac{T^4}{\lambda^2} \frac{\pi^4}{90}. \] (19.168)
The potential
\[ \frac{1}{N} \Delta_T v = \frac{\lambda^2}{2\pi} S_0 = -\frac{\lambda^2}{12\pi^2} S_2 = -T^4 \frac{\pi^2}{90} \]
(19.169)
is the analog of the free energy density \( f \) of black-body radiation. It is related to
the internal energy density \( u \) by \( f = -\frac{1}{3} u \).

Recall the analog of the Stefan-Boltzmann law for the free energy density \( f \) of hot (or massless) bosons. In this limit, \( f \) has a pure power behavior of \( T^n \), and the associated entropy density \( s = \partial f / \partial T \) is related to \( f \) by a factor \( -\frac{n^2}{T} \). For the internal energy density \( u \equiv E/V = f + TS \) and for the specific heat at constant
volume \( c_V = \partial u / \partial T \), the factors are \( 1 - n \) and \( (1 - n)n/T \), respectively.

The original Stefan-Boltzmann law for black body radiation is obtaine d from
this by accounting for the two polarization degrees of freedom with an extra factor 2 so that
\[ \Delta_T v = f = -\frac{1}{3} u = -\frac{2}{3} \pi T^4 \]
(19.170)
In physical units, this becomes
\[ f = -\frac{2}{3} \frac{\sigma}{c} T^4, \]
(19.171)
where \( \sigma \) is the Stefan-Boltzmann constant
\[ \sigma = \frac{\pi^2 k_B^4}{60 \hbar^3 c^2} \approx 5.67 \times 10^{-5} \frac{g}{\text{sec}^\circ \text{K}^4}. \]
(19.172)

The same result could, of course, have been obtained directly from (19.132),
which would have led to the four-dimensional version of Eq. (19.165). Using (19.166)
for \( D = 3 \) we obtain
\[ \frac{1}{N} \Delta_T v(\lambda) = \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} T \left[ \sum_{m=-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d\omega_m}{2\pi} \right] \log(\omega_m^2 + p^2) \]
\[ = \frac{1}{2} T \sum_{m=-\infty}^{\infty} \frac{1}{(4\pi)^{3/2}} \frac{2}{3} \Gamma(-1/2) \sqrt{\omega_m^2} = -\frac{4}{3\pi} T^4 \zeta(-3) = -T^4 \frac{\pi^2}{90}. \]
(19.173)

In the last step we have used Eq. (7.728).

At arbitrary \( D \), we use the small-\( z \) behavior
\[ K_\nu(z) \approx \frac{1}{2} \left( \frac{2}{z} \right)^\nu \Gamma(\nu) \]
in (19.161) to find
\[ S_0(\lambda_T) \xrightarrow{T \to 0} -a_D^{-1} 2\pi^{1-D/2} \frac{1}{\lambda_T^{D/2}} \zeta(D) \]
\[ = -\frac{2\pi}{\lambda_T^{D/2} a_D} \frac{1}{\pi^{D/2}} \Gamma \left( \frac{D}{2} \right) T^D \zeta(D). \]
(19.174)
Dropping the prefactor $2\pi/\lambda^{D/2}a_D$ gives the finite-temperature contribution $\Delta_T v(\lambda)/N$ to the free energy density in $D$ dimensions:

$$\Delta_T v(\lambda)/N \xrightarrow{T\to\infty} -\frac{1}{\pi^{D/2}} \Gamma\left(\frac{D}{2}\right) T^D \zeta(D).$$

(19.175)

The same result could, of course, have been obtained directly from (19.132), using

$$\sum_{m=-\infty}^{\infty} \omega_m^{D-1} = (2\pi T)^{D-1} 2\zeta(1-D),$$

(19.176)

and the well-known identity for Riemann's $\zeta$-functions

$$\Gamma\left(\frac{1}{2} - \frac{z}{2}\right) \zeta(1-z) = \pi^{1/2-z} \Gamma\left(\frac{z}{2}\right) \zeta(z).$$

