Agri non omnes frugiferi sunt
Not all fields are fruitful
CICERO (106 BC–43 BC), Tusc. Quaest., 2, 5, 13

19

Relativistic Particle Orbits

Particles moving at large velocities near the speed of light are called *relativistic* particles. If such particles interact with each other or with an external potential, they exhibit quantum effects which cannot be described by fluctuations of a single particle orbit. Within short time intervals, additional particles or pairs of particles and antiparticles are created or annihilated, and the total number of particle orbits is no longer invariant. Ordinary quantum mechanics which always assumes a fixed number of particles cannot describe such processes. The associated path integral has the same problem since it is a sum over a given set of particle orbits. Thus, even if relativistic kinematics is properly incorporated, a path integral cannot yield an accurate description of relativistic particles. An extension becomes necessary which includes an arbitrary number of mutually linked and *branching* fluctuating orbits.

Fortunately, there exists a more efficient way of dealing with relativistic particles. It is provided by *quantum field theory*. We have demonstrated in Section 7.14 that a grand-canonical ensemble of particle orbits can be described by a functional integral of a single fluctuating field. Branch points of newly created particle lines are accounted for by anharmonic terms in the field action. The calculation of their effects proceeds by perturbation theory which is systematically performed in terms of Feynman diagrams with calculation rules very similar to those in Section 3.18. There are again lines and interaction vertices, and the main difference lies in the lines which are correlation functions of fields rather than position variables $x(t)$. The lines and vertices represent direct pictures of the topology of the worldlines of the particles and their possible collisions and creations.

Quantum field theory has been so successful that it is generally advantageous to describe the statistical mechanics of many completely different types of line-like objects in terms of fluctuating fields. One important example is the polymer field theory in Section 15.12. Another important domain where field theory has been extremely successful is in the theory of line-like defects in crystals, superfluids, and superconductors. In the latter two systems, the defects occur in the form of quantized vortex lines or quantized magnetic flux lines, respectively. The entropy of their classical shape fluctuations determines the temperature where the phase transitions take place. Instead of the usual way of describing these systems as ensembles of particles with their interactions, a field theory has been developed whose Feynman diagrams are the direct pictures of the line-like defects, called *disorder field theory* [1].

The most important advantage of field theory is that it can describe most easily phase transitions, in which particles form a condensate. The disorder theory is therefore particularly suited to understand phase transitions in which defect-, vortex-, or flux-lines proliferate, which happens in the processes of crystal melting, superfluid to normal, or superconductor to normal transitions, respectively. In fact, the disorder theory is so far the only theory in which the critical behavior of the superconductor near the transition is properly understood [2].

A particular quantum field theory, called *quantum electrodynamics* describes with great success the electromagnetic interactions of electrons, muons, quarks, and photons. It has been extended successfully to include the weak interactions among these particles and, in addition, neutrinos, using only a few quantized Dirac fields and a quantized electromagnetic vector potential. The inclusion of a nonabelian gauge field, the gluon field, is a good candidate for explaining all known features of strong interactions.

It is certainly unnecessary to reproduce in an orbital formulation the great amount of results obtained in the past from the existing field theory of weak, electromagnetic, and strong interactions. The orbital formulation was, in fact, proposed by Feynman back in 1950 [3], but never pursued very far due to the success of quantum field theory. Recently, however, this program was revived in a number of publications [4, 5]. The main motivation for this lies in another field of fundamental research: the *string theory* of fundamental particles. In this theory, all elementary particles are supposed to be excitations of a single line-like object with tension, and various difficulties in obtaining a consistent theory in the physical spacetime have led to an extension by fermionic degrees of freedom, the result being the so-called *superstring*. Strings moving in spacetime form worldsurfaces rather than worldlines. They do not possess a second-quantized field theoretic formulation. Elaborate rules have been developed for the functional integrals describing the splitting and merging of strings. If one cancels one degree of freedom in such a superstring, one has a theory of splitting and merging particle worldlines. As an application of the calculation rules for strings, processes which have been known from calculations within the quantum field theory have been recalculated using these reduced superstring rules. In this textbook, we shall give a small taste of such calculations by evaluating the change in the vacuum energy of electromagnetic fields caused by fluctuating relativistic spinless and spin-1/2 particles.

It should be noted that since up to now, no physical result has emerged from superstring theory,[1] there is at present no urgency to dwell deeper into the subject.

By giving a short introduction into this subject we shall be able to pay tribute to some historic developments in quantum mechanics, where the relativistic generaliza-

[1]This theory really deserves a price for having the highest popularity-per-physicality ratio in the history of science, enjoying a great amount of financial support. The situation is very similar to the geocentric medieval picture of the world.

tion of the Schrödinger equation was an important step towards the development of quantum field theory [6]. For this reason, many textbooks on quantum field theory begin with a discussion of relativistic quantum mechanics. By analogy, we shall incorporate relativistic kinematics into path integrals.

It should be noted that an esthetic possibility to give a path Fermi statistics is based on the Chern-Simons theory of entanglement of Chapter 16. However, this approach is still restricted to $2+1$ spacetime dimensions [7], and an extension to the physical $3+1$ spacetime dimensions is not yet in sight.

19.1 Special Features of Relativistic Path Integrals

Consider a free point particle of mass M moving through the $3+1$ spacetime dimensions of Minkowski space at relativistic velocity. Its path integral description is conveniently formulated in four-dimensional Euclidean spacetime where the fluctuating worldlines look very similar to the fluctuating polymers discussed in Chapter 15.

