Nothing is mightier than an idea
whose time has come.
V. M. Hugo (1802-1885)

12

Quantum Electrodynamics

In Chapter 7 we have learned how to quantize relativistic free fields and in Chapters 10 and 11 how to deal with interactions if the coupling is small. So far, this was only done perturbatively. Fortunately, there is a large set of physical phenomena for which perturbative techniques are sufficient to supply theoretical results that agree with experiment. In particular, there exists one theory, where the agreement is extremely good. This is the quantized theory of interacting electrons and photons called *quantum electrodynamics*, or shortly QED.

12.1 Gauge Invariant Interacting Theory

The free Lagrangians of electrons and photons are known from Chapter 5 as

$$\mathcal{L}(x) = \overset{\mathrm{e}}{\mathcal{L}}(x) + \overset{\gamma}{\mathcal{L}}(x), \tag{12.1}$$

with

$$\overset{\mathrm{e}}{\mathcal{L}}(x) = \bar{\psi}(x)\,(i\partial\!\!\!/ - m)\,\psi(x), \tag{12.2}$$

$$\overset{\gamma}{\mathcal{L}}(x) = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) = -\frac{1}{4}\left[\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\right]\left[\partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)\right]. \tag{12.3}$$

When quantizing the photon field, there were subtleties due to the gauge freedom in the choice of the gauge fields A_μ:

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x). \tag{12.4}$$

For this reason, there were different ways of constructing a Hilbert space of free particles. The first, described in Subsection 7.5.1, was based on the quantization of only the two physical transverse degrees of freedom. The time component of the gauge field A^0 and the spatial divergence $\boldsymbol{\nabla}\cdot\mathbf{A}(x)$ had no canonically conjugate field and were therefore classical fields, with no operator representation in the Hilbert space. The two fields are related by Coulomb's law which reads, in the absence of charges:

$$\boldsymbol{\nabla}^2 A^0(x) = -\partial^0 \boldsymbol{\nabla}\cdot\mathbf{A}(x). \tag{12.5}$$

Only the *transverse components*, defined by

$$A_{\perp}^{i}(x) \equiv \left(\delta^{ij} - \nabla^i\nabla^j/\boldsymbol{\nabla}^2\right) A^j(\mathbf{x},t), \tag{12.6}$$

were operators. These components represent the proper dynamical variables of the system. After fulfilling the canonical commutation rules, the positive- and negative-frequency parts of these fields define creation and annihilation operators for the electromagnetic quanta. These are the photons of right and left circular polarization.

This method had an esthetical disadvantage that two of the four components of the vector field $A^\mu(x)$ require a different treatment. The components which become operators change with the frame of reference in which the canonical quantization procedure is performed.

To circumvent this, a covariant quantization procedure was developed by Gupta and Bleuler in Subsection 7.5.3. In their quantization scheme, the propagator took a pleasant covariant form. But this happened at the expense of another disadvantage, that this Lagrangian describes the propagation of four particles of which only two correspond to physical states. Accordingly, the Hilbert space contained two kinds of unphysical particle states, those with negative and those with zero norm. Still, a physical interpretation of this formalism was found with the help of a subsidiary condition that selects the physical subspace in the Hilbert space of free particles.

The final and most satisfactory successful quantization was developed by Faddeev and Popov [1] and was described in Subsection 7.5.2. It started out by modifying the initial photon Lagrangian by a gauge-fixing term

$$\mathcal{L}_{\mathrm{GF}}(x) = -D\partial^\mu A_\mu(x) + \alpha D^2(x)/2. \tag{12.7}$$

After that, the quantization can be performed in the usual canonical way.

12.1.1 Reminder of Classical Electrodynamics of Point Particles

In this chapter we want to couple electrons and photons with each other by an appropriate interaction and study the resulting interacting field theory, the famous *quantum electrodynamics* (QED). Since the coupling should not change the two physical degrees of freedom described by the four-component photon field A^μ, it is important to preserve the gauge invariance, which was so essential in assuring the correct Hilbert space of free photons. The prescription how this can be done has been known for a long time in the context of classical electrodynamics of point particles. In that theory, a free relativistic particle moving along an arbitrarily parametrized path $x^\mu(\tau)$ in four-space is described by an action

$$\mathcal{A} = -mc^2 \int d\tau \sqrt{\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau}} = -mc^2 \int dt \left[1 - \frac{\mathbf{v}^2(t)}{c^2}\right]^{1/2}, \tag{12.8}$$

where $x^0(\tau) = t$ is the time and $d\mathbf{x}/dt = \mathbf{v}(t)$ the velocity along the path. If the particle has a charge e and lies at rest at position $\mathbf{x}$, its potential energy is

$$V(t) = e\phi(\mathbf{x},t), \tag{12.9}$$

where

$$\phi(\mathbf{x},t) = A^0(\mathbf{x},t). \tag{12.10}$$

In our convention, the charge of the electron e has a negative value to agree with the sign in the historic form of the Maxwell equations

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{E}(x) &= -\boldsymbol{\nabla}^2\phi(x) = \rho(x), \\
\boldsymbol{\nabla}\times\mathbf{B}(x) - \dot{\mathbf{E}}(x) &= \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{A}(x) - \dot{\mathbf{E}}(x) \\
&= -\left[\boldsymbol{\nabla}^2\mathbf{A}(x) - \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\mathbf{A}(x)\right] - \dot{\mathbf{E}}(x) = \frac{1}{c}\mathbf{j}(x).
\end{aligned} \tag{12.11}$$

If the electron moves along a trajectory $\mathbf{x}(t)$, the potential energy becomes

$$V(t) = e\phi\left(\mathbf{x}(t),t\right). \tag{12.12}$$

In the Lagrangian $L = T - V$, this contributes with the opposite sign

$$L^{\rm int}(t) = -eA^0\left(\mathbf{x}(t),t\right), \tag{12.13}$$

adding a potential term to the interaction

$$\mathcal{A}^{\rm int}|_{\rm pot} = -e\int dt A^0\left(\mathbf{x}(t),t\right). \tag{12.14}$$

Since the time t coincides with $x^0(t)/c$ of the trajectory, this can be expressed as

$$\mathcal{A}^{\rm int}|_{\rm pot} = -\frac{e}{c}\int dx^0 A^0(x). \tag{12.15}$$

In this form it is now quite simple to write down the complete electromagnetic interaction purely on the basis of relativistic invariance. The minimal Lorentz-invariant extension of (12.15) is obviously

$$\mathcal{A}^{\rm int} = -\frac{e}{c}\int dx^\mu A_\mu(x). \tag{12.16}$$

Thus, the full action of a point particle can be written, more explicitly, as

$$\begin{aligned}
\mathcal{A} &= \int dt L(t) = -mc\int ds - \frac{e}{c}\int dx^\mu A_\mu(x) \\
&= -mc^2\int dt\left[1-\frac{\mathbf{v}^2}{c^2}\right]^{1/2} - e\int dt\left(A^0 - \frac{1}{c}\mathbf{v}\cdot\mathbf{A}\right).
\end{aligned} \tag{12.17}$$

The canonical formalism supplies the canonically conjugate momentum

$$\mathbf{P} = \frac{\partial L}{\partial \mathbf{v}} = m\frac{\mathbf{v}}{\sqrt{1-\mathbf{v}^2/c^2}} + \frac{e}{c}\mathbf{A} \equiv \mathbf{p} + \frac{e}{c}\mathbf{A}. \tag{12.18}$$

Thus the velocity is related to the canonical momentum and external vector potential via

$$\frac{\mathbf{v}}{c} = \frac{\mathbf{P} - \frac{e}{c}\mathbf{A}}{\sqrt{\left(\mathbf{P} - \frac{e}{c}\mathbf{A}\right)^2 + m^2c^2}}. \tag{12.19}$$

This can be used to calculate the Hamiltonian as the Legendre transform

$$H = \frac{\partial L}{\partial \mathbf{v}}\mathbf{v} - L = \mathbf{P}\cdot\mathbf{v} - L, \tag{12.20}$$

with the result

$$H = c\sqrt{\left(\mathbf{P} - \frac{e}{c}\mathbf{A}\right)^2 + m^2c^2} + eA^0. \tag{12.21}$$

At nonrelativistic velocities, this has the expansion

$$H = mc^2 + \frac{1}{2m}\left(\mathbf{P} - \frac{e}{c}\mathbf{A}\right)^2 + eA^0 + \dots\ . \tag{12.22}$$

The rest energy mc^2 is usually omitted in this limit.

12.1.2 Electrodynamics and Quantum Mechanics

When going over from quantum mechanics to second-quantized field theory in Chapter 2, we found the rule that a nonrelativistic Hamiltonian

$$H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x}) \tag{12.23}$$

becomes an operator

$$H = \int d^3x\, \psi^\dagger(\mathbf{x},t)\left[-\frac{\boldsymbol{\nabla}^2}{2m} + V(\mathbf{x})\right]\psi(\mathbf{x},t). \tag{12.24}$$

For brevity, we have omitted a hat on top of H and the fields $\psi^\dagger(\mathbf{x},t), \psi(\mathbf{x},t)$. Following the rules of Chapter 2, we see that the second-quantized form of the interacting nonrelativistic Hamiltonian in a static $A(\mathbf{x})$ field with the Hamiltonian (12.22) (minus mc^2),

$$H = \frac{(\mathbf{P} - e\mathbf{A})^2}{2m} + eA^0, \tag{12.25}$$

is given by

$$H = \int d^3x\, \psi^\dagger(\mathbf{x},t)\left[-\frac{1}{2m}\left(\boldsymbol{\nabla} - i\frac{e}{c}\mathbf{A}\right)^2 + eA^0\right]\psi(\mathbf{x},t). \tag{12.26}$$

The action of this theory reads

$$\mathcal{A} = \int dt L = \int dt \int d^3x \left[\psi^\dagger(\mathbf{x},t) \left(i\partial_t + eA^0 \right) \psi(\mathbf{x},t) \right.$$
$$\left. + \frac{1}{2m} \psi^\dagger(\mathbf{x},t) \left(\boldsymbol{\nabla} - i\frac{e}{c}\mathbf{A} \right)^2 \psi(\mathbf{x},t) \right]. \tag{12.27}$$

It is easy to verify that (12.26) reemerges from the Legendre transform

$$H = \frac{\partial L}{\partial \dot{\psi}(\mathbf{x},t)} \dot{\psi}(\mathbf{x},t) - L. \tag{12.28}$$

The action (12.27) holds also for time-dependent $A^\mu(x)$-fields.

These equations show that electromagnetism is introduced into a free quantum theory of charged particles following the minimal substitution rule

$$\begin{aligned} \boldsymbol{\nabla} &\to \boldsymbol{\nabla} - i\frac{e}{c}\mathbf{A}(\mathbf{x},t), \\ \partial_t &\to \partial_t + ieA^0(\mathbf{x},t), \end{aligned} \tag{12.29}$$

or covariantly:

$$\partial_\mu \to \partial_\mu - i\frac{e}{c}A_\mu(x). \tag{12.30}$$

The substituted action has the important property that the gauge invariance of the free photon action is preserved by the interacting theory: If we perform the gauge transformation

$$A^\mu(x) \to A^\mu(x) + \partial^\mu \Lambda(x), \tag{12.31}$$

i.e.,

$$\begin{aligned} A^0(\mathbf{x},t) &\to A^0(\mathbf{x},t) + \partial_t \Lambda(\mathbf{x},t), \\ \mathbf{A}(\mathbf{x},t) &\to \mathbf{A}(\mathbf{x},t) - \boldsymbol{\nabla}\Lambda(\mathbf{x},t), \end{aligned} \tag{12.32}$$

the action remains invariant provided thet we simultaneously change the fields $\psi(\mathbf{x},t)$ of the charged particles by a spacetime-dependent phase

$$\psi\left(\mathbf{x},t\right) \to e^{-i(e/c)\Lambda(\mathbf{x},t)}\psi(\mathbf{x},t). \tag{12.33}$$

Under this transformation, the space and time derivatives of the field change like

$$\begin{aligned} \boldsymbol{\nabla}\psi(\mathbf{x},t) &\to e^{-i(e/c)\Lambda(\mathbf{x},t)} \left[\boldsymbol{\nabla} - i\frac{e}{c}\boldsymbol{\nabla}\Lambda(\mathbf{x},t) \right] \psi, \\ \partial_t\psi &\to e^{-i(e/c)\Lambda(\mathbf{x},t)} \left(\partial_t - ie\partial_t\Lambda \right) \psi(\mathbf{x},t). \end{aligned} \tag{12.34}$$

The covariant derivatives in the action (12.27) have therefore the following simple transformation law:

$$\begin{aligned} \left(\boldsymbol{\nabla} - i\frac{e}{c}\mathbf{A} \right) \psi(\mathbf{x},t) &\to e^{-i(e/c)\Lambda(\mathbf{x},t)} \left(\boldsymbol{\nabla} - i\frac{e}{c}\mathbf{A} \right) \psi(\mathbf{x},t), \\ \left(\partial_t + i\frac{e}{c}A^0 \right) \psi(\mathbf{x},t) &\to e^{-i(e/c)\Lambda(\mathbf{x},t)} \left(\partial_t + ieA^0 \right) \psi(\mathbf{x},t). \end{aligned} \tag{12.35}$$

These combinations of derivatives and gauge fields are called *covariant derivatives* and are written as

$$\mathbf{D}\psi(\mathbf{x},t) \equiv \left(\boldsymbol{\nabla} - i\frac{e}{c}\mathbf{A}\right)\psi(\mathbf{x},t),$$
$$D_t\psi(\mathbf{x},t) \equiv \left(\partial_t + ieA^0\right)\psi(\mathbf{x},t), \tag{12.36}$$

or, in four-vector notation, as

$$D_\mu\psi(x) = \left(\partial_\mu + i\frac{e}{c}A_\mu\right)\psi(x). \tag{12.37}$$

Here the adjective "covariant" does not refer to the Lorentz group but to the gauge group. It records the fact that $D_\mu\psi$ transforms under local gauge changes (12.29) of ψ in the same way as ψ itself in (12.33):

$$D_\mu\psi(x) \to e^{-i(e/c)\Lambda(x)}D_\mu\psi(x). \tag{12.38}$$

With the help of such covariant derivatives, any action which is invariant under global phase changes by a constant phase angle [i.e., $U(1)$-invariant in the sense discussed in Section 8.11.1]

$$\psi(x) \to e^{-i\alpha}\psi(x), \tag{12.39}$$

can easily be made invariant under *local* gauge transformations (12.31). We merely have to replace all derivatives by covariant derivatives (12.37), and add to the field Lagrangian the gauge-invariant photon expression (12.3).

12.1.3 Principle of Nonholonomic Gauge Invariance

The minimal substitution rule can be viewed as a consequence of a more general principle of *nonholonomic gauge invariance*. The physics of the initial action (12.17) is trivially invariant under the addition of a term

$$\Delta\mathcal{A} = -\frac{e}{c}\int dt\,\dot{x}^\mu(t)\partial_\mu\Lambda(x). \tag{12.40}$$

The integral runs over the particle path and contributes only a pure surface term from the endpoints:

$$\Delta\mathcal{A} = -\frac{e}{c}\left[\Lambda(x_b) - \Lambda(x_a)\right]. \tag{12.41}$$

This does not change the particle trajectories. If we now postulate that the dynamical laws of physics remain also valid when we admit *multivalued* gauge functions $\Lambda(x)$ for which the Schwarz integrability criterion is violated, i.e., which possess noncommuting partial derivatives:

$$(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)\Lambda(x) \neq 0. \tag{12.42}$$

Then the derivatives

$$A_\mu(x) = \partial_\mu \Lambda(x) \tag{12.43}$$

have a nonzero curl $F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\mu A_\nu) = (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\Lambda(x) \neq 0$, and the action (12.41) coincides with the interaction (12.16).

Similarly we can derive the equations of motion of a wave function in an electromagnetic field from that in field-free space by noting the trivial invariance of quantum mechanics without fields under gauge transformations (12.33), and by extending the set of permissible gauge functions $\Lambda(x)$ to multivalued functions for which the partial derivatives do not commute as in (12.42).

In either case, the nonholonomic gauge transformations convert the physical laws obeyed by a particle in Euclidean spacetime without electromagnetism into those with electromagnetic fields.

This principle is discussed in detail in the literature [11]. It can be generalized to derive the equations of motion in a curved spacetime from those in flat spacetime by nonholonomic coordinate transformations which introduce defects in spacetime.

12.1.4 Electrodynamics and Relativistic Quantum Mechanics

Let us follow this rule for relativistic electrons and replace, in the Lagrangian (12.2), the differential operator $\partial\!\!\!/ = \gamma^\mu \partial_\mu$ by

$$\gamma^\mu \left(\partial_\mu + i\frac{e}{c} A_\mu \right) = \left(\partial\!\!\!/ + i\frac{e}{c} A\!\!\!/ \right) \equiv D\!\!\!\!/\,. \tag{12.44}$$

In this way we arrive at the Lagrangian of quantum electrodynamics (QED)

$$\mathcal{L}(x) = \bar{\psi}(x)\,(iD\!\!\!\!/ - m)\,\psi(x) - \frac{1}{4}F_{\mu\nu}^2. \tag{12.45}$$

The classical field equations can easily be found by extremizing the action with respect to all fields, which gives

$$\frac{\delta \mathcal{A}}{\delta \bar{\psi}(x)} = (iD\!\!\!\!/ - m)\,\psi(x) = 0, \tag{12.46}$$

$$\frac{\delta \mathcal{A}}{\delta A_\mu(x)} = \partial_\nu F^{\nu\mu}(x) - \frac{1}{c} j^\mu(x) = 0, \tag{12.47}$$

where $j^\mu(x)$ is the *current density*:

$$j^\mu(x) \equiv ec\bar{\psi}(x)\gamma^\mu\psi(x). \tag{12.48}$$

Equation (12.47) coincides with the Maxwell equation for the electromagnetic field around a classical four-dimensional vector current $j^\mu(x)$:

$$\partial_\nu F^{\nu\mu}(x) = \frac{1}{c} j^\mu(x). \tag{12.49}$$

In the Lorenz gauge $\partial_\mu A^\mu(x) = 0$, this equation reduces to

$$-\partial^2 A^\mu(x) = \frac{1}{c} j^\mu(x). \tag{12.50}$$

The current density j^μ combines the charge density $\rho(x)$ and the spatial current density $\mathbf{j}(x)$ of particles of charge e in a four-vector:

$$j^\mu = (c\rho, \mathbf{j}) \,. \tag{12.51}$$

In terms of electric and magnetic fields $E^i = F^{i0}$, $B^i = -F^{jk}$, the field equations (12.49) turn into the Maxwell equations

$$\boldsymbol{\nabla} \cdot \mathbf{E} \;=\; \rho \;=\; e\bar{\psi}\gamma^0\psi = e\psi^\dagger\psi, \tag{12.52}$$

$$\boldsymbol{\nabla} \times \mathbf{B} - \dot{\mathbf{E}} \;=\; \frac{1}{c}\mathbf{j} = \frac{e}{c}\bar{\psi}\boldsymbol{\gamma}\psi. \tag{12.53}$$

The first is Coulomb's law, the second is Ampère's law in the presence of charges and currents.

Note that the physical units employed here differ from those used in many books of classical electrodynamics [12], by the absence of a factor $1/4\pi$ on the right-hand side. The Lagrangian used in those books is

$$\begin{aligned}\mathcal{L}(x) &= -\frac{1}{8\pi} F_{\mu\nu}^2(x) - \frac{1}{c} j^\mu(x) A_\mu(x) \\ &= \frac{1}{4\pi}\left[\mathbf{E}^2(x) - \mathbf{B}^2(x)\right] - \left[\rho(x)\phi(x) - \frac{1}{c}\mathbf{j}(x)\cdot\mathbf{A}(x)\right],\end{aligned} \tag{12.54}$$

which leads to Maxwell's field equations

$$\begin{aligned}\boldsymbol{\nabla} \cdot \mathbf{E} &= 4\pi\rho, \\ \boldsymbol{\nabla} \times \mathbf{B} - \dot{\mathbf{E}} &= \frac{4\pi}{c}\mathbf{j}.\end{aligned} \tag{12.55}$$

The form employed conventionally in quantum field theory arises from this by replacing $A \to \sqrt{4\pi}A$ and $e \to -\sqrt{4\pi e^2}$. The charge of the electron in our units has therefore the numerical value

$$e = -\sqrt{4\pi\alpha} \approx -\sqrt{4\pi/137} \tag{12.56}$$

rather than $e = -\sqrt{\alpha}$.

12.2 Noether's Theorem and Gauge Fields

In electrodynamics, the conserved charge resulting from the U(1)-symmetry of the matter Lagrangian by Noether's theorem (recall Chapter 8) is the source of a massless particle, the photon. This is described by a gauge field which is minimally coupled to the conserved current. A similar structure will be seen in Chapters 27 and 28 to exist for many internal symmetries giving rise to nonabelian versions of the photon, for instance the famous W- and Z-vector mesons, which mediate the weak interactions, or the gluons which give rise to strong interactions. It is useful to recall Noether's derivation of conservation laws in such theories.

For a a locally gauge invariant theory, the conserved matter current can no longer be found by the rule (8.118), which was so useful in the globally invariant theory. Indeed, in quantum electrodynamics, the derivative with respect to the local field transformation $\epsilon(x)$ would be simply given by

$$j_\mu = \frac{\delta \mathcal{L}}{\partial \partial_\mu \Lambda}, \tag{12.57}$$

since this would be identically equal to zero, due to local gauge invariance. We may, however, subject *just* the matter field to a local gauge transformation at *fixed* gauge fields. Then we obtain the correct current

$$j_\mu \equiv \left. \frac{\partial \mathcal{L}}{\partial \partial_\mu \Lambda} \right|_\gamma . \tag{12.58}$$

Since the complete change under local gauge transformations $\delta_{\rm s}^x \mathcal{L}$ vanishes identically, we can alternatively vary *only* the gauge fields and keep the electron field fixed

$$j_\mu = -\left. \frac{\partial \mathcal{L}}{\partial \partial_\mu \Lambda} \right|_{\rm e} . \tag{12.59}$$

This is done most simply by forming the functional derivative with respect to the gauge field, and by omitting the contribution of $\overset{\gamma}{\mathcal{L}}$:

$$j_\mu = -\frac{\partial \overset{\rm e}{\mathcal{L}}}{\partial \partial_\mu \Lambda}. \tag{12.60}$$

An interesting consequence of local gauge invariance can be found for the gauge field itself. If we form the variation of the pure gauge field action

$$\delta_{\rm s} \overset{\gamma}{\mathcal{A}} = \int d^4x \ {\rm tr} \left[\delta_{\rm s}^x A_\mu \frac{\overset{\gamma}{\mathcal{A}}{}^\epsilon}{\delta A_\mu} \right], \tag{12.61}$$

and insert, for $\delta_{\rm s}^x A$, an infinitesimal pure gauge field configuration

$$\delta_{\rm s}^x A_\mu = -i \partial_\mu \Lambda(x), \tag{12.62}$$

the variation must vanish for all $\Lambda(x)$. After a partial integration, this implies the local conservation law $\partial_\mu j^\mu(x) = 0$ for the current

$$j^\mu(x) = -i \frac{\delta \overset{\gamma}{\mathcal{A}}}{\delta A_\mu}. \tag{12.63}$$

In contrast to the earlier conservation laws derived for matter fields which were valid only if the matter fields obey the Euler-Lagrange equations, the current conservation law for gauge fields is valid for *all* field configurations. It is an *identity*, often called *Bianchi identity* due to its close analogy with certain identities in Riemannian geometry.

To verify this, we insert the Lagrangian (12.3) into (12.63) and find $j^\nu = \partial_\mu F^{\mu\nu}/2$. This current is trivially conserved for any field configuration due to the antisymmetry of $F^{\mu\nu}$.

12.3 Quantization

The canonical formalism can be used to identify canonical momenta of the fields $\psi(x)$ and $A^i(x)$:

$$\pi_\psi(x) = \frac{\partial \mathcal{L}}{\partial \psi(x)} = \psi^\dagger(x), \tag{12.64}$$

and

$$\pi_{A^i}(x) \equiv \pi^i(x) = F^{0i}(x) = -E^i(x), \tag{12.65}$$

and to find the Hamiltonian density

$$\begin{aligned} \mathcal{H}(x) &= \frac{\partial L}{\partial \dot\psi(x)}\dot\psi(x) + \frac{\partial \mathcal{L}(x)}{\partial \dot A^k(x)} - \mathcal{L}(x) \\ &= \bar\psi(-i\boldsymbol{\gamma}\boldsymbol{\nabla} + m)\psi + \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) + \boldsymbol{\nabla} A^0 \cdot \mathbf{E} + e\bar\psi\gamma_\mu\psi A^\mu. \end{aligned} \tag{12.66}$$

Here, and in all subsequent discussions, we use natural units in which the light velocity is equal to unity.

The quantization procedure in the presence of interactions now goes as follows: The Dirac field of the electron has the same equal-time anticommutation rules as in the free case:

$$\begin{aligned} \{\psi_a(\mathbf{x},t), \psi_b^\dagger(\mathbf{x}',t)\} &= \delta^{(3)}(\mathbf{x}'-\mathbf{x})\delta_{ab}, \\ \{\psi(\mathbf{x},t), \psi(\mathbf{x}',t)\} &= 0, \\ \{\psi^\dagger(\mathbf{x},t)\psi^\dagger(x't)\} &= 0. \end{aligned} \tag{12.67}$$

For the photon field we first write down the naive commutation rules of the spatial components:

$$-[E_i(\mathbf{x},t), A_j(\mathbf{x}',t)] = -i\delta^{(3)}(\mathbf{x}-\mathbf{x}')\delta_{ij} \tag{12.68}$$

$$[A_i(\mathbf{x},t), A_j(\mathbf{x}',t)] = 0, \tag{12.69}$$

$$[E_i(\mathbf{x},t), E_j(\mathbf{x}',t)] = 0. \tag{12.70}$$

As in Eq. (7.346), the first commutator cannot be true here, since by Coulomb's law (12.52):

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{E} &= -\boldsymbol{\nabla}\cdot\dot{\mathbf{A}}-\boldsymbol{\nabla}^2A^0\\ &= e\psi^\dagger\psi,\end{aligned} \tag{12.71}$$

and we want the canonical fields A_i to be independent of $\psi, \psi^\dagger$, and thus to commute with them. The contradiction can be removed just as in the free case by postulating (12.68) only for the transverse parts of E_i, and using $\delta^T_{ij}(\mathbf{x}-\mathbf{x}')$ as in (7.347), while letting the longitudinal part $\boldsymbol{\nabla}\cdot\mathbf{A}(\mathbf{x},t)$ be a c-number field, since it commutes with all $A^i(\mathbf{x},t)$. The correct commutation rules are the following:

$$\begin{aligned}\left[\dot{A}_j(\mathbf{x},t), A_j(\mathbf{x}',t)\right] &= -i\delta^T_{ij}(\mathbf{x}-\mathbf{x}'),\\ \left[A_j(\mathbf{x},t), A_j(\mathbf{x}',t)\right] &= 0,\\ \left[\dot{A}_i(\mathbf{x},t), \dot{A}_j(\mathbf{x}',t)\right] &= 0.\end{aligned} \tag{12.72}$$

To calculate the temporal behavior of an arbitrary observable composed of $\psi, \psi^\dagger$, and of $A_i, \dot{A}_i$ fields, only one more set of commutation rules has to be specified which are those with A^0. This field occurs in the Hamiltonian density (12.66) and is not one of the canonical variables. Moreover, in contrast to the free-field case in Section 7.5, it is no longer a c-number. To see this, we express A^0 in terms of the dynamical fields using Coulomb's law (12.71):

$$A^0(\mathbf{x},t)=\frac{1}{4\pi}\int d^3x'\frac{1}{|\mathbf{x}'-\mathbf{x}|}\left(e\psi^\dagger\psi+\boldsymbol{\nabla}\cdot\dot{\mathbf{A}}\right)(\mathbf{x}',t). \tag{12.73}$$

This replaces Eq. (4.268) in the presence of charges. In an infinite volume with asymptotically vanishing fields, there is no freedom of adding a solution of the homogeneous Poisson equation (4.269). Hence, whereas $\boldsymbol{\nabla}\cdot\mathbf{A}$ is a c-number field, the time component of the gauge field A^0 is now a non-local operator involving the fermion fields. Since these are independent of the electromagnetic field, A^0 still commutes with the canonical A_i, $\dot{A}_j$ fields:

$$\left[A^0(\mathbf{x},t), A^i(\mathbf{x}')\right]=\left[A^0(\mathbf{x},t), \dot{A}^i(\mathbf{x}',t)\right]=0. \tag{12.74}$$

The commutator with the Fermi fields, on the other hand, is nonzero:

$$\left[A^0(\mathbf{x},t), \psi(\mathbf{x}',t)\right]=-\frac{e}{4\pi|\mathbf{x}-\mathbf{x}'|}\psi(\mathbf{x},t). \tag{12.75}$$

Note the peculiar property of A^0: It does not commute with the electron field, no matter how large the distance between the space points is. This property is called *nonlocality*. It is a typical property of the present transverse covariant quantization procedure.

Certainly, the arbitrary c-number function $\boldsymbol{\nabla}\cdot\mathbf{A}(\mathbf{x},t)$ can be made zero by an appropriate gauge transformation, as in (4.257).

In the Hamiltonian, the field A^0 can be completely removed by a partial integration:

$$\int d^3x\, \boldsymbol{\nabla} A^0 \cdot \mathbf{E} = -\int d^3x\, A^0 \boldsymbol{\nabla} \cdot \mathbf{E}, \tag{12.76}$$

if we set the surface term equal to zero. Using now the field equation

$$\boldsymbol{\nabla} \cdot \mathbf{E} = e\psi^\dagger \psi, \tag{12.77}$$

we derive

$$\begin{aligned} H &= \int d^3x\, \mathcal{H} &(12.78)\\ &= \int d^3x \left\{ \bar{\psi} \left[-i\boldsymbol{\gamma} \cdot (\boldsymbol{\nabla} - i\frac{e}{c}\mathbf{A}) + m \right] \psi + \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right) \right\}. \end{aligned}$$

When looking at this expression, one may wonder where the electrostatic interaction has gone. The answer is found by decomposing the electric field

$$E^i = -\partial^0 A^i + \partial^i A^0 \tag{12.79}$$

into longitudinal and transverse parts E_L^i and E_T^i with $\mathbf{E}_L \cdot \mathbf{E}_T = 0$:

$$\begin{aligned} E_L^i &= \partial^i \left(A^0 - \frac{\partial^i \partial^i}{\boldsymbol{\nabla}^2} A^j \right), \\ E_T^i &= -\partial^0 \left(\delta^{ij} - \frac{\partial^i \partial^j}{\boldsymbol{\nabla}^2} \right) A^j. \end{aligned} \tag{12.80}$$

Then the field energy becomes

$$\frac{1}{2}\int d^3x \left(\mathbf{E}^2 + \mathbf{B}^2\right) = \frac{1}{2}\int d^3x \left(\mathbf{E}_T^2 + \mathbf{B}^2\right) + \frac{1}{2}\int d^3x \mathbf{E}_L^2. \tag{12.81}$$

Using (12.73), we see that the longitudinal field is simply given by

$$\begin{aligned} \mathbf{E}_L(x) &= -\frac{1}{4\pi} \boldsymbol{\nabla} \int d^3x' \frac{1}{|\mathbf{x} - \mathbf{x}'|} e\psi^\dagger(\mathbf{x}', t)\psi(\mathbf{x}', t) \\ &= \boldsymbol{\nabla} \left[\frac{1}{\boldsymbol{\nabla}^2} e\psi^\dagger(x)\psi(x) \right]. \end{aligned} \tag{12.82}$$

It is the Coulomb field caused by the charge density of the electron $e\psi^\dagger(x)\psi(x)$. The field energy carried by $E_L^i(x)$ is

$$\begin{aligned} \frac{1}{2}\int d^3x\, \mathbf{E}_L^2(x) &= \frac{e^2}{2}\int d^3x \left\{ \boldsymbol{\nabla} \left[\frac{1}{\boldsymbol{\nabla}^2} \psi^\dagger(x)\psi(x) \right] \right\}^2 \\ &= -\frac{e^2}{2}\int d^3x\, \psi^\dagger(x)\psi(x) \frac{1}{\boldsymbol{\nabla}^2} \psi^\dagger(x)\psi(x) \\ &= \frac{e^2}{8\pi}\int d^3x d^3x'\, \psi^\dagger(\mathbf{x}, t)\psi(\mathbf{x}, t) \frac{1}{|\mathbf{x} - \mathbf{x}'|} \psi^\dagger(\mathbf{x}', t)\psi(\mathbf{x}', t). \end{aligned} \tag{12.83}$$

This coincides precisely with the classical Coulomb energy associated with the charge density (12.52). The term $\frac{1}{2}\int d^3x\,(\mathbf{E}_T^2+\mathbf{B}^2)$ in Eq. (12.81), on the other hand, is an operator and contains the energy of the field quanta.

In order to develop a perturbation theory for QED in this quantization, we must specify the free and interacting parts of the action. Since A^0 and $\boldsymbol{\nabla}\cdot\mathbf{A}$ are unquantized and appear only quadratically in the action, they may be eliminated in the action in the same way as in the energy, so that the action becomes

$$\mathcal{A}=\int d^4x\left\{\bar\psi(x)(i\partial\!\!\!/-M)\psi(x)+\frac{1}{2}\left[\mathbf{E}_T^2(x)-\mathbf{B}^2(x)\right]\right\}+\mathcal{A}^{\rm int}. \tag{12.84}$$

The first two terms are the actions of the Dirac field ψ and transverse electromagnetic fields $\mathbf{A}_T$, and $\mathcal{A}^{\rm int}$ denotes the interaction

$$\mathcal{A}^{\rm int}=-\frac{e^2}{8\pi}\int dt\int d^3xd^3x'\psi^\dagger(\mathbf{x},t)\psi(\mathbf{x},t)\frac{1}{|\mathbf{x}-\mathbf{x}'|}\psi(\mathbf{x}',t)\psi(\mathbf{x}',t)+\int d^4x\,\frac{1}{c}\,\mathbf{j}\cdot\mathbf{A}_T. \tag{12.85}$$

The interaction contains two completely different terms: The first is an instantaneous Coulomb interaction at a distance, which takes place without retardation and involves the charge density. It is a nontrivial field-theoretic exercise to show that the absence of retardation in the first term is compensated by current-current interaction resulting from the second term, so that it does not cause any conflicts with relativity. This will be done at the end of Section 14.16.

The special role of the Coulomb interaction is avoided from the beginning in the Gupta-Bleuler quantization procedure that was discussed in Subsection 7.5.2. There the free action was

$$\mathcal{A}=\int d^4x\left[\bar\psi\,(i\partial\!\!\!/-m)\,\psi-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-D\partial^\mu A_\mu+D^2/2\right], \tag{12.86}$$

and the interaction had the manifestly covariant form

$$\mathcal{A}^{\rm int}=-\int d^4x\,j^\mu A_\mu. \tag{12.87}$$

12.4 Perturbation Theory

Let us now set up the rules for building the Feynman diagrams to calculate the effect of the interaction. In this context, we shall from now on attach, to the free propagator, a subscript 0. The propagator of the free photon depends on the gauge. It is most simple in the Gupta-Bleuler quantization scheme, where [see (7.510)]

$$G_0^{\mu\nu}(x,x')=-g^{\mu\nu}G_0(x,x')=-g^{\mu\nu}\int\frac{d^4q}{(2\pi)^4}\frac{i}{q^2+i\eta}e^{-ik(x-x')}. \tag{12.88}$$

Since we want to calculate the effect of interactions, we shall from now on attach to the free propagator a subscript 0.

The free-particle propagator of the electrons was given in (7.289):

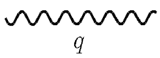
$$S_0(x - x') = \int \frac{d^4p}{(2\pi)^4} \frac{i}{\not{p} - M + i\eta} e^{-ip(x-x')}. \tag{12.89}$$

In a Wick expansion of $e^{i\mathcal{A}^{\rm int}}$, each contraction is represented by one of these two propagators:

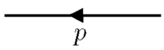
$$\overbrace{A^\mu(x)A^\nu}(x') = G_0^{\mu\nu}(x - x'), \tag{12.90}$$

$$\overbrace{\psi(x)\bar{\psi}}(x') = S_0(x - x'). \tag{12.91}$$

In the Feynman diagrams, they are pictured by the lines



$$= -g^{\mu\nu} \frac{i}{q^2 + i\epsilon}, \tag{12.92}$$

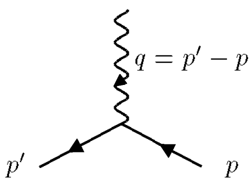


$$= \frac{i}{\not{p} - m}. \tag{12.93}$$

The interaction Lagrangian

$$\mathcal{L}^{\rm int} = -e\bar{\psi}\gamma^\mu A_\mu(x) \tag{12.94}$$

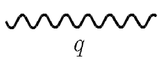
is pictured by the vertex



$$= -e\gamma^\mu. \tag{12.95}$$

With these graphical elements we must form all Feynman diagrams which can contribute to a given physical process.