(19.177)

### 19.9 Criteria for the Onset of Fluctuations in Ginzburg-Landau Theories

The understanding of the nonlinear $\sigma$-model permits us to improve our understanding of the phase transition in $\phi^4$-theories with $O(N)$-symmetry discussed in Chapter 25. We shall base the discussion on the euclidean version of the Lagrangian (16.1), which is then called Ginzburg-Landau Hamiltonian [5]:

$$\mathcal{H}(\phi_a, \partial \phi_a) = \frac{1}{2} (\partial \phi_a)^2 + \frac{m^2}{2} \phi_a^2 + \frac{g}{4} (\phi_a^2)^2.$$  

(19.178)

In the associated partition function

$$Z = \int \mathcal{D}\phi_a e^{-\int d^D x \mathcal{H}/k_BT},$$

(19.179)

a phase transition takes place at a temperature $T_c$ which lies always below the mean field temperature $T_c^{MF}$, where the mass term in (19.178) changes its sign. It is possible to give a rough estimate of the shift $\Delta T \equiv T_c^{MF} - T_c$ which is caused by fluctuations.

After an obvious renormalization of field and mass, the Ginzburg-Landau energy density in $D$ dimensions may be written as

$$\mathcal{H}(\phi_a, \partial \phi_a) = \frac{1}{2a_D^2} \left\{ \alpha^2 u^2 [\partial \phi_a(x)]^2 + \tau \phi_a^2(x) + \frac{g}{2} \left[ \phi_a^2(x) \right]^2 \right\}.$$  

(19.180)

From here on we use natural units with $k_BT_c^{MF} = 1$. The fields have zero engineering dimensions, $a$ denotes some microscopic length scale of the system, usually the size of atoms or molecules, and $g$ is some interaction strength. The parameter $\alpha$

---

specifies the zero-temperature coherence length of the system in units of \( a \) as being \( \xi_0 = \alpha a/\sqrt{2} \). This can vary greatly from system to system. In superconductors, for example, \( \alpha \) can lie anywhere between a few thousand and less than ten in high-temperature superconductors.

We shall here be concerned only with the destruction of the ordered state which lies below the critical temperature where \( \tau < 0 \). There the fields fluctuate around an ordered ground state with a constant field expectation \( \langle \phi_a \rangle \equiv \Phi_a \equiv \langle \phi \rangle N_a \equiv \Phi N_a \). The size of the field \( \phi(x) \) fluctuates around \( \Phi \) which depends on \( \tau \) as \( \Phi = \sqrt{\Phi_a^2} = \sqrt{-\tau/g} \). At that field, the energy density is minimal, and has the value \( \mathcal{H}_0 = \mathcal{H}(\Phi_a,0) = -\tau^2/4ga^D \). The directional unit vector \( N_a \) breaks spontaneously the O(\( N \))-symmetry. The temperature-dependent coherence length \( \xi = \alpha a/\sqrt{2|\tau|} \) describes the range of the size fluctuations of the order field.

### 19.9.1 Ginzburg’s Criterion

The magnitude of the fluctuations is estimated by assuming the field to live in patches on a simple cubic lattice of spacing \( \xi_l = l \xi \), choosing eventually a spacing parameter between \( l = 1 \) and 2 to ensure the independence of the patches. In the low-temperature ordered phase with \( \tau < 0 \), the small fluctuations \( \delta \phi(x) = \phi(x) - \Phi \) of the size of the order field \( \phi(x) \) around the minimum of \( (19.180) \) have a Hamiltonian

\[
\mathcal{H}_{ax}(\phi_a, \partial \phi_a) = \frac{1}{2a^D} \left\{ a^2 \alpha^2 \left[ \partial \phi(x) \right]^2 + 2|\tau| \delta \phi^2(x) \right\}.
\]  

(19.181)