Thus, time is taken to be imaginary, i.e.,

$$t = -i\tau = -ix^4/c, \tag{19.1}$$

and the length of a four-vector $x = (\mathbf{x}, x^4)$ is given by

$$x^2 = \mathbf{x}^2 + (x^4)^2 = \mathbf{x}^2 + c^2\tau^2. \tag{19.2}$$

If $x^\mu(\lambda)$ is an arbitrarily parametrized orbit, the well-known classical Euclidean action is proportional to the invariant length of the orbit in spacetime:

$$S = \int_{\lambda_a}^{\lambda_b} d\lambda \sqrt{x'^2(\lambda)}, \tag{19.3}$$

and reads

$$\mathcal{A}_{\rm cl,e} = McS, \tag{19.4}$$

or, explicitly,

$$\mathcal{A}_{\rm cl,e} = Mc\int_{\lambda_a}^{\lambda_b} ds(\lambda), \tag{19.5}$$

with

$$ds(\lambda) \equiv d\lambda\,\sqrt{x'^2(\lambda)} = d\lambda\,\sqrt{\mathbf{x}'^2(\lambda) + c^2\tau'^2(\lambda)}. \tag{19.6}$$

The prime denotes the derivative with respect to the parameter λ. The action is independent of the choice of the parametrization. If λ is replaced by a new parameter

$$\lambda \to \bar\lambda = f(\lambda), \tag{19.7}$$

then

$$x'^2 \to \frac{1}{f'^2}\, x'^2, \tag{19.8}$$

$$d\lambda \to d\lambda\, f', \tag{19.9}$$

so that ds and the action remain invariant.

We now calculate the Euclidean amplitude for the worldline of the particle to run from the spacetime point $x_a = (\mathbf{x}_a, c\tau_a)$ to $x_b = (\mathbf{x}_b, c\tau_b)$. For the sake of generality, we treat the case of D Euclidean spacetime dimensions. Before starting we observe that the action (19.5) does not lend itself easily to a calculation of the path integral over $e^{-\mathcal{A}_{\rm cl,e}/\hbar}$. There exists an alternative form for the classical action that is more suitable for this purpose. It involves an auxiliary field $h(\lambda)$ and reads:

$$\bar{\mathcal{A}}_{\rm e} = \int_{\lambda_a}^{\lambda_b} d\lambda \left[\frac{Mc}{2h(\lambda)} x'^2(\lambda) + h(\lambda)\frac{Mc}{2}\right]. \tag{19.10}$$

This has the advantage of containing the particle orbit quadratically as in a free nonrelativistic action. The auxiliary field $h(\lambda)$ has been inserted to make sure that the classical orbits of the action (19.10) coincide with those of the original action (19.5). Indeed, extremizing $\bar{\mathcal{A}}_{\rm e}$ in $h(\lambda)$ gives the relation

$$h(\lambda) = \sqrt{x'^2(\lambda)}. \tag{19.11}$$

Inserting this back into $\bar{\mathcal{A}}_{\rm e}$ renders the classical action

$$\mathcal{A}_{\rm cl,e} = Mc\int_{\lambda_a}^{\lambda_b} d\lambda \sqrt{x'^2(\lambda)}, \tag{19.12}$$

which is the same as (19.5).

At this point the reader will worry that although the new action (19.10) describes the same classical physics as the original action (19.12), it may lead to a completely different quantum physics of a relativistic partile. With a little effort, however, it can be shown that this is not so. Since the proof is quite technical, it will be given in Appendix 19A.

The new action (19.10) shares with the old action (19.12) the reparametrization invariance (19.7) for arbitrary fluctuating path configurations. We only have to assign an appropriate transformation behavior to the extra field $h(\lambda)$. If λ is replaced by a new parameter $\bar{\lambda} = f(\lambda)$, then x'^2 and $d\lambda$ transform as in (19.8) and (19.9), and the action remains invariant, if $h(\lambda)$ is simultaneously changed as

$$h \to h/f'. \tag{19.13}$$

We now set up the path integral of a relativistic particle associated with the action (19.10). First we sum over the orbital fluctuations at a *fixed* $h(\lambda)$. To find the correct measure of integration, we use the canonical formulation in which the Euclidean action reads

$$\bar{\mathcal{A}}_{\rm e}[p,x] = \int_{\lambda_a}^{\lambda_b} d\lambda \left[-ipx' + \frac{h(\lambda)}{2Mc}\left(p^2 + M^2c^2\right)\right]. \tag{19.14}$$

This must be sliced in the length parameter λ. We form $N+1$ slices as usual, choosing arbitrary small parameter differences $\epsilon_n = \lambda_n - \lambda_{n-1}$ depending on n, and write the sliced action as

$$\bar{\mathcal{A}}_{\rm e}^N[p,x] = \sum_{n=1}^{N+1}\left[-ip_n(x_n - x_{n-1}) + h_n\epsilon_n\frac{p_n^2}{2Mc} + \epsilon_n h_n \frac{Mc}{2}\right]. \tag{19.15}$$

This path integral has a universal phase space measure [recall (2.28)]

$$\int \mathcal{D}^D x \int \frac{\mathcal{D}^D p}{(2\pi\hbar)^D} e^{-\bar{\mathcal{A}}_{\rm e}[p,x]/\hbar} \approx \prod_{n=1}^{N}\left[\int d^D x_n\right] \prod_{n=1}^{N+1}\left[\int \frac{d^D p_n}{(2\pi\hbar)^D} e^{-\bar{\mathcal{A}}_{\rm e}^N[p,x]/\hbar}\right]. \tag{19.16}$$