In the transverse quantization scheme, the Feynman diagrams are much more complicated. Recalling the propagator (7.361), the photon line stands now for

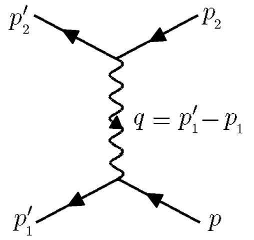


$$= P_{\rm phys}^{\mu\nu}(q) \frac{i}{q^2 + i\epsilon}, \tag{12.96}$$

with the physical off-shell polarization sum [compare (12.7)]

$$P_{\rm phys}^{\mu\nu}(q) = -g^{\mu\nu} - \frac{q_\mu q_\nu}{(q\eta)^2 - q^2} + q\eta \frac{q_\mu \eta_\nu + q_\nu \eta_\mu}{(q\eta)^2 - q^2} - \eta_\mu \eta_\nu \frac{q^2}{(q\eta)^2 - q^2}. \tag{12.97}$$

The photon propagator is very complicated due to the appearance of the frame-dependent auxiliary vector $\eta = (1, 0, 0, 0)$. As a further complication, there are diagrams from the four-fermion Coulomb interactions in (12.85). These can be pictured by a photon exchange diagram

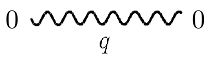


$$= \frac{i}{\mathbf{q}^2}\gamma^0 \times \gamma^0. \tag{12.98}$$

They may be derived from an auxiliary interaction

$$\mathcal{A}^{\rm int} = -\int d^4x\, j^0 A_0, \tag{12.99}$$

assuming the A_0-field to have the propagator



$$= \frac{i}{\mathbf{q}^2}\,. \tag{12.100}$$

If this propagator is added to the physical one, it cancels precisely the last term in the off-shell polarization sum (12.97), which becomes effectively

$$P^{\mu\nu}_{\rm phys,eff}(\mathbf{k}) = -g^{\mu\nu} - \frac{q_\mu q_\nu}{(q\eta)^2 - q^2} + q\eta\frac{q_\mu\eta_\nu + q_\nu\eta_\mu}{(q\eta)^2 - q^2}. \tag{12.101}$$

Of course, the final physical results cannot depend on the frame in which the theory is quantized. Thus it must be possible to drop all η-dependent terms. We shall now prove this in three steps:

First, a photon may be absorbed (or emitted) by an electron which is on their mass shell before and after the process. The photon propagator is contracted with an electron current as follows

$$\bar{u}(\mathbf{p}', s_3')\gamma_\mu u(\mathbf{p}, s_3')P^{\mu\nu}_{\rm phys,eff}(q). \quad q = p' - p. \tag{12.102}$$

Since the spinors on the right and left-hand side satisfy the Dirac equation, the current is conserved and satisfies

$$\bar{u}(\mathbf{p}', s_3')\gamma_\mu u(\mathbf{p}, s_3')q^\mu = 0. \tag{12.103}$$

This condition eliminates the terms containing the vector q^μ in the polarization sum (12.101). Only the reduced polarization sum

$$P^{\mu\nu}_{\rm red}(q) = -g^{\mu\nu} \tag{12.104}$$

survives, which is the polarization tensor of the Gupta-Bleuler propagator (12.88).

The same cancellation occurs if a photon is absorbed by an internal line, although due to a slightly more involved mechanism. An internal line may arise in two ways.

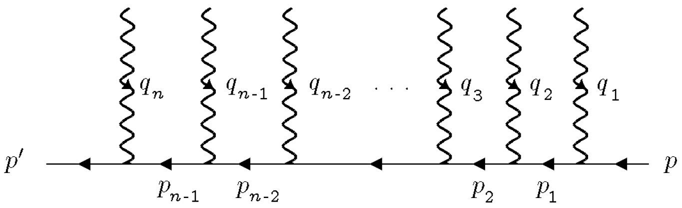


FIGURE 12.1 An electron on the mass shell absorbing several photons.

An electron may enter a Feynman diagram on the mass shell, absorb a number of photons, say n of them, and leave again on the mass shell as shown in Fig. 12.1. The associated off-shell amplitude is

$$a(p', p, q_i) = \frac{1}{\not p' - M} \not q_n \frac{1}{\not p_{n-1} - M} \not q_{n-1} \not q_{n-1} \cdots \not q_3 \frac{1}{\not p_2 - M} \not q_2 \frac{1}{\not p_1 - M} \not q_1. \quad (12.105)$$

It has to be amputated and evaluated between the initial and final spinors, which amounts to multiplying it from the left and right with $\bar{u}(\mathbf{p}', s_3')(\not p' - M)$ and with $(\not p - M)u(\mathbf{p}, s_3)$, respectively. If an additional photon is absorbed, it must be inserted as shown in Fig. 12.2. At each vertex, there is no current conservation since the

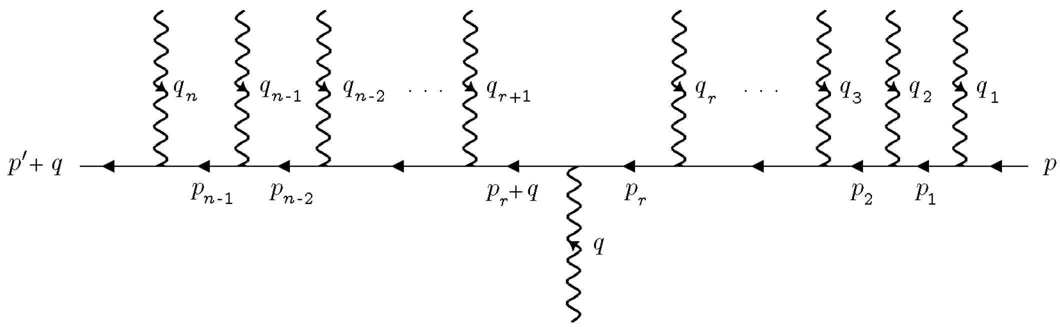


FIGURE 12.2 An electron on the mass shell absorbing several photons, plus one additional photon.

photon lines are not on their mass shell. Nevertheless, the sum of all $n+2$ diagrams does have a conserved current.

To prove this we observe the following *Ward-Takahashi identity* for free particles [2, 3]:

$$\frac{1}{\not p_r + \not q - M} \not q \frac{1}{\not p_r - M} = \frac{1}{\not p_r - M} - \frac{1}{\not p_r + \not q - M}. \quad (12.106)$$

More details on this important identity will be given in the next section.

The sum of all off-shell absorption diagrams can be written as

$$a(p', p, q_i; q) = \frac{1}{\not p' + \not q - M} \not q_n \frac{1}{\not p_{n-1} + \not q - M} \not q_{n-1} \cdots$$

$$\cdots \frac{1}{\not{p}_r + \not{q} - M} \not{q} \frac{1}{\not{p}_r - M} \cdots \not{q}_2 \frac{1}{\not{p}_1 - M} \not{q}_1 u(\mathbf{p}, s_3), \quad (12.107)$$

to be evaluated between $\bar{u}(\mathbf{p}' + \mathbf{q}, s_3')(\not{p}' + \not{q} - M)$ and $(\not{p} - M)u(\mathbf{p}, s_3)$. With the help of the Ward-Takahashi identity we can now remove the $\not{q}$ recursively from $a(p', p, q_i; q)$ and remain with the difference

$$a(p', p, q_i; q) = a(p', p, q_i) - a(p' + q, p + q, q_i). \quad (12.108)$$

When evaluating the right-hand side between the above spinors, we see that the first term in the difference vanishes since the left-hand spinor satisfies the Dirac equation. The same thing holds for the second term and the right-hand spinor. Thus the polarization sum in the photon propagator can again be replaced by the reduced expression (12.104).

Finally, the electron line can be closed to a loop as shown in Fig. 12.3. Here the

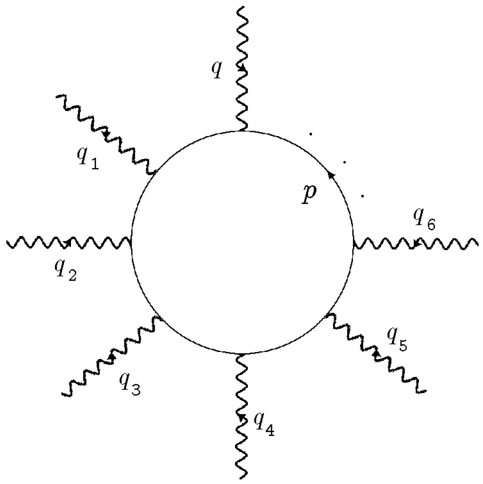


FIGURE 12.3 An internal electron loop absorbing several photons, plus an additional photon, and leaving again on the mass shell.

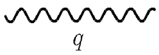
amplitude (12.108) appears in a loop integral with an additional photon vertex:

$$\int \frac{d^4p}{(2\pi)^4} \mathrm{tr}[\not{q}\, a(p', p, q_i; q)] = \int \frac{d^4p}{(2\pi)^4} \mathrm{tr}\{\not{q}\, [a(p', p, q_i) - a(p' + q, p + q, q_i)]\}. \quad (12.109)$$

If the divergence of the integral is made finite by a dimensional regularization, the loop integral is translationally invariant in momentum space and the amplitude difference vanishes. Hence, also in this case, the polarization sum can be replaced by the reduced expression (12.104).

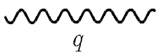
Thus we have shown that due to current conservation and the Ward-Takahashi identity, the photon propagator in all Feynman diagrams can be replaced by

(12.110)



$$= G^{\mu\nu}(q) = \frac{-ig^{\mu\nu}}{q^2 + i\eta}.$$

As a matter of fact, for the same reason, any propagator



$$= G^{\mu\nu}(q) = -\frac{i}{q^2}\left[g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}(1-\alpha)\right] \tag{12.111}$$

can be used just as well, and the parameter α is arbitrary. Indeed, this is the propagator arising when adding to the gauge-invariant Lagrangian in (12.84) the gauge-fixing expression

$$\mathcal{L}_{\rm GF} = \frac{\alpha}{2}D^2 - D\partial^\mu A_\mu. \tag{12.112}$$

For the value $\alpha = 1$ favored by Feynman, the propagator (12.4) reduces to (12.110).

12.5 Ward-Takahashi Identity

From the application of Eq. (12.106), it is apparent that the Ward-Takahashi identity plays an important role in ensuring the gauge invariance of loop diagrams. In fact, the renormalizability of quantum electrodynamics was completed only after making use of the diagonal part of it, which was the original *Ward identity*. For free particles, we observe that the Ward-Takahashi identity (12.106) can be written in terms of the electron propagator (12.90), and the free vertex function $\Gamma_0^\mu(p', p) = \gamma^\mu$, as

$$S_0^{-1}(p') - S_0^{-1}(p) = i(p-q)_\mu \Gamma_0^\mu(p', p). \tag{12.113}$$

The original Ward identity is obtained from this by forming the limit $p' \to p$:

$$\frac{\partial}{\partial p_\mu} S_0^{-1}(p) = i\Gamma_0^\mu(p, p). \tag{12.114}$$

The important contribution of Ward and Takahashi was to prove that their identity is valid for the *interacting* propagators and vertex functions, order by order in perturbation theory. Thus we may drop the subscripts zero in Eq. (12.113) and write

$$S^{-1}(p') - S^{-1}(p) = i(p'-p)_\mu \Gamma^\mu(p', p). \tag{12.115}$$

This identity is a general consequence of gauge invariance, as was first conjectured by Rohrlich [4].

For the general proof of (12.115), the key observation is that the operator version of the fully interacting electromagnetic current $j^\mu(x) = e\bar\psi(x)\gamma^\mu\psi(x)$ satisfies, at equal times, the commutation rules with the interacting electron and photon fields

$$[j^0(x), \psi(y)]\delta(x^0 - y^0) = -e\gamma^0\psi(x)\,\delta(x^0 - y^0), \tag{12.116}$$

$$[j^0(x), \bar\psi(y)]\delta(x^0 - y^0) = e\bar\psi(x)\gamma^0\,\delta(x^0 - y^0), \tag{12.117}$$

$$[j^0(x), \bar A^\mu(y)]\delta(x^0 - y^0) = 0. \tag{12.118}$$

This follows directly from the canonical equal-time anticommutation rules of the electrons written in the form

$$\{\psi(x), \psi^\dagger(y)\}\delta(x^0 - y^0) = \delta^{(4)}(x - y). \tag{12.119}$$

As a consequence of (12.116)–(12.118), we find for any local operator $O(x)$:

$$\partial_\mu \left(\hat{T}\, j^\mu(x) O(y)\right) = \left(j^0(x) O(y_i)\right) \delta(x^0 - y^0) + \hat{T}\left(\partial_\mu j^\mu(x) O(y)\right). \tag{12.120}$$

The first term on the right-hand side arises when the derivative is applied to the Heaviside functions in the definition (2.232) of the time-ordered product. The generalization to many local operators reads:

$$\partial_\mu\left(\hat{T} j^\mu(x) O(y_1)\cdots O(y_i)\cdots O(y_n)\right) = \sum_{i=1}^{n} \hat{T}\, O(y_1)\cdots\left[j^0(x), O(y_i)\right]\cdots O(y_n)\delta(x^0 - y_i^0) + \hat{T}\left(\partial_\mu j^\mu(x) O(y_1)\cdots O(y_i)\cdots O(y_n)\right). \tag{12.121}$$

Since the electromagnetic current is conserved, the last term vanishes.

A particular case of (12.121) for a conserved current is the relation

$$\partial_\mu\left(\hat{T} j^\mu(x)\bar{\psi}(y_1)\psi(y_2)\right) = e\hat{T}\left(\bar{\psi}(y_1)\psi(y_2)\right)\left[\delta(x - y_1) - \delta(x - y_2)\right]. \tag{12.122}$$

Taking this between single-particle states and going to momentum space yields an identity that is valid to all orders in perturbation theory [5]

$$-i(p' - p)_\mu S(p')\Gamma^\mu(p', p) S(p) = S(p') - S(p). \tag{12.123}$$

This is precisely the Ward-Takahashi identity (12.113).

12.6 Magnetic Moment of Electron

For dimensional reasons, the magnetic moment of the electron is proportional to the Bohr magnetic moment

$$\mu_{\rm B} = \frac{e\hbar}{2Mc}. \tag{12.124}$$

Since it is caused by the spin of the particle, it is proportional to it and can be written as

$$\boldsymbol{\mu} = g\mu_{\rm B}\frac{\mathbf{S}}{\hbar}. \tag{12.125}$$

The proportionality factor g is called the *gyromagnetic ratio*. If the spin is polarized in the z-direction, the z-component of $\boldsymbol{\mu}$ is

$$\mu = g\mu_{\rm B}\frac{1}{2} = g\frac{e\hbar}{2Mc}\frac{1}{2}. \tag{12.126}$$

We have discussed in Subsec. 4.15 that, as a result of the *Thomas precession*, an explanation of the experimental fine structure will make the g-factor of the electron magnetic moment to have a value near 2. This is twice as large as that of a charged rotating sphere of angular momentum L, whose magnetic moment is

$$\boldsymbol{\mu} = \frac{e\hbar}{2Mc}\frac{\mathbf{L}}{\hbar}, \tag{12.127}$$

i.e., whose g-value is unity. The result $g = 2$ has been found also in Eq. (6.119) by bringing the Dirac equation in an electromagnetic field to the second-order Pauli form (6.110).

Let us convince ourselves that a Dirac particle possesses the correct gyromagnetic ratio $g = 2$. Consider an electron of momentum $\mathbf{p}$ in a electromagnetic field which changes the momentum to $\mathbf{p}'$ (see Fig. 12.95). The interaction Hamiltonian is given by the matrix element

$$H^{\rm int} = \int d^3x A_\mu(\mathbf{x})\langle \mathbf{p}'|j^\mu(x)|\mathbf{p}\rangle, \tag{12.128}$$

where in Dirac's theory:

$$\langle \mathbf{p}', s_3'|j^\mu(x)|\mathbf{p}, s_3\rangle = e\langle 0|a(\mathbf{p}', s_3')\,\bar\psi(x)\gamma^\mu\psi(x)\,a^\dagger(\mathbf{p}, s_3)|0\rangle. \tag{12.129}$$

Inserting the field expansion (7.224) in terms of creation and annihilation operators

$$\psi(x) = \sum_{\mathbf{p},s_3} \frac{1}{\sqrt{VE_\mathbf{p}/M}}\left[e^{-ipx}u(\mathbf{p}, s_3)a_{\mathbf{p},s_3} + e^{ipx}v(\mathbf{p}, s_3)b^\dagger(\mathbf{p}, s_3)\right], \tag{12.130}$$

and using the anticommutators (7.228) and (7.229), we obtain

$$\langle \mathbf{p}', s_3'|j^\mu(x)|\mathbf{p}, s_3\rangle = e\bar u(\mathbf{p}', s_3)\gamma^\mu u(\mathbf{p}, s_3)\frac{e^{i(p'-p)x}}{\sqrt{VE_{\mathbf{p}'}/M}\sqrt{VE_\mathbf{p}/M}}. \tag{12.131}$$

The difference between final and initial four-momenta

$$q' \equiv p' - p \tag{12.132}$$

is the momentum transfer caused by the incoming photon.

In order to find the size of the magnetic moment we set up a constant magnetic field in the third space direction by assuming the second component of the vector potential to be the linear function $A_2(x) = x^1 B_3$. Then we put the final electron to rest, i.e., $p'^\mu = (M, \mathbf{0})$, and let the initial electron move slowly in the 1-direction. We create an associated spinor $u(\mathbf{p}, s_3)$ by applying a small Lorentz-boost $e^{-i\zeta^1(i\gamma^0\gamma^1)/2}$ to the rest spinors (4.676), and expanding the matrix element (12.131) up to the first order in $\mathbf{p}$. In zeroth order, we see that

$$\bar u(\mathbf{0}, s_3')\gamma^\mu u(\mathbf{0}, s_3) = \chi^\dagger(s_3')\chi(s_3)\delta^\mu{}_0, \tag{12.133}$$

showing that the charge is unity. The linear term in q^1 gives rise to a 2-component:

$$\langle \mathbf{p}', s_3'|j^2(x)|\mathbf{p}, s_3\rangle = e\bar u(\mathbf{0}, s_3)\gamma^2 e^{-i\zeta^1(i\gamma^0\gamma^1)/2}u(\mathbf{0}, s_3)\frac{e^{iq^1x^1}}{V}. \tag{12.134}$$

The two normalization factors on the right-hand side of (12.131) differ only by second-order terms in q^1. Now, since $\bar{u}(\mathbf{0}, s_3)\gamma^2 u(\mathbf{0}, s_3) = 0$ and $i\gamma^1\gamma^2/2 = S_3$, the spinors on the right-hand side reduce to

$$-ie\zeta^1 u^\dagger(\mathbf{0}, s_3) S_3 u(\mathbf{0}, s_3) = -ie\zeta^1 s_3. \tag{12.135}$$

Momentum conservation enforces $\zeta^1 = -q^1/M$, so that we find

$$\langle \mathbf{p}', s_3' | j^2(x) | \mathbf{p}, s_3 \rangle = ie\frac{q^1}{M} s_3 \frac{e^{iq^1x^1}}{V}. \tag{12.136}$$

Inserting this into the interaction Hamiltonian (12.128), we obtain

$$\begin{aligned} E^{\rm int} &= \lim_{q^1\to 0} \int d^3x A_2(\mathbf{x}) iq^1 e^{iq^1x^1} \frac{e}{M} s_3 \frac{1}{V} = \lim_{q^1\to 0} \int d^3x\, A_2(\mathbf{x}) \partial_1 e^{iq^1x^1} \frac{e}{M} s_3 \frac{1}{V} \\ &= -\int d^3x\, \partial_1 A_2(\mathbf{x}) \frac{e}{M} s_3 \frac{1}{V}. \end{aligned} \tag{12.137}$$

Inserting here the above vector potential $A_2(x) = x^1 B_3$, we obtain the magnetic interaction Hamiltonian

$$H^{\rm int} = -B_3 \frac{e}{M} s_3. \tag{12.138}$$

Since a magnetic moment μ interacts, in general, with a magnetic field via the energy $-\boldsymbol{\mu}\mathbf{B}$, we identify the magnetic moment as being (12.126), implying a gyromagnetic ratio $g = 2$.

Note that the magnetic field caused by the orbital motion of an electron leads to a coupling of the orbital angular momentum $\mathbf{L} = \mathbf{x} \times \mathbf{p}$ with a g-factor $g = 1$. In order to see this relative factor 2 most clearly, consider the interaction Hamiltonian

$$H^{\rm int} = -\int d^3x \mathbf{A}(x) \langle \mathbf{p}', s_3' | \mathbf{j}(x) | \mathbf{p}, s_3 \rangle, \tag{12.139}$$

and insert the Dirac current (12.131). For slow electrons we may neglect quantities of second order in the momenta, so that the normalization factors E/M are unity, and we obtain

$$H^{\rm int} = -e \int d^3x\, \mathbf{A}(x) \bar{u}(\mathbf{p}', s_3) \boldsymbol{\gamma}\, u(\mathbf{p}, s_3) e^{-i(\mathbf{p}'-\mathbf{p})\mathbf{x}}. \tag{12.140}$$

At this place we make use of the so-called *Gordon decomposition formula*

$$\bar{u}(\mathbf{p}', s_3')\gamma^\mu u(\mathbf{p}, s_3) = \bar{u}(\mathbf{p}', s_3') \left[\frac{1}{2M}(p'^\mu + p^\mu) + \frac{i}{2M}\sigma^{\mu\nu} q_\nu \right] u(\mathbf{p}, s_3), \tag{12.141}$$

where $q \equiv p' - p$ is the momentum transfer. This formula follows directly from the anticommutation rules of the γ-matrices and the Dirac equation.

An alternative decomposition is

$$\langle \mathbf{p}' | j^\mu | \mathbf{p} \rangle = e\bar{u}(\mathbf{p}') \left[\frac{1}{2M}(p'^\mu + p^\mu) F_1(q^2) + \frac{i}{2M}\sigma^{\mu\nu} q_\nu F_2(q^2) \right] u(\mathbf{p}'), \tag{12.142}$$

with the form factors $F_1(q^2)$, $F_2(q^2)$ related to $F(q^2)$, $G(q^2)$ via (12.141) by

$$F(q^2) = F_1(q^2), \quad G(q^2) = F_1(q^2) + F_2(q^2). \tag{12.143}$$

Then we rewrite the interaction Hamiltonian as

$$H^{\rm int} = -\frac{e}{M}\int d^3x \mathbf{A}(x)\,\bar{u}(\mathbf{p}', s_3)\,(\mathbf{p}+\mathbf{q}-i\mathbf{q}\times\mathbf{S})\,u(\mathbf{p}, s_3)e^{-i\mathbf{q}\mathbf{x}}, \tag{12.144}$$

where we have used the relations (4.518) and (4.515). We now replace $\mathbf{q}$ by the derivatives $i\partial_{\mathbf{x}}$ in front of the exponential $e^{-i\mathbf{q}\mathbf{x}}$, and perform an integration by parts to make the derivatives act on the vector potential $\mathbf{A}(x)$, with the opposite sign. In the transverse gauge, the term $\mathbf{A}(x)\cdot\mathbf{q}$ gives zero while $-i\mathbf{A}(x)\cdot(\mathbf{q}\times\mathbf{S})$ becomes $\mathbf{B}\cdot\mathbf{S}$. For equal incoming and outgoing momenta, this leads to the interaction Hamiltonian

$$H^{\rm int} = -\frac{e}{M}\int d^3x\,[\mathbf{A}(x)\cdot\mathbf{p}+\mathbf{B}(x)\cdot\mathbf{S}]\,. \tag{12.145}$$

We now express the vector potential in terms of the magnetic field as

$$\mathbf{A}(x) = \frac{1}{2}\mathbf{B}\times\mathbf{x}, \tag{12.146}$$

and rewrite (12.145) in the final form

$$H^{\rm int} = -\frac{e}{2M}\mathbf{B}\cdot(\mathbf{L}+2\mathbf{S})\,, \tag{12.147}$$

where $\mathbf{L} = \mathbf{x}\times\mathbf{p}$ is the orbital angular momentum. The relative factor 2 discovered by Alfred Landé in 1921 between orbital and spin angular momentum gives rise to a characteristic splitting of atomic energy levels in an external magnetic field. If the field is weak, both orbital and spin angular momenta will precess around the direction of the total angular momentum. Their averages will be, for example,

$$\bar{\mathbf{L}} = \mathbf{J}\,\frac{\mathbf{J}\cdot\mathbf{L}}{\mathbf{J}^2}, \quad \bar{\mathbf{S}} = \mathbf{J}\,\frac{\mathbf{J}\cdot\mathbf{S}}{\mathbf{J}^2}. \tag{12.148}$$

By rewriting

$$\mathbf{J}\cdot\mathbf{L} = \frac{1}{2}\left(\mathbf{J^2}+\mathbf{L}^2-\mathbf{S}^2\right), \quad \mathbf{J}\cdot\mathbf{S} = \frac{1}{2}\left(\mathbf{J^2}-\mathbf{L}^2+\mathbf{S}^2\right), \tag{12.149}$$

we see that

$$\bar{\mathbf{L}} = f_{LJ}\,\mathbf{J}, \quad \bar{\mathbf{S}} = f_{SJ}\,\mathbf{J} \tag{12.150}$$

with the factors

$$f_{LJ} = [J(J+1)+L(L+1)-S(S+1)]\,/2J(J+1), \tag{12.151}$$
$$f_{SJ} = [J(J+1)-L(L+1)+S(S+1)]\,/2J(J+1). \tag{12.152}$$

Inserting this into (12.147), we obtain the interaction energies of an atomic state $|JM\rangle$:

$$H^{\rm int} = -g_{LS}\frac{e}{2M}BM, \tag{12.153}$$

where

$$g_{LS} = f_{LJ} + 2f_{SJ} = 1 + [J(J+1) - L(L+1) + S(S+1)]/2J(J+1) \quad (12.154)$$

is the *gyromagnetic ratio* of the coupled system. This has been measured in many experiments as *Zeeman effect*, if the external field is small, and as *anomalous Zeeman effect* or *Paschen-Back effect*, if the external field strength exceeds the typical field strength caused by the electron orbit. Then orbital and spin angular momenta decouple and precess independently around the direction of the external magnetic field.

12.7 Decay of Atomic State

The first important result of quantum electrodynamics is the explanation of the decay of an atom. In quantum mechanics, this decay can only be studied by means of the correspondence principle.

Consider an electron in an atomic state undergoing a transition from a state n with energy E_n to a lower state n' with energy $E_{n'}$, whereby a photon is emitted with a frequency $\omega = (E_{n'} - E_n)/\hbar$ (see Fig. 12.4). According to the correspondence

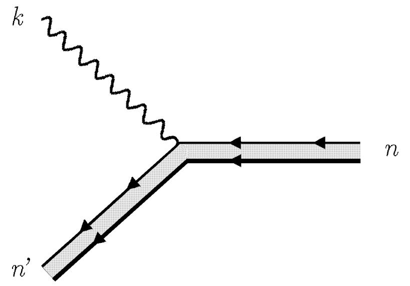


FIGURE 12.4 Transition of an atomic state from a state n with energy E_n to a lower state n' with energy $E_{n'}$, thereby emitting a photon with a frequency $\omega = (E_{n'} - E_n)/\hbar$.

principle, this is the frequency with which the center of charge of the electronic cloud oscillates back and forth along the direction $\hat{\boldsymbol{\epsilon}}$ with an amplitude:

$$\mathbf{x}_0 = \langle n'|\boldsymbol{\epsilon}\cdot\mathbf{x}|n\rangle = \epsilon\langle n'|\hat{\boldsymbol{\epsilon}}\cdot\mathbf{x}|n\rangle. \quad (12.155)$$

The oscillating charge emits antenna radiation. The classical theory of this process has been recapitulated in Section 5.1, where we have given in Eq. (5.37) the radiated power per solid angle. Its directional integral led to the Larmor formula (5.38), and reduced to (5.38) for a harmonic oscillator.

Quantum mechanically, the antenna radiation formula (5.37) can be applied to an atom that decays from level n to n', if we replace $|\mathbf{x}_0|^2$ by the absolute square of the quantum mechanical matrix element (12.155):

$$|\mathbf{x}_0|^2 \to |\langle n'|\boldsymbol{\epsilon}\cdot\mathbf{x}|n\rangle|^2 . \quad (12.156)$$

Then formula (5.37) yields the radiated power per unit solid angle

$$\frac{d\dot{\mathcal{E}}_{n'n}}{d\Omega} = \frac{e^2}{8\pi^2}\frac{\omega^4}{c^3}\left|\langle n'|\boldsymbol{\epsilon}\cdot\mathbf{x}|n\rangle\right|^2\sin^2\theta. \tag{12.157}$$

Integrating over all $d\Omega$ gives the total radiated power, and if we divide this by the energy per photon $\hbar\omega$, we obtain the decay rate

$$\dot{\Gamma}_{n'n} = \frac{4}{3}\frac{e^2}{4\pi}\frac{\omega^3}{\hbar c^3}\left|\langle n'|\boldsymbol{\epsilon}\cdot\mathbf{x}|n\rangle\right|^2 = \frac{4}{3}\alpha\,\omega|\mathbf{k}|^2\left|\langle n'|\boldsymbol{\epsilon}\cdot\mathbf{x}|n\rangle\right|^2. \tag{12.158}$$

Let us now confirm this result by a proper calculation within quantum electrodynamics. Consider a single nonrelativistic electron in a hydrogen-like atom with central charge Ze. For an electron of mass M moving in a Coulomb potential

$$V_{\rm C}(\mathbf{x}) \equiv -\frac{Z\alpha}{|\mathbf{x}|}, \tag{12.159}$$

the Hamiltonian reads in the transverse gauge with $\boldsymbol{\nabla}\mathbf{A}(x) = 0$,

$$H = \frac{\mathbf{p}^2}{2M} - \frac{1}{M}\mathbf{A}(\mathbf{x},t)\cdot\mathbf{p} + \frac{e^2}{2M}\mathbf{A}^2(\mathbf{x},t) + V_{\rm C}(\mathbf{x}) + eA^0(\mathbf{x},t) = H_{\rm C} + H^{\rm int}. \tag{12.160}$$

where

$$H_{\rm C} = \frac{\mathbf{p}^2}{2M} + \mathbf{V}_{\rm C}(\mathbf{x}) \tag{12.161}$$

is the Hamiltonian of the hydrogen-like atom by itself, and $H^{\rm int}$ contains the interaction of the electron with the vector potential $A^\mu(\mathbf{x},t) = 0$. Its magnitude is determined by the electronic charge distribution via the Coulomb law as shown in Eq. (12.73). The radiation field $\mathbf{A}(\mathbf{x},t)$, has an expansion in terms of photon creation and annihilation operators given in Eq. (7.350):

$$\mathbf{A}(\mathbf{x},t) = \sum_{\mathbf{k},h}\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}\left[e^{-ikx}\boldsymbol{\epsilon}(\mathbf{k},h)a_{\mathbf{k},h} + \text{h.c.}\right]. \tag{12.162}$$

Let $|n\rangle$ be an excited initial state of an atom with an electron orbit having the principal quantum number n, and suppose that it decays into lower state $a^\dagger(\mathbf{k},h)|n'\rangle$ with a principal quantum number n'. In addition to the electron, the lower atomic state contains a photon with wave vector $\mathbf{k}$, energy $\omega = ck$, and helicity h. According to Eq. (9.235), the decay probability of the initial state per unit time is given by Fermi's golden rule (in the remainder of this section we use physical units):

$$\frac{dP_{n'n}}{dt} = \int\frac{d^3kV}{(2\pi)^3}\,2\pi\hbar\delta(E_n + \hbar\omega - E_{n'})\left|\frac{1}{\hbar}\langle n'|a(\mathbf{k},h)T|n\rangle\right|^2, \tag{12.163}$$

where T is the T-matrix which coincides, in lowest order perturbation theory, with the matrix $H_{\rm int}\int d^3x\,\mathcal{H}_{\rm int}\mathbf{x})$ [see (9.132) and (9.288)]. The matrix element is obviously

$$\langle n'|a(\mathbf{k},h)H^{\rm int}|n\rangle = \frac{c}{\sqrt{2V\omega}}\frac{e}{Mc}\langle n'|e^{-i\mathbf{kx}}\boldsymbol{\epsilon}^*(\mathbf{k},h)\cdot\mathbf{p}|n\rangle. \tag{12.164}$$

Performing the integral over the photon momentum (neglecting recoil) we find from (12.163) the differential decay rate [compare (9.338)]

$$\frac{d\Gamma_{n'n}}{d\Omega} = \frac{e^2}{8\pi^2\hbar}\frac{\omega}{M^2c^3}|\langle n'|e^{-i\mathbf{kx}}\boldsymbol{\epsilon}^*(\mathbf{k},h)\cdot\mathbf{p}|n\rangle|^2. \tag{12.165}$$

Further calculations are simplified by the observation that the wavelength of the emitted photons is the inverse of their energy (in massless units), and thus of the order of $\hbar/Z\alpha^2Mc$, about 100 times larger than the atomic diameter which is of the order of the Bohr radius $a_{\rm B} = \hbar/Z\alpha Mc$ for an atom of charge Z. The exponential $e^{-i\mathbf{kx}}$ is therefore almost unity and can be dropped. This yields the *dipole approximation* to the atomic decay rate:

$$\frac{d\Gamma_{n'n}}{d\Omega} \approx \frac{e^2}{8\pi^2\hbar}\frac{\omega}{M^2c^3}|\boldsymbol{\epsilon}^*(\mathbf{k},h)\cdot\langle n'|\mathbf{p}|n\rangle|^2. \tag{12.166}$$

Another way of writing this result is

$$\frac{d\Gamma_{n'n}}{d\Omega} \approx \alpha\,\frac{\omega}{2\pi\hbar}\frac{\omega^2}{c^2}|\boldsymbol{\epsilon}^*(\mathbf{k},h)\cdot\langle n'|\mathbf{x}|n\rangle|^2. \tag{12.167}$$

The momentum operator $\mathbf{p}$ can be replaced by $M\dot{\mathbf{x}} = i[\hat{H},\mathbf{x}]/\hbar$, and thus, in the matrix element $\langle n'|\mathbf{p}|n\rangle|^2$, by $iM(E_{n'}-E_n)\mathbf{x}/\hbar = -iM\omega\mathbf{x}$. Multiplying the decay rate by the energy of the photon $\hbar\omega$ to get the rate of radiated energy, the result (12.167) coincides with the classical result (12.158).

It is customary to introduce the so-called *oscillator strength* for an oscillator in the direction ϵ:

$$f^{\boldsymbol{\epsilon}}_{n'n} \approx \frac{2M\omega}{\hbar}\left|\sum_\nu\langle n'|\boldsymbol{\epsilon}\cdot\mathbf{x}|n\rangle\right|^2 = 2\frac{|\mathbf{k}|}{\lambda_e}\left|\sum_\nu\langle n'|\boldsymbol{\epsilon}\cdot\mathbf{x}|n\rangle\right|^2. \tag{12.168}$$

This quantity fulfills the *Thomas-Reiche-Kuhn sum rule*:

$$\sum_{n'} f^{\boldsymbol{\epsilon}}_{n'n} = 1. \tag{12.169}$$

For an atom with Z electrons, the right-hand side is equal to Z.

To derive this sum rule (and a bit more) we define the operator

$$\hat{E}^{\boldsymbol{\epsilon}} \equiv \frac{1}{|\mathbf{k}|}e^{-i|\mathbf{k}|\boldsymbol{\epsilon}\cdot\mathbf{x}}, \tag{12.170}$$

whose time derivative is

$$\dot{\hat{E}}^{\boldsymbol{\epsilon}} = \frac{i}{\hbar}[\hat{H},\hat{E}^{\boldsymbol{\epsilon}}] = \frac{\hbar}{2M}\left(e^{-i|\mathbf{k}|\boldsymbol{\epsilon}\cdot\mathbf{x}}\boldsymbol{\epsilon}\cdot\boldsymbol{\nabla} + \boldsymbol{\epsilon}\cdot\boldsymbol{\nabla}e^{-i|\mathbf{k}|\boldsymbol{\epsilon}\cdot\mathbf{x}}\right). \tag{12.171}$$

According to the canonical commutator $[\hat{p}_i,\hat{x}_j] = -i\hbar\delta_{ij}$, the Hermitian-conjugate of $\dot{\hat{E}}^{\boldsymbol{\epsilon}}$ commutes with $\hat{E}^{\boldsymbol{\epsilon}}$ like

$$[\dot{\hat{E}}^{\boldsymbol{\epsilon}\dagger},\hat{E}^{\boldsymbol{\epsilon}}] = -i\frac{\hbar}{M}. \tag{12.172}$$

Taking this commutator between states $\langle n|$ and $|n\rangle$, and inserting a completeness relation $\sum_{n'} |n'\rangle\langle n'| = 1$ in the middle, we may go to small $\mathbf{k}$, to find indeed the sum rule (12.169).