Their size is therefore given by

\[
\langle [\phi(x) - \Phi]^2 \rangle = \int \frac{d^D k}{(2\pi)^D} \frac{a^D}{\alpha^2 a^2 k^2 + 2|\tau|} = \frac{a^{D-2}}{\xi_l^{D-2}} \frac{1}{\alpha^2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + \frac{l^2}{\xi_l^2}},
\]  

(19.182)

where \( q \) is the dimensionless reduced momentum \( \xi_l k \). Inserting \( \xi_l \), the right-hand side can be rewritten as

\[
\frac{l^{2-D}}{\alpha^D} (2|\tau|)^{D/2-1} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + \frac{l^2}{\xi_l^2}}.
\]  

(19.183)

The relative size of the fluctuations can therefore be written as

\[
\frac{\langle [\phi(x) - \Phi]^2 \rangle}{\Phi^2} = 2l^{2-D} (2|\tau|)^{D/2-2} \alpha^{-D} v_{l^2}^D(0),
\]  

(19.184)

where \( v_{l^2}^D(0) \) is the lattice Yukawa potential of reduced mass \( l \) of Eq. (19.67). Its size is calculated most easily with the help of the integral formula (19.72). Mean-field behavior breaks down if (19.184) is of the order unity, which for \( D < 4 \) happens at the reduced Ginzburg temperature

\[
|\tau_G| \approx [l^{2-D} K v_{l^2}^D(0)]^{2/(4-D)},
\]  

(19.185)
19.9 Criteria for the Onset of Fluctuations in Ginzburg-Landau Theories

where

\[ K \equiv 2^{D/2-1} g/\alpha^D, \]  

(19.186)
i.e., at a Ginzburg temperature \( T_G \equiv T_{c}^{MF}(1 - |\tau_G|) \). In his original paper [6], Ginzburg estimated \( l \approx 1 \) and \( v_D^1(0) \) in three dimensions by an integral carried up to \( |p| = \pi \):

\[ v_D^1(0) \approx \int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2 + 1} \approx \frac{1}{2\pi^2} \int_0^\pi dp \frac{p^2}{p^2 + 1} \approx \frac{1}{4\pi}. \]  

(19.187)

This led him to the estimate

\[ |\tau_G| \approx \frac{1}{8\pi^2} g^2 \alpha^3. \]  

(19.188)

In old-fashioned type-II superconductors, \( |\tau_G| \) can be as small as \( 10^{-8} \) [7], which explains why conventional superconductors are well described by mean-field theory. In modern high-\( T_c \) superconductors, on the other hand, Ginzburg’s estimate leads to \( |\tau_G| \approx 0.01 \) [8], such that critical exponents should become observable.

Ginzburg’s estimate \( l \approx 1 \) for a rough determination of the critical temperature interval is based on the assumption that the order field \( \Phi(x) \) is properly defined only up to a length scale of the order of the coherence length. In an ordinary superconductor, this is indeed the case. There the order field describes the Cooper pairs of electrons whose wave function extends over a coherence length, such that the cutoff in all momentum integrals would be of this order. In modern superconductors, where the phase transition occurs at higher critical temperature being of the order of 100 K, however, the Cooper pairs could have a much smaller diameter than the coherence length [9]. In this case, \( l \) would be considerably smaller than unity. If the Cooper pairs are bound very strongly, another effect appears: The phase transition is caused by quantum fluctuations. In this limit, the Cooper pairs form an almost free gas of almost point-like bosons which undergo Bose-Einstein condensation of the type described in Subsection 2.15.3. The relevant length scale is then the de Broglie wavelength of thermal motion

\[ \lambda = \frac{2\pi\hbar}{\sqrt{2Mk_BT}}, \]  

(19.189)

where \( M \) is the mass of the Cooper pairs.