The momentum variables p_n are integrated out to give the configuration space integrals (setting $\lambda_{N+1} \equiv \lambda_b$, $h_{N+1} \equiv h_b$) [compare (2.79)]

$$\frac{1}{\sqrt{2\pi\hbar\epsilon_b h_b/Mc}^D} \prod_{n=1}^{N}\left[\int \frac{d^D x_n}{\sqrt{2\pi\hbar\epsilon_n h_n/Mc}^D}\right] \exp\left(-\frac{1}{\hbar}\bar{\mathcal{A}}_{\rm e}^N[x]\right), \tag{19.17}$$

with the time-sliced action in configuration space

$$\bar{\mathcal{A}}_{\rm e}^N[x] = \sum_{n=1}^{N+1}\left[\frac{Mc}{2h_n\epsilon_n}(\Delta x_n)^2 + \epsilon_n h_n \frac{Mc}{2}\right]. \tag{19.18}$$

The Gaussian integrals over x_n in (19.17) can now be done successively using Formula (2.75), and we find [as in (2.80)]

$$\frac{1}{\sqrt{2\pi\hbar S/Mc}^D} \exp\left[-\frac{Mc}{2\hbar}\frac{(x_b - x_a)^2}{S} - \frac{Mc}{2\hbar}S\right], \tag{19.19}$$

where S is the total sliced length of the orbit

$$S \equiv \sum_{n=1}^{N+1} \epsilon_n h_n, \tag{19.20}$$

whose continuum limit is

$$S = \int_{\lambda_a}^{\lambda_b} d\lambda\, h(\lambda). \tag{19.21}$$

The result (19.19) does not depend on the function $h(\lambda)$ but only on S, this being a consequence of the reparametrization invariance of the path integral. While the total λ-interval changes under the transformation, the total length S of (19.21) is invariant under the joint transformations (19.7) and (19.13). This invariance permits only the invariant length S to appear in the integrated expression (19.19), and the path integral over $h(\lambda)$ can be reduced to a simple integral over S. The appropriate path integral for the time evolution amplitude reads

$$(x_b|x_a) = \mathcal{N} \int_0^\infty dS \int \mathcal{D}h\,\Phi[h] \int \mathcal{D}^D x\, e^{-\bar{\mathcal{A}}_{\rm e}/\hbar}, \tag{19.22}$$

where $\mathcal{N}$ is some normalization factor and $\Phi[h]$ an appropriate *gauge-fixing functional*.

19.1.1 Simplest Gauge Fixing

The simplest choice for the gauge-fixing functional is a δ-functional,

$$\Phi[h] = \delta[h-1], \tag{19.23}$$

which fixes $h(\lambda)$ to be equal to the light velocity everywhere, and relates

$$S = \lambda_b - \lambda_a. \tag{19.24}$$

This Lorentz-invariant length parameter is the so-called proper length of special relativity, equal to c times the *proper time*. By analogy with the discussion of thermodynamics in Chapter 2 we shall then denote $\lambda_b - \lambda_a$ by $c\hbar\beta$ and write (19.24) as

$$S = \lambda_b - \lambda_a \equiv c\hbar\beta. \tag{19.25}$$

If we further use translational invariance to set $\lambda_a = 0$, we arrive at the gauge-fixed path integral

$$(x_b|x_a) = \mathcal{N} c\hbar \int_0^\infty d\beta\, e^{-\beta Mc^2/2} \int \mathcal{D}^D x\, e^{-\mathcal{A}_{0,\rm e}/\hbar}, \tag{19.26}$$

with

$$\mathcal{A}_{0,\rm e} = \int_0^{\hbar\beta} d\lambda\, \frac{M}{2}\dot{x}^2. \tag{19.27}$$

Here we have gone to a timelike paramter $\tau = \lambda/c$ und therefore used a dot to denote the derivative: $\dot{x}(\tau) \equiv dx(\lambda)/d\tau$. Remarkably, the gauge-fixed action coincides with the action of a free nonrelativistic particle in D Euclidean spacetime dimensions. Having taken the trivial term $\int_0^{\hbar\beta} d\tau\, Mc^2/2\hbar$ out of the action (19.14), the expression (19.26) contains a Boltzmann weight $e^{-\beta Mc^2/2}$ for each particle orbit of mass M.

The solution of the path integral is then given by

$$(x_b|x_a) = \mathcal{N} c\hbar \int_0^\infty d\beta\, \frac{1}{\sqrt{2\pi\hbar^2\beta/M}^D} \exp\left[-\frac{M}{2\hbar}\frac{(x_b-x_a)^2}{\hbar\beta} - \beta\frac{Mc^2}{2}\right]. \tag{19.28}$$

By Fourier-transforming the x-dependence, this amplitude can also be written as

$$(x_b|x_a) = \mathcal{N} c\hbar \int_0^\infty d\beta\, e^{-\beta Mc^2/2} \int \frac{d^Dk}{(2\pi)^D} \exp\left[ik(x_b - x_a) - \beta\frac{\hbar^2 k^2}{2M}\right], \tag{19.29}$$

and evaluated to

$$(x_b|x_a) \;=\; \mathcal{N}\frac{2Mc}{\hbar}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2+M^2c^2/\hbar^2}e^{ik(x_b-x_a)}. \tag{19.30}$$

Upon setting $\mathcal{N} = \lambda_M^{\rm C}/2$, where

$$\lambda_M^{\rm C} \equiv \hbar/Mc, \tag{19.31}$$

is the well-known Compton wavelength of a particle of mass M [recall Eq. (4.377)], this becomes the Green function of the Klein-Gordon field equation in Euclidean time:

$$(-\partial_b^2 + M^2c^2/\hbar^2)(x_b|x_a) = \delta^{(D)}(x_b - x_a). \tag{19.32}$$