Let us calculate the angular properties of the radiation in more detail. The decomposition of the hydrogen wave functions into radial and angular parts is

$$\langle \mathbf{x}|nlm\rangle = R_{nl}(r)Y_{lm}(\theta,\varphi). \tag{12.173}$$

Then the matrix elements of $\mathbf{x}$ factorize:

$$\langle n'l'm'|\mathbf{x}|nlm\rangle = r_{n'l';nl} \times \langle l'm'|\hat{\mathbf{x}}|lm\rangle. \tag{12.174}$$

The matrix elements of r,

$$r_{n'l';nl} \equiv \int_0^\infty dr r^2 R_{n'l'}(r) r R_{nl}(r), \tag{12.175}$$

have been calculated by Gordon [25]:

$$r_{n'l';nl} = \frac{(-1)^{n'-l}}{4(2l-1)!}\sqrt{\frac{(n+l)!(n'+l-1)!}{(n'-l)!(n-l-1)!}}(4nn')^{l+1}\frac{(n-n')^{n+n'-2l-2}}{(n+n')^{n+n'}} \tag{12.176}$$
$$\times\left\{F\left(-n'_r,2l,-\frac{4nn'}{(n-n')^2}\right)-\left(\frac{n-n'}{n+n'}\right)^2F\left(-n_r-2,-n'_r,2l,-\frac{4nn'}{(n-n')^2}\right)\right\},$$

with $F(a,b,c;z)$ being hypergeometric functions. The angular matrix elements of the unit vector in (12.174),

$$\langle l'm'|\hat{\mathbf{x}}|lm\rangle \equiv \int d\hat{\mathbf{x}}\, Y^*_{l'm'}(\theta,\varphi)\hat{\mathbf{x}}Y_{lm}(\theta,\varphi), \tag{12.177}$$

are easily calculated since $\hat{x}_3 = \cos\theta$, and the spherical harmonics satisfy the recursion relation

$$\cos\theta\, Y_{lm}(\theta,\varphi)=\sqrt{\frac{(l+1)^2-m^2}{(2l+2)(2l+1)}}Y_{l+1\,m}(\theta,\varphi)+\sqrt{\frac{l^2-m^2}{(2l+1)(2l-1)}}Y_{l-1\,m}(\theta,\varphi). \tag{12.178}$$

On account of the orthonormality relation [recall (4F.3)]

$$\int d\hat{\mathbf{x}}\, Y^*_{l'm'}(\theta,\varphi)Y_{lm}(\theta,\varphi) = \delta_{l'l}\delta_{m'm}, \tag{12.179}$$

we obtain immediately the angular matrix elements of $\hat{x}_3$:

$$\langle l+1\ m|\hat{x}_3|lm\rangle = \sqrt{\frac{(l+1)^2-m^2}{(2l+2)(2l+1)}},\quad \langle l-1\ m'|\hat{x}_3|lm\rangle = \sqrt{\frac{l^2-m^2}{(2l+1)(2l-1)}}, \tag{12.180}$$

with all others vanishing. The matrix elements of $\hat{x}_1$ and $\hat{x}_2$ are found with the help of the commutation rule

$$[\hat{L}_i,\hat{x}_j] = i\epsilon_{ijk}\hat{x}_k, \tag{12.181}$$

which states that $\hat{x}_i$ is a *vector operator*. As a consequence, the matrix elements satisfy the Wigner-Eckart theorem,

$$\langle l'm' \pm 1|\hat{x}^M|lm\rangle = \langle l'm'|1M; lm\rangle \hat{x}_{l'l}, \tag{12.182}$$

where $\langle l'm'|1m''; lm\rangle$ are Clebsch-Gordan coefficients (see Appendix 4E) and $\hat{x}^{m''}$ are the spherical components of $\hat{\mathbf{x}}$ [recall the definition in Eq. (4.893)]:

$$\hat{x}^3 = \cos\theta, \quad \hat{x}^{\pm} = \mp(\hat{x}^2 \pm \hat{x}^2)/\sqrt{2} = \sin\theta e^{\pm i\varphi}/\sqrt{2}. \tag{12.183}$$

Explicitly:

$$\begin{aligned} \langle l+1\ m\pm 1|\hat{x}^{\pm}|lm\rangle &= \sqrt{\frac{(l\pm m+2)(l\pm m+1)}{2(2l+3)(2l+1)}}, \\ \langle l-1\ m\pm 1|\hat{x}^{\pm}|lm\rangle &= \sqrt{\frac{(l\mp m)(l\mp m-1)}{2(2l+1)(2l-1)}}. \end{aligned} \tag{12.184}$$

12.8 Rutherford Scattering

The scattering of electrons on the Coulomb potential of nuclei of charge Ze,

$$V_{\mathrm{C}}(r) = -\frac{ZE^2}{4\pi r} = -\frac{Z\alpha}{r}, \tag{12.185}$$

was the first atomic collision observed experimentally by Rutherford.

The associated scattering cross section can easily be calculated in an estimated classical approximation.

12.8.1 Classical Cross Section

In a Coulomb potential the electronic orbits are hyperbola. If an incoming electron runs along the z-direction and is deflected by a scattering angle θ towards the x direction (see Fig. 12.5), the nucleus has the coordinates

$$(x_{\mathrm{F}}, z_{\mathrm{F}}) = (b, -a), \tag{12.186}$$

where

$$\frac{a}{b} = \tan\frac{\theta}{2}, \qquad \frac{b}{d} = \cos\frac{\theta}{2}. \tag{12.187}$$

The parameter d is equal to $a\epsilon$, where $\epsilon > 1$ is the excentricity of the hyperbola. The distance of closest approach to the nucleus is

$$r_{\mathrm{c}} = d - a. \tag{12.188}$$

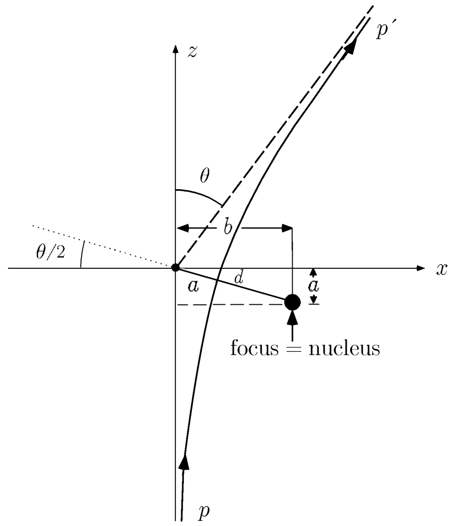


FIGURE 12.5 Kinematics of Rutherford scattering.

It is determined by the conservation of the nonrelativistic energy:

$$E^{\rm nr} = \frac{p^2}{2M} = \frac{p'^2}{2M} = \frac{p_{\rm c}^2}{2M} - \frac{Z\alpha}{r_{\rm c}}, \tag{12.189}$$

where $p_{\rm c}$ is the momentum at the point of closest approach. This momentum is determined by the conservation of orbital angular momentum:

$$l = pb = p'b = p_c r_c. \tag{12.190}$$

From these equations we find

$$b = \frac{Z\alpha}{2E\tan(\theta/2)}. \tag{12.191}$$

The quantity b is called the *impact parameter* of the scattering process. It is the closest distance which an particle would have from the nucleus if it were not deflected at all. Particles which would pass through a thin annular ring with the radii b and $b + db$ are, in fact, scattered into a solid angle $d\Omega$ given by

$$db = -\frac{1}{4\pi}\frac{Z\alpha}{4E}\frac{1}{\sin^3(\theta/2)\cos(\theta/2)}d\Omega. \tag{12.192}$$

The current density of a single randomly incoming electron is $j = v/V$. It would pass through the annular ring, with a probability per unit time

$$d\dot{P} = j\, 2\pi b db. \tag{12.193}$$

With this probability it winds up in the solid angle $d\Omega$. Inserting Eq. (12.192), we find the differential cross section [recall the definition Eq. (9.242)]

$$\frac{d\sigma}{d\Omega} = \frac{d\dot{P}}{d\Omega} = 2\pi b\frac{db}{d\Omega} = \frac{1}{4\sin^4(\theta/2)}\left(\frac{Z\alpha}{2E}\right)^2. \tag{12.194}$$

12.8.2 Quantum-Mechanical Born Approximation

Somewhat surprisingly, the same result is obtained in quantum mechanics within the Born approximation. According to Eqs. (9.147) and (9.248), the differential cross section is

$$\frac{d\sigma}{d\Omega} \approx \frac{M^2V^2}{(2\pi)^2}|V_{\mathbf{p}'\mathbf{p}}|^2, \tag{12.195}$$

where

$$V_{\mathbf{p}'\mathbf{p}} \;=\; -\frac{1}{V}\int d^3x\, e^{i(\mathbf{p}'-\mathbf{p})\mathbf{x}}\frac{Z\alpha}{r} = -\frac{4\pi}{V}\frac{Z\alpha}{|\mathbf{q}^2|^2}. \tag{12.196}$$

The quantity

$$\mathbf{q}^2 \equiv |\mathbf{p}'-\mathbf{p}|^2 = 2p^2(1-\cos\theta) = 4p^2\sin^2(\theta/2) = 8EM\sin^2(\theta/2) \tag{12.197}$$

is the momentum transfer of the process. Inserting this into (12.196), the differential cross section (12.195) coincides indeed with the classical expression (12.194).

12.8.3 Relativistic Born Approximation: Mott Formula

Let us now see how the above cross section formula is modified in a relativistic calculation involving Dirac electrons. The scattering amplitude is, according to Eq. (10.142),

$$S_{\rm fi} = -ie\langle\mathbf{p}', s_3'|\int d^4x\,\bar\psi(x)\gamma^\mu\psi(x)|\mathbf{p}, s_3\rangle A_\mu(x), \tag{12.198}$$

where $A_\mu(x)$ has only the time-like component

$$A^0(x) = -\frac{Ze}{4\pi r} = -\int\frac{d^3q}{(2\pi)^3}\,e^{i\mathbf{q}\mathbf{x}}\frac{Ze}{|\mathbf{q}|^2}. \tag{12.199}$$

The time-ordering operator has been dropped in (12.198) since there are no operators at different times to be ordered in first-order perturbation theory. By evaluating the matrix element of the current in (12.198), and performing the spacetime integral we obtain

$$S_{\rm fi} = i2\pi\delta(E'-E)\sqrt{\frac{M^2}{V^2E'E}}\bar u(\mathbf{p}', s_3')\gamma^0 u(\mathbf{p}, s_3)\frac{Ze^2}{|\mathbf{q}|^2}, \tag{12.200}$$

where E and E' are the initial and final energies of the electron, which are in fact equal in this elastic scattering process.

Comparing this with (9.130) we identify the T-matrix elements

$$T_{\mathbf{p}'\mathbf{p}} = \frac{M}{VE}\bar u(\mathbf{p}', s_3')\gamma^0 u(\mathbf{p}, s_3)\frac{Ze^2}{|\mathbf{q}|^2}, \tag{12.201}$$

In the nonrelativistic limit this is equal to

$$T_{\mathbf{p}'\mathbf{p}} = -\frac{1}{V}\frac{Ze^2}{|\mathbf{q}|^2}. \tag{12.202}$$

Its relativistic extension contains a correction factor

$$C = \frac{M}{E}\bar{u}(\mathbf{p}', s_3)\gamma^0 u(\mathbf{p}, s_3). \tag{12.203}$$

Its absolute square multiplies the nonrelativistic differential cross section (12.194). Apart from this, the relativistic cross section contains an extra kinematic factor E^2/M^2 accounting for the different phase space of a relativistic electron with respect to the nonrelativistic one [the ratio between (9.247) and (9.244)]. The differential cross section (12.194) receives therefore a total relativistic correction factor

$$\frac{E^2}{M^2}|C|^2 = |\bar{u}(\mathbf{p}', s_3')\gamma^0 u(\mathbf{p}, s_3)|^2. \tag{12.204}$$

If we consider the scattering of unpolarized electrons and do not observe the final spin polarizations, this factor has to be summed over s_3' and averaged over s_3, and the correction factor is

$$\frac{E^2}{M^2}|C|^2 = \frac{1}{2}\sum_{s_3', s_3} \bar{u}(\mathbf{p}', s_3')\gamma^0 u(\mathbf{p}, s_3)\bar{u}(\mathbf{p}, s_3)\gamma^0 \bar{u}(\mathbf{p}', s_3'). \tag{12.205}$$

To write the absolute square in this form we have used the general identity in the spinor space, valid for any 4×4 spinor matrix M:

$$(\bar{u}' M u)^* = \bar{u}\bar{M}u', \tag{12.206}$$

where the operation *bar* is defined for a spinor matrix in complete analogy to the corresponding operation for a spinor:

$$\bar{M} \equiv \gamma^0 M^\dagger \gamma^{0-1}. \tag{12.207}$$

The Dirac matrices themselves satisfy

$$\bar{\gamma}^\mu = \gamma^\mu. \tag{12.208}$$

The "bar" operation has the typical property of an "adjoining" operation. If it is applied to a product of matrices, the order is reversed:

$$\overline{M_1 \cdots M_n} = \overline{M}_n \cdots \overline{M}_1. \tag{12.209}$$

We use now the semi-completeness relation (4.702) for the u-spinors and rewrite (12.205) as

$$\frac{E^2}{M^2}|C|^2 = \frac{1}{2}\mathrm{tr}\left(\gamma^0 \frac{\not{p}+M}{2M}\gamma^0 \frac{\not{p}+M}{2M}\right). \tag{12.210}$$

The trace over product of gamma matrices occurring in this expression is typical for quantum electrodynamic calculations. Its evaluation is somewhat tedious, but follows a few quite simple algebraic rules.

The first rule states that a trace containing an odd number of gamma matrices vanishes. This is a simple consequence of the fact that γ^μ and any product of an odd number of gamma matrices change sign under the similarity transformation $\gamma_5\gamma^\mu\gamma_5 = -\gamma^\mu$, while the trace is invariant under *any* similarity transformation.

The second rule governs the evaluation of a trace containing an even number of gamma matrices. It is a recursive rule which makes essential use of the invariance of the trace under cyclic permutations

$$\mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\cdots\gamma^{\mu_{n-1}}\gamma^{\mu_n}) = \mathrm{tr}(\gamma^{\mu_2}\gamma^{\mu_3}\cdots\gamma^{\mu_{n-1}}\gamma^{\mu_n}\gamma^{\mu_1}). \tag{12.211}$$

This leads to an explicit formula that is a close analog of Wick's expansion formula for time-ordered products of fermion field operators. To find this, we define a pair contraction between $\not{a}$ and $\not{b}$ as

$$\overbracket{\not{a}\not{b}} \equiv \frac{1}{4}\mathrm{tr}(\not{a}\,\not{b}) = ab. \tag{12.212}$$

Then we consider a more general trace

$$\frac{1}{4}\mathrm{tr}(\not{a}_1\cdots\not{a}_n) \tag{12.213}$$

and move the first gamma matrix step by step to the end, using the anticommutation rules between gamma matrices (4.566), which imply that

$$\not{a}_1\not{a}_i = -\not{a}_i\not{a}_1 + 2a_1a_i = -\not{a}_i\not{a}_1 + 2\,\overbracket{\not{a}_1\not{a}_i}\,. \tag{12.214}$$

Having arrived at the end, it can be taken back to the front, using the cyclic invariance of the trace. This produces once more the initial trace, except for a minus sign, thus doubling the initial trace on the left-hand side of the equation if n is even. In this way, we find the recursion relation

$$\begin{aligned}\frac{1}{4}\mathrm{tr}(\not{a}_1\,\not{a}_2\,\not{a}_3\,\cdots\not{a}_{n-1}\,\not{a}_n) &= \frac{1}{4}\mathrm{tr}(\overbracket{\not{a}_1\not{a}_2}) + \frac{1}{4}\mathrm{tr}(\overbracket{\not{a}_1\not{a}_2\not{a}_3}\cdots\not{a}_{n-1}\,\not{a}_n)\\ &+\ldots+\frac{1}{4}\mathrm{tr}(\overbracket{\not{a}_1\not{a}_2\not{a}_3\cdots\not{a}_{n-1}}\,\not{a}_n)+\ldots+\frac{1}{4}\mathrm{tr}(\overbracket{\not{a}_1\not{a}_2\not{a}_3\cdots\not{a}_{n-1}\not{a}_n}).\end{aligned} \tag{12.215}$$

The contractions within the traces are defined as in (12.212), but with a minus sign for each permutation necessary to bring the Dirac matrices to adjacent positions. Performing these operations, the result of (12.215) is

$$\begin{aligned}\frac{1}{4}\mathrm{tr}(\not{a}_1\,\not{a}_2\,\not{a}_3\cdots\not{a}_{n-1}\,\not{a}_n) &= (a_1a_2)\frac{1}{4}\mathrm{tr}(\not{a}_3\,\not{a}_4\cdots\not{a}_{n-1}\,\not{a}_n) + (a_1a_3)\frac{1}{4}\mathrm{tr}(\not{a}_2\,\not{a}_4\cdots\not{a}_{n-1}\,\not{a}_n)\\ &+\ldots+(a_1a_{n-1})\frac{1}{4}\mathrm{tr}(\not{a}_2\,\not{a}_3\,\cdots\not{a}_{n-2}\,\not{a}_n) + (a_1a_n)\frac{1}{4}\mathrm{tr}(\not{a}_2\,\not{a}_3\,\cdots\not{a}_{n-1}\,\not{a}_n).\end{aligned} \tag{12.216}$$

By applying this formula iteratively, we arrive at the expansion rule of the Wick type:

$$\frac{1}{4}\mathrm{tr}(\not{a}_1\,\cdots\not{a}_n) = \sum_{\text{pair contractions}} (-)^P\,(a_{p(1)}a_{p(2)})(a_{p(3)}a_{p(4)})\cdots(a_{p(n-1)}a_{p(n)}), \tag{12.217}$$

where P is the number of permutations to adjacent positions.

The derivation of the rule is completely parallel to that of the thermodynamic version of Wick's rule in Section 4.14, whose Eqs. (7.857) and (7.858) can directly be translated into anticommutation rules between gamma matrices and the cyclic invariance of their traces, respectively.

Another set of useful rules following from (12.214) and needed later is

$$\begin{aligned}
\gamma^\mu \not{a}\, \gamma^\nu &= \not{a} + 2a^\mu \gamma^\nu, && (12.218)\\
\gamma^\mu \gamma_\mu &= 4, && (12.219)\\
\gamma^\mu \not{a}\, \gamma_\mu &= -2\not{a}\,, && (12.220)\\
\gamma^\mu \not{a}\, \not{b}\, \gamma_\mu &= -2\not{a}\,, && (12.221)\\
\gamma^\mu \not{a}\, \not{b}\, \not{b}\, \gamma_\mu &= -2\not{c}\, \not{b}\, \not{a}\,. && (12.222)
\end{aligned}$$

Following the Wick rule (12.253) for γ-matrices, we now calculate the expression (12.210) as

$$\begin{aligned}
\frac{E^2}{M^2}|C|^2 &= \frac{1}{2M^2}\left[\frac{1}{4}\mathrm{tr}(\gamma^0 \not{p}\, \gamma^0 \not{p}') + M^2 \mathrm{tr}(\gamma^{02})\right]\\
&= \frac{1}{2M^2}\left(2p^0 p'^0 - pp' + M^2\right). && (12.223)
\end{aligned}$$

Inserting $p^0 = p'^0 = E$ and, in the center-of-mass frame,

$$pp' = E^2 - |\mathbf{p}|^2 \cos\theta = M^2 + 2|\mathbf{p}|^2 \sin^2\frac{\theta}{2}, \tag{12.224}$$

the total cross section in the center-of-mass frame becomes

$$\frac{d\sigma}{d\Omega_{\mathrm{CM}}} = \frac{Z^2\alpha^2 M^2}{4|\mathbf{p}|^4 \sin^4(\theta/2)} \times \frac{E^2}{M^2}|C|^2. \tag{12.225}$$

This relativistic version of Rutherford's formula is known as *Mott's formula*.

In terms of the incident electron velocity, the total modification factor reads

$$\frac{E^2}{M^2}|C|^2 = \frac{1}{1-(v/c)^2}\left[1 - \left(\frac{v}{c}\right)^2 \sin^2\frac{\theta}{2}\right]. \tag{12.226}$$

It is easy to verify that the same differential cross section is valid for positrons. In the nonrelativistic case, this follows directly from the invariance of (12.195) under $e \to -e$. But also the relativistic correction factor remains the same under the interchange of electrons and positrons, where (12.205) becomes

$$\frac{E^2}{M^2}|C|^2 = \frac{1}{2}\sum_{s_3', s_3} \bar{v}(\mathbf{p}', s_3')\gamma^0 v(\mathbf{p}, s_3)\bar{v}(\mathbf{p}, s_3)\gamma^0 \bar{v}(\mathbf{p}', s_3'). \tag{12.227}$$

Inserting the semi-completeness relation (4.703) for the spinors $v(\mathbf{p}, s_3)$, this becomes

$$\frac{E^2}{M^2}|C|^2 = \frac{1}{2}\mathrm{tr}\left(\gamma^0 \frac{\not{p} - M}{2M}\gamma^0 \frac{\not{p}' - M}{2M}\right), \tag{12.228}$$

which is the same as (12.205), since only traces of an even number of gamma matrices contribute.

12.9 Compton Scattering

A simple scattering process, whose cross section can be calculated to a good accuracy by means of the above diagrammatic rules, is photon-electron scattering, also referred to as *Compton scattering.* It gives an important contribution to the blue color of the sky.[1]

Consider now a beam of photons with four-momentum $k_{\rm i}$ and polarization $\lambda_{\rm i}$ impinging upon an electron target of four-momentum $p_{\rm i}$ and spin orientation $\sigma_{\rm i}$. The two particles leave the scattering regime with four-momenta $k_{\rm f}$ and $p_{\rm f}$, and spin indices $\lambda_{\rm f}, \sigma_{\rm f}$, respectively. Adapting formula (10.103) for the scattering amplitude to this situation we have

$$\begin{aligned} S_{\rm fi} &= S\left(\mathbf{p}', s_3'; \mathbf{k}', h'|\mathbf{k}, h; \mathbf{p}, s_3\right) \\ &\equiv \frac{\langle 0|a(\mathbf{p}', s_3')a(\mathbf{k}', h')\hat{T}e^{-i\int_{-\infty}^{\infty} dt V_I(t)}a^\dagger(\mathbf{k}, h)a^\dagger(\mathbf{p}, s_3)|0\rangle}{\langle 0|e^{-i\int_{-\infty}^{\infty} dt V_I(t)}|0\rangle} \\ &\equiv S_N\left(\mathbf{p}', s_3'; \mathbf{k}', h'|\mathbf{k}, h; \mathbf{p}, s_3\right)/Z[0], \end{aligned} \tag{12.229}$$

with

$$e^{-i\int_{-\infty}^{\infty} dt\, V_I(t)} = e^{-ie\int d^4x\, \bar\psi(x)\gamma^\mu\psi(x)A^\mu(x)}. \tag{12.230}$$

Expanding the exponential in powers of e, we see that the lowest-order contribution to the scattering amplitude comes from the second-order term which gives rise to the two Feynman diagrams shown in Fig. 12.6. In the first, the electron s_1 of momentum p absorbs a photon of momentum k, and emits a second photon of momentum k', to arrive in the final state of momentum p'. In the second diagram, the acts of emission and absorption have the reversed order. Before we calculate the scattering cross section associated with these Feynman diagrams, let us first recall the classical result.

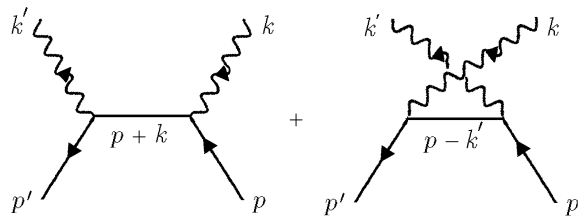


FIGURE 12.6 Lowest-order Feynman diagrams contributing to Compton Scattering and giving rise to the Klein-Nishina formula.

[1]The blue color is usually attributed to *Rayleigh scattering.* This arises from generalizing Thomson's formula (12.233) for the scattering of light of wavelength λ on electrons to that on droplets of diameter d with refractive index n. That yields $\sigma_{\rm Rayleigh} = 2\pi^5 d^6[(n^2-1)/(n^2+2)]^2/3\lambda^4$.

12.9.1 Classical Result

Classically, the above process is described as follows. A target electron is shaken by an incoming electromagnetic field. The acceleration of the electron causes an emission of antenna radiation. For a weak and slowly oscillating electromagnetic field of amplitude, the electron is shaken nonrelativistically and moves with an instantaneous acceleration

$$\ddot{\mathbf{x}} = \frac{e}{M}\mathbf{E} = \frac{e}{M}\boldsymbol{\epsilon}\, E_0 e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}, \tag{12.231}$$

where ω is the frequency and E_0 the amplitude of the incoming field.

The acceleration of the charge gives rise to antenna radiation following Larmor's formula. Inserting (12.231) into (5.37), and averaging over the temporal oscillations, the radiated power per unit solid angle is

$$\frac{d\dot{\mathcal{E}}}{d\Omega} = \frac{1}{2}\left(\frac{e^2}{4\pi M}\right)^2 E_0^2 \sin^2\beta, \tag{12.232}$$

where β is the angle between the direction of polarization of the incident light and the direction of the emitted light.

For a later comparison with quantum electrodynamic calculations we associate this emitted power with a differential cross section of the electron with respect to light. According to the definition in Chapter 6, a cross section is obtained by dividing the radiated power per unit solid angle by the incident power flux density $c\mathbf{E}_0^2/2$. This yields

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{4\pi M c^2}\right)^2 \sin^2\beta = r_e^2 \sin^2\beta, \tag{12.233}$$

where

$$r_e = \frac{e^2}{4\pi M c^2} = \frac{\hbar\alpha}{Mc} \approx 2.82\times 10^{-13}\ \text{cm} \tag{12.234}$$

is the *classical electron radius*. Formula (12.233) describes the so-called *Thomson scattering cross section*. It is applicable to linearly polarized incident waves. For unpolarized waves, we have to form the average between the cross section (12.233) and another one in which the plane of polarization is rotated by 90%. Suppose the incident light runs along the z-axis, and the emitted light along the direction $\hat{\mathbf{k}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. For a polarization direction $\boldsymbol{\epsilon} = \hat{\mathbf{x}}$ along the x-axis, the angle β is found from

$$\sin^2\beta = (\hat{\mathbf{k}}\times\boldsymbol{\epsilon})^2 = \cos^2\theta + \sin^2\theta\sin^2\phi. \tag{12.235}$$

For a polarization direction $\boldsymbol{\epsilon} = \hat{\mathbf{x}}$ along the y-axis, it is

$$\sin^2\beta = (\hat{\mathbf{k}}\times\boldsymbol{\epsilon})^2 = \cos^2\theta + \sin^2\theta\cos^2\phi. \tag{12.236}$$

The average is

$$\overline{\sin^2\beta} = \cos^2\theta + \frac{1}{2}\sin^2\theta = \frac{1}{2}(1+\cos^2\theta). \tag{12.237}$$

Integrating this over all solid angles yields the Thomson cross section for unpolarized light

$$\sigma_{\rm tot} = \frac{8\pi}{3} r_e^2. \qquad (12.238)$$

12.9.2 Klein-Nishina Formula

The scattering amplitude corresponding to the two Feynman diagrams in Fig. 12.6 is obtained by expanding formula (10.103) up to second order in e:

$$S_{\rm fi} = -e^2 \int d^4x d^4y \langle 0|a_{\mathbf{p}',s_3'} \hat{T} \bar{\psi}(y) \not\epsilon' \psi(y) \bar{\psi}(x) \not\epsilon \psi(x) a^\dagger_{\mathbf{p},s_3}|0\rangle, \qquad (12.239)$$

and by performing the Wick contractions of Section 7.8:

$$\langle 0|a_{\mathbf{p}',s_3'} \hat{T} \bar{\psi}(y) \not\epsilon' \psi(y) \bar{\psi}(x) \not\epsilon \psi(x) a^\dagger_{\mathbf{p},s_3}|0\rangle = \langle 0|\hat{T}\, a_{\mathbf{p}',s_3'} \bar{\psi}(y)\, \not\epsilon'\, \psi(y) \bar{\psi}(x)\, \not\epsilon\, \psi(x) a^\dagger_{\mathbf{p},s_3}\,|0\rangle$$
$$+ \langle 0|\hat{T}\, a_{\mathbf{p}',s_3'} \bar{\psi}(y) \not\epsilon' \psi(y) \bar{\psi}(x) \not\epsilon \psi(x) a^\dagger_{\mathbf{p},s_3}\,|0\rangle|0\rangle. \qquad (12.240)$$

After Fourier-expanding the intermediate electron propagator,

$$G_0(y,x) = \int \frac{d^4p^{\rm i}}{(2\pi)^4} e^{-ip^{\rm i}(y-x)} \frac{i}{\not p^{\,\rm i} - M}, \qquad (12.241)$$

we find

$$\begin{aligned} S_{\rm fi} &= -e^2 \int d^4x d^4y e^{k'y-kx} \int \frac{d^4p^{\rm i}}{(2\pi)^4} \frac{M}{\sqrt{V^2 E E' 2\omega 2\omega'}} \\ &\times \Bigg[e^{i(p'-p^{\rm i})y-(p-p^{\rm i})x} \bar{u}(\mathbf{p}', s_3') \not\epsilon'^* \frac{i}{\not p^{\,\rm i} - M} \not\epsilon\, u(\mathbf{p}, s_3) \\ &\quad + e^{i(p'-p^{\rm i})x-(p-p^{\rm i})y} \bar{u}(\mathbf{p}', s_3') \not\epsilon \frac{i}{\not p^{\,\rm i} - M} \not\epsilon'\, u(\mathbf{p}, s_3) \Bigg]. \end{aligned} \qquad (12.242)$$

One of the spatial integrals fixes the intermediate momentum in accordance with energy-momentum conservation, the other yields a $\delta^{(4)}$-function for overall energy-momentum conservation. The result is

$$S_{\rm fi} = -i(2\pi)^4 \delta^{(4)}(p' + k' - p - k) e^2 \frac{M}{\sqrt{V^2 E E' 2\omega 2\omega'}} \bar{u}(\mathbf{p}', s_3') H u(\mathbf{p}, s_3), \qquad (12.243)$$

where H is the 4×4-matrix in spinor space

$$H \equiv \not\epsilon'^* \frac{(\not p + \not k) + M}{(p+k)^2 - M^2} \not\epsilon + \not\epsilon \frac{(\not p - \not k') + M}{(p-k')^2 - M^2} \not\epsilon'^*. \qquad (12.244)$$

We have written $\epsilon(\mathbf{k}, h)$, $\omega_{\mathbf{k}}$ as ϵ, ω, and $\epsilon(\mathbf{k}', h')$, $\omega_{\mathbf{k}'}$ as ϵ', ω', respectively, with a similar simplification for E and E'. The second term of the matrix H arises from the first by the *crossing symmetry*

$$\epsilon \leftrightarrow \epsilon', \qquad k \leftrightarrow -k'. \qquad (12.245)$$

Simplifications arise from the properties (12.246). It can, moreover, be simplified by recalling that external electrons and photons are on their mass shell, so that

$$p^2 = p'^2 = M^2, \quad k^2 = k'^2 = 0, \tag{12.246}$$

$$k\,\epsilon = k'\epsilon' = 0. \tag{12.247}$$

A further simplification arises by working in the laboratory frame in which the initial electron is at rest, $p = (M, 0, 0, 0)$. Also, we choose a gauge in which the polarization vectors have only spatial components. Then

$$p\,\epsilon = p\,\epsilon' = 0, \tag{12.248}$$

since p has only a temporal component and ϵ only space components. We also use the fact that H stands between spinors which satisfy the Dirac equation $(\not{p} - M)u(\mathbf{p}, s_3) = 0$, $\bar{u}(\mathbf{p}, s_3)(\not{p} - M) = 0$. Further we use the commutation rules (4.566) for the gamma matrices to write [as in (12.214)]

$$\not{\epsilon}\not{p} = -\not{p}\not{\epsilon} + 2p\epsilon. \tag{12.249}$$

The second term vanishes by virtue of Eq. (12.248). Similarly, we see that $\not{p}$ anticommutes with $\not{\epsilon}'$. Using these results, we may eliminate the terms $\not{p} + M$ occurring in M. Finally, using Eq. (12.247), we obtain

$$\bar{u}(\mathbf{p}', s_3')Hu(\mathbf{p}, s_3) \;=\; \bar{u}(\mathbf{p}', s_3')\left\{\not{\epsilon}'^* \frac{\not{k}}{2pk}\not{\epsilon} + \not{\epsilon}\frac{\not{k}'}{2pk'}\not{\epsilon}'^*\right\}u(\mathbf{p}, s_3). \tag{12.250}$$

To obtain the transition probability, we must take the absolute square of this. If we do not observe initial and final spins, we may average over the initial spin and sum over the final spin directions. This produces a factor 1/2 times the sum over both spin directions, which is equal to

$$F = \sum_{s_3', s_3} |\bar{u}(\mathbf{p}', s_3')Hu(\mathbf{p}, s_3)|^2 = \sum_{s_3', s_3} \bar{u}(\mathbf{p}', s_3')Hu(\mathbf{p}, s_3)\bar{u}(\mathbf{p}, s_3)Hu(\mathbf{p}', s_3'). \tag{12.251}$$

Here we apply the semi-completeness relation (4.702) for the spinors to find

$$F \;=\; \frac{1}{4M^2}\mathrm{tr}\left[(\not{p}' + M)H(\not{p} + M)H\right]. \tag{12.252}$$

The trace over the product of gamma matrices can be evaluated according to the Wick-type of rules explained on p. 830.

$$\frac{1}{4}\mathrm{tr}(\not{a}_1 \cdots \not{a}_n) = \sum_{\text{pair contractions}} (-)^P \not{a}_1 \cdots \not{a}_n. \tag{12.253}$$

After some lengthy algebra (see Appendix 9A), we find

$$F = \frac{1}{2M^2}\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - 2 + 4|\boldsymbol{\epsilon}'^* \cdot \boldsymbol{\epsilon}|^2\right]. \tag{12.254}$$

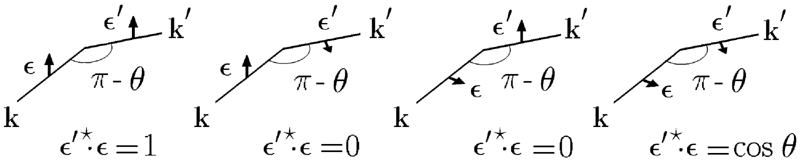


FIGURE 12.7 Illustration of the photon polarization sum $\sum_{h,h'} |\boldsymbol{\epsilon}'^* \boldsymbol{\epsilon}|^2 = 1 + \cos^2\theta$ in Compton scattering in the laboratory frame. Incoming and outgoing photon momenta with scattering angle θ are shown in the scattering plane, together with their transverse polarization vectors.

This holds for specific polarizations of the incoming and outgoing photons. If we sum over all final polarizations and average over all initial ones, we find (see Fig. 12.7)

$$\frac{1}{2}\sum_{h,h'} |\boldsymbol{\epsilon}'^* \boldsymbol{\epsilon}|^2 = \frac{1}{2}(1 + \cos^2\theta), \tag{12.255}$$

This can also be found more formally using the transverse completeness relation (4.334) of the polarization vectors:

$$\frac{1}{2}\sum_{h,h'} |\boldsymbol{\epsilon}'^* \boldsymbol{\epsilon}|^2 = \frac{1}{2}\sum_{h,h'} \epsilon'^{i*}(\mathbf{k}', h')\epsilon^i(\mathbf{k}, h)\epsilon^{*j}(\mathbf{k}, h)\epsilon'^i(\mathbf{k}', h') \tag{12.256}$$

$$= \frac{1}{2}\left(\delta^{ij} - \frac{k^i k^j}{\mathbf{k}^2}\right)\left(\delta^{ji} - \frac{k'^j k'^i}{\mathbf{k}'^2}\right) = \frac{1}{2}\left[1 + \frac{(\mathbf{k}\cdot\mathbf{k}')^2}{\mathbf{k}^2\mathbf{k}'^2}\right] = \frac{1}{2}(1 + \cos^2\theta).$$

The average value of F is therefore

$$\bar{F} = \frac{1}{2}\sum_{h,h'} \frac{1}{2M^2}\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - 2 + 4|\boldsymbol{\epsilon}'^* \cdot \boldsymbol{\epsilon}|^2\right] = \frac{1}{M^2}\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right]. \tag{12.257}$$

We are now ready to calculate the transition rate, for which we obtain from Eq. (9.298):

$$\frac{dP}{dt} = V(2\pi)^4 \int \frac{d^3p'V}{(2\pi)^3} \int \frac{d^3k'V}{(2\pi)^3} \delta^{(4)}(p' - p)|t_{\mathrm{fi}}|^2, \tag{12.258}$$

with the squared t-matrix elements

$$|t_{\mathrm{fi}}|^2 = e^4 \frac{1}{V^4} \frac{M^2}{EE'2\omega 2\omega'} \frac{1}{2} F. \tag{12.259}$$

The spatial part of the δ-function removes the momentum integral over $\mathbf{p}'$. The temporal part of the δ-function enforces energy conservation. This is incorporated into the momentum integral over $\mathbf{k}'$ as follows. We set $E_{\mathrm{f}} = p'_0 + k'_0$, and write

$$d^3k' = k'^2 dk' d\Omega = \omega'^2 \frac{d\omega'}{dE_{\mathrm{f}}} d\Omega dE_{\mathrm{f}}, \tag{12.260}$$

where Ω is the solid angle into which the photon has been scattered. Then (12.258) becomes

$$\frac{dP}{dtd\Omega} = Ve^4(2\pi)^4 \frac{V^2}{(2\pi)^6} \left(\frac{\omega'}{\omega}\right) \frac{d\omega'}{dE_{\rm f}}\bigg|_{E_{\rm f}=E_{\rm i}} \frac{1}{4V^4} \frac{M^2}{EE'} \frac{1}{2} F. \tag{12.261}$$

For an explicit derivative $d\omega'/dE_{\rm f}$, we go to the laboratory frame and express the final energy as

$$E_{\rm f} = \omega' + \sqrt{\mathbf{p}'^2 + M^2} = \omega' + \sqrt{(\mathbf{k} - \mathbf{p}')^2 + M^2} = \omega' + \sqrt{\omega^2 - 2\omega\omega' \cos\theta + \omega'^2 + M^2}, \tag{12.262}$$

where θ is the scattering angle in the laboratory. This yields the derivative

$$\frac{dE_{\rm f}}{d\omega'} = 1 + \frac{\omega' - \omega\cos\theta}{E'}. \tag{12.263}$$

By equating $E_{\rm i} = M + \omega$ with $E_{\rm f}$, we derive the *Compton relation*

$$\omega\omega'(1 - \cos\theta) = M(\omega - \omega'), \tag{12.264}$$

or

$$\omega' = \frac{\omega}{1 + \omega(1 - \cos\theta)/M}, \tag{12.265}$$

and therefore

$$\frac{dE_{\rm f}}{d\omega'} = 1 + \frac{\omega' - \omega\cos\theta}{E'} = \frac{E' + \omega' - \omega\cos\theta}{E'} = \frac{M + \omega - \omega\cos\theta}{E'} = \frac{M}{E'}\frac{\omega}{\omega'}. \tag{12.266}$$

Since $E = M$ in the laboratory frame, Eq. (12.261) yields the differential probability rate

$$\frac{dP}{dtd\Omega} = Ve^4(2\pi)^4 \frac{V^2}{(2\pi)^6} V \left(\frac{\omega'}{\omega}\right)^2 \frac{1}{4V^4} \frac{1}{2} F. \tag{12.267}$$

To find the differential cross section, this has to be divided by the incoming particle current density. According to Eq. (9.315), this is given by

$$j = \frac{v}{V}, \tag{12.268}$$

where v is the velocity of the incoming particles. The incoming photons move with light velocity, so that (in natural units with $c = 1$)

$$j = \frac{1}{V}. \tag{12.269}$$

This leaves us with the Klein-Nishina formula for the differential cross section

$$\frac{d\sigma}{d\Omega} = \alpha^2 \left(\frac{\omega'}{\omega}\right)^2 \frac{1}{2} F. \tag{12.270}$$

In the nonrelativistic limit where $\omega \ll M$, the Compton relation (12.265) shows that $\omega' \approx \omega$, and Eq. (12.257) reduces to

$$F \to \frac{1}{M^2}(1+\cos^2\theta). \tag{12.271}$$

Expressed in terms of the classical electron radius $r_0 = \alpha/M$, the differential scattering cross section becomes, expressed in terms of the classical electron radius $r_0 \equiv \alpha/M$,

$$\frac{d\sigma}{d\Omega} \approx {r_0}^2\,\frac{1}{2}(1+\cos^2\theta). \tag{12.272}$$

This is the Thomson formula for the scattering of low energy radiation by a static charge. To find the total Thomson cross section, we must integrate (12.272) over all solid angles and obtain:

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} \approx 2\pi \int_{-1}^{1} d\cos\theta \; r_0^2\,\frac{1}{2}(1+\cos^2\theta). \tag{12.273}$$

In the low-energy limit we identify

$$\sigma_{\rm Thomson} \equiv {r_0}^2\frac{8\pi}{3}. \tag{12.274}$$

Let us also calculate the total cross section for relativistic scattering, integrating (12.270) over all solid angles:

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = 2\pi \int_{-1}^{1} d\cos\theta\,\frac{\alpha^2}{2M^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega}+\frac{\omega}{\omega'}-\sin^2\theta\right]. \tag{12.275}$$

Inserting

$$\omega' = \frac{\omega}{1+\omega(1-\cos\theta)/M}, \tag{12.276}$$

which follows from (12.265), the angular integral yields

$$\sigma = \sigma_{\rm Thomson} f(\omega), \tag{12.277}$$

where $f(\omega)$ contains the relativistic corrections to Thomson's cross section:

$$f(\omega) = \frac{3}{8\omega^3}\left\{\frac{2\omega[\omega(\omega+1)(\omega+8)+2]}{(2\omega+1)^2} - (2+2\omega-\omega^2)\log(1+2\omega)\right\} \tag{12.278}$$

For small ω, $f(\omega)$ starts out like (see Fig. 12.8a):

$$f(\omega) = 1 - 2\omega + \mathcal{O}(\omega^2). \tag{12.279}$$

For large ω, $\omega f(\omega)$ increases like (see Fig. 12.8b):

$$\omega f(\omega) = \frac{3}{8\omega}\log 2\omega + \mathcal{O}(1/\log\omega). \tag{12.280}$$

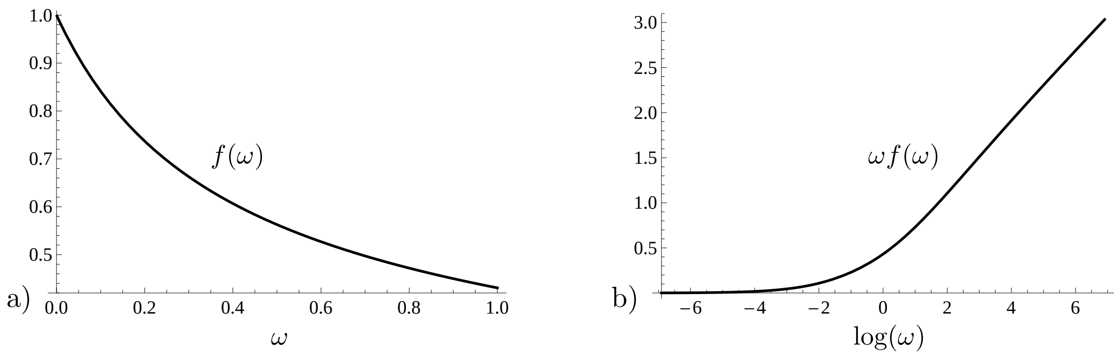


FIGURE 12.8 Ratio between total relativistic Compton cross section and nonrelativistic Thomson cross section.