For \( D > 4 \), the right-hand side in (19.184) decreases when approaching the critical point, so only mean-field behavior is observed. If \( D = 4 - \varepsilon \) lies only slightly below four, the right-hand side of (19.184) behaves like \( |\tau|^{-\varepsilon/2} \), implying a good mean-field description as long as \( |\tau| \) is sufficiently small.

19.9.2 Azimuthal Correction to Ginzburg’s Criterion

If a \( \phi^4 \)-theory has only a single real field, Ginzburg’s criterion gives a rough estimate of the temperature at which fluctuations become important. In an \( O(N) \)-symmetric system with Nambu-Goldstone modes, however, it grossly underestimates the temperature shift \( \Delta T \) of the transition. A better estimate is based on the observation
that the kinetic term defines a second, completely independent, energy scale of the system. To identify it, we split the fields according to size and direction in $O(N)$ field space as $\phi_a = \phi n_a$, $n_a^2 = 1$. The directions $n_a$ describe the long-range fluctuations of the Goldstone modes. Sufficiently far from the critical regime, we may neglect the gradient term of the size $\phi(x)$, and approximate the energy density by

$$H(\phi, \partial n_a) = \frac{1}{2a^D} \left\{ \alpha^2 a^2 \phi^2(x) [\partial n_a(x)]^2 + \tau \phi^2(x) + \frac{g}{2} \phi^4(x) \right\}.$$  

The fluctuations of the Goldstone modes are controlled by the gradient term whose magnitude depends on the size $\Phi$ of $\phi$ at the minimum of the potential. The gradient energy density is

$$H_{n_a}(\partial n_a) = \frac{\beta}{2 \xi_l^{D-2}} [\partial n_a(x)]^2,$$  

with

$$\beta = \beta(\Phi) = \alpha^2 \left( \frac{\xi_l}{a} \right)^{D-2} \Phi^2 = \frac{\alpha^D l^{D-2}}{2(2|\tau|)^{D/2-2} g} = \frac{l^{D-2}}{K |\tau|^{D/2-2}}.$$  

This is the second energy scale. It measures how much energy is spent when reversing the direction vector $n_a$ over the distance $\xi_l$. It is the continuous version of the stiffness of the directional field defined for a lattice model in Eq. (19.64).

From the discussion of the spherical model in Section 19.6 we know that directional fluctuations disorder a system if the bending stiffness drops below a certain critical value $\beta_{cr}$. For the $O(N)$-symmetry with large $N$, this critical value was found on a cubic lattice in three, four, and large dimensions $D$, respectively, to have the values [see Table 19.1 and the hopping expansion (19.74)]

$$\beta_{cr} = N v_0 D(0) \approx N \ 0.2527, \ \ N \ 0.1549, \ N/2.$$

We have remarked below Eq. (19.76) that, although these values were derived only for large $N$, Monte Carlo simulations show that they can be trusted already for $D = 3$ and $N = 2$ with an error of only 10%.

The simulations are done by putting the Heisenberg model on a lattice of unit spacing, so that the energy density for $N = 2$ takes the XY-model form

$$H_{XY}(\partial n_a) \approx \beta \sum_{\mu=1,\ldots,D} [1 - \cos \nabla_{\mu} \gamma(x)],$$  

where $\nabla_{\mu}$ denotes the lattice gradient in the $\mu$th coordinate direction, and $\gamma \equiv \arctan n_2/n_1$. Since the quality of the approximation increases with $N$ and $D$, we can trust Eq. (19.192) within 10% for all $N$ and $D \geq 3$. This accuracy will be sufficient for the criterion to be derived here.