In the Fourier representation (19.30), the integral over k [or the integral over β in (19.28)] can be performed with the explicit result for the Green function

$$(x_b|x_a) = \frac{1}{(2\pi)^{D/2}}\left(\frac{Mc}{\hbar\sqrt{x^2}}\right)^{D/2-1} K_{D/2-1}\left(Mc\sqrt{x^2}/\hbar\right), \tag{19.33}$$

where $K_\nu(z)$ denotes the modified Bessel function and $x \equiv x_b - x_a$. In the nonrelativistic limit $c \to \infty$, the asymptotic behavior $K_\nu(z) \to \sqrt{\pi/2z}e^{-z}$ [see Eq. (1.357)] leads to

$$(x_b|x_a) = (\mathbf{x}_b\tau_a|\mathbf{x}_a\tau_a) \xrightarrow{c\to\infty} \frac{\hbar}{2Mc}e^{-Mc^2(\tau_b-\tau_a)/\hbar}\,(\mathbf{x}_b\tau_b|\mathbf{x}_a\tau_a)_{\rm Schr}\,, \tag{19.34}$$

with the usual Euclidean time evolution amplitude of the free Schrödinger equation

$$(\mathbf{x}_b\tau_b|\mathbf{x}_a\tau_a)_{\rm Schr} = \frac{1}{\sqrt{2\pi\hbar(\tau_b-\tau_a)/M}^{D-1}}\exp\left\{-\frac{M}{2\hbar}\frac{(\mathbf{x}_b-\mathbf{x}_a)^2}{\tau_b-\tau_a}\right\}. \tag{19.35}$$

The exponential prefactor in (19.34) contains the effect of the rest energy Mc^2 which is ignored in the nonrelativistic Schrödinger theory.

Note that the same limit may be calculated conveniently in the saddle point approximation to the β-integral (19.28). For $c \to \infty$, the exponent has a sharp extremum at

$$\beta = \frac{\sqrt{(x_b-x_a)^2}}{c\hbar} = \frac{\sqrt{(\mathbf{x}_b-\mathbf{x}_a)^2+c^2(\tau_b-\tau_a)^2}}{c\hbar} \underset{c\to\infty}{\to} \frac{\tau_b-\tau_a}{\hbar} + \frac{(\mathbf{x}_b-\mathbf{x}_a)^2}{2c^2\hbar(\tau_b-\tau_a)} + \ldots\,, \tag{19.36}$$

and the β-integral can be evaluated in a quadratic approximation around this value. This yields once again (19.34).

19.1.2 Partition Function of Ensemble of Closed Particle Loops

The diagonal amplitude (19.26) with $x_b = x_a$ contains the sum over all lengths and shapes of a *closed* particle loop in spacetime. This sum can be made a partition function of a closed loop if we remove a degeneracy factor proportional to $1/L$ from the integral over L. Then all cyclic permutations of the points of the loop are counted only once. Apart from an arbitrary normalization factor to be fixed later, the partition function of a single closed loop reads

$$Z_1 = \int_0^\infty \frac{d\beta}{\beta}\, e^{-\beta Mc^2/2} \int \mathcal{D}^D x\, e^{-\mathcal{A}_{0,\mathrm{e}}/\hbar}. \tag{19.37}$$

Inserting the right-hand integral in (19.29) for the path integral (with $x_b = x_a$), this becomes

$$Z_1 = V_D \int_0^\infty \frac{d\beta}{\beta}\, e^{-\beta Mc^2/2} \int \frac{d^D k}{(2\pi)^D} \exp\left(-\beta \frac{\hbar^2 k^2}{2M}\right), \tag{19.38}$$

where V_D is the total volume of spacetime. This can be evaluated immediately. The Gaussian integral gives for each of the D dimensions a factor $1/\sqrt{2\pi\hbar^2\beta/M}$, after which formula (2.498) leads to

$$Z_1 = V_D \int_0^\infty \frac{d\beta}{\beta}\, e^{-\beta Mc^2/2} \frac{1}{\sqrt{2\pi\hbar^2\beta/M}^{\,D}} = \frac{V_D}{\lambda_M^{\mathrm{C}\,D}} \frac{\Gamma(1-D/2)}{(4\pi)^{D/2}}, \tag{19.39}$$

where λ_M^{C} is the Compton wavelength (19.31). With the help of the sloppy formula (2.506) of analytic regularization which implies the minimal subtraction explained in Subsection 2.15.1, the right-hand side of (19.38) can also be written as

$$Z_1 = -V_D \int \frac{d^D k}{(2\pi)^D} \log\left(k^2 + M^2c^2/\hbar^2\right). \tag{19.40}$$

The right-hand side can be expressed in functional form as

$$Z_1 = -\mathrm{Tr}\,\log\left(-\partial^2 + M^2c^2/\hbar^2\right) = -\mathrm{Tr}\,\log\left(-\hbar^2\partial^2 + M^2c^2\right), \tag{19.41}$$

the two expressions being equal in the analytic regularization of Section 2.15, since a constant inside the logarithm gives no contribution by Veltman's rule (2.508).

The partition function of a grand-canonical ensemble is obtained by exponentiating this:

$$Z = e^{Z_1} = e^{-\mathrm{Tr}\,\log\left(-\hbar^2\partial^2 + M^2c^2\right)}. \tag{19.42}$$

In order to interprete this expression physically, we separate the integral $\int d^D k/(2\pi)^D$ into an integral over the temporal component k^D and a spatial remainder, and write

$$k^2 + M^2c^2/\hbar^2 = \left(k^D\right)^2 + \omega_{\mathbf{k}}^2/c^2, \tag{19.43}$$

with the frequencies

$$\omega_{\mathbf{k}} \equiv c\sqrt{\mathbf{k}^2 + M^2c^2/\hbar^2}. \tag{19.44}$$

Recalling the result (2.505) of the integral (2.491) we obtain

$$Z_1 = -2V_D \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{\hbar\omega_{\mathbf{k}}}{2c}. \tag{19.45}$$

The exponent is the sum of two ground state energies of oscillators of energy $\hbar\omega_{\mathbf{k}}/2$, which are the vacuum energies associated with two relativistic particles. In quantum field theory one learns that these are particles and antiparticles. Many neutral particles are identical to their antiparticles, for example photons, gravitons, and the pion with zero charge. For these, the factor 2 is absent. Then the integral (19.38) contains a factor 1/2 accounting for the fact that paths running along the same curve in spacetime but in the opposite sense are identified.