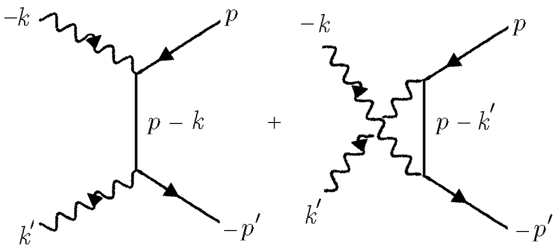


FIGURE 12.9 Lowest-order Feynman diagrams contributing to electron-positron annihilation. It arises from the Compton diagram by the crossing operation $p' \to -p'$, $k \to -k$, $\epsilon \to \epsilon^*$.

12.10 Electron-Positron Annihilation

The Feynman diagrams in Fig. 12.6 can also be read from bottom to top in which case they describe annihilation processes. In order to see this, we use as much as possible previous results and reinterpret the Feynman diagrams as shown in Fig. 12.9. Instead of an outgoing electron with momentum p', we let a negative-energy electron go out with momentum $-p'$. Instead of an incoming photon with momentum k we let a negative-energy photon go out with momentum $-k$. The former are represented by spinors $\bar{v}(\mathbf{p}', s_3)$, which are negative energy solutions of the Dirac equation with inverted momenta and spin directions. The S-matrix element to lowest order is found from a slightly modified (12.239):

$$S_{\rm fi} = -e^2 \int d^4x d^4y \langle 0|\hat{T}\bar{\psi}(y)\not{\epsilon}'\psi(y)\bar{\psi}(x)\not{\epsilon}\psi(x) b^{\dagger}_{\mathbf{p}',s_3'} a_{\mathbf{p},s_3}|0\rangle. \qquad (12.281)$$

Performing the Wick contractions as in (12.240) we obtain

$$\begin{aligned} S_{\rm fi} &= -e^2 \int d^4x d^4y e^{k'y+kx} \int \frac{d^4p^{\rm i}}{(2\pi)^4} \frac{M}{\sqrt{V^2 E E' 2\omega 2\omega'}} \\ &\times \left[e^{i(-p'-p^{\rm i})y-(p-p^{\rm i})x} \bar{v}(\mathbf{p}', s_3')\not{\epsilon}'^* \frac{i}{\not{p}^{\rm i} - M}\not{\epsilon}^* u(\mathbf{p}, s_3) \right. \end{aligned}$$

$$+ e^{i(-p'-p^{\rm i})x-(p-p^{\rm i})y}\bar{v}(\mathbf{p}', s_3')\not\epsilon^* \frac{i}{\not p^{\rm i} - M}\not\epsilon' u(\mathbf{p}, s_3)\Bigg] . \tag{12.282}$$

Note that this arises from the Compton expression (12.242) by the crossing operation

$$p' \to -p',\ \bar{u}(\mathbf{p}, s_3) \to \bar{v}(\mathbf{p}, s_3), \qquad k \to -k,\ \epsilon(k,h) \to \epsilon^*(k,h). \tag{12.283}$$

As before, one of the spatial integrals fixes the intermediate momentum in accordance with energy-momentum conservation, the other yields a $\delta^{(4)}$-function for overall energy-momentum conservation. The result is

$$S_{\rm fi} = -i(2\pi)^4\delta^{(4)}(k + k' - p - p')e^2 \frac{M}{\sqrt{V^2 EE'2\omega 2\omega'}}\bar{v}(\mathbf{p}', s_3')Hu(\mathbf{p}, s_3), \tag{12.284}$$

where H is the 4×4-matrix in spinor space

$$H \equiv \not\epsilon'^* \frac{(\not p - \not k) + M}{(p-k)^2 - M^2}\not\epsilon^* + \not\epsilon^* \frac{(\not p - \not k') + M}{(p-k')^2 - M^2}\not\epsilon'^*. \tag{12.285}$$

As before, we have written $\epsilon(\mathbf{k}, h),\ \omega_{\mathbf{k}}$ as ϵ, ω, and $\epsilon(\mathbf{k}', h'),\ \omega_{\mathbf{k}'}$ as ϵ', ω', respectively, with a similar simplification for E and E'. The second term of the matrix H arises from the first by the *Bose symmetry*

$$\epsilon \leftrightarrow \epsilon', \qquad k \leftrightarrow k'. \tag{12.286}$$

Simplifications arise from the mass shell properties (12.246), the gauge conditions (12.247), and the other relations (12.248), (12.249). We also work again in the laboratory frame in which the initial electron is at rest, $p = (M, 0, 0, 0)$ and the positron comes in with momentum $\mathbf{p}'$ and energy $E' = \sqrt{\mathbf{p}'^2 + M^2}$. This leads to

$$\bar{v}(\mathbf{p}', s_3')Hu(\mathbf{p}, s_3) \;=\; \bar{v}(\mathbf{p}', s_3')\left\{\epsilon'^* \frac{\not k}{2pk}\not\epsilon^* + \not\epsilon^* \frac{\not k'}{2pk'}\not\epsilon'^*\right\} u(\mathbf{p}, s_3). \tag{12.287}$$

To obtain the transition probability, we must take the absolute square of this. If we do not observe initial spins, we may average over the initial spin components which gives a factor $1/4$ times the spin sum

$$F = \sum_{s_3', s_3} |\bar{v}(\mathbf{p}', s_3')Hu(\mathbf{p}, s_3)|^2 = \sum_{s_3', s_3} \bar{v}(\mathbf{p}', s_3')Hu(\mathbf{p}, s_3)\bar{u}(\mathbf{p}, s_3)Hu(\mathbf{p}', s_3'). \tag{12.288}$$

Now we use the semi-completeness relations (4.702) and (4.703) for the spinors to rewrite (12.288) as

$$F \;=\; -\frac{1}{4M^2}{\rm tr}\left[(-\not p' + M)H(\not p + M)H\right]. \tag{12.289}$$

The trace over a product of gamma matrices is now evaluated as before and we obtain, for specific polarizations of the two outgoing photons, almost the same expression as before in Eq. (12.257) (see Appendix 12A):

$$F = \frac{1}{2M^2}\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + 2 - 4|\boldsymbol{\epsilon}'^* \cdot \boldsymbol{\epsilon}^*|^2\right]. \tag{12.290}$$

This result can be deduced directly from the previous polarization sum (12.254) by the crossing operation (12.283), apart from an overall minus sign, whose origin is the negative sign in front of (12.289).

We are now ready to calculate the transition rate, for which we obtain from Eq. (9.298)

$$\frac{dP}{dt} = V(2\pi)^4 \int \frac{d^3k'V}{(2\pi)^3} \int \frac{d^3k'V}{(2\pi)^3} \delta^{(4)}(p'-p)|t_{\rm fi}|^2, \tag{12.291}$$

with the squared t-matrix elements

$$|t_{\rm fi}|^2 = e^4 \frac{1}{V^4} \frac{M^2}{EE'2\omega 2\omega'} \frac{1}{4} F. \tag{12.292}$$

The spatial part of the δ-function removes the momentum integral over $\mathbf{k}'$. The temporal part of the δ-function enforces energy conservation. This is incorporated into the momentum integral over $\mathbf{k}$ as follows. We set $E_{\rm f} = \omega + \omega'$, and write

$$d^3k = k^2 dk d\Omega = \omega^2 \frac{d\omega}{dE_{\rm f}} d\Omega dE_{\rm f}, \tag{12.293}$$

where Ω is the solid angle into which the photon with momentum k emerges. Then (12.291) becomes

$$\frac{dP}{dtd\Omega} = Ve^4(2\pi)^4 \frac{V^2}{(2\pi)^6} \left(\frac{\omega}{\omega'}\right) \frac{d\omega}{dE_{\rm f}}\bigg|_{E_{\rm f}=E_{\rm i}} \frac{1}{4V^4} \frac{M^2}{EE'} \frac{1}{4} F. \tag{12.294}$$

To calculate $d\omega/dE_{\rm f}$ explicitly, we express the final energy in the laboratory frame as

$$E_{\rm f} = \omega + \sqrt{\mathbf{k}'^2} = \omega + \sqrt{(\mathbf{p}'-\mathbf{k})^2} = \omega + \sqrt{\mathbf{p}'^2 - 2\omega|\mathbf{p}'|\cos\theta + \omega^2}, \tag{12.295}$$

where θ is the scattering angle in the laboratory. Hence:

$$\begin{aligned} \frac{dE_{\rm f}}{d\omega} &= 1 + \frac{1}{2\omega'} \frac{d}{d\omega}\left(\mathbf{p}'^2 - 2\omega|\mathbf{p}'|\cos\theta + \omega^2\right) = \frac{1}{\omega\omega'}\left(\omega\omega' - 2\omega|\mathbf{p}'|\cos\theta + \omega^2\right) \\ &= \frac{1}{\omega\omega'}\left[\omega\left(M + \sqrt{\mathbf{p}'^2 + M^2}\right) - 2\omega|\mathbf{p}'|\cos\theta\right] = \frac{k(p+p')}{\omega\omega'} = \frac{M(M+E')}{\omega\omega'}. \end{aligned} \tag{12.296}$$

We must now divide (12.294) by the incoming positron current density $j = |\mathbf{p}'|/E'V$ [recall (9.315)], and find the differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{|\mathbf{p}'|(M+E')} \left(\frac{\omega}{M}\right)^2 \frac{1}{4} M^2 F, \tag{12.297}$$

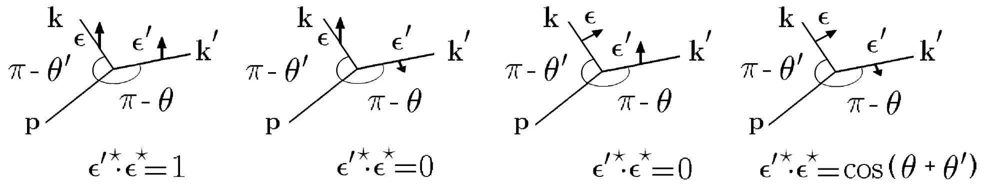


FIGURE 12.10 Illustration of the photon polarization sum $\sum_{h,h'} |\boldsymbol{\epsilon}'^{*}\boldsymbol{\epsilon}|^2$ in electron-positron annihilation in the laboratory frame. Incoming positron and outgoing photon momenta with scattering angles θ and θ' are shown in the scattering plane, together with their transverse polarization vectors.

where

$$\frac{M}{\omega} = \frac{M + E' - |\mathbf{p}'| \cos\theta}{M + E'}, \tag{12.298}$$

as follows from equating the right-hand sides of

$$kk' = \frac{1}{2}(k + k')^2 = \frac{1}{2}(p + p')^2 = M(M + E'), \tag{12.299}$$

and

$$kk' = k(p + p' - k) = \omega(M + E' - |\mathbf{p}'| \cos\theta). \tag{12.300}$$

If the incoming positron is very slow, then $\omega \approx \omega' \approx M$, and the two photons share equally the rest energies of the electron and the positron. We can now substitute in F of Eq. (12.290):

$$\frac{\omega'}{\omega} = \frac{E' - |\mathbf{p}'|\cos\theta}{M}, \tag{12.301}$$

and sum over all photon polarizations to obtain [see Fig. 12.10 and Eq. (12.257)]

$$\sum_{h,h'} |\boldsymbol{\epsilon}'^{*}\boldsymbol{\epsilon}|^2 = 1 + \frac{(\mathbf{k}\cdot\mathbf{k}')^2}{\mathbf{k}^2\mathbf{k}'^2} = 1 + \cos^2(\theta + \theta') = \left(1 - \frac{M}{\omega} - \frac{M}{\omega'}\right)^2. \tag{12.302}$$

The last expression is found by observing that

$$k\,k' = \omega\omega'[1 - \cos(\theta + \theta')], \tag{12.303}$$

so that we can express

$$\cos(\theta + \theta') = 1 - \frac{k\,k'}{\omega\omega'} = 1 - \frac{1}{2}\frac{(k + k')^2}{\omega\omega'} = 1 - \frac{1}{2}\frac{(p + p')^2}{\omega\omega'} = 1 - \frac{M(M + E')}{\omega\omega'}.$$

Energy conservation leads to

$$\cos(\theta + \theta') = 1 - \frac{k\,k'}{\omega\omega'} = 1 - \frac{M(\omega + \omega')}{\omega\omega'}, \tag{12.304}$$

thus obtainig the right-hand side of (12.302). Note that from (12.303) and (12.304) we find a relation

$$k\,k' = M(\omega + \omega'). \tag{12.305}$$

We now introduce the relativistic factor of the incoming positron $\gamma \equiv E'/M$ and write the relation (12.298) as

$$\frac{M}{\omega} = 1 - \sqrt{\frac{\gamma-1}{\gamma+1}}\cos\theta, \quad \frac{M}{\omega'} = \frac{1-\sqrt{\frac{\gamma-1}{\gamma+1}}\cos\theta}{\gamma-\sqrt{\gamma^2-1}\cos\theta}. \tag{12.306}$$

The result is integrated over all solid angles, and divided by 2 to account for Bose statistics of the two final photons. This yields the cross section expressed in terms of the classical electron radius $r_0 = \alpha/M$:

$$\sigma(\gamma) = \frac{1}{2}\int d\Omega \frac{d\sigma}{d\Omega} = \frac{\pi r_0^2}{1+\gamma}\left[\frac{\gamma^2+4\gamma+1}{\gamma^2-1}\log\left(\gamma+\sqrt{\gamma^2-1}\right) - \frac{\gamma+3}{\sqrt{\gamma^2-1}}\right]. \tag{12.307}$$

For small incoming positron energy, the cross section diverges like

$$\sigma(\gamma) \xrightarrow[\text{small } \mathbf{p}]{} \sigma_{\text{low-energy}}(\gamma) \equiv \frac{\pi r_0^2\,\gamma}{\sqrt{\gamma^2-1}} \approx \frac{\pi r_0^2 c}{v}, \tag{12.308}$$

whereas in the high-energy limit it behaves like

$$\sigma \xrightarrow[|\mathbf{p}|\to\infty]{} \frac{\pi r_0^2}{\gamma}\left[\log(2\gamma)-1\right]. \tag{12.309}$$

The detailed behavior is shown in Fig. 12.11.

The above result can be used to estimate the lifetime of a positron moving through matter. We simply have to multiply the cross section σ by the incident current density $j = |\mathbf{p}|/EV$ of a single positron and by the number N of target electrons, which is Z per atom. For slow positrons we may use Eq. (12.308) to find the decay rate

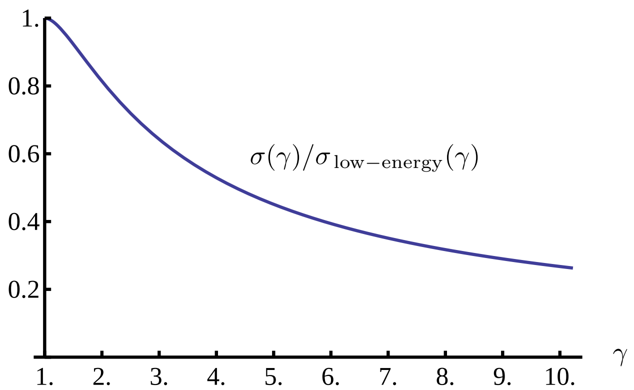


FIGURE 12.11 Electron-positron annihilation cross section divided by its low-energy limiting expression (12.308) as a function of $\gamma = 1/\sqrt{1-v^2/c^2}$ of the incoming positron in the laboratory frame.

$$\dot{P} = \pi r_0^2 v\, Z\frac{N}{V}. \tag{12.310}$$

For lead, this yields a lifetime $\tau = 1/\dot{P} \approx 10^{-10}$s.

The formula (12.313) is actually not very precise, since the incoming positron wave is strongly distorted towards the electron by the Coulomb attraction. This is the so-called initial-state interaction. It is closely related to the problem to be discussed in the next section.

12.11 Positronium Decay

The previous result can be used to calculate the lifetime of positronium. Since the momenta in positronium are nonrelativistic, the annihilation cross section (12.308) is relevant. The wave function of positronium at rest in an s-wave is approximately given by [recall (7.311)]

$$|\psi^{S,S_3}\rangle = \int \frac{d^3p}{(2\pi)^3}\tilde{\psi}(\mathbf{p}) a^\dagger_{\mathbf{p},s_3} b^\dagger_{-\mathbf{p},s_3'}|0\rangle\langle S,S_3|s,s_3;s,s_3'\rangle,\ \ \tilde{\psi}(\mathbf{p}) = \frac{8\sqrt{\pi a^3}}{(1+a^2|\mathbf{p}|^2)^2}, \tag{12.311}$$

where $S=1$ for ortho-positronium and $S=0$ for para-positronium, and a is the Bohr radius of positronium, which is twice the Bohr radius of hydrogen: $a = 2\,a_H = 2/\alpha M$, the factor 2 being due to the reduced mass μ being half the electron mass [recall the general formula (6.132)]. This amplitude can be pictured by the Feynman diagrams shown in Fig. 12.12.

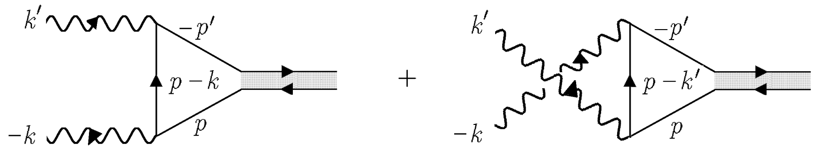


FIGURE 12.12 Lowest-order Feynman diagrams contributing to the decay of the spin singlet para-positronium, i.e., the ground state.

In Eq. (7.310) we have calculated the charge conjugation parity η_C of these states to be ∓ 1, respectively. Since a photon is odd under charge conjugation [recall (7.550)], ortho-positronium can only decay into three photons (one is forbidden by energy-momentum conservation). Thus only para-positronium decays into two photons and the cross section calculated in the last section arises only from the spin singlet contribution of the initial state. Since it was obtained from the average of a total of four states, three of which do not decay at all, the decay rate of the singlet state is four times as big as calculated previously from (12.308). The integral over all momenta in the amplitude (12.281) can be factored out since the low-energy decay rate is approximately independent of v and hence of $\mathbf{p}$. Thus we obtain directly the decay rate [15]

$$\Gamma_{\text{para}\to 2\gamma} \equiv \dot{P}_{\text{para}} = 4\pi r_0^2 c \left|\int \frac{d^3p}{(2\pi)^3}\tilde{\psi}(\mathbf{p})\right|^2 = 4\pi r_0^2 c\,|\psi(\mathbf{0})|^2 = 4\pi r_0^2 c\,\frac{1}{\pi a^3}$$

$$= \frac{Mc^2}{\hbar}\frac{\alpha^5}{2}. \tag{12.312}$$

Using the electron energy $Mc^2 \approx 0.510$ MeV and the Planck constant $\hbar \approx 6.682 \times 10^{-16}$eV s, the ratio $\hbar/Mc^2$ is equal to 1.288×10^{-22} s. Together with the factor $2/\alpha^5$, this leads to a lifetime $\tau = 1/\Gamma \approx 0.13 \times 10^{-9}$ s.

The decay of the spin triplet ortho-positronium state proceeds at a roughly thousand times slower rate

$$\Gamma_{\text{orth}\to 3\gamma} \equiv \dot{P}_{\text{orth}} = \frac{Mc^2}{\hbar}\frac{\alpha^6}{2}\frac{4(\pi^2-9)}{9\pi}, \tag{12.313}$$

with a lifetime of 140×10^{-9} s.

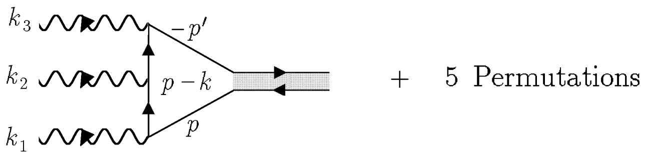


FIGURE 12.13 Lowest-order Feynman diagrams contributing to decay of the spin-triplet ortho-positronium, the first excited state which lies 203.5 GHz above the ground state.

Decays into 4 and 5 photons have also been calculated and measured experimentally. The theoretical rates are [14, 15]:

$$\Gamma_{\text{para}\to 4\gamma} \approx 0.0138957\frac{Mc^2}{\hbar}\alpha^7\left[1-14.5\frac{\alpha}{\pi}+\mathcal{O}(\alpha^2)\right] \approx 1.43 \times 10^{-6} \times \Gamma_{\text{para}\to 2\gamma}. \tag{12.314}$$

$$\Gamma_{\text{orth}\to 5\gamma} \approx 0.0189\,\alpha^2 \times \Gamma_{\text{orth}\to 3\gamma} \approx 0.959 \times 10^{-6} \times \Gamma_{\text{orth}\to 3\gamma}. \tag{12.315}$$

Experimentally, the branching ratios are $1.14(33) \times 10^{-6}$ and $1.67(99) \times 10^{-6}$, in reasonable agreement with the theoretical numbers. The validity of C-invariance has also been tested by looking for the forbidden decays of para-positronium into an odd and ortho-positronium into an even number of photons. So far, there is no indication of C-violation.

12.12 Bremsstrahlung

If a charged particle is accelerated or slowed down, it emits an electromagnetic radiation called *Bremsstrahlung*. This is a well-known process in classical electrodynamics and we would like to find the quantum field theoretic generalization of it. First, however, we shall recapitulate the classical case.

12.12.1 Classical Bremsstrahlung

Consider a trajectory in which a particle changes its momentum abruptly from p to p' [see Fig. 12.14].

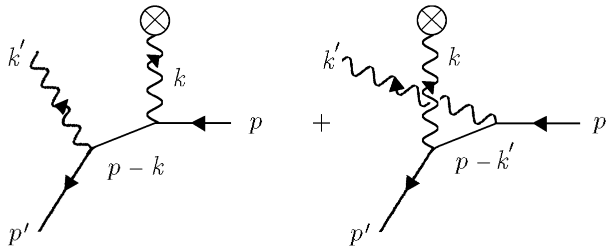


FIGURE 12.14 Trajectories in the simplest classical Bremsstrahlung process: An electron changing abruptly its momentum.

The trajectory may be parametrized as:

$$x^\mu(\tau) = \begin{cases} \tau p/M & \text{for} \quad \tau < 0, \\ \tau p'/M & \quad\quad\; \tau > 0, \end{cases} \tag{12.316}$$

where τ is the proper time. The electromagnetic current associated with this trajectory is

$$\begin{aligned} j^\mu(x) &= e\int d\tau \frac{dx^\mu(\tau)}{d\tau}\delta^{(4)}(x - x(\tau)) \\ &= \frac{e}{M}\int d\tau \left[\Theta(\tau)p^\mu\delta^{(4)}(x - \tau p/M) + \theta(-\tau)p'^\mu\delta^{(4)}(x - \tau p'/M)\right]. \end{aligned} \tag{12.317}$$

After a Fourier decomposition of the δ-functions, this can be written as

$$j^\mu(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} j^\mu(k), \tag{12.318}$$

with the Fourier components

$$j^\mu(k) = -ie\left(\frac{p^\mu}{pk} - \frac{p'^\mu}{p'k}\right). \tag{12.319}$$

The vector potential associated with this current is found by solving the Maxwell equation Eq. (12.50). Under the initial condition that at large negative time, the vector potential describes the retarded Coulomb potential of the incident particle, we obtain

$$A^\mu(x) = i\int dx' G_R(x - x') j^\mu(x'), \tag{12.320}$$

where $G_R(x - x')$ is the retarded Green function defined in Eq. (7.162). At very large times the particle has again a Coulomb field associated with it which can be found by using the advanced Green function of Eq. (7.168):

$$A^\mu_{\text{out}}(x) = i\int dx' G_A(x - x') j^\mu(x'). \tag{12.321}$$

As the acceleration takes place, the particle emits radiation which is found from the difference between the two fields,

$$A^\mu_{\text{rad}} = \frac{1}{i}\int dx'[G_R(x, x') - G_A(x, x')] j^\mu(x'). \tag{12.322}$$

Remembering the list (7.212) of Fourier transforms of the various Green functions, we see that the Fourier components of the radiation field are given by

$$A^\mu_{\rm rad}(k) = -i2\pi\epsilon(k^0)\delta(k^2)j^\mu(k). \tag{12.323}$$

The energy of the electromagnetic field is, by Eq. (7.427):

$$H = \int d^3x \frac{1}{2}(-\dot{A}^\mu \dot{A}_\mu - \boldsymbol{\nabla} A^\mu \boldsymbol{\nabla} A_\mu). \tag{12.324}$$

A classical field which solves the field equations can be Fourier decomposed into positive- and negative-frequency components as in (7.390),

$$A^\mu(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} \left(e^{-ikx} a^\mu_{\mathbf{k}} + e^{ikx} a^{\mu\dagger}_{\mathbf{k}} \right) , \tag{12.325}$$

where

$$a^\mu_{\mathbf{k}} \equiv \sum_{\lambda=0}^{3} \epsilon^\mu(\mathbf{k},\lambda) a_{\mathbf{k},\lambda} \tag{12.326}$$

are classical Fourier components. Then the combination of time and space derivatives in (12.324) eliminates all terms of the form $a_{\mathbf{k}}a_{\mathbf{k}}$ and $a^*_{\mathbf{k}}a^*_{\mathbf{k}}$, and we find

$$H = \sum_{\mathbf{k}} \frac{\omega_{\mathbf{k}}}{2} \left(-a^{\mu *}_{\mathbf{k}} a^\nu_{\mathbf{k}} - a^\mu_{\mathbf{k}} a^{\nu *}_{\mathbf{k}}\right) g_{\mu\nu}, \tag{12.327}$$

just as in the calculation of the energy in (7.428). The radiation field (12.323) corresponds to

$$a^\mu_{\mathbf{k}} \;\; = \;\; -ij^\mu(k)|_{k^0=|\mathbf{k}|}. \tag{12.328}$$

Inserting everything into (12.327) we derive, for large t, an emitted energy

$$\mathcal{E} = -\int \frac{d^3k}{2k^0(2\pi)^3} k^0 \sum_{h=1,2} j^*_\mu(k) j^\mu(k). \tag{12.329}$$

Inserting (12.319) we obtain the energy emitted into a momentum space element d^3k:

$$d\mathcal{E} = \frac{1}{2}\frac{d^3k}{(2\pi)^3} e^2 \left[\frac{2pp'}{(pk)(p'k)} - \frac{M^2}{(pk)^2} - \frac{M^2}{(p'k)^2} \right], \tag{12.330}$$

where polarization vectors have vanishing zeroth components. Dividing out the energy per photon k^0, this can be interpreted as the probability of omitting a photon into d^3k:

$$dP = \frac{d^3k}{2k^0(2\pi)^3} e^2 \left[\frac{2pp'}{(pk)(p'k)} - \frac{M^2}{(pk)^2} - \frac{M^2}{(p'k)^2} \right]. \tag{12.331}$$

If we are interested in the polarization of the radiated electromagnetic field, we make use of the local current conservation law $\partial_\mu j^\mu$ which reads, in momentum space,

$$\mathbf{k} \cdot \mathbf{j} = k^0 j^0. \tag{12.332}$$

This allows us to rewrite

$$j^*_\mu(k)j^\mu(k) = |j^0(k)|^2 - |\mathbf{j}(k)|^2 = \frac{1}{\mathbf{k}^2}|\mathbf{k}\cdot\mathbf{j}(k)|^2 - |\mathbf{j}(k)|^2 = -|\mathbf{j}_T(\mathbf{k})|^2, \qquad (12.333)$$

where

$$j^i_T(k) \equiv \left(\delta^{ij} - \frac{k^ik^j}{\mathbf{k}^2}\right) j^j(k) \qquad (12.334)$$

is the transverse part of the current. In the second-quantized description of the energy, the transverse projection is associated with a sum over the two outgoing photon polarization vectors [recall (4.334)]:

$$\delta_{ij} - \frac{k_ik_j}{\mathbf{k}^2} = \sum_{h=1,2} \epsilon^*_i(\mathbf{k},h)\epsilon_i(\mathbf{k},h). \qquad (12.335)$$

The emitted energy (12.329) can therefore be resolved with respect to the polarization vectors as

$$\mathcal{E} = \int \frac{d^3k}{2k^0(2\pi)^3}k^0 \sum_{h=1,2} |\boldsymbol{\epsilon}(\mathbf{k},h)\cdot\mathbf{j}(k)|^2. \qquad (12.336)$$

This leads to an energy emitted into a momentum space element d^3k:

$$d\mathcal{E} = \frac{d^3k}{2(2\pi)^3}e^2 \sum_{h=1,2} \left|\frac{p\epsilon}{pk} - \frac{p'\epsilon}{p'k}\right|^2, \qquad (12.337)$$

and a corresponding probability of omitting a photon into d^3k, by analogy with (12.331).

Let us calculate the angular distribution of the emitted energy in Eq. (12.330). Denote the direction of $\mathbf{k}$ by $\mathbf{n}$:

$$\mathbf{n} \equiv \frac{\mathbf{k}}{|\mathbf{k}|}. \qquad (12.338)$$

Then we can write pk as

$$pk = E|\mathbf{k}|(1-\mathbf{v}\cdot\mathbf{n}) \qquad (12.339)$$

and find

$$d\mathcal{E} = \frac{d^3k}{(2\pi)^3 2}\frac{e^2}{|\mathbf{k}|^2}\left[\frac{2(1-\mathbf{v}\cdot\mathbf{v}')}{(1-\mathbf{v}\cdot\mathbf{n})(1-\mathbf{v}'\cdot\mathbf{n})} - \frac{M^2}{E^2(1-\mathbf{v}\cdot\mathbf{n})^2} - \frac{M^2}{E'^2(1-\mathbf{v}'\cdot n)^2}\right]. \qquad (12.340)$$

The radiation is peaked around the directions of the incoming and outgoing particles.

12.12.2 Bremsstrahlung in Mott Scattering

We now turn to the more realistic problem of an electron scattering on a nucleus. Here the electron changes its momentum within a finite period of time rather than abruptly. Still, the Bremsstrahlung will be very similar to the previous one. Let us consider immediately a Dirac electron, i.e., we study the Bremsstrahlung emitted

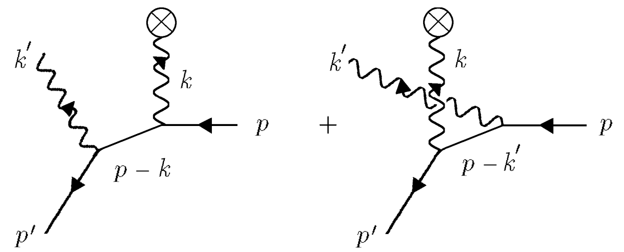


FIGURE 12.15 Lowest-order Feynman diagrams contributing to Bremsstrahlung. The vertical photon line indicates the nuclear Coulomb potential.

in Mott scattering. The lowest-order Feynman diagrams governing this process are shown in Fig. 12.15. The vertical photon line indicates the nuclear Coulomb potential

$$V_{\rm C}(\mathbf{x}) = -\frac{Z\alpha}{4\pi r}. \tag{12.341}$$

The scattering amplitude is found from the Compton amplitude by simply interchanging the incoming photon field

$$eA^\mu(x) = \epsilon^\mu_{k,\lambda} e^{-ikx}\sqrt{2Vk_0}$$

with the static vector potential

$$\delta^\mu{}_0 V_{\rm C}(r) = -\delta^\mu{}_0\, Z\alpha \int \frac{d^4q}{(2\pi)^4} 2\pi\delta(q^0) e^{-iqx}\frac{1}{|\mathbf{q}|^2}. \tag{12.342}$$

The scattering amplitude is therefore

$$S_{\rm fi} = i\frac{4\pi Z\alpha e}{|\mathbf{q}|^2} 2\pi\delta(p'^0 + k' - p^0)\frac{M}{\sqrt{V^2E'E}}\frac{1}{\sqrt{2Vk^{0\prime}}} \\ \times \bar{u}(\mathbf{p}', s_3')\left[\not\epsilon'^* \frac{1}{\not p' + \not k' - M}\gamma^0 - \gamma^0 \frac{1}{\not p' - \not k' - M}\not\epsilon'\right] u(\mathbf{p}, s_3), \tag{12.343}$$

where

$$\mathbf{q} = \mathbf{p}' + \mathbf{k}' - \mathbf{p} \tag{12.344}$$

is the spatial momentum transfer. The amplitude conserves only energy, not spatial momentum. The latter is transferred from the nucleus to the electron without any restriction. The unpolarized cross section following from $S_{\rm fi}$ is

$$d\sigma = M^2 Z^2 (4\pi\alpha)^3 \frac{1}{2k^{0\prime}E'E}\frac{1}{\mathbf{v}}\int \frac{d^3p'd^3k'}{(2\pi)^6} 2\pi\delta(p'^0 + k'^0 - p^0)\frac{F}{|\mathbf{q}|^2}, \tag{12.345}$$

where we have used the incoming particle current density $\mathbf{v}/V = \mathbf{p}/EMV$ and set

$$F \equiv \frac{1}{2}\sum_h \mathrm{tr}\left[\left(\not\epsilon'^* \frac{\not p' + \not k' + M}{2p'k'}\gamma^0 - \gamma^0 \frac{\not p - \not k' + M}{2pk}\not\epsilon'\right)\frac{\not p + M}{2M}\right. \\ \left.\left(\gamma^0 \frac{\not p + \not k + M}{2p'k}\not\epsilon' - \not\epsilon' \frac{\not p - \not k + M}{2pk}\gamma^0\right)\frac{\not p' + M}{2M}\right]. \tag{12.346}$$

Let $d\Omega_e$ and $d\Omega_\gamma$ be the solid angles of outgoing electrons and photons $d\Omega_e$ and $d\Omega_\gamma$, respectively. If we drop the prime on the emitted photon variables, so that ϵ' and k' are written as ϵ and k, and write for the energy k^0 of the outgoing photon the variable ω, the differential cross section becomes

$$\frac{d\sigma}{d\Omega_e d\Omega_\gamma d\omega} = \frac{Z^2\alpha^3}{\pi^2}\frac{M^2}{|\mathbf{q}|^4}\frac{|\mathbf{p}'|}{|\mathbf{p}|}\omega F. \tag{12.347}$$

The calculation of F is as tedious as that of the trace (12.289) in the Klein-Nishina cross section. It can be done again with the help of the formulas in Appendix 12A. Let us introduce the angles θ and θ' between the outgoing photon momentum and the initial and final electron momenta p and p', respectively (see Fig. 12.16). Then

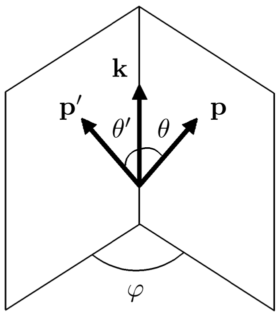


FIGURE 12.16 The angles θ', θ, φ in the Bethe-Heitler cross section formula.

we calculate the polarization sums

$$\begin{aligned}\sum_h (\boldsymbol{\epsilon}\cdot\mathbf{p}')^2 &= |\mathbf{p}'^2|\sin^2\theta', \\ \sum_h (\boldsymbol{\epsilon}\cdot\mathbf{p})^2 &= |\mathbf{p}|^2\sin^2\theta, \end{aligned}\tag{12.348}$$

$$\sum_h (\epsilon\cdot\mathbf{p}')(\boldsymbol{\epsilon}\cdot\mathbf{p}) = |\mathbf{p}'||\mathbf{p}|\sin\theta'\sin\theta\cos\varphi, \tag{12.349}$$

and obtain

$$\begin{aligned} F = \frac{1}{4\omega^2}\Bigg[&\frac{p'^2\sin^2\theta'}{(E'-p'\cos\theta')^2}(4E^2-q^2) + \frac{p^2\sin^2\theta}{(E-p\cos\theta)^2}(4E'^2-q^2) \\ &+ 2\omega^2\frac{p^2\sin^2\theta + p'^2\sin^2\theta'}{(E'-p'\cos\theta')(E-p\cos\theta)} \\ &- 2\frac{p'p\sin\theta'\sin\theta\cos\varphi}{(E'-p'\cos\theta')(E-p\cos\theta)}(4E'E-q^2+2\omega^2)\Bigg]. \end{aligned}\tag{12.350}$$

With this form of the function F, Eq. (12.347) is known as the *Bethe-Heitler cross section* formula. For soft photon emission, $\omega \to 0$, the cross section becomes:

$$\frac{d\sigma}{d\Omega_e} \approx \left.\frac{d\sigma}{d\Omega_e}\right|_{\text{elastic}} \times e^2\frac{d^3k}{2\omega(2\pi)^3}\left|\frac{\epsilon p'}{kp'} - \frac{\epsilon p}{kp}\right|^2. \tag{12.351}$$

It consists of the elastic cross section that is multiplied by the cross section of the classical Bremsstrahlung.