The critical stiffness can, incidentally, be also estimated by calculating its renormalized version from a sum of an infinite number of terms in a perturbation expansion. Expanding the cos-function in the energy (19.193) into a Taylor series,
and calculating the harmonic expectation values of quartic, sextic, etc. terms, we find, in a self-consistent approximation of the Hartree-Fock-Bogoliubov type, that the stiffness has a renormalized value \( \beta_R = \beta e^{-1/2D\beta_R} \) [12]. This softens with increasing temperature \( 1/\beta \), until \( \beta \) reaches a critical value \( \beta_{cr} = \epsilon/2D \), where \( \beta_R \) drops to zero. In \( D = 3 \) dimensions, this happens at \( \beta_{cr} = 0.4530 \ldots \), a value which is in excellent agreement with the Monte Carlo number \( \beta_{cr}^{MC} \approx 0.45 \). The prediction of such sharp drop is true only in two dimensions, as shown by Kosterlitz and Thouless [11]. For \( D > 2 \) it is an artefact of the approximations, and the exact stiffness goes to zero like \( |T_e - T|^{(D-2)/\nu} \), with a critical exponent \( \nu \approx 1/2 + (4 - D)/10 + \ldots \).

The estimate for the critical stiffness (19.192) leads now directly to the announced criterion: The phase fluctuations disorder the system if the stiffness \( \beta \) in Eq. (19.190) drops below the critical value (19.192), which happens at a reduced temperature

\[
|\tau_K| \approx |N\tau^{2-D}_K v_0^D(0)|^{2/(4-D)}, \quad D < 4, \; N \geq 2. \tag{19.194}
\]

Thus we obtain the important result that

\[
|\tau_K| \approx |Nv_0^D(0)/v_{\beta}^2(0)|^{2/(4-D)}|\tau_G|, \quad D < 4, \; N \geq 2. \tag{19.195}
\]

This implies that, for all systems with \( N \geq 2 \), directional fluctuations destroy the order before size fluctuations become large. They cause a phase transition below the Ginzburg temperature, at \( T_K \equiv T^{MF}_G(1 - |\tau_K|) \). For \( D = 3 \), and \( l = (1,3/2,2) \), the relation becomes \( |\tau_K| \approx N^2|\tau_G| \times (2.20, 3.48, 5.56) \). Thus, if the critical regime is approached in a \( \phi^4 \)-theory with a well-formed mean-field regime, the transition is always initiated by directional fluctuations. In particular, the estimates for the critical regime of the high-\( |T_e| \), superconductors [8] will receive a factor \( \approx 9 \).

The dominance of directional fluctuations is, of course, most prominent in the limit of large \( N \), and it is therefore not surprising that the critical exponents of the \( \phi^4 \)-theory and the Heisenberg model have the same \( 1/N \)-expansions in any dimension \( D > 2 \), this being a pleasant demonstration of the universality of critical phenomena.

By adding the energy density of directional fluctuations with the field-dependent stiffness \( \beta = \beta(\phi) = \alpha^D \phi^{2(D-2)}/(2|\tau|)^{D/2-1} \) to the field energy density \( \mathcal{H}(\phi, \partial n_a) \), we can study, as in Ref. [13], the combined energy density in the disordered phase where the symmetry is restored but the average \( \bar{\Phi} \) of the size of the order field \( \phi \) is nonzero.

The directional fluctuations play a crucial role in pion physics, as pointed out in Ref. [13] and discussed in detail in Section 23.10.

### 19.9.3 Experimental Consequences

How do we determine experimentally the fluctuation parameter \( K \) to estimate \(|\tau_G|\) and \(|\tau_K|\)? In magnetic systems, one measures the susceptibility tensor \( \chi_{AB}(k) \equiv \int d^Dx e^{ikx}\langle \phi_a(x)\phi_b(0) \rangle \) at wave vector \( k \), and decomposes it into parallel and perpendicular parts as \( \chi_{AB}(k) = (\Phi_a\Phi_b/\Phi^2)\chi_\parallel(k) + (\delta_{AB} - \Phi_a\Phi_b/\Phi^2)\chi_\perp(k) \). The mean-field behavior of these quantities is \( \chi_\parallel(k) \approx a^D/(\alpha^2a^2k^2 + 2|\tau|) \) and \( \chi_\perp(k) \approx a^D/\alpha^2a^2k^2 \). Combining these at \( k = 0 \) with the mean-field behavior of the spontaneous magnetization \( \Phi = \sqrt{|\tau|}/g \), and with the temperature-dependent coherence length \( \xi \), we see
that the size of $K$ can immediately be estimated from a plot, versus $t \equiv T/T_c - 1$, of either of the dimensionless experimental quantities

$$
K_{\text{exp}} \approx |t|^{-2/D/2} \frac{k^2}{\xi^{D-2} k_B T \Phi^2} \chi(k)_{k \to 0},
$$