Comparing (19.42) with (3.559) and (3.622) for $j = 0$ we identify $-Z_1\hbar$ with $-W[0]$ and the Euclidean effective action Γ of the ensemble of loops:

$$-Z_1 = -W[0]/\hbar = \Gamma_{\rm e}/\hbar. \tag{19.46}$$

19.1.3 Fixed-Energy Amplitude

The fixed-energy amplitude is related to (19.22) by a Laplace transformation:

$$(\mathbf{x}_b|\mathbf{x}_a)_E \equiv -i\int_{\tau_a}^{\infty} d\tau_b \, e^{E(\tau_b-\tau_a)/\hbar} \, (x_b|x_a) \, , \tag{19.47}$$

where τ_b, τ_a are once more the time components in x_b, x_a. As explained in Chapter 9, the poles and the cut along the energy axis in this amplitude contain all information on the bound and continuous eigenstates of the system. The fixed-energy amplitude has the reparametrization-invariant path integral representation which reads with the conventions of Eq. (19.10):

$$(\mathbf{x}_b|\mathbf{x}_a)_E = \frac{\hbar}{2Mc}\int_0^{\infty} dL \int \mathcal{D}h \, \Phi[h] \int \mathcal{D}^D x \, e^{-\bar{\mathcal{A}}_{{\rm e},E}/\hbar}, \tag{19.48}$$

with the Euclidean action

$$\bar{\mathcal{A}}_{{\rm e},E} = \int_{\lambda_a}^{\lambda_b} d\lambda \left[\frac{Mc}{2h(\lambda)}\mathbf{x}'^2(\lambda) - h(\lambda)\frac{E^2}{2Mc^3} + h(\lambda)\frac{Mc}{2} \right]. \tag{19.49}$$

To prove this, we write the temporal x^D-part of the sliced D-dimensional action (19.18) in the canonical form (19.15). In the associated path integral (19.16), we integrate out all x_n^D-variables, producing N δ-functions. These remove the integrals over N momentum variables p_n^D, leaving only a single integral over a common p^D. The Laplace transform (19.47), finally, eliminates also this integral making p^D equal to $-iE/c$. In the continuum limit, we thus obtain the action (19.49).

The path integral (19.48) forms the basis for studying relativistic potential problems. Only the physically most relevant example will be treated here.

19.2 Tunneling in Relativistic Physics

Relativistic harbors several new tunneling phenomena, of which we want to discuss two especially interesting ones.

19.2.1 Decay Rate of Vacuum in Electric Field

In relativistic physics, an empty Minkowski space with a constant electric field $\mathbf{E}$ is unstable. There is a finite probability that a particle-antiparticle pair can be created. For particles of mass M, this requires the energy

$$E_{\rm pair} = 2Mc^2. \tag{19.50}$$

This energy can be supplied by the external electric field. If the pair of charge $\pm e$ is separated by a distance which is roughly twice the Compton wavelength $\lambda_M^{\rm C} = \hbar/Mc$ of Eq. (19.31), it gains an energy $2|\mathbf{E}|\lambda_M^{\rm C}e$. The decay will therefore become significant when

$$|\mathbf{E}| > E_c = \frac{M^2c^3}{e\hbar}. \tag{19.51}$$

Euclidean Action

In Chapter 17 we have shown that in the semiclassical limit where the decay-rate is small, it is proportional to a Boltzmann-like factor $e^{-\mathcal{A}_{\rm cl,e}/\hbar}$, where $\mathcal{A}_{\rm cl,e}$ is the action of a Euclidean classical solution mediating the decay. Such a solution is easily found. We use the classical action in the form (19.12) and choose the parameter λ to measure the imaginary time $\lambda = \tau = it = x^4/c$. Then the action takes the form

$$\mathcal{A}_{\rm cl,e} = \int_{\tau_a}^{\tau_b} d\tau \left[Mc^2\sqrt{1+\dot{\mathbf{x}}^2(\tau)/c^2} - e\,\mathbf{E}\cdot\mathbf{x}(\tau)\right]. \tag{19.52}$$

This is extremized by the classical equation of motion

$$M\frac{d}{d\tau}\frac{\dot{\mathbf{x}}(\tau)}{\sqrt{1+\dot{\mathbf{x}}^2(\tau)/c^2}} = -e\mathbf{E}, \tag{19.53}$$

whose solutions are circles in spacetime of a fixed E-dependent radius l_E:

$$(\mathbf{x}-\mathbf{x}_0)^2 + c^2(\tau-\tau_0)^2 = l_E^2 \equiv \left(\frac{Mc^2}{eE}\right)^2, \quad E \equiv |\mathbf{E}|. \tag{19.54}$$

To calculate their action we parametrize the circles in the $\hat{\mathbf{E}} - \tau$ -plane, where $\hat{\mathbf{E}}$ is the unit vector in the direction of $\mathbf{E}$, by an angle θ as

$$\mathbf{x}(\theta) = l_E\hat{\mathbf{E}}\,\cos\theta + \mathbf{x}_0, \qquad \tau(\theta) = \frac{l_E}{c}\sin\theta + \tau_0. \tag{19.55}$$