12.13 Electron-Electron Scattering

The leading Feynman diagrams are shown in Fig. 12.17. The associated scattering

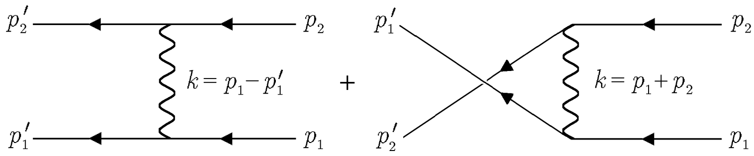


FIGURE 12.17 Lowest-order Feynman diagrams contributing to electron-electron scattering.

amplitude is given by

$$\begin{aligned} S_{fi} = {} & (2\omega)^4 \delta^4(p_1' + p_2' - p_1 - p_2)(-ie)^2 \\ & \times \left[-\bar{u}(p_1', \varepsilon_1')\gamma^\nu u(p_1, \varepsilon_1) \frac{-ig_{\nu\rho}}{(p_1 - p_1')^2} \bar{u}(p_2', \varepsilon_2')\gamma^\rho u(p_2, \varepsilon_2) \right. \\ & \left. + \bar{u}(p_2', \varepsilon_2')\gamma^\nu u(p_1, \varepsilon_1) \frac{-ig_{\nu\rho}}{(p_1 - p_2')^2} \bar{u}(p_1', \varepsilon_1')\gamma^\rho u(p_2, \varepsilon_2) \right]. \end{aligned} \tag{12.352}$$

For the scattering amplitude t_{fi} defined by

$$S_{fi} \equiv -ie^2 (2\pi)^4 \delta^4(p_1' + p_2' - p_1 - p_2) t_{fi}, \tag{12.353}$$

we find

$$t_{fi} = \frac{\bar{u}(p_1', \varepsilon_1') u(p_1, \varepsilon_1) \bar{u}(p_2', \varepsilon_2')\gamma_\nu u(p_2, \varepsilon_2)}{(p_1 - p_1')^2} - \frac{\bar{u}(p_2', \varepsilon_2')\gamma^\nu u(p_1, \varepsilon_1) \bar{u}(p_1', \varepsilon_1')\gamma_\nu u(p_2, \varepsilon_2)}{(p_1 - p_2')^2}. \tag{12.354}$$

There is a manifest antisymmetry of the initial or final states accounting for the Pauli principle. Due to the identity of the electrons, the total cross section is obtained by integrating over only half of the final phase space.

Let us compute the differential cross section for unpolarized initial beams, when the final polarizations are not observed. The kinematics of the reaction in the center of mass frame is represented in Fig. 12.18, where θ is the scattering angle in this frame. The energy E is conserved, and we denote $|\mathbf{p}| = |\mathbf{p}'| = p = \sqrt{E^2 - m^2}$. Using the general formula (9.311) with the covariant fermion normalization $V \to 1/E$, we obtain

$$\frac{d\sigma}{d\Omega_{\text{CM}}} = \frac{M^2 e^4}{4E^2 (2\pi)^2} \overline{|t_{fi}|^2}. \tag{12.355}$$

The bar on the right-hand side indicates an average over the initial polarizations and a sum over the final polarizations. More explicitly, we must evaluate the traces1

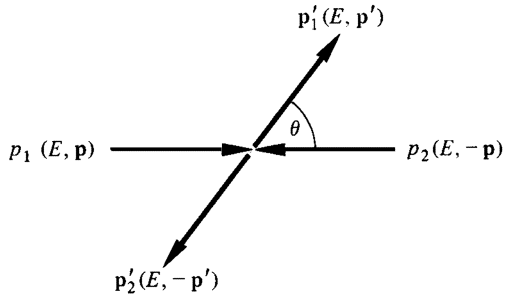


FIGURE 12.18 Kinematics of electron-electron scattering in the center of mass frame.

$$\begin{aligned}\overline{|t_{fi}|^2} &= \frac{1}{4}\sum_{\varepsilon_1\varepsilon_2\varepsilon_1'\varepsilon_2'}|t_{fi}|^2\\ &= \frac{1}{4}\left\{\ \mathrm{tr}\left(\gamma_\nu\frac{\not p_1+M}{2M}\gamma_p\frac{\not p_1'+M}{2M}\right)\mathrm{tr}\left(\gamma^\nu\frac{\not p+M}{2M}\gamma^\rho\frac{\not p_2+M}{2M}\right)\frac{1}{[(p_1'-p_1)^2]^2}\right.\\ &\quad -\mathrm{tr}\left(\gamma_\nu\frac{\not p+M}{2M}\gamma_\rho\frac{\not p_2+M}{2M}\gamma^\nu\frac{\not p+M}{2M}\gamma^\rho\frac{\not p_1'+M}{2M}\right)\frac{1}{(p_1'-p_1)^2(p_2'-p_1)^2}\\ &\quad \left.+(p_1'\leftrightarrow p_2')\right\}. \qquad (12.356)\end{aligned}$$

This is done using the formulas

$$\begin{aligned}&\mathrm{tr}\,[\gamma_\nu(\not p_1+M)\gamma_\rho(\not p_1+M)] = 4(p_{1\nu}p_{1\rho}'-g_{\nu\rho}p_1\cdot p_1'+p_{1\rho}p_{1\nu}'+M^2g_{\nu\rho}),\\ &\mathrm{tr}\,[\gamma_\nu(\not p_1+M)\gamma_\pi(\not p_1'+M)]\times\mathrm{tr}\,[\gamma^\nu(\not p_2+M)\gamma^\rho(\not p_2+M)]\\ &\qquad = 32[(p_1\cdot p_2)^2+(p_1\cdot p_2')^2+2M^2(p_1\cdot p_2'-p_1\cdot p_2)], \qquad (12.357)\end{aligned}$$

and further

$$\gamma_\nu(\not p_1+M)\gamma_\rho(\not p_2+M)\gamma^\nu = -2\ \not p_2'\gamma_\rho\ \not p_1+4M(p_{2\rho}'+p_{1\rho})-2M^2\gamma_\rho, \quad (12.358)$$

leading to

$$\mathrm{tr}[\gamma_\nu(\not p_1+M)\gamma_\rho(\not p_2'+M)\gamma^\nu(\not p_2+M)\gamma^\rho(\not p_1'+M)] = -32(p_1\,p_2)^2-2M^2p_1\,p_2],$$

and thus to

$$\begin{aligned}\overline{|t_{fi}|^2} &= \frac{1}{2M^4}\left\{\frac{(p_1\,p_2)^2+(p_1\,p_2')^2+2M(p_1\,p_2'-p_1\,p_2)}{[(p_1'-p_1)^2]^2}\right.\\ &\quad +\frac{(p_1\,p_2)^2+(p_1\,p_1')^2+2M^2(p_1\,p_1'-p_1\,p_2)}{[(p_2'-p_1)^2]^2}\\ &\quad \left.+\,2\,\frac{(p_1\,p_2)^2-2M^2p_1\,p_2}{(p_1'-p_1)^2(p_2'-p_1)^2}\right\}. \qquad (12.359)\end{aligned}$$

This can be expressed in terms of the Mandelstam variables s, t, u whose properties were discussed in Eqs. (9.318)–(9.331), yielding

$$\begin{aligned}\overline{|t_{fi}|^2} &= \frac{1}{2M^4}\left\{\frac{1}{t^2}\left[\frac{s^2+u^2}{4}+2m^2(t-m^2)\right]+\frac{1}{u^2}\left[\frac{s^2+t^2}{4}+2m^2(u-m^2)\right]\right.\\ &\quad \left.+\frac{1}{tu}\left[\left(\frac{s}{2}-m^2\right)\left(\frac{s}{2}-3m^2\right)\right]\right\}. \qquad (12.360)\end{aligned}$$

We may now easily express all invariants in terms of the center-of-mass energy $E_{\rm CM}$ and the scattering angle θ:

$$\begin{aligned} p_1\, p_2 &= 2E_{\rm CM}^2 - M^2, \\ p_1\, p_1' &= E_{\rm CM}^2(1-\cos\theta) + M^2\cos\theta, \\ p_1\, p_2' &= E_{\rm CM}^2(1-\cos\theta) - M^2\cos\theta. \end{aligned} \tag{12.361}$$

This leads to the *Møller formula* (1932):

$$\frac{d\sigma}{d\Omega_{\rm CM}} = \frac{\alpha^2(2E_{\rm CM}^2 - M^2)^2}{4E_{\rm CM}^2(E_{\rm CM}^2 - M^2)^2}\left[\frac{4}{\sin^4\theta} - \frac{3}{\sin^2\theta} + \frac{(E_{\rm CM}^2 - M^2)^2}{(2E_{\rm CM}^2 - M^2)^2}\left(1 + \frac{4}{\sin^2\theta}\right)\right]. \tag{12.362}$$

In the ultrarelativistic limit of high incident energies $M/E_{\rm CM} \to 0$, we have

$$\left.\frac{d\sigma}{d\Omega_{\rm CM}}\right|_{\rm ur} \approx \frac{\alpha^2}{E_{\rm CM}^2}\left(\frac{4}{\sin^4\theta} - \frac{2}{\sin^2\theta} + \frac{1}{4}\right) = \frac{\alpha^2}{4E_{\rm CM}^2}\left(\frac{1}{\sin^4\theta/2} + \frac{1}{\cos^4\theta/2} + 1\right). \tag{12.363}$$

For small energies where $E_{\rm CM}^2 \simeq M^2$, $v^2 = (p^2 - M^2)/E_{\rm CM}^2$, we obtain the nonrelativistic formula

$$\begin{aligned} \left.\frac{d\sigma}{d\Omega_{\rm CM}}\right|_{\rm nr} &= \left(\frac{\alpha}{M}\right)^2 \frac{1}{4v^4}\left(\frac{4}{\sin^4\theta} - \frac{3}{\sin^2\theta}\right) \\ &= \left(\frac{\alpha}{M}\right)^2 \frac{1}{16v^4}\left(\frac{1}{\sin^4\theta/2} + \frac{1}{\cos^4\theta/2} - \frac{1}{\sin^2\theta/2\cos^2\theta/2}\right), \end{aligned} \tag{12.364}$$

that was first derived by Mott in 1930.

Comparing (12.364) with the classical Rutherford formula for Coulomb scattering in Eq. (12.194), we see that the forward peak is the same for both if we set $Z = 1$ and replace M by the reduced mass $M/2$. The particle identity yields, in addition, the backward peak.

12.14 Electron-Positron Scattering

Let us now consider electron-positron scattering. The kinematics and lowest-order diagrams are depicted in Figs. 12.19 and 12.20. Polarization indices are omitted

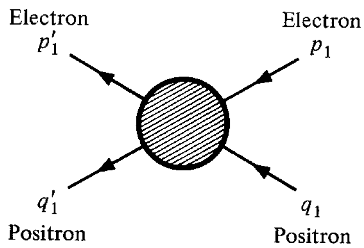


FIGURE 12.19 General form of diagrams contributing to electron-positron scattering.

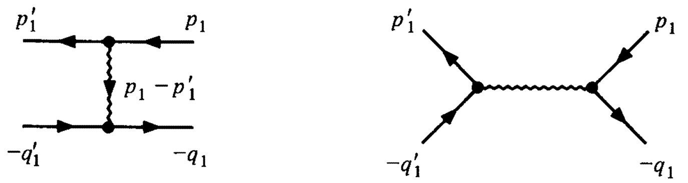


FIGURE 12.20 Lowest-order contributions to electron-positron scattering.

and in Fig. 12.20 four-momenta are oriented according to the charge flow. The scattering amplitude may then be obtained from (12.354) by substituting

$$
\begin{aligned}
p_2 &\to q_1', \qquad u(p_2) \to v(q_1'),\\
p_2' &\to -q_1', \quad u(p_2') \to v(q_1'),
\end{aligned}
$$

and by changing the sign of the amplitude. The center of mass cross section is then given by the formula

$$
\frac{d\sigma}{d\Omega} = \frac{M^4 e^4}{4E_{\rm CM}^2 (2\pi)^2} \overline{|t_{fi}|^2} \tag{12.365}
$$

with

$$
\begin{aligned}
\overline{|t_{fi}|^2} \;=\; \frac{1}{2M^4} \Bigg\{ & \frac{(p_1\, q_1')^2 + (p_1\, q_1)^2 - 2M^2 (p_1\, q_1 - p_1\, q_1')}{[(p_1' + p_1)^2]^2} \\
& + \frac{(p_1\, q_1')^2 + (p_1\, p_1')^2 + 2M^2 (p_1\, p_1' + p_1\, q_1')}{[(p_1 + q_1)^2]^2} \\
& + 2 \frac{(p_1\, q_1')^2 + 2M^2 p_1\, q_1'}{(p_1 - p_1')^2 (p_1 + q_1)^2} \Bigg\}.
\end{aligned} \tag{12.366}
$$

This can be expressed in terms of the Mandelstam variables s, t, u whose properties were discussed in Eqs. (9.318)–(9.331) as follows:

$$
\begin{aligned}
\overline{|t_{fi}|^2} \;=\; \frac{1}{2M^4} \Bigg\{ & \frac{1}{t^2} \left[\frac{u^2 + s^2}{4} + 2m^2 (t - m^2) \right] + \frac{1}{s^2} \left[\frac{u^2 + t^2}{4} + 2m^2 (s - m^2) \right] \\
& + \frac{1}{st} \left[\left(\frac{u}{2} - m^2 \right) \left(\frac{u}{2} - 3m^2 \right) \right] \Bigg\}.
\end{aligned} \tag{12.367}
$$

It is then straightforward to derive the cross section formula first obtained by Bhabha (1936):

$$
\begin{aligned}
\frac{d\sigma^{e^- e^+}}{d\Omega} = \frac{\alpha}{2E_{\rm CM}^2} \Bigg[& \frac{5}{4} - \frac{8E_{\rm CM}^4 - M^4}{E_{\rm CM}^2 (E_{\rm CM}^2 - M^2)(1 - \cos\theta)} + \frac{(2E_{\rm CM}^2 - M^2)^2}{2(E_{\rm CM}^2 - M^2)^2 (1 - \cos\theta)^2} \qquad (12.368) \\
& + \frac{2E_{\rm CM}^4 (-1 + 2\cos\theta + \cos^2\theta) + 4E_{\rm CM}^2 M^2 (1 - \cos\theta)(2 + \cos\theta) + 2M^4 \cos^2\theta}{16 E_{\rm CM}^4} \Bigg].
\end{aligned}
$$

In the ultrarelativistic limit, this becomes

$$\frac{d\sigma^{e^-e^+}}{d\Omega} \underset{\text{ur}}{=} \frac{\alpha^2}{8E_{\text{CM}}^2}\left[\frac{1+\cos^4\theta/2}{\sin^4\theta/2}+\frac{1}{2}(1+\cos^2\theta)-2\frac{\cos^4\theta/2}{\sin^2\theta/2}\right]. \quad (12.369)$$

The nonrelativistic limit is simply

$$\frac{d\sigma^{e^-e^+}}{d\Omega} \underset{\text{nr}}{=} \left(\frac{\alpha}{M}\right)^2 \frac{1}{16v^4\sin^4\theta/2}. \quad (12.370)$$

This agrees again with the classical Rutherford cross section (12.194). The annihilation diagram does not contribute in this limit,

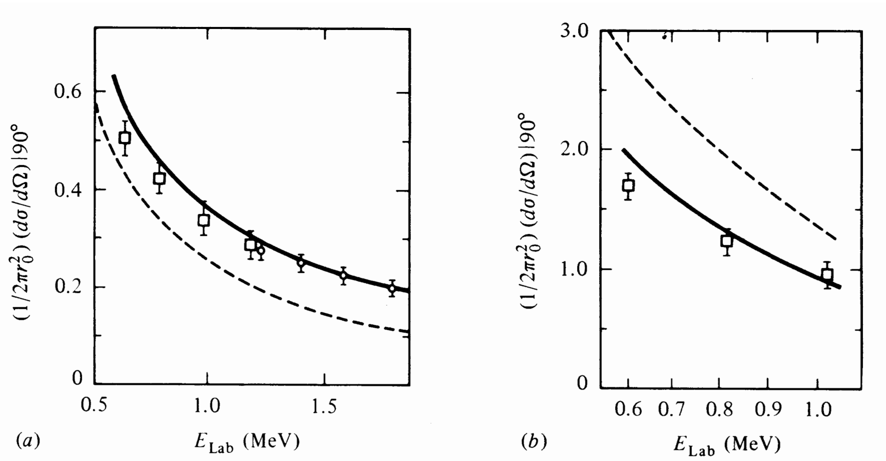


FIGURE 12.21 Experimental data for electron-electron and electron-positron scattering at $\theta = 90^0$ as a function of the incident electron energy in the laboratory frame. (a) Electron-electron scattering. The solid line represents the Møller formula, the broken one the Møller formula when the spin terms are omitted. (b) Electron-positron scattering. The solid line follows the Bhabha formula, the broken one the prediction when annihilation terms are deleted. Data are from A. Ashkin, L.A. Page, and W.M. Woodward, Phys. Rev. *94*, 357, (1974).

The results of Eqs. (12.362) and (12.368) may be compared with experimental data. At low energies we show in Fig. 12.21 some experimental data for electron-electron scattering at 90 degrees [7]. Møller's formula (12.362) is a good agreement with the data. The agreement confirms the fact that the spin of the electron is really 1/2. If it was zero, the agreement would have been bad (see the dashed curve in Fig. 12.21).

Electron-positron scattering data are fitted well by Bhabha's cross section, and the annihilation term is essential for the agreement. The energy of the incident particle in the laboratory frame plotted on the abscissa is chosen in the intermediate range where neither the nonrelativistic nor the ultrarelativistic approximation is valid. The numerical values show a significant departure from the ratio 2:1 between e^-e^- and e^-e^+ cross sections, expected on the basis of a naive argument of indistinguishability of the two electrons.

12.15 Anomalous Magnetic Moment of Electron and Muon

The most directly observable effect of loop diagrams is a change in the magnetic moment of the electron. Recall the precession equation (6.74).

As a consequence of loop diagrams in quantum electrodynamics, the gyromagnetic ratio g in the relation (12.125) receives a correction and becomes $g = 2(1+a)$. The number

$$a = (g-2)/2 > 0 \tag{12.371}$$

is called the *anomalous magnetic moment* of the electron. It has been measured experimentally with great accuracy [16]:

$$a = 1\,159\,652\,188.4(4.3) \times 10^{-12}. \tag{12.372}$$

The numbers in parentheses indicate the error estimate in the last two digits. For the positron, the result is

$$\bar{a} = 1\,159\,652\,187.9(4.3) \times 10^{-12}. \tag{12.373}$$

Quantum electrodynamics has been able to explain these numbers up to the last digits – a triumph of quantum field theory.

To lowest order in α, the anomalous magnetic moment can easily be calculated. Interestingly enough, it is found to be a *finite* quantity; no divergent integrals occur in its calculation. The Feynman diagram responsible for it is the vertex correction shown in Fig. 12.23. This diagram changes the electromagnetic current of an electron from

$$\langle \mathbf{p}', s_3'|j^\mu|\mathbf{p}, s_3\rangle = e\bar{u}(\mathbf{p}')\gamma^\mu u(\mathbf{p}) \tag{12.374}$$

to

$$\langle \mathbf{p}', s_3'|j^\mu|\mathbf{p}, s_3\rangle = e\bar{u}(\mathbf{p}')\left[\gamma^\mu + \Lambda^\mu(p',p)\right]u(\mathbf{p}), \tag{12.375}$$

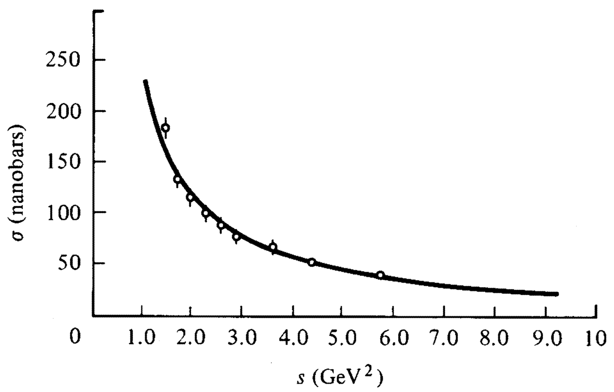


FIGURE 12.22 Cross section for Bhabha scattering at high energy, for scattering angle $45^0 < \theta < 135^0$ as a function of total energy-momentum square s. The solid line is calculated from quantum electrodynamics with first-order radiative corrections [8].

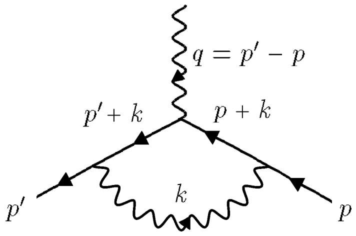


FIGURE 12.23 Vertex correction responsible for the anomalous magnetic moment.

where the vertex correction $\Lambda^\mu(p',p)$ is given by the Feynman integral

$$\Lambda^\mu(p',p) = -i4\pi\alpha \int \frac{d^4k}{(2\pi)^4}\gamma^\nu \frac{1}{\not p' + \not k - M}\gamma^\mu \frac{1}{\not p + \not k - M}\gamma_\nu \frac{1}{k^2}. \qquad (12.376)$$

This integral is logarithmically divergent at large momenta k_μ. It can be regularized by cutting the integration off at some large but finite momentum Λ which is later removed by a renormalization of the charge in the Lagrangian. There is also an infrared divergence which is kept finite by cutting off the k-integration at a small mass value $k^2 = \mu^2$, much smaller than the electron mass, i.e., $\mu^2 \ll M^2$.

To do the integral, we rewrite the integrand as

$$\gamma^\nu \frac{\not p' + \not k + M}{(p'+k) - M^2}\gamma^\mu \frac{\not p + \not k + M}{(p+k)^2 - M^2}\gamma_\nu \frac{1}{k^2 - \mu^2}, \qquad (12.377)$$

where we have introduced a small photon mass μ to avoid infrared divergencies at small momenta. We now collect the product of denominators into a single denominator with the help of Feynman's formula (11.158) for three denominators

$$\frac{1}{ABC} = 2\int_0^1 dx \int_0^x dy [Ay + B(x-y) + C(1-x)]^{-3}, \qquad (12.378)$$

so that

$$\frac{1}{(p'+k) - M^2}\frac{1}{(p+k)^2 - M^2}\frac{1}{k^2} \qquad (12.379)$$
$$= 2\int_0^1 dx \int_0^x dy \left\{ \left[(p'+k)^2 - M^2\right] y + \left[(p+k)^2 - M^2\right](x-y) + (k^2-\mu^2)(1-x)\right\}^{-3}.$$

This can be simplified to

$$2\int_0^1 dx \int_0^x dy \left\{ [k - p'y - p(x-y)]^2 - \mu^2(1-x) - M^2x^2 + q^2y(x-y)\right\}^{-3}. \qquad (12.380)$$

After performing a shift of the integration variable

$$k \to k + p'y + p(x-y),$$

the vertex correction (12.376) takes the form

$$\Lambda^\mu(p',p) = -i4\pi\alpha\ 2\int_0^1 dx\int_0^x dy\int\frac{d^4k}{(2\pi)^4} \tag{12.381}$$
$$\times\gamma^\nu\frac{[\not p'(1-y)-\not k-\not p(x-y)+M]\gamma^\mu[\not p(1-x-y)-\not k-\not p' y+M]}{\left[k^2-\mu^2(1-x)-M^2x^2+q^2y(x-y)\right]^3}\ \gamma_\nu.$$

Instead of calculating this general expression, we shall restrict ourselves to matrix elements of the current between electron states and evaluate $\bar u(\mathbf{p}',s_3')\Lambda^\mu(p',p)u(\mathbf{p},s_3)$. Then we can use the mass shell conditions $p^2 = p'^2 = M^2$ and the Dirac equations $\not p\,u(\mathbf{p},s_3) = Mu(\mathbf{p},s_3)$ and $\bar u(\mathbf{p}',s_3')\not p = M\bar u(\mathbf{p}',s_3')$. We now employ appropriately the anticommutation rules of the gamma matrices using the formulas (12.220)–(12.222), perform a Wick rotation $k^0 \to ik^4$, and integrate over the Euclidean four-momenta $d^4k_E = 2\pi^2 dk\,k^3$, setting $p^2=-p_E^2,\ p'^2=-p_E'^2$. Cutting off the k-integral at $k_E=\Lambda$, we arrive at the triple integral

$$\bar u(\mathbf{p}',s_3')\Lambda^\mu(p',p)u(\mathbf{p},s_3)=\alpha\int_0^1 dx\int_0^x dy\int_0^\Lambda dk_E k_E^3\frac{k_E^3}{\left[k_E^2+\mu^2(1-x)+M^2x^2+q_E^2y(x-y)\right]^3}$$
$$\times\bar u(\mathbf{p}',s_3')\left\{\gamma^\mu\left[k_E^2-2M^2(x^2-4x+2)+2q^2\left(y(x-y)+1-x\right)\right]\right. \tag{12.382}$$
$$\left.-4Mp'^\mu(y-x+xy)-4Mp^\mu(x^2-xy-y)\right\}u(\mathbf{p},s_3).$$

The denominator is symmetric under the exchange $y\to x-y$. Under this operation, the coefficients of p^μ and p'^μ are interchanged, showing that the vertex function is symmetric in p^μ and p'^μ. We can therefore replace each of these coefficients by the common average

$$\left\{\begin{array}{c} y-x+xy \\ x^2-xy-y\end{array}\right\}\to\frac{1}{2}[(y-x+xy)+(x^2-xy-y)] = -\frac{1}{2x(1-x)},$$

and rewrite (12.382) as

$$\bar u(\mathbf{p}',s_3')\Lambda^\mu(p',p)u(\mathbf{p},s_3)=\alpha\int_0^1 dx\int_0^x dy\int_\mu^\Lambda dk_E k_E^3\frac{k_E^3}{\left[k_E^2+\mu^2(1-x)+M^2x^2-q^2y(x-y)\right]^3}$$
$$\times\bar u(\mathbf{p}',s_3')\left\{\gamma^\mu\left[k_E^2-2M^2(x^2-4x+2)+2q^2\left(y(x-y)+1-x\right)\right]\right. \tag{12.383}$$
$$\left.-2M(p'^\mu+p^\mu)x(1-x)\right\}u(\mathbf{p},s_3).$$

This expression may be decomposed as follows:

$$\bar u(\mathbf{p}',s_3')\Lambda^\mu(p',p)u(\mathbf{p},s_3) = \bar u(\mathbf{p}',s_3')\left[\gamma^\mu H(q^2)-\frac{1}{2M}(p'^\mu+p^\mu)G(q^2)\right]u(\mathbf{p},s_3), \tag{12.384}$$

with the invariant functions

$$H(q^2)=\frac{\alpha}{\pi}\int_0^1 dx\int_0^x dy\int_\mu^\Lambda dk_E\,k_E^3\frac{k_E^2-2M^2(x^2-4x+2)+2q^2[y(x-y)+1-x]}{\left[k_E^2+\mu^2(1-x)+M^2x^2-q^2y(x-y)\right]^3} \tag{12.385}$$

and

$$G(q^2) = \frac{\alpha}{\pi} M^2 \int_0^1 dx \int_0^x dy \int_\mu^\Lambda dk_E\, k_E^3 \frac{4M^2 x(1-x)}{[k_E^2 + \mu^2(1-x) + M^2x^2 - q^2 y(x-y)]^3}. \tag{12.386}$$

The momentum integral in the second invariant function $G(q^2)$ is convergent at small and large momenta, such that we can set the photon mass μ to zero and ultraviolet to infinity. Using the integral formula

$$\int_0^\infty dk_E^2\, k_E^2 \frac{1}{(k_E^2 + M_1^2)^3} = \frac{1}{2M_1^2}, \tag{12.387}$$

we obtain

$$G(q^2) = \frac{\alpha}{\pi} M^2 \int_0^1 dx \int_0^x dy \frac{x(1-x)}{M^2x^2 - q^2 y(x-y)}. \tag{12.388}$$

The integral over y yields

$$G(q^2) = \frac{\alpha}{\pi} M^2 \int_0^1 dx\, 4(1-x) \frac{1}{\sqrt{q^2(4M^2 - q^2)}} \arctan \sqrt{\frac{q^2}{4M^2 - q^2}}, \tag{12.389}$$

leading to

$$G(q^2) = \frac{\alpha}{\pi} \frac{2M^2}{\sqrt{q^2(4M^2 - q^2)}} \arctan \sqrt{\frac{q^2}{4M^2 - q^2}}, \tag{12.390}$$

which can be rewritten as

$$G(q^2) = \frac{\alpha}{2\pi} \frac{2\theta}{\sin 2\theta}, \quad \text{with} \quad \sin^2\theta \equiv \frac{q^2}{4M^2}. \tag{12.391}$$

For small q^2, it has the expansion

$$G(q^2) = \frac{\alpha}{2\pi} \left(1 + \frac{q^2}{6M^2} + \dots\right). \tag{12.392}$$

In the first invariant function $H(q^2)$, both the cutoff and the photon mass are necessary to obtain a finite result. The divergence can be isolated by a subtraction of the integrand, separating

$$H(q^2) = H(0) + \Delta H(q^2), \tag{12.393}$$

with a divergent integral

$$H(0) = \frac{\alpha}{\pi} \int_0^1 dx \int_0^x dy \int_0^\infty dk_E\, k_E^3 \frac{k_E^2 - 2M^2(x^2 - 4x + 2)}{[k_E^2 + \mu^2(1-x) + M^2x^2]^3} \tag{12.394}$$

and a convergent one at large momenta

$$\Delta H(q^2) = \frac{\alpha}{\pi}\int_0^1 dx \int_0^x dy \int_0^\infty dk_E\, k_E^3 \left\{ \frac{k_E^2 - 2M^2(x^2-4x+2) + 2q^2[y(x-y)+1-x]}{[k_E^2+\mu^2(1-x)+M^2x^2 - q^2y(x-y)]^3} - \frac{k_E^2 - 2M^2(x^2-4x+2)}{[k_E^2+\mu^2(1-x)+M^2x^2]^3} \right\}. \tag{12.395}$$

The divergent momentum integral in (12.394) must be performed with the help of some regularization scheme, for which we choose the Pauli-Villars regularization, replacing the photon propagator as follows:

$$\frac{1}{k^2-\mu^2} \to \frac{1}{k^2-\mu^2} - \frac{1}{k^2-\Lambda^2}, \tag{12.396}$$

where Λ is a large cutoff mass. Then we may use the formula

$$\int_0^\infty dk_E^2\, k_E^2 \left\{ \left[\frac{k_E^2 + M_2^2}{(k_E^2+\mu^2(1-x)+M_1^2)^3} \right] - [\mu^2 \to \Lambda^2] \right\} = -\log\frac{\mu^2(1-x)+M_1^2}{\Lambda^2(1-x)} + \frac{1}{2}\frac{M_2^2}{\mu^2(1-x)+M_1^2}. \tag{12.397}$$

With this, the convergent momentum integral (12.395) yields

$$\Delta H(q^2) = \frac{\alpha}{2\pi}\int_0^1 dx \int_0^x dy \left\{ -\frac{M^2(x^2-4x+2) - q^2[y(x-y)+1-x]}{\mu^2(1-x)+M^2x^2 - q^2y(x-y)} + \frac{M^2(x^2-4x+2)}{\mu^2(1-x)+M^2x^2} - \log\left[\frac{\mu^2(1-x)+M^2x^2 - q^2y(x-y)}{\mu^2(1-x)+M^2x^2}\right] \right\}. \tag{12.398}$$

If the photon mass is set equal to zero, the integral is divergent at $x = 0$. The physical meaning of this infrared divergence will be explained later.

Let us first understand $\Delta H(q^2)$ for small q^2 by expanding $\Delta H(q^2) = \Delta H'(0)q^2 + \mathcal{O}(q^4)$. Then $\Delta H'(0)$ is given by the integral

$$\Delta H'(0) = \frac{\alpha}{2\pi}\int_0^1 dx \int_0^x dy \left\{ \frac{y(x-y)+1-x}{\mu^2(1-x)+M^2x^2} - \frac{M^2(x^2-4x+2)y(x-y)}{[\mu^2(1-x)+M^2x^2]^2} + \frac{y(x-y)}{\mu^2(1-x)+M^2x^2} \right\} = \frac{\alpha}{2\pi}\left(\frac{1}{3}\log\frac{M^2}{\mu^2} - \frac{1}{12}\right), \tag{12.399}$$

where we have discarded all terms which go to zero for $\mu \to 0$.

The full result is

$$H(q^2) = H(0) + \frac{\alpha}{2\pi}\left[\left(\log\frac{M^2}{\mu^2} - 2\right)\left(1 - \frac{2\theta}{\tan 2\theta}\right) + \theta\tan\theta + \frac{4}{\tan 2\theta}\int_0^\theta dx\, x\tan x + \frac{2\theta}{\sin 2\theta}\right], \tag{12.400}$$

which has precisely the first Taylor coefficient (12.399).

12.15.1 Form Factors

We now introduce the customary Lorentz-invariant decompositions of the matrix elements of the current (12.375) of a spin-1/2 particle as follows:

$$\langle \mathbf{p}'|j^\mu|\mathbf{p}\rangle = e\bar{u}(\mathbf{p}')\left[\gamma^\mu F(q^2) + \frac{i}{2M}\sigma^{\mu\nu}q_\nu G(q^2)\right]u(\mathbf{p}'). \tag{12.401}$$

The invariant functions $F(q^2)$ and $G(q^2)$ are the standard *form factors* of the electron. The relation between this and (12.384) follows directly from Gordon's decomposition formula (12.141), showing that $G(q^2)$ in (12.386) coincides with $G(q^2)$ in (12.401), whereas

$$F(q^2) = 1 + H(q^2) - G(q^2). \tag{12.402}$$

The *charge form factor* $F(q^2)$ at $q^2 = 0$ specifies the charge of the electron. Inserting (12.400), we obtain

$$\begin{aligned} F(q^2) &= F(0) + \frac{\alpha}{2\pi}\left[\left(\log\frac{M^2}{\mu^2} - 2\right)\left(1 - \frac{2\theta}{\tan 2\theta}\right)\right. \\ &\qquad\qquad \left. + \theta\tan\theta + \frac{4}{\tan 2\theta}\int_0^\theta dx\, x\tan x\right], \end{aligned} \tag{12.403}$$

where

$$F(0) \equiv 1 + \frac{\alpha}{2\pi}\left(\log\frac{\Lambda}{M} + \frac{9}{4} - \log\frac{M^2}{\mu^2}\right). \tag{12.404}$$

The value $F(0)$ contains both the ultraviolet and the infrared cutoff. The subtracted function $\Delta F(q^2) \equiv F(q^2) - F(0)$ has only an infrared divergence. In writing down the expressions (12.403) and (12.404), we have ignored all contributions which vanish for $\mu \to 0$ and $\Lambda \to \infty$.

Due to the loop integral, the charge is changed to the new value

$$e_1 = eF(0) = e\left[1 + \frac{\alpha}{2\pi}\left(\log\frac{\Lambda}{M} + \frac{9}{4} - 2\log\frac{M}{\mu}\right)\right]. \tag{12.405}$$

The factor is commonly denoted as the renormalization constant Z_1^{-1}. To order α, it is

$$Z_1 \equiv F^{-1}(0) = 1 - \frac{\alpha}{2\pi}\left(\log\frac{\Lambda}{M} + \frac{9}{4} - 2\log\frac{M}{\mu}\right). \tag{12.406}$$

According to the theory of renormalization, this has to be equated with the experimentally observed charge. After this we can replace, in Eq. (12.403), the number $F(0)$ by 1, and the bare fine-structure constant α by the physical one (keeping the notation, for simplicity). The latter substitution is also done in $G(q)$.