(19.196)

or

$$
K_{\text{exp}} \approx |t|^{-2/D/2} \frac{1}{\xi^{D} k_B T \Phi^2} \chi(0),
$$

(19.197)

these being written down in physical units. Note that $t$ measures the temperature distance from the experimental $T_c$, in contrast to $\tau \equiv T/T_c^{\text{MF}} - 1$. In the mean-field regime, where $t \approx \tau$, $K_{\text{exp}}$ is constant and can be inserted into Eq. (19.194) to find the temperature $T_K$ where directional fluctuations destroy the order.

In superfluid helium we may plot, in analogy to the transverse susceptibility expression for $K_{\text{exp}}$, the quantity $K_{\text{exp}} \approx |t|^{-2/D/2} M^2 k_B T / \xi^{D-2} h^2 \rho_s$, where $M$ is the atomic mass and $\rho_s$ the superfluid mass density, which at the mean-field level is defined by writing the gradient energy (19.190) as $(\rho_s/2 k_B T) (h^2/M^2) [\partial n_a(x)]^2$. In the critical regime, the three expressions for $K_{\text{exp}}$ go universally to zero like $|t|^{-2/D/2}$, since $\xi \propto |t|^{-\nu}$, $\chi(0) \approx |t|^{(D-2)\nu}$, $k^2 \chi(k)_{k \to 0} \approx |t|^{\nu}$, $\Phi^2 \approx |t|^4 (D-2+\eta)$, $\rho_s \approx |t|^{(D-2)\nu}$, with $\eta \approx [(N+2)/2(N+8)^2](4-D)^2 + \ldots$.

Experimentally, the superfluid density of helium for $D = 3$ shows no mean-field behavior à la Ginzburg-Landau down to $T \approx T_c/4$, such that the above formulas cannot properly be applied. Let us nevertheless estimate orders of magnitude of a would-be mean-field behavior: $\rho_s/\rho \approx 2|\tau|$ [14], where $\rho = M/a^3$ is the total mass density, with $a \approx 3.59$ Å [15]. Then the factor $k_B T c$ at $T_c = 2.18$ K can be expressed as $k_B T_c \approx 2.35 h^2/M a^3$ [15]. With $\xi_0 \approx 2$ Å, we obtain an estimate $K \approx 1.2 a/\xi_0 \approx 2$. Inserting this into Eq. (19.194) and the relation (19.195), we obtain, for $l = 1$ and $2$,

$$
|\tau_K| \approx 1, \quad |\tau_G| \approx 0.12 \quad \text{and} \quad |\tau_K| \approx 0.255, \quad |\tau_G| \approx 0.03,
$$

(19.198)

respectively. The large size of $|\tau_K|$ reflects the bad quality of a mean-field description. The larger $l$ gives the more physical estimate.

Notes and References

For an introduction into the subject of this chapter read the textbook by

The particular citations in this chapter refer to:

[1] L.P. Kadanoff, Physics 2, 263 (1966);
K.G. Wilson, Phys. Rev. B 4, 3174, 3184 (1971);
See also


See Eq. (3.24) on p. 315 of the textbook in [3] where \( g/\alpha^3 \approx 111(T_c/T_F)^2 \), with \( T_F \approx 10^3 T_c \).


See Section 7.8 in the textbook [3].


See Fig. 5.3 on p. 428 in the textbook [3].

See pp. 256–257 in the textbook [3].