A closed circle has an action

$$\mathcal{A}_{\rm cl,e}=Mc^2\frac{l_E}{c}\int_0^{2\pi}d\theta\cos\theta\left[\frac{1}{\cos\theta}-\cos\theta\right]=Mc\,l_E\pi=\hbar\frac{E_c}{E}\pi. \tag{19.56}$$

The decay rate of the vacuum is therefore proportional to

$$\Gamma\propto e^{-\pi E_c/E}. \tag{19.57}$$

The circles (19.54) are, of course, the space-time pictures of the creation and the annihilation of particle-antiparticle pairs at times $\tau_0 - l_E/c$ and $\tau_0 + l_E/c$ and positions $\mathbf{x}_0$, respectively (see Fig. 19.1). A particle can also run around the circle

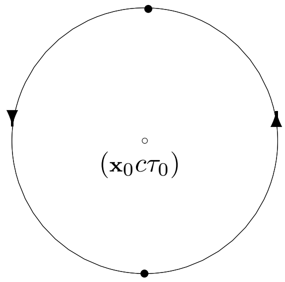

Figure 19.1 Spacetime picture of pair creation at the point $\mathbf{x}_0$ and time $\tau_0 - l_E/c$ and annihilation at the later time $\tau_0 + l_E/c$.

repeatedly. This leads to the formula

$$\Gamma\propto\sum_{n=1}^{\infty}F_n\,e^{-n\pi E_c/E}, \tag{19.58}$$

with fluctuation factors F_n which we are now going to determine.

Fluctuations

As explained above, fluctuations must be calculated with the relativistic Euclidean action (19.10), in which we have to include the electric field by minimal coupling:

$$\bar{\mathcal{A}}_{\rm e}=\int_{\lambda_a}^{\lambda_b}d\lambda\left[\frac{M}{2h(\lambda)}x'^2(\lambda)+h(\lambda)\frac{Mc}{2}-i\frac{e}{c}A(x(\lambda))\,x'(\lambda)\right]. \tag{19.59}$$

The vacuum will decay by the creation of an ensemble of pairs which in Euclidean spacetime corresponds to an ensemble of particle loops. For free particles, the partition function Z was given in (19.42) as an exponential of the one-loop partition function Z_1 in (19.37). The corresponding Z_1 in the presence of an electromagnetic field has the one-loop partition function

$$Z_1=\int_0^\infty\frac{d\beta}{\beta}\,e^{-\beta Mc^2/2}\int\mathcal{D}^4x\,e^{-\bar{\mathcal{A}}_{\rm e}/\hbar}, \tag{19.60}$$

with the Euclidean action

$$\bar{\mathcal{A}}_{\rm e} = \int_0^{\hbar\beta} d\tau \left[\frac{M}{2}x'^2(\tau) - i\frac{e}{c}A(x(\tau))\,x'(\tau)\right]. \tag{19.61}$$

The equations of motion are now

$$\mathbf{x}'' = \frac{e}{Mc}\left(x_4'\,\mathbf{E} - i\,\mathbf{x}'\times\mathbf{B}\right), \qquad x_4'' = -\frac{e}{Mc}\mathbf{x}'\cdot\mathbf{E}, \tag{19.62}$$

If both a constant electric and a constant magnetic field are present, the vector potential is $A_\mu = -F_{\mu\nu}x^\nu/2$ and the action (19.61) takes the simple quadratic form

$$\bar{\mathcal{A}}_{\rm e} = \int_0^{\hbar\beta} d\tau \left[\frac{M}{2}x'^2 - i\,\frac{e}{2c}\,F_{\mu\nu}x^\mu x'^\nu\right], \tag{19.63}$$

and the equations of motion (19.62) are

$$x''^\mu = -i\,\frac{e}{c}\,F^{\mu\nu}x'_\nu, \quad \text{with} \;\; F_{ij} = -\epsilon_{ijk}B^k, \;\; F^{i4} = iF^{i0} = iE^i. \tag{19.64}$$

In a purely electric field, the solutions are circular orbits:

$$\mathbf{x}(\tau) = \hat{\mathbf{E}}\,A\cos\omega_L^E(\tau-\tau_0) + c_2, \quad x_4(\tau) = A\sin\omega_L^E(\tau-\tau_0) + c_4, \tag{19.65}$$

with the electric version of the Landau or cyclotron frequency (2.648)

$$\omega_L^E \equiv \frac{eE}{Mc}. \tag{19.66}$$

The circular orbits are the same as those in the previous formulation (19.55). If $\mathbf{E}$ points in the z-direction, the action (19.63) decomposes into two decoupled quadratic actions $\bar{\mathcal{A}}_{\rm e}^{(12)} + \bar{\mathcal{A}}_{\rm e}^{(34)}$ for the motion in the $x^1 - x^2$ and $x^3 - x^4$ -planes, respectively, and the one-loop partition function (19.60) factorizes as follows:

$$\begin{aligned} Z_1 &= \int_0^\infty \frac{d\beta}{\beta}\,e^{-\beta Mc^2/2}\int \mathcal{D}^2x^{(12)}\,e^{-\bar{\mathcal{A}}_{\rm e}^{(12)}/\hbar}\int \mathcal{D}^2x^{(34)}\,e^{-\bar{\mathcal{A}}_{\rm e}^{(34)}/\hbar} \\ &\equiv \int_0^\infty \frac{d\beta}{\beta}\,e^{-\beta Mc^2/2}Z^{(12)}(0)Z^{(34)}(E). \end{aligned} \tag{19.67}$$

The path integral for $Z^{(12)}(0)$ collecting the fluctuations in the $x^1 - x^2$ -plane have the trivial action

$$\bar{\mathcal{A}}_{\rm e}^{(12)} = \int_0^{\hbar\beta} d\tau\,\frac{M}{2}({x_1'}^2 + {x_2'}^2), \tag{19.68}$$

with the trivial fluctuation determinant Det $(-\partial_\tau^2) = 1$, so that we obtain for $Z^{(12)}(0)$ the free-particle partition function in two dimensions

$$Z^{(12)}(0) = \Delta x_1 \Delta x_2 \sqrt{\frac{M}{2\pi\hbar^2\beta}}^{\,2}. \tag{19.69}$$

The factor $\Delta x_1 \Delta x_2$ is the total area of the system in the $x^1 - x^2$ -plane. Note that upper and lower indices are the same in the present Euclidean metric.