12.15.2 Charge Radius

For small momentum transfers q^2, the renormalized charge form factor has the Taylor series expansion

$$F_R(q^2) \;=\; 1+q^2\frac{\alpha}{3\pi M^2}\left(\log\frac{M}{\mu}-\frac{3}{8}\right)+\mathcal{O}(q^4). \tag{12.407}$$

The form factor of the anomalous magnetic moment is, for small q,

$$G_R(q^2) \;=\; q^2\frac{\alpha}{3\pi M^2}\frac{1}{4}+\mathcal{O}(q^4). \tag{12.408}$$

Due to the emission and absorption of virtual photons, the electron is shaken over a finite range. It is customary to define here a charge radius R_e of the electron by the first term in the expansion of the charge form factor[2]

$$F(q^2)=1-q^2\frac{R_e^2}{6}+\ldots\;. \tag{12.409}$$

The factor 1/6 is due to the fact that for a uniformly charged shell of radius R_e, the Fourier transform of the charge density has the expansion

$$\begin{aligned} F(q^2) \;&=\; \int d^3x\, e^{i\mathbf{qx}}\rho(\mathbf{x})=\int d^3x\left[1+i\mathbf{qx}-\frac{1}{2}\left(\mathbf{qx}\right)^2+\ldots\right]\rho(\mathbf{x}) \\ &=\; 1-\frac{1}{6}\mathbf{q}^2\int d^3x\, r^2\rho(\mathbf{x})=1-\mathbf{q}^2\frac{R_e^2}{6}+\ldots\;. \end{aligned} \tag{12.410}$$

Setting $q^0=0$ in Eq. (12.407), we find that due to the vertex correction, the charge radius $R_e^{\rm vc}$ of the electron is given by

$$R_e^{2\,\rm vc}=\frac{\alpha}{\pi}\frac{2}{M^2}\left(\log\frac{M}{\mu}-\frac{3}{8}\right). \tag{12.411}$$

If the zeroth component of the current (12.401) couples to a static electric potential $A_0(\mathbf{x})$, the q^2-term in (12.407) yields a correction factor

$$\left[1+\frac{R_e^2}{6}\boldsymbol{\nabla}^2\right]A_0(\mathbf{x}). \tag{12.412}$$

The fluctuations of the electron position leads to an extra term proportional to the harmonic average of the potential. Since the potential obeys Gauss' law, this is nonzero only where there are charges. In an atom of nuclear charge Ze at the origin, the potential is

$$eA_0(\mathbf{x})=V_{\rm C}(\mathbf{x})=-\frac{Z\alpha}{r} \tag{12.413}$$

[2]Note the difference of this quantity with respect to the classical electron radius (12.234).

and the harmonic average produces an additional δ-function at the origin:

$$e\boldsymbol{\nabla}^2 A_0(\mathbf{x}) = 4\pi Z\alpha\, \delta^{(3)}(\mathbf{x}). \tag{12.414}$$

Thus the correction factor changes the Coulomb potential into an effective potential

$$V_{\rm C}^{\rm eff} = -\frac{Z\alpha}{r} + \frac{R_{\rm e}^{2\,\rm vc}}{6} Z\alpha\, 4\pi\delta^{(3)}(\mathbf{x}). \tag{12.415}$$

The prediction of this additional term is the origin of one of the early triumphs of QED. The δ-function leads to an energy shift of s-wave orbits with respect to those of nonzero orbital angular momentum, called the *Lamb shift*. It removes the degeneracy between the $2S_{1/2}$ and the $2P_{1/2}$ predicted by Dirac's equation in an external Coulomb field. It will be discussed in detail below.

12.15.3 Anomalous Magnetic Moment

On the right-hand side of the current matrix element (12.401) we replace the γ^μ-term by a combination of vectors $p'^\mu + p^\mu$ and $\sigma^{\mu\nu}q_\nu$, as before in (12.141), and see that the magnetic moment of the free electron acquires a correction factor $1+G(0)$. Thus the gyromagnetic ratio g in Eq. (12.126) is changes from $g = 2$ to

$$g \equiv 2(1+a) = 2[1 + G(0) + \dots\ ] \tag{12.416}$$

[recall (12.371)]. The number $G(0)$ yields therefore directly the anomalous magnetic moment:

$$a = G(0). \tag{12.417}$$

In contrast to the charge, this quantity is finite. Its value is [17]

$$a = G(0) = \frac{\alpha}{2\pi} \sim 1\,161\,409\,74292 \times 10^{-12}. \tag{12.418}$$

This result was first calculated by Schwinger [18]. It is about 1.5% larger than the experimental values (12.372) and (12.373). The difference can be explained by higher-order electrodynamic and strong-interaction corrections [19]. By including all diagrams of sixth order in perturbation theory one finds, after a considerable effort (there are 72 Feynman diagrams to sixth order), the expansion

$$a = \frac{\alpha}{2\pi} + c_2\left(\frac{\alpha}{\pi}\right)^2 + c_3\left(\frac{\alpha}{\pi}\right)^3 \tag{12.419}$$

with the coefficients [20]

$$c_2 = \frac{197}{144} + \left(\frac{1}{2} - 3\log 2\right)\zeta(2) + \frac{3}{4}\zeta(4) = -0.328\,478\,965\,\dots\ , r \tag{12.420}$$

$$c_3 = 1.176\,11(42). \tag{12.421}$$

Up to c_2, the theory is lower than the experimental number by only 1 part in 10^5. Adding the c_3-term, the theoretical value becomes

$$a = 1\,159\,652\,140.4(27.1) \times 10^{-12}, \tag{12.422}$$

reducing the discrepancy to 1 part in 10^8. The error is mainly due to the uncertainty in the fine-structure constant [17].

The above calculation may be used to show that the anomalous magnetic moment of an antiparticle is the same as for a particle. The matrix element of the initial current between positron states is

$$\langle \mathbf{p}', s_3' | j^\mu(x) | \mathbf{p}, s_3 \rangle = e\langle 0 | b(\mathbf{p}', s_3') \bar{\psi}(x) \gamma^\mu \psi(x) b^\dagger(\mathbf{p}, s_3) | 0 \rangle = -e\bar{v}(\mathbf{p}', s_3)\gamma^\mu u(\mathbf{p}, s_3) e^{iqx}, \tag{12.423}$$

rather than (12.129). The minus sign is due to the odd number of exchanges of field operators necessary to evaluate the anticommutators. If both momenta are zero, only the zeroth component survives, showing that the charge of the positron is $-e$. For small momentum transfers, we calculate the second spatial component of (12.423) more explicitly as

$$\langle \mathbf{p}', s_3' | j^2(x) | \mathbf{p}, s_3 \rangle = -e\bar{v}(\mathbf{0}, s_3)\gamma^2 e^{i\zeta^1(i\gamma^0\gamma^1)/2} v(\mathbf{0}, s_3), \tag{12.424}$$

and compare this with (12.134). The linear term in ζ^1 contains the contribution of the charge form factor to the magnetic moment. We see that (12.424) has the opposite overall sign of (12.134) which is compensated by an opposite sign in the exponent of the Lorentz transformation. Thus we obtain the matrix element of j^2 between positrons:

$$-ie\zeta^1 \bar{v}(\mathbf{0}, s_3) S_3 v(\mathbf{0}, s_3) = ie\zeta^1 s_3, \tag{12.425}$$

with an opposite sign in comparison with the electron in (12.135). The sign change at the end is caused by the opposite spin orientation of the two-spinors $\xi(s_3)$ contained in the spinors $v(\mathbf{0}, s_3)$ [recall (4.684),(4.685) and (4.676)].

An opposite sign is also found for the contribution from the second form factor where the matrix element of the electron is

$$e\bar{u}(\mathbf{0}, s_3)\frac{i}{2M}\sigma^{21} q_1 G(0) u(\mathbf{0}, s_3) = -ie\bar{u}(\mathbf{0}, s_3) S_3 \zeta^1 G(0) u(\mathbf{0}, s_3) = -ie\zeta^1 s_3 G(0). \tag{12.426}$$

This produces the correction to the g-factor

$$g = 2[1 + G(0)]. \tag{12.427}$$

For a positron, the matrix element of the current is

$$-e\bar{v}(\mathbf{p}', s_3')[\gamma^2 F(q^2) + \frac{i}{2M}\sigma^{2\nu} q_\nu G(q^2)] v(\mathbf{p}, s_3), \tag{12.428}$$

and the second term becomes, for $\mathbf{p}' = 0$ and small $\mathbf{p}$ in the x-direction,

$$-e\bar{v}(\mathbf{0}, s_3)\frac{i}{2M}\sigma^{21} q_1 G(0) v(\mathbf{0}, s_3) = ie\bar{v}(\mathbf{0}, s_3) S_3 \zeta^1 G(0) v(\mathbf{0}, s_3) = ie\zeta^1 s_3 G(0), \tag{12.429}$$

which is exactly the opposite of (12.426). Thus a positron has the same correction (12.427) to the g-factor as an electron. Note that the sign change found in (12.425) is present also here, but it is compensated by a minus sign from an extra γ^0-matrix acting upon $v^\dagger$.

For a muon, the coefficients are [21]

$$c_2 = 0.765\,857\,376(27), \qquad c_3 = 24.050\,508\,98(44),$$
$$c_4 = 24.050\,508\,98(44), \qquad c_5 = +930(170), \tag{12.430}$$

leading to

$$a_\mu = 1\,165\,847\,057(29) \times 10^{-12}. \tag{12.431}$$

The strong interactions change this slightly via the composite Feynman diagrams

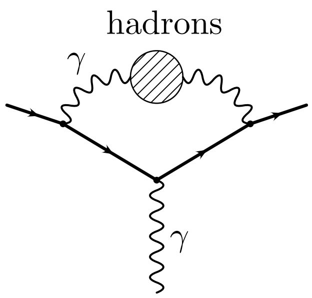


FIGURE 12.24 Leading hadronic vacuum polarization corrections to a_μ.

indicated in Fig. 12.24. Their contribution was estimated in [22] to amount to

$$a_\mu^{\rm str} = 6\,924(62) \times 10^{-11} \quad \text{to} \quad 6\,988(111) \times 10^{-11}. \tag{12.432}$$

The amplitude is calculated from the formula

$$a_\mu^{\rm str}(\text{vac. pol.}) = \frac{1}{4\pi^3}\int_{4m_\pi^2}^\infty ds\, K\,(s)\;\sigma^0(s)_{e^+e^-\to\text{hadrons}}, \tag{12.433}$$

where $\sigma^0(s)_{e^+e^-\to\text{hadrons}}$ is the cross section for the process indicated in the subscript. Some radiative corrections have been taken care of, such as initial state radiation by a subtraction from the measured cross sections. The function $K(s)$ is equal to

$$K(s) = x^2\left(1-\frac{x^2}{2}\right) + (1+x)^2\left(1+\frac{1}{x^2}\right)\left[\ln(1+x)-x+\frac{x^2}{2}\right] + \frac{1+x}{1-x}\,x^2\ln x, \tag{12.434}$$

with

$$x \;=\; \frac{1-\sqrt{1-4m_\mu^2/s}}{1+\sqrt{1-4m_\mu^2/s}}. \tag{12.435}$$

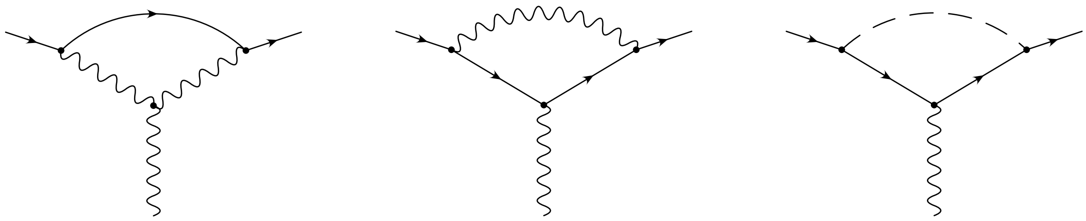
$\sqrt{s}$ (GeV)	$a_\mu^{\rm str}$(vac. pol.) $\times 10^{11}$
$2m_\pi - 1.8$	6343 ± 60
$1.8 - 3.7$	338.7 ± 4.6
$3.7 - 5 + \psi(1S, 2S)$	143.1 ± 5.4
$5 - 9.3$	68.7 ± 1.1
$9.3 - 12$	12.1 ± 0.5
$12 - \infty$	18.0 ± 0.1
Total	6924 ± 62

TABLE 12.1 Different contributions to $a_\mu^{\rm str}$(vac. pol.) in the integral 12.433.

It comes from the remaining part of the diagram. The contributions from the different parts of the cross sections to $a_\mu^{\rm str}$(vac. pol.) in the integral (12.433) are separately listed in Table 12.15.3. The weak interactions in the standard model add to this [23]

$$a_\mu^{\rm EW}(1\,{\rm loop}) = \frac{5}{3}\frac{{\rm G}_\mu {\rm m}_\mu^2}{8\sqrt{2}\pi^2}\left[1 + \frac{1}{5}(1 - 4\sin^2\theta_W)^2 + \mathcal{O}\left(\frac{m_\mu^2}{M^2}\right)\right] \approx 195 \times 10^{-11}, \quad (12.436)$$

where $G_\mu = 1.16637(1) \times 10^{-5}\,{\rm GeV}^{-2}$, $\sin^2\theta_W \equiv 1 - M_W^2/M_Z^2 \simeq 0.223$. and $M = M_W$ or $M_{\rm Higgs}$. See Fig. 12.25 for the Feynman diagrams.



(a) (b) (c)

FIGURE 12.25 One-loop electroweak radiative corrections to a_μ. The wiggly lines are gluons.

Two-loop corrections change this slightly by

$$a_\mu^{\rm weak}(2\ {\rm loop}) = -43(4) \times 10^{-11}, \quad (12.437)$$

if we assume a Higgs particle mass of $m_H \simeq 150$ GeV (with little sensitivity to the exact value). Altogether, we obtain

$$a_\mu^{\rm theory} = 1\,165\,915\,97(67) \times 10^{-11}, \quad (12.438)$$

in agreement with the experimental numbers [24]

$$a_\mu^{\exp} = 11\,659\,204(7) \times 10^{-9},\ 11\,659\,202(22) \times 10^{-9},\ 11\,659\,204(7) \times 10^{-9}. \quad (12.439)$$

See Fig. 12.26 for a comparison of theory and experiment.

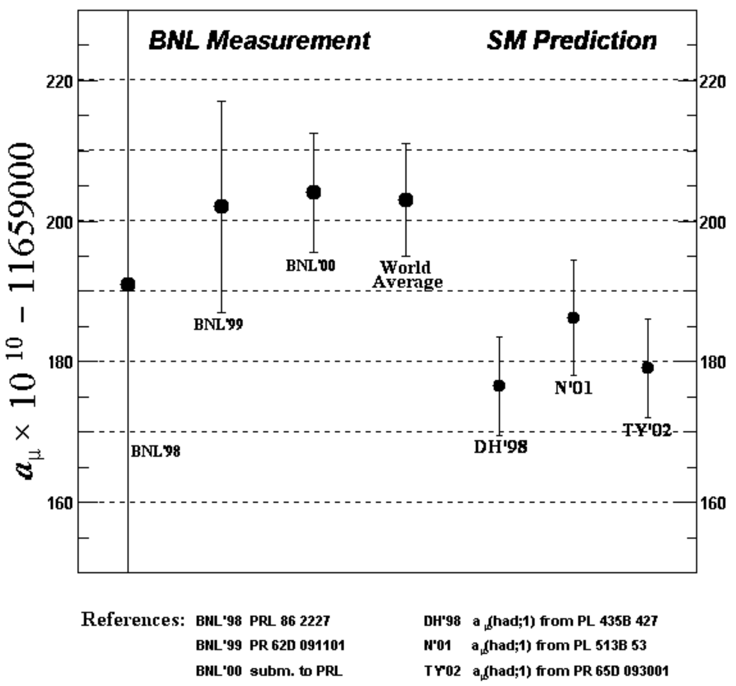


FIGURE 12.26 Measured values of a_μ and prediction of the Standard Model (SM). The small error bars of the theoretical value come from the left-hand estimate for the hadron contribution in Eq. (12.432). For sources see Ref. [24].

12.16 Vacuum Polarization

Let us now turn to the vacuum polarization. The lowest-order Feynman diagram is shown in Fig. 12.27.

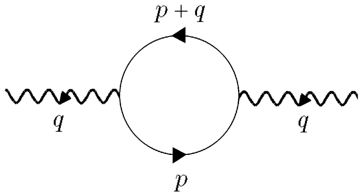


FIGURE 12.27 Lowest-order Feynman diagram for the vacuum polarization.

In general, the propagator of the photon has the form

$$G^{\mu\nu}(q) = i\frac{P^{\mu\nu}(q)}{q^2}, \tag{12.440}$$

where the polarization tensor $P^{\mu\nu}(q)$ depends on the gauge [see (12.111)]. From the lowest-order Feynman diagram in Fig. 12.27, the propagator receives a lowest-order correction

$$G_0^{\mu\nu}(q)\,[-i\Pi_{\lambda\kappa}(q)]\,G_0^{\kappa\nu}(q), \tag{12.441}$$

where $-\Pi_{\mu\nu}(k)$ is given by the Feynman integral.

$$-i\Pi_{\mu\nu}(q) = e^2\int\frac{d^4p}{(2\pi)^4}\frac{\mathrm{tr}[\gamma_\mu(\not p+\not q+M)\gamma_\nu(\not p+M)]}{[(p+q)^2-M^2](p^2-M^2)}. \tag{12.442}$$

Using Feynman's formula (11.156)

$$\frac{1}{AB} = \int_0^1 dz\frac{1}{[Az+B(1-z)]^2}, \tag{12.443}$$

we rewrite (12.442) as

$$-i\Pi_{\mu\nu}(q) = e^2\int_0^1 dz\int\frac{d^4p}{(2\pi)^4}\frac{\mathrm{tr}[\gamma_\mu(\not p+\not q+M)\gamma_\nu(\not p+M)]}{[(p+qz)^2+q^2(z-z^2)-M^2]^2}. \tag{12.444}$$

This expression is symmetric in μ and ν. A shift in the p-integration by an amount qz brings it to the form

$$-i\Pi_{\mu\nu}(q) = e^2\int_0^1 dz\int\frac{d^4p}{(2\pi)^4}\frac{\mathrm{tr}\{\gamma_\mu[\not p+\not q(1-z)+M]\gamma_\nu(\not p-\not q z+M)\}}{[p^2+q^2(z-z^2)-M^2]^2}. \tag{12.445}$$

After evaluating the trace, dropping odd terms in p, and using the symmetry in μ and ν, we obtain

$$-i\Pi_{\mu\nu}(q) = 4e^2\int_0^1 dz\int\frac{d^4p}{(2\pi)^4}\frac{-(-g_{\mu\nu}q^2+2q_\nu q_\nu)(z-z^2)-g_{\mu\nu}(p^2-M^2)+2p_\mu p_\nu}{[p^2+q^2(z-z^2)-M^2]^2}. \tag{12.446}$$

Because of the rotational symmetry of the integrand we can use the first replacement rule in Eq. (11.136) to replace $-g_{\mu\nu}(p^2-M^2)+2p_\mu p_\nu$ by $-g_{\mu\nu}(p^2/2-M^2)$.

The momentum integral is quadratically divergent, since there are two more powers of integration variables in the numerator than in the denominator. The situation is improved by imposing the requirement of gauge invariance, $q^\mu\Pi_{\mu\nu}(q) = 0$. This makes the final results independent of the gauge choice in the photon propagator. Thus we postulate, for the moment, the vanishing of the divergent integral

$$-iq^\mu\Pi_{\mu\nu}(q) = e^2 q_\nu\, 4\int_0^1 dz\int\frac{d^4p}{(2\pi)^4}\frac{-q^2(z-z^2)-(p^2/2-M^2)}{[p^2+q^2(z-z^2)-M^2]^2}. \tag{12.447}$$

We shall verify in the next Subsection that this is guaranteed if we calculate the Feynman diagrams by analytic regularization in D dimensions rather than $D = 4$.

Thus we are left with the logarithmically divergent integral

$$-i\Pi_{\mu\nu}(q) = -(-g_{\mu\nu}q^2 + q_\mu q_\nu)\, 8e^2 \int_0^1 dz(z - z^2) \int \frac{d^4p}{(2\pi)^4} \frac{1}{[p^2 + q^2(z - z^2) - M^2]^2}. \tag{12.448}$$

Let us define the invariant function that accompanies the tensor $(-g_{\mu\nu}q^2 + q_\mu q_\nu) = q^2 P_{\mu\nu}(q)$ as $-i\Pi(q^2)$, i.e., we write

$$-i\Pi_{\mu\nu}(q) \equiv -iP_{\mu\nu}(q)q^2\Pi(q^2). \tag{12.449}$$

If we expand

$$\Pi(q^2) = \Pi(0) + \Pi'(0)q^2 + \dots\ , \tag{12.450}$$

we see that only $\Pi(0)$ is logarithmically divergent:

$$-i\Pi(0) = -8e^2 \int_0^1 dz(z - z^2) \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 - M^2)^2}, \tag{12.451}$$

whereas all remaining terms in the expansion (12.450) are finite, for example the first term:

$$-i\Pi'(0) = 16e^2 \int_0^1 dz(z - z^2)^2 \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 - M^2)^3}. \tag{12.452}$$

Since the mass M carries a small negative imaginary part, we now perform a Wick rotation of the integration contour, setting $p^0 = ip^4$ and letting p^4 run from $-\infty$ to ∞. Thus we substitute $p^2 = -p_E^2$ and $d^4p/(2\pi)^4 = id^4p_E/(2\pi)^4$, and calculate from (12.452):

$$\Pi'(0) = \frac{e^2}{(2\pi)^4} \frac{4}{15} \frac{\pi^2}{M^2} = \frac{\alpha}{6\pi M^2} \frac{1}{5}. \tag{12.453}$$

In the divergent quantity $\Pi(0)$, we perform a Wick rotation so that the integral $d^4p/(2\pi)^4$ turns into an integral over all Euclidean four-momenta $d^4p_E/(2\pi)^4 = S_4 dp_E p_E^3 = dp_E^2\, p_E^2/16\pi^2$. If we introduce an ultraviolet momentum cutoff at Λ^2, we obtain

$$\begin{aligned}\Pi(0) &= 8e^2 \frac{1}{6} \int \frac{d^4p_E}{(2\pi)^4} \frac{1}{(p_E^2 + M^2)^2} = 8e^2 \frac{1}{6} \frac{1}{16\pi^2} \int_0^\infty dp_E^2 \frac{p_E^2}{(p_E^2 + M^2)^2} \\ &= \frac{\alpha}{\pi} \frac{1}{3} \left(\log \frac{\Lambda^2}{M^2} - 1 \right).\end{aligned} \tag{12.454}$$

The complete invariant function of the polarization tensor is therefore

$$\Pi(q^2) = \frac{\alpha}{\pi} \left[\frac{1}{3} \left(\log \frac{\Lambda^2}{M^2} - 1 \right) + \left(\frac{2}{3} + \frac{1}{3\sin^2\theta} \right) \left(1 - \frac{\theta}{\tan\theta} \right) - \frac{1}{9} \right]. \tag{12.455}$$

For small q^2, we expand

$$\Pi(q^2) = \left(\frac{2}{3} + \frac{1}{3\sin^2\theta}\right)\left(1 - \frac{\theta}{\tan\theta}\right) = \frac{1}{9} + \frac{4}{15}\theta^2 + \dots\ , \qquad (12.456)$$

and find the bracket to behave like

$$\Pi(0) + \Pi'(0)q^2 + \dots = \frac{\alpha}{\pi}\left[\frac{1}{3}\left(\log\frac{\Lambda^2}{M^2} - 1\right) + \frac{1}{15}\frac{q^2}{M^2} + \mathcal{O}(q^4)\right]. \qquad (12.457)$$

The last term agrees with the result (12.453) of a direct calculation of $\Pi'(0)$. It is gratifying to find out that the condition (12.447) is fulfilled by the dimensional regularization in $d = 4 - \epsilon$ dimensions.

Note that the expression (12.455) is real only for $q^2 < M^2$. When $q^2 > (2M)^2$ the external field can produce electron-positron pairs and $\Pi(q^2)$ acquires an imaginary part. The imaginary part causes a decrease of the probability amplitude for the occurrence of a pure scattering process below the threshold of pair production.

A string of vacuum polarization diagrams produces the geometric series

$$G^{\mu\nu} = G_0^{\mu\nu} + G_0^{\mu\lambda}\left[-i\Pi_{\lambda\kappa}\right]G_0^{\kappa\nu} + \dots + G_0^{\mu\lambda}[-i\Pi_{\lambda\kappa}]G_0^{\kappa\sigma}[-i\Pi_{\sigma\tau}]G_0^{\tau\nu} + \dots\ , \qquad (12.458)$$

which can be summed up to

$$G^{\mu\nu}(q) = \{[G_{\mu\nu}^{-1}(q) + i\Pi_{\mu\nu}(q)]^{-1}\}^{-1}. \qquad (12.459)$$

Inserting the tensor decomposition (12.449), this can be written as

$$G^{\mu\nu}(q) = i\frac{P^{\mu\nu}(q)}{q^2[1 - \Pi(q^2)]} \approx i\frac{P^{\mu\nu}(q)}{q^2[1 + \Pi(0) + \Pi'(0)q^2 - \dots]} \qquad (12.460)$$

For small q^2, the photon propagator is, therefore,

$$G^{\mu\nu}(q) \approx i\frac{P_{\mu\nu}(q)}{q^2}\left\{1 + \Pi(0) + \frac{\alpha}{15\pi M^2}q^2\right\}^{-1}. \qquad (12.461)$$

The divergent number $1+\Pi(0)$ can be absorbed into the field renormalization factor, after defining

$$A_R^\mu(x) = Z_3^{-1/2}A(x) \qquad (12.462)$$

with

$$Z_3 = [1 + \Pi(0)]^{-1} = 1 - \frac{\alpha}{3\pi}\log\frac{\Lambda^2}{m^2} + \dots\ . \qquad (12.463)$$

This corresponds to renormalizing the charge to

$$\alpha_R = Z_3\alpha. \qquad (12.464)$$

The propagator of the renormalized field A_R^μ is then

$$G_R^{\mu\nu}(q) = i\frac{P_{\mu\nu}(q)}{q^2}\left(1 + \frac{\alpha_R}{15\pi}\frac{q^2}{M^2}\right)^{-1}. \qquad (12.465)$$

For a given static source of charge Ze at the origin, the Coulomb potential

$$V_{\mathrm{C}}(x) = -\frac{Z\alpha}{r} \tag{12.466}$$

is obtained by applying the free propagator $G_0^{\mu\nu}$ to the current $j^\mu(x) = \delta^\mu{}_0 \, Ze\delta^{(3)}(\mathbf{x})$. To lowest order in α, the vacuum polarization changes $G_0{}^{\mu\nu}(q)$ by a factor

$$\left(1 - \frac{\alpha_R}{15\pi}\frac{q^2}{M^2}\right)$$

which is equivalent to multiplying the potential $V_{\mathrm{C}}(\mathbf{x})$ by a factor

$$\left(1 - \frac{\alpha_R}{15\pi}\frac{\boldsymbol{\nabla}^2}{M^2}\right).$$

The potential is therefore modified to

$$V_{\mathrm{C}}^R(x) = -Z\alpha\left[\frac{1}{r} + \frac{\alpha}{15\pi M^2}4\pi\delta^{(3)}(\mathbf{x})\right]. \tag{12.467}$$

A comparison with (12.407)–(12.415) shows that the vacuum polarization decreases the effective radius of the electron (12.411), derived from the vertex correction, by

$$R_e^{2\,\mathrm{vp}} = -\frac{\alpha}{3\pi M^2}\frac{1}{5}, \tag{12.468}$$

thus giving rise to a total effective radius determined to lowest order in α by:

$$R_e^{2\,\mathrm{vp}} = -\frac{\alpha}{3\pi M^2}\frac{1}{5}\left(\log\frac{M}{\mu} - \frac{3}{8} - \frac{1}{5}\right). \tag{12.469}$$

Since the finite radius of the electron gives rise to the Lamb shift to be derived below, the vacuum polarization *decreases* the Lamb shift.

12.17 Dimensional Regularization

We still must show that the divergent integral (12.447) is really zero, to ensure the gauge invariance of the self-energy (12.446). In D dimensions, the Dirac matrices have the dimension $2^{D/2}$, and (12.446) reads [26]

$$-i\Pi_{\mu\nu}(q) = 2^{D/2}e^2\int_0^1 dz \int \frac{d^Dp}{(2\pi)^D}\frac{-(-g_{\mu\nu}q^2+2q_\nu q_\nu)(z-z^2)-g_{\mu\nu}(p^2-M^2)+2p_\mu p_\nu}{[p^2+q^2(z-z^2)-M^2]^2}. \tag{12.470}$$

Then the previous replacement under the integral $p_\mu p_\nu \to g_{\mu\nu}/4$ becomes $p_\mu p_\nu \to g_{\mu\nu}/D$ [recall (11.136)]. The integral d^Dp over all four-momenta may now be Wick-rotated to

$$i\int d^Dp_E/(2\pi)^D = iS_D/2(2\pi)^D\int_0^\infty dp_E^2(p_E^2)^{D/2-1}, \tag{12.471}$$

where S_D is the surface of the unit sphere (11.126). Hence we can rewrite the integral (12.447) in Euclidean spacetime as

$$-iq^\mu\Pi_{\mu\nu}(q) = e^2 q_\nu\, 2^{D/2} i \int_0^1 dz \int \frac{d^D p_E}{(2\pi)^D} \frac{-q^2(z-z^2)+(1-2/D)p_E^2+M^2}{[p_E^2+m^2]^2}, \quad (12.472)$$

where

$$m^2 = m^2(q^2, z) \equiv M^2 - q^2 z(1-z), \quad (12.473)$$

and further as

$$-iq^\mu\Pi_{\mu\nu}(q) = e^2 q_\nu\, 2^{D/2} i \int_0^1 dz \int \frac{d^D p_E}{(2\pi)^D} \frac{2m^2/D+(1-2/D)(p_E^2+m^2)}{[p_E^2+m^2]^2}. \quad (12.474)$$

Now we use Formula (11.130) to replace

$$\int \frac{d^D p_E}{(2\pi)^D} \frac{1}{(p_E^2+m^2)^\nu} = \bar{S}_D \frac{\Gamma(D/2)\Gamma(\nu-D/2)}{2} \Gamma(\nu)[m^2]^{D/2-\nu}, \quad (12.475)$$

so that

$$-iq^\mu\Pi_{\mu\nu}(q) = e^2 q_\nu\, 2^{D/2} i \bar{S}_D \frac{\Gamma(D/2)}{2}$$
$$\times \left[\frac{2}{D}\Gamma(1-D/2)\Gamma(2) + \left(1-\frac{2}{D}\right)\Gamma(2-D/2)\Gamma(1)\right] \int_0^1 dz[m^2]^{D/2-1} = 0, \quad (12.476)$$

thus guaranteeing the gauge invariance of the vacuum polarization tensor in any dimension D [10].

12.18 Two-Dimensional QED

If the bare electron mass is zero, an interesting phenomenon arises in two dimensions: The self-energy of the photon makes the photon massive. This happens in spite of gauge invariance of the Lagrangian, as pointed out by Schwinger [6] in 1962. Since this was a surprise, the two-dimensional massless QED is called the *Schwinger model* [27].

Consider the self-energy (12.448) in D spacetime dimensions where the prefactor 4 in Eq. (12.446) is replaced by $2^{D/2}$, so that (12.448) reads

$$-i\Pi_{\mu\nu}(q) = -(-g_{\mu\nu}q^2 + q_\mu q_\nu)\, 2^{D/2+1} e^2 \int_0^1 dz(z-z^2) \int \frac{d^D p}{(2\pi)^D} \frac{1}{[p^2+q^2(z-z^2)-M^2]^2}. \quad (12.477)$$

Here the integral over $d^D p$ in momentum spacetime is Wick-rotated, as in (12.471), into the Euclidean momentum integral

$$\frac{S_D}{2(2\pi)^D} i \int_0^\infty dp_E^2 \frac{1}{(p_E^2+m^2)^2} = \frac{S_D}{2(2\pi)^D} i [m^2]^{D/2-2} \frac{\Gamma(D/2)\Gamma(2-D/2)}{\Gamma(2)}, \quad (12.478)$$

with m^2 determined by Eq. (12.473). Inserting this into (12.477) yields, for $-i\Pi_{\mu\nu}(q)$ of massless electrons, the tensor

$$-(-g_{\mu\nu}q^2+q_\mu q_\nu)\,2^{D/2+1}e^2 i\int_0^1 dz[z(1-z)]^{D/2-1}(-q^2)^{D/2-2}\frac{S_D}{2(2\pi)^D}\frac{\Gamma(D/2)\Gamma(2-D/2)}{\Gamma(2)}, \tag{12.479}$$

which is equal to[3]

$$-(-g_{\mu\nu}q^2+q_\mu q_\nu)2^{D/2+1}e^2 i\frac{\Gamma^2(D/2)}{\Gamma(D)}(-q^2)^{D/2-2}\frac{S_D}{2(2\pi)^D}\frac{\Gamma(D/2)\Gamma(2-D/2)}{\Gamma(2)}, \tag{12.480}$$

so that we obtain, in $D=2$ dimensions,

$$-i\Pi_{\mu\nu}(q)=-i(-g_{\mu\nu}q^2+q_\mu q_\nu)\,\frac{4e^2}{-q^2}\frac{S_2}{2(2\pi)^2}=-i(-g_{\mu\nu}q^2+q_\mu q_\nu)\frac{e^2}{-\pi q^2}=-iP_{\mu\nu}\frac{e^2}{-\pi q^2}, \tag{12.481}$$

or, recalling (12.449),

$$\Pi(q)=-\frac{e^2}{\pi q^2}. \tag{12.482}$$

Inserting this into (12.460), we obtain the renormalized photon propagator

$$G^{\mu\nu}(q)=i\frac{P^{\mu\nu}(q)}{q^2+q^2\Pi(q)}=i\frac{P^{\mu\nu}(q)}{q^2-e^2/\pi}. \tag{12.483}$$

This shows that the photon has acquired a nonzero mass $m_\gamma^2=e^2/\pi$. The effective Lagrangian of the photon to this order is

$$\mathcal{L}_{\rm eff}=-\tfrac{1}{4}F^{\mu\nu}(1+m_\gamma^2/\Box)F_{\mu\nu}. \tag{12.484}$$

The Schwinger model illustrates the remarkable fact that in spite of the gauge invariance of the theory, a mass term can be generated for the photon by a loop diagram of a massless fermion.

12.19 Self-Energy of Electron

The lowest-order Feynman diagram for the self-energy of the electron is shown in Fig. 12.28. It adds to the electron propagator a term

$$G_0(p)[-i\Sigma(p)]G_0(p), \tag{12.485}$$

where

$$G_0(p)=\frac{i}{\not p-M} \tag{12.486}$$

[3]I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, New York, 1980, Formula 3.191.3.

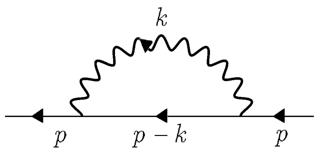


FIGURE 12.28 Lowest-order Feynman diagram for the self-energy of the electron.

is the free Dirac propagator and the self-energy $-i\Sigma(p)$ of the electron is given by the diagram in Fig. 12.28. Explicitly:

$$-i\Sigma(p) = e^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{\not{p} - \not{k} + M}{(p-k)^2 - M^2} \gamma_\mu \frac{1}{k^2}. \tag{12.487}$$

Using the anticommutation rule (12.214), we can simplify, in the numerator,

$$\gamma^\mu(\not{p} - \not{k} + M)\gamma_\mu = -2(\not{p} - \not{k}). \tag{12.488}$$

The integral is logarithmically divergent at large k. It also has an infrared divergence. To enforce convergence, we employ the Pauli-Villars regularization method and modify the photon propagator as follows:

$$\frac{1}{k^2} \to \frac{1}{k^2 - \mu^2} - \frac{1}{k^2 - \Lambda^2}, \tag{12.489}$$

where Λ is a large cutoff mass and μ is a small photon mass. The self-energy becomes

$$-i\Sigma(p) = e^2 \int \frac{d^4k}{(2\pi)^4} \frac{-2(\not{p} - \not{k}) + 4M}{(p-k)^2 - M^2} \left(\frac{1}{k^2 - \mu^2} - \frac{1}{k^2 - \Lambda^2} \right). \tag{12.490}$$

The mass parameter Λ cuts off the contribution of short-wave photons with $k \gg \Lambda^2$. At the end we shall take the cutoff to infinity.

By adding the same Feynman diagram repeatedly to an electron line, we obtain the geometric series

$$G_0(p) + G_0(p)[-i\Sigma(p)]G_0(p) + G_0(p)[-i\Sigma(p)]G_0(p)[-i\Sigma(p)]G_0(p) + \ldots\ , \tag{12.491}$$

which can be summed up to

$$G(p) = \frac{i}{\not{p} - M - \Sigma(p)}. \tag{12.492}$$

Using Feynman's formula (12.443), this can be rewritten as

$$\begin{aligned} \Sigma(p) &= -ie^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \\ &\times \left\{ \frac{-2(\not{p} - \not{k}) + 4M}{[-(k-px)^2 - p^2x(1-x) + M^2x + \Lambda^2(1-x)]^2} - (\Lambda = \mu) \right\}. \end{aligned} \tag{12.493}$$

A simplification occurs by shifting the integration variable from k to $k+px$. Then the terms proportional to k are off in k and can be dropped. After performing a Wick rotation of the integration contour, we obtain

$$\Sigma(p)=\frac{2\pi^2e^2}{(2\pi)^4}\int_0^1 ds\int_0^1 dk_E k_E^3\left\{\frac{-2\not p(1-x)+4M}{\left[k_E^2-p^2x(1-x)+M^2x+\Lambda^2(1-x)\right]^2}-(\Lambda=0)\right\}. \tag{12.494}$$

The k_E-integral is easily done, and yields

$$\begin{aligned}\Sigma(p) &= \frac{\pi^2e^2}{(2\pi)^4}\int_0^1 dx\,[-2\not p(1-x)+4M]\\ &\times\left\{\log\left[\frac{-p^2x(1-x)+M^2x+\Lambda^2(1-x)}{-p^2x(1-x)+M^2x}\right]-(\Lambda=\mu)\right\}.\end{aligned} \tag{12.495}$$

The right-hand side is a 4×4-matrix in spinor space which may be decomposed into invariant functions

$$\Sigma(p)=(\not p-M)A(p^2)+B(p^2), \tag{12.496}$$

where

$$\begin{aligned}A(p^2) &= \frac{\pi^2e^2}{(2\pi)^4}\int_0^1 dx[-2(1-x)]\left\{\log f(p^2,x,\Lambda)-(\Lambda=\mu)\right\},\\ B(p^2) &= \frac{\pi^2e^2}{(2\pi)^4}\int_0^1 dx\,2M(1+x)\left\{\log f(p^2,x,\Lambda)-(\Lambda=\mu)\right\},\end{aligned} \tag{12.497}$$

with

$$f(p^2,x,\Lambda)=\frac{-p^2x(1-x)+M^2x+\Lambda^2(1-x)}{-p^2x(1-x)+M^2x}. \tag{12.498}$$

The invariant functions are logarithmically divergent for large Λ.

We may expand them around the mass shell $p^2=M^2$ in powers of p^2-M^2. Then only the lowest expansion terms carry the logarithmic divergence:

$$\begin{aligned}A(M^2) &= \frac{\pi^2e^2}{(2\pi)^4}\int_0^1 dx\,[-2(1-x)]\log\left[\frac{M^2x^2+\Lambda^2(1-x)}{M^2x^2}\right],\\ B(M^2) &= \frac{\pi^2e^2}{(2\pi)^4}\int_0^1 dx\,2M(1+x)\log\left[\frac{M^2x^2+\Lambda^2(1-x)}{M^2x^2}\right],\\ B'(M^2) &= \frac{\pi^2e^2}{(2\pi)^4}\int_0^1 dx\,2M(1+x)(1-x)\\ &\times\left[\frac{\Lambda^2(1-x)}{M^2x[M^2x^2+\Lambda^2(1-x)]}-\frac{\mu^2(1-x)}{M^2x[M^2x^2+\mu^2(1-x)]}\right].\end{aligned} \tag{12.499}$$

All higher expansion terms are finite and can be evaluated with an infinite cutoff Λ. We have also dropped all terms that vanish in the limit of zero photon mass μ. Omitting the regular parts of the self-energy, the propagator becomes

$$G(p) = \frac{i}{(\not p-M)[1+A(M^2)]+B(M^2)+B'(M^2)(p^2-M^2)}$$

$$= \frac{1}{1+A(M^2)+2MB'(M^2)}\,\frac{i}{\not p - M - \delta M}, \tag{12.500}$$

where

$$\delta M \equiv -\frac{B(M^2)}{1+A(M^2)+2MB'(M^2)}. \tag{12.501}$$

The prefactor in the denominator is commonly denoted by

$$Z_2 \equiv 1 + A(M^2) + 2MB'(M^2). \tag{12.502}$$

It can be removed by renormalizing the field $\psi(x)$ to $\psi_R(x) = Z_2^{1/2}\psi(x)$. The renormalized field has then a propagator with a pole term $i/(\not p - M)$. For large Λ^2, we can simplify the results for Z_2 and δM^2 to order α:

$$\begin{aligned} Z_2 - 1 &= -\frac{\alpha}{2\pi}\int_0^1 dx \left\{(1-x)\left[\log\frac{\Lambda^2}{M^2} + \log\frac{1-x}{x^2}\right] - 2\frac{1-x^2}{x}\left[1 - \frac{\mu^2(1-x)}{M^2x^2+\mu^2(1-x)}\right]\right\}, \\ \delta M^2 &= \frac{\alpha}{2\pi}\int_0^1 dx(1+x)\left[\log\frac{\Lambda^2}{M^2} + \log\frac{1-x}{x^2}\right]. \end{aligned} \tag{12.503}$$

Performing the integrals over x yields

$$\begin{aligned} Z_2 - 1 &= -\frac{\alpha}{2\pi}\left[\log\frac{\Lambda}{M} + \frac{5}{4} + 1 - \log\frac{M^2}{\mu^2}\right], \\ \delta M^2 &= \frac{\alpha}{2\pi}\int_0^1 dx\left[3\log\frac{\Lambda}{M} + \frac{3}{4}\right]. \end{aligned} \tag{12.504}$$

12.20 Ward-Takahashi Identity

It is important to realize that Z_2 coincides with the renormalization constant $Z_1 = F^{-1}(0)$ defined in Eq. (12.406) by the charge form factor to make the current matrix element finite. This equality is a consequence of the *Ward identity* fulfilled by the vertex correction $\Lambda^\mu(p',p)$ defined in Eq. (12.375):

$$\Lambda^\mu(p,p) = -\frac{\partial}{\partial p_\mu}\Sigma(p). \tag{12.505}$$

For the total Dirac matrix $\Gamma^\mu(p',p) \equiv \gamma^\mu + \Lambda^\mu(p',p)$ in the current (12.375), this implies that

$$\Gamma^\mu(p,p) = \frac{\partial}{\partial p_\mu}\left[\not p - M - \Sigma(p)\right]. \tag{12.506}$$

This relation holds to all orders in α. For off-diagonal matrix elements of the current, there also exists a more general relation:

$$(p'-p)_\mu\Gamma^\mu(p',p) = [\not p' - M - \Sigma(p)] - [\not p - M - \Sigma(p)], \tag{12.507}$$

from which (12.506) can be derived in the limit $p' \to p$. This is the famous *Ward-Takahashi identity*. Its validity for a free theory is obvious. For the interacting theory it will be proved at the end of this section..

A first important consequence of the Ward-Takahashi identity is the equality of the renormalization constants $Z_1 = Z_2$. To derive it, we use (12.506) to rewrite

$$\begin{aligned}\Gamma^\mu(p',p) &= -\frac{\partial}{\partial p_\mu}\Sigma(p) + \Gamma^\mu(p',p) - \Gamma^\mu(p,p) \\ &= -\gamma^\mu + \frac{\partial}{\partial p_\mu}\left[\not p - M - \Sigma(p)\right] + \Gamma^\mu(p',p) - \Gamma^\mu(p,p). \quad (12.508)\end{aligned}$$

From the renormalization equation of the electron propagator (12.500), we see that

$$\not p - M - \Sigma(p) = Z_2^{-1}\left[\not p - M_R - \Sigma_R(p)\right], \quad (12.509)$$

where $M_R = M + \delta M$. This leads to

$$\Gamma^\mu(p',p) = \gamma^\mu(Z_2^{-1} - 1) - Z_2^{-2}\frac{\partial}{\partial p_\mu}\Sigma_R(p) + \Gamma^\mu(p',p) - \Gamma^\mu(p,p). \quad (12.510)$$

This must be compared with the definition of the renormalized vertex function

$$\Gamma^\mu(p',p) = \gamma^\mu(Z_1^{-1} - 1) + Z_1^{-1}\Gamma_R^\mu(p',p), \quad (12.511)$$

to conclude that

$$\begin{aligned}Z_1 &= Z_2, \\ \Gamma_R^\mu(p',p) &= Z_1\left[\Gamma^\mu(p',p) - \Gamma^\mu(p,p)\right] - \frac{\partial}{\partial p_\mu}\Sigma_R(p). \quad (12.512)\end{aligned}$$

The proof of the Ward-Takahashi identity (12.507) follows from the canonical commutation relations of the current with the fields. These are certainly true in the presence of interactions [28]:

$$\begin{aligned}[j^0(\mathbf{x},x_0), \psi(\mathbf{x}',x_0)] &= -e\delta^{(3)}(\mathbf{x}-\mathbf{x}')\psi(\mathbf{x},x_0), \\ [j^0(\mathbf{x},x_0), \bar\psi(\mathbf{x}',x_0)] &= e\delta^{(3)}(\mathbf{x}-\mathbf{x}')\bar\psi(\mathbf{x},x_0). \quad (12.513)\end{aligned}$$

These illustrate Noether's theorem, which makes j^0 the generator of phase transformations [recall Eq. (8.280)]. We now form the derivative of the time-ordered expectation

$$\partial_z^\mu\langle 0|\hat T\psi(x')j_\mu(z)\bar\psi(x)|0\rangle = e\langle 0|\hat T\psi(x')\bar\psi(x)|0\rangle\left[\delta^{(4)}(x'-z) - \delta^{(4)}(z-x)\right]. \quad (12.514)$$

Expressed in terms of the full propagators, this equation becomes

$$iS(x'-z)\,\partial_z^\mu\Gamma_\mu(z)\,iS(z-x) = iS(z-x) - iS(x'-z). \quad (12.515)$$

After a Fourier transformation, it reads

$$S(p')(p'-p)^{\mu}\Gamma_{\mu}(p',p)S(p) = S(p) - S(p'), \tag{12.516}$$

or

$$(p'-p)^{\mu}\Gamma_{\mu}(p',p) = S^{-1}(p) - S^{-1}(p'), \tag{12.517}$$

which is precisely the Ward-Takahashi identity (12.507).

As a consequence of the equality $Z_1 = Z_2$, we find that the matrix elements of the current between one-loop corrected electron states exhibit a unit charge, rather than the divergent charge that is contained in the charge form factor $F(0) = Z_1^{-1}$. Thus we may conclude the important property of QED that the electric charge remains unrenormalized to all orders in perturbation theory.

12.21 Lamb Shift

One of the most important early confirmations of the correctness of quantum electrodynamics of electrons and photons came from the atomic Lamb shift. According to Dirac's theory, the energy spectrum of an electron in an external Coulomb field is

$$E_{nl} = Mc^2 \left\{ 1 + \frac{\alpha^2 Z^2}{n' + \sqrt{(j+1/2)^2 - \alpha^2 Z^2}} \right\}^{1/2}, \tag{12.518}$$

where $n' = n - j - 1/2 = 0, 1, 2, \ldots$ is the radial quantum number, and $j = 1/2, 3/2, \ldots$ the total angular momentum. Up to lowest order in the fine-structure constant $\alpha = e^2/\hbar c$, this is approximately equal to

$$E_{nl} = Mc^2 - \frac{1}{2}\alpha^2 Mc^2 \frac{Z^2}{n^2}\left[1 + \frac{\alpha^2 Z^2}{n}\left(\frac{1}{j+1/2} - \frac{3}{4n}\right) + \ldots\right]. \tag{12.519}$$

These formulas show that the Schrödinger degeneracy, of all levels with the same principal quantum number and different values of the orbital angular l, is modified in Dirac's theory, where levels with the same quantum numbers n and j are degenerate for different l's. The lowest states, where this degeneracy can be compared with experimental data, are the $n = 2$ - states $2S_{1,2}$ and $2P_{1/2}$. They are found to have slightly different energies, thus contradicting the simple Dirac theory.

The energy difference is explained by quantum electrodynamics, and may be attributed to three physical effects. First, the electron encircling the nucleus is shaken by the vacuum fluctuations of the electromagnetic field over a range of the order of the Compton wavelength. Thus it sees a harmonic average of the Coulomb potential over this length scale. This lifts the level $2S_{1/2}$ against the level $2P_{1/2}$ by roughly 27 MHz. Second, the anomalous magnetic moment of the electron changes slightly the Coulomb attraction. Third, the photon running through the vacuum can excite an electron-positron pair. These three effects together cause an *upwards* shift of the level $2S_{1/2}$ with respect to the level $2P_{1/2}$ equal to [29] $\Delta E = 1\,010$ MHz, 68 MHz, and -27 MHz, respectively. The sum is roughly $1\,052$ MHz, a number which was confirmed experimentally around 1950 [30]. The calculation of these effects will now be reviewed.

12.21.1 Rough Estimate of the Effect of Vacuum Fluctuations

In order to estimate the first effect, consider a free nonrelativistic electron of mass M in the vacuum. It is shaken by the zero-point oscillation of the electromagnetic field, which causes an acceleration

$$M\dot{\mathbf{x}} = e\mathbf{E}. \tag{12.520}$$

For a given frequency ω, the electron is shaken around its average position by a displacement

$$\Delta\mathbf{x} = -\frac{e}{M}\omega^2\mathbf{E}(\omega). \tag{12.521}$$

Its square-average is

$$\langle(\Delta\mathbf{x})^2\rangle = \frac{e^2}{M^2}\int_0^\infty \frac{d\omega}{\omega^4}\langle\mathbf{E}^2(\omega)\rangle. \tag{12.522}$$

The right-hand side can be estimated from the energy (7.341) of the free electromagnetic field in the vacuum, where it has the value [recall (7.434)]

$$E = \frac{1}{2}\sum_{\mathbf{k},\lambda}\omega_{\mathbf{k}}. \tag{12.523}$$

The polarization sum runs over the two helicities. Hence, with the usual limiting phase space integral (7.21) for the momentum sum, we have

$$E = \int \frac{d^3k}{(2\pi)^3}\omega_{\mathbf{k}} = \frac{1}{2\pi}\int_0^\infty \frac{d\omega}{\pi}\omega^3. \tag{12.524}$$

Since the vacuum energy is equally distributed between electric and magnetic fields, we find

$$\langle E^2\rangle = \frac{1}{2\pi}\int_0^\infty \frac{d\omega}{\pi}\omega^3, \tag{12.525}$$

and hence

$$\langle(\Delta\mathbf{x})^2\rangle = \frac{2\alpha}{M^2}\int_0^\infty \frac{d\omega}{\pi}\frac{1}{\omega}. \tag{12.526}$$

The integral is divergent at small and large frequencies ω. A priori, it is unclear which are the relevant frequencies that will contribute in a proper calculation. If we consider only electromagnetic waves with wavelength shorter than the Bohr radius $a_{\rm B} = 1/\alpha M$, the integral starts at $\omega_{\rm min} = M\alpha$. Alternatively, we may expect the energy of the atomic electrons to supply the relevant cutoff. Then $\omega_{\rm min} = M\alpha^2$. On the high-frequency end, we omit wavelengths shorter than the Compton wavelength of the electron, where classical considerations become invalid. Thus we cut off the integral at $\omega_{\rm max} = M$. In this way we obtain a mean square somewhere in the range

$$\langle \Delta \mathbf{x} \rangle^2 = \frac{2\alpha}{\pi M} C, \tag{12.527}$$

with a constant C somewhere in the range

$$C \in \left(\log \frac{1}{\omega_{\rm min}}, 2 \log \frac{1}{\omega_{\rm min}} \right) \approx 4.92 \times (1, 2). \tag{12.528}$$

The electric interaction energy of an electron shaken over this region is modified as follows. The Hamiltonian at the position $\mathbf{x} + \Delta\mathbf{x}(t)$ is

$$H = e \int d^3x A_0(\mathbf{x} + \Delta\mathbf{x}(t)). \tag{12.529}$$

Averaging over $\Delta\mathbf{x}$ gives

$$\begin{aligned} H &= e \int d^3x \left[A_0(\mathbf{x}) + \frac{1}{2} \langle \Delta x_i \Delta x_j \rangle \partial_i \partial_j A_0(\mathbf{x}) \right] \\ &= e \int d^3x \left[1 + \frac{1}{6} \langle \Delta \mathbf{x} \cdot \Delta \mathbf{x} \rangle \boldsymbol{\nabla}^2 \right] A_0(\mathbf{x}) \\ &= e \int d^3x \left[1 + \frac{1}{3\pi} \frac{\alpha}{M^2} C\, \boldsymbol{\nabla}^2 \right] A_0(\mathbf{x}). \end{aligned} \tag{12.530}$$

In an atom of nuclear charge Ze with a Coulomb potential (12.413), the Laplace operator yields (12.414), and the potential is changed into an effective one:

$$V_{\rm C}^{\rm eff} = -\frac{Z\alpha}{r} + \frac{1}{3\pi} \frac{\alpha}{M^2} C Z\alpha 4\pi \delta^{(3)}(\mathbf{x}). \tag{12.531}$$

For an atomic s-state with wave function ψ_n, we treat the extra potential $V_{\rm C}^{\rm eff}$ perturbatively. We evaluate its expectation value in a state of principal quantum number n, and find the *positive* energy shift

$$\Delta E_n = \frac{4\alpha}{3M^2} Z\alpha C |\psi_n(\mathbf{0})|^2. \tag{12.532}$$

Thus, the present rough estimate of the effect of vacuum fluctuations produces the same term as before in (12.415), except for a different logarithmic factor (12.411).

For a hydrogen atom we insert

$$\psi_n(\mathbf{0}) = \frac{1}{\sqrt{n^3\pi}} \left(\frac{1}{a_{\rm B}} \right)^{3/2}, \tag{12.533}$$

where $a_{\rm B} = 1/M\alpha$ is the Bohr radius. If the nuclear charge is Z, then $a_{\rm B}$ is diminished by this factor. Thus, we obtain the energy shift

$$\Delta E_n = \frac{4\alpha^2 Z}{3M^2} (MZ\alpha)^3 \frac{1}{n^3\pi} C. \tag{12.534}$$

For a hydrogen atom with $n = 2$, this becomes

$$\Delta E_2 = \frac{\alpha^3}{6\pi}\alpha^2 MC. \tag{12.535}$$

The quantity $M\alpha^2$ is the unit energy of atomic physics determining the hydrogen spectrum to be $E_n = -M\alpha^2/2n^2$. Thus

$$M\alpha^2 \quad = \quad 4.36 \times 10^{-11}\text{erg} = 27.21\text{eV} = 2\,\text{Ry} = 2 \cdot 3.288 \times 10^{15}\text{Hz}. \tag{12.536}$$

Inserting this, together with $\alpha \approx 1/137.036$, into (12.542) yields[4]

$$\Delta E_2 \approx 135.6\text{MHz} \times C, \tag{12.537}$$

which amounts to the theoretical estimate

$$667.15 \text{ MHz} < \Delta E_2 < 1334.3 \text{ MHz}. \tag{12.538}$$

The experimental Lamb shift[5]

$$\Delta E_{\text{Lamb shift}} \approx 1\,057 \text{ MHz}, \tag{12.539}$$

is indeed contained in this range.

12.21.2 Relativistic Estimate

The above simple estimate of the effect of vacuum fluctuations produces the same type of correction to the Coulomb potential as the vertex correction in Eq. (12.415), and the vacuum polarization in (12.467). Those two corrections yield an energy shift in an s-state $\psi_n(\mathbf{x})$ of principal quantum number n:

$$\Delta E_n = \frac{4\alpha}{3M^2} Z\alpha|\psi_n(\mathbf{0})|^2 C_{\text{rel}} = \frac{4\alpha^3 Z^4}{3\pi n^3}\alpha^2 M C_{\text{rel}}, \tag{12.540}$$

with the constant C_{rel}:

$$C_{\text{rel}} \equiv \log\frac{M}{\mu} - \frac{3}{8} - \frac{1}{5}. \tag{12.541}$$

For a hydrogen atom in an s-state of principal quantum number $n = 2$ with $n = 2$, this implies

$$\Delta E_2 = \frac{\alpha^3}{6\pi} M\alpha^2 C_{\text{rel}} \approx 135.641 \text{ MHz} \times C_{\text{rel}}. \tag{12.542}$$

The result is not completely determined since it contains an infrared cutoff μ in the constant C_{rel} of Eq. (12.541). In a first approximation, we may imagine the atomic energy $M\alpha^2$ to provide the infrared cutoff for the photon energies. This leads to an estimate for the energy shift of the $2S_{1/2}$ levels with respect to the $2P_{1/2}$ levels of

[4]The precise value of the *Lamb constant* $\alpha^4 M/6\pi$ is 135.641± 0.004 MHz.

[5]See Notes and References.

the hydrogen atom that is about 6% smaller than the larger of the nonrelativistic estimates (12.538):

$$\Delta E_2 \approx (1\,334.3 - 51 - 27.3)\text{ MHz.} \approx 1\,256\text{ MHz.} \tag{12.543}$$

The intermediate expression exhibits the contribution of the terms $-3/8$ and $-1/5$ in (12.541) (thus showing that vacuum polarization gives a negative shift -27.31 MHz). This shift was first calculated by Uehling [31], who initially thought that vacuum polarization was the main cause of the Lamb shift. He was disappointed to see it contributing only with about 3%. In muonic atoms, however, vacuum polarization does produce the dominant contribution to the Lamb shift for a simple reason: While the above-calculated energy shifts contain a factor $1/M^2$ in formulas such as (12.593), where M is the mass of the muon, the leading vacuum polarization graph still involves an electron loop containing the electron mass, thus being enhanced by a factor $(M_\mu/M_e)^2 \approx 210^2$.

12.21.3 Effect of Wave Functions

In the above calculations the finite size of the electron was derived from a one-loop Feynman-diagram in which the electron lines describe free particles. In an atom, however, the electrons move in a Coulomb potential. The electron is bound to the nucleus. A more accurate calculation should take into account the atomic wave functions of the electron. This is most simply done in an approximation which treats the electrons as nonrelativistic particles. Such an approximation carries an intrinsic error caused by the fact that if a nonrelativistic electron emits a photon with energy of the order Mc^2 and larger, the recoil will necessarily make the electron relativistic. Such an error can, however, be avoided by separating the relativistic from the nonrelativistic contributions. In the first contribution, the effect of the atomic binding of the electrons is negligible, so that the electrons can be treated as free relativistic particles. In the second contribution, the electrons remain approximately nonrelativistic. There exists a natural energy scale $K = M\alpha$ which is much larger than the atomic energy $M\alpha^2$, but much smaller than the rest energy of the electron M. The energy scale K serves to make the separation quantitative. For photons in the upper energy regime, to be referred to as the *hard-photon regime*, we may equip the photon with a mass $\alpha^2 M \gg \mu \ll \alpha M$, and deduce the Lamb shift from Eq. (12.540) to be

$$\Delta E_n = \frac{4\alpha}{3M^2} Z\alpha |\psi_n(\mathbf{0})|^2 C_{\text{hard}} = \frac{4\alpha^3 Z^4}{3\pi n^3}\alpha^2 M C_{\text{hard}}, \tag{12.544}$$

with the constant

$$C_{\text{hard}} \equiv \log\frac{M}{\mu} - \frac{3}{8} - \frac{1}{5}. \tag{12.545}$$

The renormalization procedure has removed the ultraviolet divergences occuring in the calculation of vertex corrections and vacuum polarizations. In the *soft-photon regime*, this leaves us with the task of calculating the contribution from the photons of low energy where the electrons stay nonrelativistic. In the transverse gauge with

$\nabla \mathbf{A}(x) = 0$, the Hamiltonian has the form (12.160), with the Coulomb Hamiltonian (12.161). The radiation field (12.162) contains, in the soft-photon regime under consideration, only photon energies within the limited interval $\omega \in (0, K)$ where K satisfies $M\alpha^2 \ll K \ll M$. The radiation field is expanded in terms of photon creation and annihilation operators as in Eq. (7.350):

$$\hat{A}^\mu(x) = \sum_{\mathbf{k},\lambda} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} \left[e^{-ikx} \epsilon^\mu(\mathbf{k}, \lambda) \hat{a}_{\mathbf{k},\lambda} + \text{h.c.} \right]. \tag{12.546}$$

The hats above the field operators are displayed, for clarity. To estimate the expected effects we consider first the influence of the radiation field upon a free electron.

Effect of Vacuum Oscillations upon Free Electron

Let us first calculate the size of the energy shift for a free electron. This is the quantum-mechanical version of the calculation in Subsection 9.10.1.

The Hamilton operator of a free electron of momentum $\mathbf{p}$ is $\hat{H}_0 = \hat{\mathbf{p}}^2/2M$, and the wave functions is a plane wave

$$\langle \mathbf{x}|\mathbf{p}\rangle = \frac{1}{\sqrt{V}} e^{i\mathbf{p}\mathbf{x}} \tag{12.547}$$

of energy $E_{\mathbf{p}} = \mathbf{p}^2/2M$. If the electromagnetic field is quantized, there is a ground state for the fluctuating vector potential, which is the vacuum state $|0\rangle$ of the photon field. The combined state will be denoted by $|\mathbf{p}; 0\rangle$. The calculation is done perturbatively. Thus we expand the energy shift in powers of the coupling constant:

$$\Delta E_{\mathbf{p}} = \Delta E_{\mathbf{p}}^{(1)} + \Delta E_{\mathbf{p}}^{(2)} + \ldots\ , \tag{12.548}$$

assuming the charge e to be sufficiently small. The first-order shift $\Delta E_{\mathbf{p}}^{(1)}$ is simply the expectation value of the interaction operator

$$\hat{H}^{\text{int}} = -\frac{e}{Mc} \hat{\mathbf{p}} \cdot \hat{\mathbf{A}}(\mathbf{x}) + \frac{e^2}{M^2c^2} \hat{\mathbf{A}}^2(\mathbf{x}). \tag{12.549}$$

Since this is odd in the field $\mathbf{A}(\mathbf{x})$ which has no expectation value, the first-order energy shift vanishes. Thus we turn to the second-order shift $\Delta E_{\mathbf{p}}^{(2)}$, in which we may ignore the second term in the interaction (called the *seagull term*) that contributes equally to all atomic levels. Then we have

$$\Delta E_{\mathbf{p}} = \frac{e^2}{(Mc)^2} \langle \mathbf{p}; 0|\hat{\mathbf{p}} \cdot \hat{\mathbf{A}}(\mathbf{x}) \frac{1}{E_{\mathbf{p}} - \hat{H}_0} \hat{\mathbf{p}} \cdot \hat{\mathbf{A}}(\mathbf{x})|\mathbf{p}; 0\rangle. \tag{12.550}$$

Inserting a complete set of intermediate electron-plus-single photon states $|\mathbf{p}; k\rangle$, we find, in natural units with $c = 1, \hbar = 1$,

$$\Delta E_{\mathbf{p}} = \frac{e^2}{M^2} \int d^3x \sum_{\substack{\mathbf{k},\mathbf{k}' \\ \lambda,\lambda'=1,2}} \frac{1}{\sqrt{2V|\mathbf{k}|}} \frac{1}{\sqrt{2V|\mathbf{k}'|}} \frac{\left[\mathbf{p} \cdot \boldsymbol{\epsilon}(\mathbf{k}, \lambda) e^{i\mathbf{k}\mathbf{x}}\right] \left[\mathbf{p} \cdot \boldsymbol{\epsilon}^*(\mathbf{k}', \lambda') e^{-i\mathbf{k}'\mathbf{x}}\right]}{\dfrac{\mathbf{p}^2}{2M} - \dfrac{1}{2M}(\mathbf{p} - \mathbf{k}')^2 - |\mathbf{k}'|}$$

$$= \frac{e^2}{M^2} \sum_{\mathbf{k},\lambda=1,2} \frac{1}{2V|\mathbf{k}|} \frac{|\mathbf{p}\cdot\boldsymbol{\epsilon}(k,\lambda)|^2}{\frac{\mathbf{p}\cdot\mathbf{k}}{M} - \frac{\mathbf{k}^2}{2M} - |\mathbf{k}|}. \tag{12.551}$$

Performing the polarization sum

$$\sum_{\lambda=1,2} |\mathbf{p}\cdot\boldsymbol{\epsilon}(k,\lambda)|^2 = \mathbf{p}^2 - \frac{(\mathbf{p}\cdot\mathbf{k})^2}{\mathbf{k}^2}, \tag{12.552}$$

and replacing the sum over momenta by a phase space integral, $V\int d^3k/(2\pi)^3$, we obtain

$$\Delta E_{\mathbf{p}} = \frac{\mathbf{p}^2}{2M} J_M(p^2) \tag{12.553}$$

with

$$J_M(p^2) = \frac{e^2}{M}\int \frac{d^3k}{(2\pi)^3}\frac{1}{|\mathbf{k}|}\left[1 - \frac{(\mathbf{p}\cdot\mathbf{k})^2}{\mathbf{p}^2\mathbf{k}^2}\right]\frac{1}{\frac{\mathbf{p}\cdot\mathbf{k}}{M} - \frac{\mathbf{k}^2}{2M} - |\mathbf{k}|}. \tag{12.554}$$

Writing the last factor as

$$-\frac{1}{k} + \frac{1}{k}\frac{\frac{k^2}{2M} - \frac{\mathbf{p}\cdot\mathbf{k}}{M}}{\frac{k^2}{2M} - \frac{\mathbf{p}\cdot\mathbf{k}}{M} + k}, \tag{12.555}$$

we obtain

$$J_M(p^2) = J_M(0) + J'_M(p^2) \tag{12.556}$$

with

$$J_M(0) = -\frac{e^2}{M}\int \frac{d^3k}{(2\pi)^3}\frac{1}{k}\left[1 - \frac{(\mathbf{p}\cdot\mathbf{k})^2}{p^2k^2}\right]\frac{1}{k} \tag{12.557}$$

and

$$J'_M(p^2) = \frac{e^2}{M}\int \frac{d^3k}{(2\pi)^3}\frac{1}{k}\left[1 - \frac{(\mathbf{p}\cdot\mathbf{k})^2}{p^2k^2}\right]\frac{\frac{k^2}{2M} - \frac{\mathbf{p}\cdot\mathbf{k}}{M}}{\frac{k^2}{2M} - \frac{\mathbf{p}\cdot\mathbf{k}}{M} + k}. \tag{12.558}$$

The first term can easily be calculated:

$$\begin{aligned} J_M(0) &= -\frac{e^2}{4\pi^2 M}\int_0^\infty dk \int_0^\infty d\cos\theta(1-\cos^2\theta) \\ &= -\frac{4\alpha}{3\pi M}\int_0^\infty dk. \end{aligned} \tag{12.559}$$

Being in the soft-photon regime, a quadratic divergence at large k is avoided. The integral is cut off at $k = K << M$. With the resulting finite $J_M(0)$, the kinetic energy of the electron is changed from $E(\mathbf{p}) = \mathbf{p}^2/2M$ to the renormalized energy

$$E_R(\mathbf{p}) = \frac{\mathbf{p}^2}{2M}[1 + J_M(0)]. \tag{12.560}$$

Such a factor may be absorbed in the mass of the electron, by defining a renormalized mass

$$M_R = \frac{M}{1 + J_M(0)} \sim M - MJ_M(0). \tag{12.561}$$

In terms of this and the subtracted function $J'_M(p^2)$ of Eq. (12.558), we find the full momentum dependence of the energy to order α:

$$E_R(\mathbf{p}) = \frac{\mathbf{p}^2}{2M_R}[1 + J'_{M_R}(p^2)]. \tag{12.562}$$

Effect of Vacuum Oscillations upon Bound Electron

We now perform the same calculation once more in the presence of the Coulomb potential $V_{\mathrm{C}}(\mathbf{x})$, and study an electron in an orbit of principal quantum number n with a wave function $\psi_n(\mathbf{x})$, moving through a photon vacuum $|0\rangle$. Then Eq. (30B.18) becomes

$$\Delta E_n = \frac{e^2}{M^2}\langle n;0|\hat{\mathbf{p}}\cdot\hat{\mathbf{A}}(\mathbf{x})\frac{1}{E_n - \hat{H}_{\mathrm{C}}}\hat{\mathbf{p}}\cdot\hat{\mathbf{A}}(\mathbf{x})|n;0\rangle. \tag{12.563}$$

Inserting a complete set of states $|n;0\rangle$ between the operators in Eq. (12.563) leads to

$$\Delta E_n = \frac{e^2}{M^2}\sum_n\sum_{\lambda,\lambda'}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2V|\mathbf{k}|}\frac{(\hat{\mathbf{p}}e^{i\mathbf{kx}})_{nn'}\boldsymbol{\epsilon}(\mathbf{k},\lambda)(\hat{\mathbf{p}}e^{-i\mathbf{kx}})_{n'n}\boldsymbol{\epsilon}^*(\mathbf{k},\lambda)}{E_n - E_{n'} - k} \tag{12.564}$$

where $(\hat{\mathbf{p}}e^{i\mathbf{kx}})_{nn'}$ denotes the matrix elements

$$(\hat{\mathbf{p}}e^{i\mathbf{kx}})_{nn'} \equiv \int d^3x\psi_n(\mathbf{x})(\hat{\mathbf{p}}e^{i\mathbf{kx}})_{nn'}\psi_{n'}(\mathbf{x}). \tag{12.565}$$

A Schrödinger wave function corresponds graphically to an infinite set of static photon exchanges. The energy shift to be calculated from Eq. (12.563) has therefore the following graphical representation: The additional photon provides a radiative

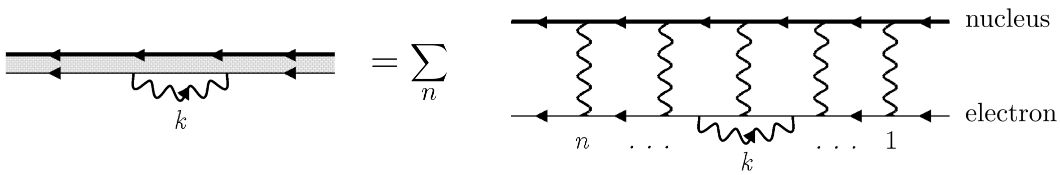


FIGURE 12.29 Diagrammatic content in the calculation of the energy shift with the help of Schrödinger wave functions. A hydrogen atom is represented by the fat line on the left which results from an infinite sum of photon exchanges.

correction to the static photons which creates the bound state. The sum over n' must, of course, include also an integral over the continuous wave functions. By rewriting, as in (12.555),

$$\frac{1}{E_n - E_{n'} - k} = -\frac{1}{k} + \frac{E_{n'} - E_n}{k(E_{n'} - E_n + k)}, \tag{12.566}$$

we obtain

$$\Delta E_n = \frac{e^2}{M^2}\sum_{n'}(\hat{p}_i e^{i\mathbf{kx}})_{nn'}\left(\hat{p}_j e^{-i\mathbf{kx}}\right)_{n'n} \times \int \frac{d^3k}{(2\pi)^3}\frac{1}{2k^2}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\left[-1+\frac{E_{n'}-E_n}{E_{n'}-E_n+k}\right]. \tag{12.567}$$

The $|\mathbf{k}|$-integration is again restricted to the soft-photon regime $|\mathbf{k}| < K << M$. In the integral involving only the first term in the brackets of (12.567), the energies $E_{n'}$ are absent and we can replace the sum over all wave functions $|n'\rangle$ by an integral over plane waves. Then we obtain the expectation value of the energy calculated for a free electron in Eq. (12.560):

$$-\frac{1}{2M}\langle n|\mathbf{p}^2|n\rangle J_M(0). \tag{12.568}$$

This produces the same mass renormalization as before. Subtracting this from ΔE_n, we are left with

$$\Delta E'_n = \frac{e^2}{2M^2}\sum_{n'}(\hat{p}_i e^{i\mathbf{k}x})_{nn'}(\hat{p}_j e^{-i\mathbf{kx}})_{n'n} \times \int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\frac{E_{n'}-E_n}{E_{n'}-E_n+k}. \tag{12.569}$$

In order to understand the behavior of the integral, it is useful to further split the k-integral in the soft-photon regime. Thus we introduce the eneries $M\alpha^2 \ll K' \ll K = M\alpha$, and distinguish an upper regime with wave number $K' < k < K$ and a lower regime with $M\alpha^2 \ll k < K'$. The corresponding energy shifts are denoted by $\Delta_{\rm low}E'_n$ and $\Delta_{\rm up}E'_n$, respectively. In the upper regime we may approximate (12.569) by

$$\Delta_{\rm up}E'_n = \frac{e^2}{4M^2}\left[\sum_{n'}(\hat{p}_i e^{i\mathbf{kx}})_{nn'}(\hat{p}_j e^{-i\mathbf{kx}})_{n'n}(E_{n'}-E_n)+(i\leftrightarrow j)\right] \times \int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right). \tag{12.570}$$

The sum over n' can now be expressed with the help of the Hamilton operator $\hat{H}_{\rm C}$ of the Coulomb system as

$$\sum_{n'}\left(\hat{p}_i e^{i\mathbf{kx}}\right)_{nn'}\left(\hat{p}_j e^{-i\mathbf{kx}}\right)_{n'n}(E_{n'}-E_n)+(i\leftrightarrow j) = \langle n|[\hat{p}_i e^{i\mathbf{kx}}[\hat{H}_{\rm C},\hat{p}_j e^{-i\mathbf{kx}}]]|n\rangle. \tag{12.571}$$

Working out the commutators, this reduces to

$$[\hat{p}_i e^{i\mathbf{kx}},[\hat{H}_{\rm C},\hat{p}_j e^{-i\mathbf{kx}}]] = \partial_i\partial_j V_{\rm C}(\mathbf{x}) + \dots\ , \tag{12.572}$$

where the omitted terms contain a factor k_i, k_j, or both. Due to their longitudinal nature, they do not contribute to (12.571), where they are contracted with the transverse projection tensor $\delta_{ij} - k_i k_j/k^2$. Thus we find

$$\Delta_{\rm up} E'_n = \frac{e^2}{4M^2} \langle n|\partial_i \partial_j V_{\rm C}(\mathbf{x})|n\rangle \int_{K>|\mathbf{k}|>K'} \frac{d^3k}{(2\pi)^3} \frac{1}{k^3} \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right). \tag{12.573}$$

Doing the integral over all $\mathbf{k}$-directions yields

$$\Delta_{\rm up} E'_n = \frac{e^2}{4M^2} \langle n|\boldsymbol{\nabla}^2 V_{\rm C}(\mathbf{x})|n\rangle \frac{1}{3\pi^2} \int_{K'}^{K} \frac{dk}{k}. \tag{12.574}$$

Inserting

$$\boldsymbol{\nabla}^2 V_{\rm C}(x) = 4\pi Z\alpha \delta^{(3)}(\mathbf{x}), \tag{12.575}$$

this can be rewritten as

$$\Delta_{\rm up} E'_n = \frac{4\alpha Z}{3\pi M^2} |\psi_n(\mathbf{0})|^2 \log \frac{K}{K'}. \tag{12.576}$$

Note that this expression matches smoothly with the energy shift (12.544) caused by hard photons. By adding the two results, the intermediate energy scale K cancels, producing an energy shift depending only on the separation parameter K':

$$(\Delta_{\rm hard} + \Delta_{\rm up}) E'_n = \frac{4\alpha}{3\pi M^2} |\psi_n(\mathbf{0})|^2 \left(\log \frac{M}{K'} - \frac{3}{8} - \frac{1}{5}\right). \tag{12.577}$$

Consider now the lower soft-photon part of the integral (12.569). Here we take advantage of the fact that, due to the presence of the atomic probability distribution $|\psi_n(\mathbf{x})|^2$, the integration over $\mathbf{x}$ is limited to a range $|\mathbf{x}| \lesssim a_{\rm B} = 1/\alpha M$. Since $|\mathbf{k}| < K' \ll \alpha M$ we see that

$$|\mathbf{kx}| \ll 1, \tag{12.578}$$

so that we can neglect the exponential $e^{i\mathbf{kx}}$ in the matrix elements $(\hat{\mathbf{p}} e^{i\mathbf{kx}})_{nn'}$. Thus we may evaluate the simpler expression

$$\Delta_{\rm low} E'_n \approx \frac{e^2}{2M^2} \sum_{n'} (p_i)_{nn'} (p_j)_{n'n} \int_0^K \frac{d^3k}{(2\pi)^3} \frac{1}{k^2} \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \frac{E_{n'} - E_n}{E_{n'} - E_n + k}. \tag{12.579}$$

This approximation amounts to neglecting the recoil of the atom. Performing the integral over all $\mathbf{k}$-directions yields

$$\Delta_{\rm low} E'_n \approx \sum_{n'} \frac{|\mathbf{p}_{nn'}|^2}{2M} J_M(n, n') \tag{12.580}$$

with

$$J_M(n, n') = \frac{2\alpha}{\pi M} \frac{2}{3} \int_0^{K'} dk \frac{E_{n'} - E_n}{E_{n'} - E_n + k}. \tag{12.581}$$

After doing the k-integral, this becomes

$$J_M(n,n') \approx \frac{4\alpha}{3\pi M}(E_{n'}-E_n)\left[\log\frac{K'}{|E_{n'}-E_n|}+i\pi\Theta(E_n-E_{n'})\right]. \tag{12.582}$$

The imaginary part contributes to the decay rate of an atom from the state n into a lower state n'. It can be dropped in the final formula for the energy shift.