For the motion in the $x^3 - x^4$ -plane, the quadratic fluctuations with periodic boundary conditions have a functional determinant

$$\mathrm{Det}\begin{pmatrix} -\partial_\tau^2 & -\omega_L^E\partial_\tau \\ \omega_L^E\partial_\tau & -\partial_\tau^2 \end{pmatrix} = \mathrm{Det}\left(-\partial_\tau^2\right)\times\mathrm{Det}\left(-\partial_\tau^2 - \omega_L^{E2}\right) = 1\times\frac{\sin\hbar\omega_L^E\beta/2}{\hbar\omega_L^E\beta/2}, \tag{19.70}$$

leading to the partition function

$$Z^{(34)}(E) = \Delta x_3 \Delta x_4 \sqrt{\frac{M}{2\pi\hbar^2\beta}}^2 \frac{\sin\hbar\omega_L^E\beta/2}{\hbar\omega_L^E\beta/2},. \tag{19.71}$$

This result can, of course, also be obtained without calculation from the observation that the Euclidean electric path integral is completely analogous to the real-time magnetic path integral solved in Section 2.18. Indeed, with the **E**-field pointing in the z-direction, the action (19.63) becomes for the motion in the $x^3 - x^4$ -plane

$$\bar{\mathcal{A}}_{\mathrm{e}}^{(34)} = \int_0^{\hbar\beta} d\tau \frac{M}{2}\left[{x_3'}^2 + {x_4'}^2 + \frac{e}{c}E(x_3x_4' - x_4x_3')\right]. \tag{19.72}$$

This coincides with the magnetic action (2.635) in real time, if we insert there the magnetic vector potential (2.636) and replace B by E. The equations of motion (19.62) reduces to

$$x_3'' = \omega_L^E x_4', \qquad x_4'' = -\omega_L^E x_3', \tag{19.73}$$

in agreement with the magnetic equations of motion (2.672) in real time. Thus the motion in the $x_3 - x_4$ -plane as a function of the pseudotime τ is the same as the real-time motion in the $x - y$ -plane, if the magnetic field B in the z-direction is exchanged by an E-field of the same size pointing in the x_3-axis. The amplitude can therefore be taken directly from (2.668), — we must merely replace the real time difference $t_b - t_a$ by $\hbar\beta$. Anyhow, it is zero for any closed orbit.

Yet another way of obtaining the result (19.71) is by summing over the imaginary-time Boltzmann factors $e^{i\beta\mathcal{E}_m}$ of the energies $\mathcal{E}_m = (m+\frac{1}{2})\hbar\omega_L^E$ of the Hamiltonian associated with the Euclidean action (19.72):

$$Z^{(34)}(E) = \Delta x_3 \Delta x_4 \sqrt{\frac{M}{2\pi\hbar^2\beta}}^2 \hbar\omega_L^E\beta \sum_{m=0}^{\infty} e^{-i(m+\frac{1}{2})\hbar\omega_L^E}. \tag{19.74}$$

Inserting (19.69) and (19.71) into (19.67), we obtain the partition function of a single closed particle orbit in four Euclidean spacetime dimensions:

$$Z_1 = \Delta x_4\, V \int_0^\infty \frac{d\beta}{\beta} \sqrt{\frac{M}{2\pi\hbar^2\beta}}^4 \frac{\omega_L^E\hbar\beta/2}{\sin\omega_L^E\hbar\beta/2} e^{-\beta Mc^2/2}, \tag{19.75}$$

where $V \equiv \Delta x_1 \Delta x_2 \Delta x_3$ is the total spatial volume. We now go over to real times by setting $\Delta x_4 = ic\Delta t$. By exponentiating the subtracted expression (19.75) as in Eq. (19.42), we go to a grand-canonical ensemble, and may identify Z_1 with i times the effective electromagnetic action caused by fluctuating ensemble of particle loops

$$Z_1 = i\Delta\mathcal{A}^{\rm eff}/\hbar = i\Delta t\, V\, \Delta\mathcal{L}^{\rm eff}/\hbar. \tag{19.76}$$

The integral over β in (19.75) is divergent. In order to make it convergent, we perform two subtractions. In the subtracted expression we change the integration variable to the dimensionless $\zeta = \beta Mc^2/2$, and obtain the effective Lagrangian density

$$\Delta\mathcal{L}^{\rm eff} = \hbar c\left(\frac{Mc}{\hbar}\right)^4 \frac{1}{4(2\pi)^2}\int_0^\infty \frac{d\zeta}{\zeta^3}\left[\frac{E\zeta/E_c}{\sin E\zeta/E_c} - 1 - \frac{1}{6}\left(\frac{E}{E_c}\right)^2\right] e^{-\zeta}. \tag{19.77}$$