Let us decompose

$$\log\frac{K'}{E_{n'}-E_n}=\log\frac{K'}{2E_{1S}}-\log\frac{E_{n'}-E_n}{2E_{1S}}, \tag{12.583}$$

thus separating (12.580) into two sums $\Delta^{(1)}_{\rm low}E_n$ and $\Delta_{\rm low}E_n$. The first of these sums can be treated the same way as in (12.571) and yields a contribution

$$\Delta^{(1)}_{\rm low}E'_n=\frac{4\alpha Z}{3\pi M^2}|\psi_n(\mathbf{0})|^2\log\frac{K'}{2E_{1S}}. \tag{12.584}$$

Together with the energy shift (12.577), this becomes

$$(\Delta_{\rm hard}+\Delta_{\rm up}+\Delta^{(1)}_{\rm low})E'_n=\frac{4\alpha Z}{3\pi M^2}|\psi_n(\mathbf{0})|^2\bar{C}=\frac{4\alpha^3Z^4}{3\pi n^3}\alpha^2M\bar{C} \tag{12.585}$$

with the constant

$$\bar{C}=\log\frac{M}{2E_{1S}}-\frac{3}{8}-\frac{1}{5}=\log\frac{1}{\alpha^2}-\frac{3}{8}-\frac{1}{5}\approx 9.265. \tag{12.586}$$

The separation parameter K' has disappeared, and the result is unambiguous.

For the $n=2$ -state of the hydrogen atom, the numerical value is

$$\begin{aligned}(\Delta_{\rm hard}+\Delta_{\rm up}+\Delta^{(1)}_{\rm low})E'_n &\approx 135.6\text{ MHz}\times\bar{C}\\ &\approx (1\,334-51)\text{ MHz}\approx 1\,283\text{ MHz}.\end{aligned} \tag{12.587}$$

This is still larger than the experimental value 1 057 MHz. The relativistic treatment of the hard regime together with the upper soft-photon regime have produced a number which lies about 4% below the upper rough estimate (12.538).

It remains to calculate the second sum $\Delta_{\rm low}E'_n$ in the lower soft-photon regime which contains the effect of the wave functions in an essential way. This sum is slightly more involved and requires evaluating a detailed spectral sum. We shall write it as

$$\Delta_{\rm low}E'_n=-\frac{2\alpha}{3\pi}M^2S_n \tag{12.588}$$

where S_n denotes the sum

$$S_n=\sum_{n'}|\mathbf{p}_{nn'}|^2(E_{n'}-E_n)\log\frac{|E_{n'}-E_n|}{2E_{1S}}. \tag{12.589}$$

It is convenient to define an average energy $E^{a\nu}$ by the relation

$$S_n = \left[\sum_{n'} |\mathbf{p}_{nn'}|^2 (E_{n'} - E_n)\right] \log \frac{E^{\text{av}}}{2E_{1S}}. \tag{12.590}$$

The bracket is, by virtue of (12.571), equal to

$$\begin{aligned}\sum_{n'} |\mathbf{p}_{nn'}|^2 (E_{n'} - E_n) &= \frac{1}{2}\langle n|[\hat{\mathbf{p}} e^{i\mathbf{kx}}[\hat{H}_{\text{C}}, \hat{\mathbf{p}} e^{-i\mathbf{kx}}]]|n\rangle = \frac{1}{2}\langle n|\boldsymbol{\nabla}^2 V_{\text{C}}|n\rangle \\ &= \frac{1}{2} 4\pi Z\alpha |\psi_n(\mathbf{0})|^2 = \frac{2Z^4\alpha}{n^3}.\end{aligned} \tag{12.591}$$

This provides a useful check for the convergence of the calculation. An explicit evaluation of the sum gives, for the $2S$ level [32]:

$$E^{\text{av}} = 8.320\ \alpha^2 M. \tag{12.592}$$

By writing S_n in the form (12.590), the effect of the correction is to subtract a term $\log(2E_{1S}/E^{\text{av}})$ from $\bar{C}$ in (12.585), thus producing the result

$$\Delta E_n = \frac{4\alpha}{3\pi M^2} |\psi_n(\mathbf{0})|^2 C_2 = \frac{4\alpha^3 Z^4}{3\pi n^3} \alpha^2 M C_2, \tag{12.593}$$

with

$$C^{\text{tot}}_{2S^{1/2}} = \bar{C} - \log \frac{2E_{1S}}{E^{\text{av}}} = \log \frac{1}{\alpha^2} - \frac{3}{8} - \frac{1}{5} - \log \frac{2E_{1S}}{E^{\text{av}}} \approx 7.146. \tag{12.594}$$

In combining the hard-photon with the two soft-photon results we have been a bit careless since the first result (12.544) was derived with a finite photon mass $\alpha^2 M \gg \mu \ll \alpha M$ as an infrared cutoff parameter. The calculation of the finite correction (12.595) should be done in the same way, i.e., we should integrate

$$\begin{aligned}\Delta_{\text{low}} E'_n \approx\ & \frac{e^2}{2M^2} \sum_{n'} (p_i)_{nn'} (p_j)_{n'n} \\ & \times \int_0^K \frac{d^3k}{(2\pi)^3} \frac{1}{k^2 + \mu^2} \left(\delta_{ij} - \frac{k_i k_j}{k^2 + \mu^2}\right) \frac{E_{n'} - E_n}{E_{n'} - E_n + \sqrt{k^2 + \mu^2}}.\end{aligned} \tag{12.595}$$

The photon mass changes (12.581) into

$$J'_M(n, n') = \frac{2\alpha}{\pi M} \int_0^{K'} dk \left(1 - \frac{1}{3}\frac{k^2}{k^2 + \mu^2}\right) \frac{E_{n'} - E_n}{E_{n'} - E_n + \sqrt{k^2 + \mu^2}}. \tag{12.596}$$

The difference between the two integrals is

$$J'_M(n, n') - J_M(n, n') = \frac{4\alpha}{3\pi M}(E_{n'} - E_n) \left[\log \frac{\mu}{2|E_{n'} - E_n|} + \frac{5}{6} + i\pi\Theta(E_n - E_{n'})\right]. \tag{12.597}$$

A comparison with (12.582) shows that the constant $C^{\rm tot}_{2S^{1/2}}$ in (12.594) receives an additional correction of $5/6 - \log 2 \approx 0.140$, bringing it up to

$$C^{\rm tot}_{2S^{1/2}} = \log\frac{M}{2E_{1S}} - \frac{3}{8} - \frac{1}{5} - \log\frac{2E_{1S}}{E^{\rm av}} + \frac{5}{6} - \log 2 \approx 7.286, \tag{12.598}$$

corresponding to an energy shift

$$\Delta E_{2S^{1/2}} \approx (1\,334.3 - 51 - 27.3 - 287 + 20)\ \text{MHz} \approx 989\ \text{MHz}, \tag{12.599}$$

which is smaller than the experimental value 1 057 MHz.

An important correction is missing in this calculation: the one caused by the anomalous magnetic moment of the electron. This produces an energy shift of about 68.5 MHz, which brings the Lamb shift from the value (12.599) up to 1 057.5 MHz, in excellent agreement with the experimental number 1 057 MHz.

This contribution will be calculated in the next subsection. Before we come to that, however, we want to observe that, while the relativistic Lamb shift was applied only to s-waves, the effect of the wave functions changes also the energy of states with orbital angular momenta $l \geq 0$. For such wave functions, we may define an average energy analog to (12.590) as

$$S_n = \left[\sum_{n'} |\mathbf{p}_{nn'}|^2 (E_{n'} - E_n)\right]_{l=0} \log\frac{E_{\rm av}}{2E_{1S}}. \tag{12.600}$$

The bracket must be taken for $l = 0$ since it vanishes for $l > 0$ by virtue of the same commutator calculation as in (12.591), to be evaluated between $l \neq 0$-states. By doing the spectral sum one finds the average energy [33]

$$E^{\rm av}_{2P} = 0.9704\ \alpha^2 M. \tag{12.601}$$

This raises the p-wave slightly by

$$\Delta^{\rm av} E_{2P} = \frac{\alpha^3}{6\pi}\alpha^2 M C^{\rm av}_{2P}, \tag{12.602}$$

with

$$C^{\rm av}_{2P} = \log\frac{2E_{1S}}{2E^{\rm av}_{2P}} \approx 0.03, \tag{12.603}$$

i.e., by

$$\Delta^{\rm av} E_{2P} \approx 135.6\ \text{MHz} \times C^{\rm av}_{2P} \approx 4\ \text{MHz}. \tag{12.604}$$

12.21.4 Effect of the Anomalous Magnetic Moment

The relativistic current of the electron was found in Eq. (12.401) to have the form

$$\langle \mathbf{p}'|j^\mu|\mathbf{p}\rangle = e\bar{u}(\mathbf{p}')\left[\gamma^\mu F(q^2) + \frac{i}{2M}\sigma^{\mu\nu} q_\nu G(q^2)\right] u(\mathbf{p}'), \tag{12.605}$$

with the form factors $F(q^2)$ and $G(q^2)$ given by Eqs. (12.403) and (12.391). For small momentum transfers, these can be approximated by [recall (12.407) and (12.418)]

$$\begin{aligned} F(q^2) &\approx 1 + q^2 R_e^{\mathrm{vc}\,2}, \quad R_e^{\mathrm{vc}\,2} = \frac{\alpha}{3\pi M^2}\left(\log\frac{M}{\mu} - \frac{3}{8}\right), \\ G(q^2) &\approx \frac{\alpha}{2\pi}. \end{aligned} \tag{12.606}$$

In configuration space, this amounts to an effective extra electromagnetic interaction of the Dirac field of the electron which can be written as a Lagrangian density

$$\mathcal{L}_{\mathrm{eff}} = -e\bar{\psi}(x)\left\{\gamma^\mu\left[1 - R_e^{\mathrm{vc}\,2}\ \Box\right]A_\mu(x) + \frac{i}{2M}\sigma^{\mu\nu} i\frac{\alpha}{2\pi}\partial_\nu A_\mu(x)\right\}\psi(x). \tag{12.607}$$

In a static electric field, the radiative corrections provide a solution of the Dirac equation $\psi_{njm}(\mathbf{x})$ with an additional energy

$$\Delta E = e\bar{\psi}(\mathbf{x})\left[\gamma^0 R_e^{\mathrm{vc}\,2}\ \boldsymbol{\nabla}^2 A_0(\mathbf{x}) + \frac{i\alpha}{4\pi M}\boldsymbol{\gamma}\cdot\mathbf{E}(\mathbf{x})\right]\psi(\mathbf{x}). \tag{12.608}$$

To lowest order in α, we approximate the solutions to the Dirac equation by combinations of the nonrelativistic Schrödinger wave functions ψ_{nlm} with rest spinors $u(\mathbf{0}, s_3)$, combining them to state of total angular momentum j with the help of Clebsch-Gordan coefficients, as shown in Eq. (6.184). The first term in (12.608) leads precisely to the relativistic energy shift $\Delta_{\mathrm{hard}}E$ calculated in (12.544). However, that calculation lacked the energy shift due to the second term, which arises from the anomalous magnetic moment:

$$\Delta E^{\mathrm{a}} = i\frac{e\alpha}{4\pi M}\int d^3x\, \bar{\psi}(\mathbf{x})\boldsymbol{\gamma}\psi(\mathbf{x})\mathbf{E}(\mathbf{x}). \tag{12.609}$$

In order to calculate this we have to approximate the solutions of the Dirac equations a little better than so far. We go to the Dirac representation of the gamma matrices and decompose the bispinor into two simple spinors, as in Eq. (6.184),

$$\psi(\mathbf{x}) = \begin{pmatrix} \xi(x) \\ \eta(x) \end{pmatrix}. \tag{12.610}$$

These satisfy the Dirac equation (4.500). We observed in (4.580) that the lower, small components are related to the upper large ones by [see also (6.113)]

$$\eta(\mathbf{x}) = -i\frac{\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}}{M}\xi(\mathbf{x}) + \mathcal{O}(\alpha). \tag{12.611}$$

Neglecting the corrections of the order of α and inserting (12.611) into (12.610) and (12.609), thereby using the explicit form of the electric field

$$\mathbf{E}(\mathbf{x}) = -Z\alpha\frac{\mathbf{x}}{|\mathbf{x}|^3}, \tag{12.612}$$

the integrand is found to contain an expression

$$\bar{\psi}(\mathbf{x})\boldsymbol{\gamma}\cdot\frac{\mathbf{x}}{|\mathbf{x}|^3}\psi(\mathbf{x}) = \xi^\dagger(\mathbf{x})\frac{\boldsymbol{\sigma}\cdot\mathbf{x}}{|\mathbf{x}|^3}\eta(\mathbf{x}) - \eta^\dagger(\mathbf{x})\gamma\frac{\boldsymbol{\sigma}\cdot\mathbf{x}}{|\mathbf{x}|^3}\xi(\mathbf{x}) \approx \frac{1}{2Mi}\left[\xi^\dagger(\mathbf{x})\frac{\boldsymbol{\sigma}\cdot\mathbf{x}}{|\mathbf{x}|^3}\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\xi(\mathbf{x}) + \text{h.c.}\right]. \tag{12.613}$$

After an integration by parts this leads to

$$\int d^3x\bar{\psi}(\mathbf{x})\boldsymbol{\gamma}\cdot\frac{\mathbf{x}}{|\mathbf{x}|^3}\psi(x) \approx \xi^\dagger(\mathbf{x})\left[\boldsymbol{\sigma}\cdot\boldsymbol{\nabla},\frac{\boldsymbol{\sigma}\cdot\mathbf{x}}{|\mathbf{x}|^3}\right]\xi(\mathbf{x}). \tag{12.614}$$

We now observe that

$$\boldsymbol{\nabla}\cdot\frac{\mathbf{x}}{|\mathbf{x}|^3} = -\boldsymbol{\nabla}^2\frac{1}{|\mathbf{x}|} = 4\pi\delta^{(3)}(\mathbf{x}), \tag{12.615}$$

which allows us to calculate the commutator, using the chain rule, as being

$$\left[\boldsymbol{\sigma}\cdot\boldsymbol{\nabla},\frac{\boldsymbol{\sigma}\cdot\mathbf{x}}{|\mathbf{x}|^3}\right] = -\boldsymbol{\nabla}^2\frac{1}{|\mathbf{x}|} - 2i\boldsymbol{\sigma}\cdot(\mathbf{x}\times\boldsymbol{\nabla})\frac{1}{|\mathbf{x}|^3} = 4\pi\delta^{(3)}(\mathbf{x}) + 4\frac{1}{|\mathbf{x}|^3}\mathbf{L}\cdot\mathbf{S}. \tag{12.616}$$

The first term gives rise to another energy shift for s-waves, which may be written in the same general form as the previous ones:

$$\Delta E_n = \frac{4\alpha}{3M^2}Z\alpha|\psi_n(\mathbf{0})|^2C^{\text{a}}, \tag{12.617}$$

with the constant being now

$$C^{\text{a}} \equiv \frac{3}{8}. \tag{12.618}$$

The second term depends on the angular momentum of the Dirac wave function. Writing $2\mathbf{L}\cdot\mathbf{S} = \mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2$, the eigenvalues of $\mathbf{L}\cdot\mathbf{S}$ are $j(j+1) - l(l+1) - 3/4$. The expectation value of $1/|\mathbf{x}|^3$ in $l \neq 0$ -states is

$$\langle nlm|\frac{1}{|\mathbf{x}|^3}|nlm\rangle = \frac{2}{l(l+1)(2l+1)n^3}(Z\alpha M)^3. \tag{12.619}$$

Thus we obtain an energy shift for $l \neq 0$-states:

$$\Delta E_n = \frac{4\alpha}{3M^2}Z\alpha|\psi_n(\mathbf{0})|^2C^{\text{a}}_{j,lj} = \frac{4\alpha^3Z^4}{3\pi n^3}\alpha^2MC^{\text{a}}_{j,l} \tag{12.620}$$

with

$$C^{\text{a}}_{j,l} \equiv \frac{3}{8}\frac{1}{2l+1}\begin{cases}\dfrac{1}{l+1} & \text{for} \quad j = l + \dfrac{1}{2}, \\ -\dfrac{1}{l} & \qquad\; j = l - \dfrac{1}{2}; \quad (l \geq 1).\end{cases} \tag{12.621}$$

This leads for a level shift of the s-wave

$$\Delta^{\mathrm{a}} E_{2S^{1/2}} = \frac{\alpha^3}{6\pi}\alpha^2 M C^{\mathrm{a}}_{1/2,0}, \tag{12.622}$$

with

$$C^{\mathrm{a}}_{1/2,0} = \frac{3}{8}. \tag{12.623}$$

Remarkably, the anomalous magnetic moment causes a small energy shift also in states of higher angular momenta. The $2P^{1/2}$-state of the relativistic hydrogen atom is shifted opposite to the $2S^{1/2}$-state by

$$\Delta^{\mathrm{a}} E_{2P^{1/2}} = \frac{\alpha^3}{6\pi}\alpha^2 M C^{\mathrm{a}}_{1/2,1}, \tag{12.624}$$

with

$$C^{\mathrm{a}}_{1/2,1} \equiv -\frac{1}{8}. \tag{12.625}$$

Together with the s-wave splitting (12.623) this leads to the total *Lamb shift*:

$$\Delta^{\mathrm{a}} E = \frac{\alpha^3}{6\pi}\alpha^2 M (C^{\mathrm{a}}_{1/2,0} - C^{\mathrm{a}}_{1/2,1}) = \frac{\alpha^3}{6\pi}\alpha^2 M C^{\mathrm{a}}_{\mathrm{Lamb}}, \tag{12.626}$$

with

$$C^{\mathrm{a}}_{\mathrm{Lamb}} = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}, \tag{12.627}$$

i.e., by

$$\Delta^{\mathrm{a}} E \approx 135.6\ \mathrm{MHz} \times C^{\mathrm{a}}_{\mathrm{Lamb}} \approx 68.9\ \mathrm{MHz}. \tag{12.628}$$

Recently, Lamb shifts have been measured also in heavy atoms where the atomic levels are accessible to X-ray spectroscopy. There the theoretical analysis is much more difficult due to the narrowness of the atomic wave functions and the large size of the nucleus [35].

Appendix 12A Calculation of the Dirac Trace in the Klein-Nishina Formula

The trace (12.289) can be expanded as

$$F = \frac{1}{M^2}\left[\frac{t_1^8 + M^2 t_1^6}{(2pk)^2} + \frac{t_2^8 + M^2 t_2^6}{(2pk)(2p'k')} + \frac{t_3^8 + M^2 t_3^6}{(2pk)(2pk')} + \frac{t_4^8 + M^2 t_4^6}{(2pk')^2}\right], \tag{12A.1}$$

where t_j^i denotes the following Dirac traces involving i gamma matrices:

$$\begin{aligned}
t_1^8 &= \tfrac{1}{4}\mathrm{tr}(\not\epsilon'\not k\not\epsilon\not p\not\epsilon\not k\not\epsilon'\not p'), & t_2^8 &= \tfrac{1}{4}\mathrm{tr}(\not\epsilon'\not k\not\epsilon\not p\not\epsilon'\not k'\not\epsilon\not p'),\\
t_3^8 &= \tfrac{1}{4}\mathrm{tr}(\not\epsilon\not k'\not\epsilon'\not p\not\epsilon\not k\not\epsilon'\not p'), & t_4^8 &= \tfrac{1}{4}\mathrm{tr}(\not\epsilon\not k'\not\epsilon'\not p\not\epsilon'\not k'\not\epsilon\not p'),\\
t_1^6 &= \tfrac{1}{4}\mathrm{tr}(\not\epsilon'\not k\not\epsilon\not\epsilon\not k\not\epsilon'), & t_2^6 &= \tfrac{1}{4}\mathrm{tr}(\not\epsilon'\not k\not\epsilon\not\epsilon'\not k'\not\epsilon),\\
t_3^6 &= \tfrac{1}{4}\mathrm{tr}(\not\epsilon\not k'\not\epsilon'\not\epsilon\not k\not\epsilon'), & t_4^6 &= \tfrac{1}{4}\mathrm{tr}(\not\epsilon\not k'\not\epsilon'\not\epsilon'\not k'\not\epsilon).
\end{aligned} \tag{12A.2}$$

For brevity, we have omitted the symbols of complex conjugation on the outgoing polarization vectors, which may be taken to be real and purely spatial, corresponding to linear polarizations.

Traces involving an odd number of gamma matrices have been omitted, since they vanish. We now use the crossing symmetry (12.286) to rewrite F as

$$F = \frac{1}{M^2}\left[\frac{t_1^8 + M^2 t_1^6}{(2pk)^2} + \frac{t_2^8 + M^2 t_2^6}{(2pk)(2p'k')} + (\epsilon \leftrightarrow \epsilon', k \leftrightarrow -k')\right]. \tag{12A.3}$$

The traces are evaluated using the Wick expansion (12.253), together with the properties

$$p^2 = p'^2 = M^2, \quad k^2 = k'^2 = 0, \quad \epsilon^2 = \epsilon'^2 = -1. \tag{12A.4}$$

Using further the properties in the laboratory frame with $p = (M, 0, 0, 0)$:

$$\begin{aligned} \epsilon p &= \epsilon' p = 0, \\ \epsilon' p' &= \epsilon'(p + k - k') = \epsilon' k, \end{aligned}$$

we calculate t_1^8 as follows: First we use $\epsilon^2 = -1$, and the Dirac relation (12.249) to reduce

$$\not\epsilon \not p \not\epsilon = -\not p \not\epsilon \not\epsilon = \not p. \tag{12A.5}$$

Similarly, $k^2 = 0$ allows us to rewrite

$$\not k \not p \not k = -\not k \not k \not p + 2(kp)\not k = 2(kp)\not k. \tag{12A.6}$$

Then t_1^8 becomes

$$t_1^8 = 2(pk)\frac{1}{4}\mathrm{tr}(\not\epsilon' \not k \not\epsilon' \not p'). \tag{12A.7}$$

Now there are only three Wick contractions in the expansion à la (12.253):

$$t_1^8 = 2(pk)\left[(\epsilon' k)(\epsilon' p') - (\epsilon'\epsilon')(kp') + (\epsilon' p')(k\epsilon')\right], \tag{12A.8}$$

which yield

$$t_1^8 = 2(pk)\left[2(\epsilon' k)(\epsilon' p') + (kp')\right]. \tag{12A.9}$$

With the help of the substitution

$$kp' = -\tfrac{1}{2}[(p' - k)^2 - M^2] = -\tfrac{1}{2}[(p - k')^2 - M^2] = pk', \tag{12A.10}$$

this becomes

$$t_1^8 = 4(pk)(\epsilon' k)^2 + 2(pk')(pk). \tag{12A.11}$$

Note that the similar relation

$$kp = \tfrac{1}{2}[(p + k)^2 - M^2] = \tfrac{1}{2}[(p' + k')^2 - M^2] = p'k', \tag{12A.12}$$

leads to

$$t_4^8 = -4(pk')(\epsilon k')^2 + 2(pk')(pk) \tag{12A.13}$$

arising from t_1^8 via the crossing operation (12.286). The other traces are

$$t_1^6 = t_4^6 = 0, \tag{12A.14}$$

since they contain, in the middle, the products $\not k \not\epsilon \not\epsilon \not k = -\not k \not k = -k^2 = 0$ and $\not k' \not\epsilon' \not\epsilon' \not k' = -\not k' \not k' = -k'^2 = 0$. We further find

$$t_2^6 = t_3^6 = \tfrac{1}{4}\mathrm{tr}(\not\epsilon' \not\epsilon \not\epsilon' \not\epsilon \not k' \not k) = M^2\left[2(kk')(\epsilon\epsilon')^2 - 2(\epsilon\epsilon')(k\epsilon')(k'\epsilon) - (kk')\right], \tag{12A.15}$$

and finally,

$$\begin{aligned} t_2^8 = t_3^8 = &-4(\epsilon\epsilon')^2(kp)(k'p) + 2(\epsilon\epsilon')^2(kk')M^2 - 2(\epsilon\epsilon')(\epsilon k')(\epsilon' k)M^2 - 2(\epsilon k')^2(kp) \\ &+ 2(\epsilon k)^2(k'p) - (kk')M^2 + 2(kp)(k'p). \end{aligned} \tag{12A.16}$$

Hence

$$\frac{1}{2M^2}\left[\frac{t_1^8+M^2t_1^6}{(2pk)^2}+\frac{t_2^8+M^2t_2^6}{(2pk)(2p'k')}\right](\epsilon\leftrightarrow\epsilon',k\leftrightarrow k')=\frac{1}{2M^2}\left\{2(\epsilon\epsilon')^2\right.$$
$$\left.+\left[\frac{1}{2(pk)(pk')}\left((\epsilon k')^2(kp)+(\epsilon' k)^2(k'p)-(kk')(k'p)\right)+(\epsilon\leftrightarrow\epsilon',k\leftrightarrow k')\right]\right\}.$$

The bracket is, explicitly,

$$\frac{1}{2(pk)(pk')}\left\{(kk')[(kp)-(k'p)]\right\}, \tag{12A.17}$$

which can be simplified to

$$\frac{1}{2(pk)(pk')}(kk')^2, \tag{12A.18}$$

using the equation

$$kk'=k(p+k-p')=kp-kp'=kp-k'p. \tag{12A.19}$$

Thus we obtain [34]

$$F=\frac{1}{2M^2}\left[\frac{(k'k)^2}{(pk)(pk')}+4(\epsilon'\epsilon)^2\right]. \tag{12A.20}$$

In the laboratory frame where

$$pk'=M\omega',\quad pk=M\omega, \tag{12A.21}$$

and Compton's relation ensures that (12.264),

$$k'k=\omega'\omega(1-\cos\theta)=M\omega'\omega\left(\frac{1}{\omega'}-\frac{1}{\omega}\right)=M\left(\omega-\omega'\right),$$

$$pk'=M\omega',\quad pk=M\omega, \tag{12A.22}$$

and expression (12A.20) reduces to (12.254).

In the electron-positron annihilation process, the trace in Eq. (12.289) becomes

$$F=\frac{1}{M^2}\left[\frac{t_1^8-M^2t_1^6}{(2pk)^2}+\frac{t_2^8-M^2t_2^6}{(2pk)(2p'k')}+(\epsilon\leftrightarrow\epsilon',k\leftrightarrow k')\right], \tag{12A.23}$$

and we find, after the replacement $p'=k+k'-p$,

$$F=\frac{1}{2M^2}\left\{\frac{(k'k)\left[(kp)+(k'p)\right]}{(kp)(k'p)}-4(\epsilon'\epsilon)^2\right\}. \tag{12A.24}$$

Inserting (12.305) and using $k\,p=\omega M$ and $k'\,p=\omega' M$, this becomes

$$F=\frac{1}{2M^2}\left[\frac{(k'k)^2}{4(kp)(k'p)}-4(\epsilon'\epsilon)^2\right], \tag{12A.25}$$

instead of (12A.20), thus yielding Eq. (12.290) which we wanted to derive.

Notes and References

More on this subject can be found in the textbooks
S. Schweber, *Relativistic Quantum Fields*, Harper and Row, N.Y., N.Y., 1961;
J.D. Bjorken and S.D. Drell, Vol. I: *Relativistic Quantum Mechanics*, Vol. II: *Relativistic Quantum Fields*, McGraw-Hill, New York, 1965;
J.M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons*, Springer, Second Edition, Berlin 1975;
L.H. Ryder, *Quantum Field Theory*, Second edition, Cambridge University Press, 1996, Cambridge;
M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory*, Addison-Wesley, New York, 1995;
and, most of all,
C. Itzykson and J.-B. Zuber, *Quantum Field Theory*, McGraw-Hill, New York (1985).

The individual citations refer to:

[1] L.D. Faddeev and V.N. Popov, Phys. Lett. B *25*, 29 (1967).

[2] J.C. Ward, Phys. Rev. **78**, 182 (1959).

[3] Y. Takahashi, Nuovo Cimento **6**, 371 (1957).

[4] F. Rohrlich, Phys. Rev. **80**, 666 (1950).

[5] See the third of the above textbooks;

[6] J. Schwinger, Phys. Rev. **125**, 397 (1962); ibid. **128**, 2425 (1962).

[7] A. Ashkin, L.A. Page, and W.M. Woodward, Phys. Rev. **94**, 1974 (p. 357).

[8] K. Strauch in *Proceedings of the Sixth Symposium on Electron and Photon Interactions at High Energies*, edited by H. Rollnik and W. Pfeil, North-Holland, Amsterdam, 1974.

[9] S. Coleman, R. Jackiw, and L. Susskind, Ann. Phys. **93**, 267 (1975);
S. Coleman, ibid. **101**, 239 (1976);
C. Adam, Nucl. Phys. B **54**, 198 (1997).

[10] G. 't Hooft and M.T. Veltman, Nucl. Phys. B **44**, 189 (1972).
For earlier related publications see:
C. Bollini, J. Giambagi, and A.G. Dominguez, Nuovo Cimento **31**, 550 (1964);
P. Breitenlohner and H. Mitter, Nucl. Phys. B **7**, 443 (1968);
C. Bollini and J. Giambagi, Nuovo Cimento B **12** (1972);
J.F. Ashmore, Lett. Nuovo Cim. **4**, 289 (1972).

[11] See the textbook
H. Kleinert, *Multivalued Fields in Condensed Matter, Electromagnetism, and Gravitation*, World Scientific, Singapore 2008, pp. 1-497, (`klnrt.de/b11`)

[12] See for example
J.D. Jackson, *Classical Electrodynamics*, Wiley and Sons, New York, 1967, or
H. Ohanian, *Classical Electrodynamics*, Allyn and Bacon, Boston, Mass., 1988.

[13] W. Gordon, Ann. d. Phys. **2**, 1031 (1929).
The matrix elements (12.177) can also be found very simply by group theory, as shown in H. Kleinert, Fortschr. Phys. **6**, 1 (1968), and *Group Dynamics of the Hydrogen Atom*, Lectures presented at the 1967 Boulder Summer School, published in *Lectures in Theoretical Physics*, Vol. X B, pp. 427–482, ed. by A.O. Barut and W.E. Brittin, Gordon and Breach, New York, 1968.

[14] G.S. Atkins and E.D. Pfahl, Phys. Rev. A **59**, R915 (1999);
G.S. Atkins and F.R. Brown, Phys. Rev. A **28**, 1164 (1983).

[15] For a review of the experimental situation see the internet page http://int.phys.washington.edu/~int_talk/WorkShops/int_02_3/People/Vetter_P/pstalk.pdf.

[16] See
R.S. Van Dyck, in *Quantum Electrodynamics*, ed. by T. Kinoshita, World Scientific, Singapore, 1990, p. 322.

[17] The numerical value uses the fine-structure constant $\alpha^{-1} = 137.035\,997\,9(3\,2)$ found with the help of the quantum Hall effect. See M.E. Cage et al., IEEE Trans. Instrum. Meas. **38**, 284 (1989).

[18] J. Schwinger, Phys. Rev. **73**, 416 (1948).

[19] For a discussion see the review by
J. Gailey and E. Picasso, Progr. Nucl. Phys. **12**, Part 1, eds. D.M. Brink and J.H. Mulvey, Pergamon, Oxford, 1970, pp. 43-.

[20] The calculation of c_2 was performed in the Feynman gauge by
A. Petermann, Helv. Phys. Acta **30**, 407 (1957);
C.M. Sommerfield, Phys. Rev. **107**, 328 (1957);
and in another gauge by
G.S. Adkins, Phys. Rev. D **39**, 3798 (1989).
The number for c_3 was published in T. Kinoshita, IEEE Trans. Instrum. Meas. **38**, 172 (1989).
See also the review articles by T. Kinoshita as well as by
R.Z. Roskies, E. Remiddi, and M.J. Levine, in *Quantum Electrodynamics*, ed. by T. Kinoshita, World Scientific, Singapore, 1990, p. 218 and p. 162, respectively.

[21] T. Kinoshita in *Quantum Electrodynamics*, ed. by T. Kinoshita, World Scientific, Singapore, 1990, p. 419; A. Czarnecki and W.J. Marciano (hep-ph/0102122).

[22] The small errors are estimated by
M. Davier, Nucl. Phys. B (Proc. Suppl.) **76**, 327 (1999) (hep-ex/9912044).
The second, more conservative estimate is due to
F. Jegerlehner, in *Radiative Corrections*, edited by J. Solà (World Scientific, Singapore, 1999), pp. 75–89. See also his lectures in a seminar at New York University in honor of A. Sirlin's 70th Birthday.

[23] R. Jackiw and S. Weinberg, Phys. Rev. D **5**, 2473 (1972);
W.A. Bardeen, R. Gastmans, and B.E. Lautrup, Nucl. Phys. B **46**, 315 (1972).

[24] J. Bailey et al., Phys. Lett. B **68**, 191 (1977);
F.J.M. Farley and E. Picasso, in *Quantum Electrodynamics*, ed. by T. Kinoshita, World Scientific, Singapore, 1990, p. 479.
H.N. Brown et al., Phys. Rev. Lett. **86**, 2227 (2001) (hep-ex/0102017);
G.W. Bennett et al., Phys. Rev. Lett. **89**, 101804 (2002); Erratum-ibid. **89**, 129903 (2002) (hep-ex/0208001);
See also http://phyppro1.phy.bnl.gov/g2muon.

[25] W. Gordon, Ann. d. Phys. **2**, 1031 (1929).
The matrix elements (12.177) can also be found very simply by group theory, as shown by H. Kleinert, Fortschr. Phys. **6**, 1, (1968), and *Group Dynamics of the Hydrogen Atom*, Lectures presented at the 1967 Boulder Summer School, published in *Lectures in Theoretical Physics*, Vol. X B, pp. 427–482, ed. by A.O. Barut and W.E. Brittin, Gordon and Breach, New York, 1968.

[26] For details the reader is referred to Bernard de Wit's lecture on this subject in http://www.phys.uu.nl/~bdewit/ftip/AppendixE.pdf.

[27] The Schwinger model with a nonzero initial electron mass is discussed in Ref. [9]

[28] Y. Takahashi, Nuovo Cimento **6**, 370 (1957).

[29] H.A. Bethe, Phys. Rev. **72**, 339 (1947);
J. Schwinger and V. Weisskopf, Phys. Rev. **73**, 1272A (1948);
R.P. Feynman, Phys. Rev. **76**, 939 (1948);
N.M. Kroll and W.E. Lamb, Phys. Rev. **75**, 388 (1949);
J.B. French and V. Weisskopf, Phys. Rev. **75**, 1240 (1949);
H.A. Bethe, L.M. Brown, and J.R. Stehn, Phys. Rev. **77**, 370 (1950);
A.J. Layzer, Phys. Rev. Lett. **4**, 580 (1960);
G.W. Erickson and D.R. Yennie, Annals of Phys. **35**, 271, 447 (1965);
G.W. Erickson, Phys. Rev. Lett. **27**, 780 (1971);
P.J. Mohr, Phys. Rev. Lett. **34**, 1050 (1975).

[30] The first experiment was done by
W.E. Lamb and R.C. Retherford, Phys. Rev. **72**, 241 (1947); **79**, 549 (1950); **81**, 222 (1951); **86**, 1014 (1951).
See also the review
W.E. Lamb, Rep. Progr. Phys. **14**, 19 (1951).
A more accurate value $\Delta E = 1\,057.8 \pm 0.1$ MHz was found by
S. Triebwasser, E.E. Dayhoff, and W.E. Lamb, Phys. Rev. **89**, 98 (1953).

[31] E.A. Uehling, Phys. Rev. **49**, 55 (1935).

[32] H.A. Bethe and E.E. Salpeter in *Encyclopedia of Physics (Handbuch der Physik)*, Springer, Berlin, 1957, p. 405.

[33] For more average energy values see J.M. Harriman, Phys. Rev. **101**, 594 (1956).

[34] The calculation can actually be performed with a simple reduce program which can be downloaded from `http://www.physik.fuberlin.de/~kleinert/b6/programs/reduce`.

[35] For a review see
P.J. Mohr, G. Plunien, and G. Seff, *QED-Corrections in Heavy Atoms*, Phys. Rep **293**, 227 (1998).