The first subtraction has removed the divergence coming from the $1/\zeta^3$-singularity in the integrand. This produces a real infinite field-independent contribution to the effective action which can be omitted since it is unobservable by electromagnetic experiments.[2] After subtracting this divergence, the integral still contains a logarithmic divergence. It can be interpreted as a contribution proportional to E^2 to the Lagrangian density

$$\Delta\mathcal{L}^{\rm eff}_{\rm div} = \hbar c\left(\frac{Mc}{\hbar}\right)^4 \frac{1}{24(2\pi)^2}\left(\frac{E}{E_c}\right)^2 \int_0^\infty \frac{d\zeta}{\zeta} e^{-\zeta} = \frac{\alpha}{24\pi}\int_0^\infty \frac{d\zeta}{\zeta}e^{-\zeta}\ , \tag{19.78}$$

which changes the original Maxwell term $E^2/2$ to $Z_A E^2/2$ with

$$Z_A = 1 + \frac{\alpha}{12\pi}\int_0^\infty \frac{d\zeta}{\zeta}e^{-\zeta}. \tag{19.79}$$

According to the rules of renormalization theory, the prefactor is removed by renormalizing the field strength, replacing $E \to E/Z_A^{1/2}$, and identifying the replaced field with the physical, *renormalized* field $E/Z_A^{1/2} \equiv E_R$.

Due to the presence of the affective action, the vacuum is no longer time independent, put depends on time like $e^{-i(H-i\Gamma\hbar/2)\Delta t/\hbar}$. Thus the decay rate of the vacuum per unit volume is given by the imaginary part of the effective Lagrangian density

$$\frac{\Gamma}{V} = \frac{2}{\hbar}\,{\rm Im}\,\Delta\mathcal{L}^{\rm eff}. \tag{19.80}$$

In order to calculate this we replace $E\zeta/E_c$ by z in (19.77), and expand in the integrand

$$\frac{z}{\sin z} = 1 + 2\sum_{n=1}^\infty (-1)^n \frac{z^2}{z^2 - n^2\pi^2} = 2\sum_{n=1}^\infty (-1)^n \frac{\zeta^2}{\zeta^2 - \zeta_n^2}, \quad \zeta_n \equiv n\pi\frac{E_c}{E}\,. \tag{19.81}$$

[2]This energy would, however, be observable in the cosmological evolution to be discussed in Subsection 19.2.3.

Adding to the poles the usual infinitesimal $i\eta$-shifts in the complex plane (see p. 115), we replace with the help of the decomposition (1.329)

$$\frac{\zeta}{\zeta^2-\zeta_n^2} \to \frac{\zeta}{\zeta^2-\zeta_n^2+i\eta} \;=\; i\,\frac{\pi}{2}\,\delta(\zeta+\zeta_n) - i\,\frac{\pi}{2}\,\delta(\zeta-\zeta_n) + \zeta\,\frac{\mathcal{P}}{\zeta^2-\zeta_n^2}. \tag{19.82}$$

The δ-functions yield imaginary parts and thus directly the decay rate

$$\begin{aligned}\frac{\Gamma}{V} &= \frac{2}{\hbar}\,\mathrm{Im}\,\mathcal{L}^{\rm eff} = c\left(\frac{Mc}{\hbar}\right)^4\left(\frac{E}{E_c}\right)^2\frac{1}{4\pi^3}\,\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(-1)^{n-1}}{n^2}e^{-n\pi E_c/E}\\ &= \frac{1}{8\pi^3}\frac{e^2}{\hbar c}E^2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}e^{-n\pi E_c/E}.\end{aligned} \tag{19.83}$$

The principal values produce a real effective Lagrangian density

$$\Delta\mathcal{L}^{\rm eff}_{\mathcal{P}} \;=\; \hbar c\left(\frac{Mc}{\hbar}\right)^4\frac{1}{4(2\pi)^2}\,\mathcal{P}\int_0^\infty\frac{d\zeta}{\zeta^3}\left(\frac{E\zeta/E_c}{\sin E\zeta/E_c}-1-\frac{E^2\zeta^2}{6E_c^2}\right)e^{-\zeta}. \tag{19.84}$$

If we expand

$$\frac{z}{\sin z} = 1+\frac{z^2}{6}+\frac{7}{360}z^4+\frac{31}{15120}z^6+\dots \tag{19.85}$$

and perform the integrals over ζ, we find

$$\Delta\mathcal{L}^{\rm eff} = \hbar c\left(\frac{Mc}{\hbar}\right)^4\frac{1}{4(2\pi)^2}\left[\frac{7}{360}\left(\frac{E}{E_c}\right)^4+\frac{31}{2520}\left(\frac{E}{E_c}\right)^6+\dots\right], \tag{19.86}$$

or

$$\mathcal{L}^{\rm eff} = \frac{1}{2}\left\{E^2+\frac{7\alpha^2}{180}\frac{(\hbar c)^3}{(Mc^2)^4}E^4+\frac{31\pi\alpha^3}{315}\left[\frac{(\hbar c)^3}{(Mc^2)^4}\right]^2E^6+\dots\right\}, \tag{19.87}$$

The subscript $\mathcal{P}$ of $\mathcal{L}$ has been omitted since the imaginary part of the effective Lagrangian density (19.83) has a vanishing Taylor expansion. Each coefficient in (19.87) is exact to leading order in α.

The extra expansion terms in (19.87) imply that the physical vacuum has a nontrivial E-dependent dielectric constant $\epsilon(E)$. This is caused by the virtual creation and annihilation of particle-antiparticle pairs. Since the dielectric displacement $D(E)$ is obtained from the first derivative of $\mathcal{L}^{\rm eff}$, the dielectric constant is given by $\epsilon(E) = D(E)/E = \partial\mathcal{L}^{\rm eff}/E\partial E$. From (19.87) we find the lowest expansion terms

$$\epsilon(E) = 1+\frac{7\alpha^2}{90}\frac{(\hbar c)^3}{(Mc^2)^4}E^2+\frac{31\pi\alpha^3}{105}\left[\frac{(\hbar c)^3}{(Mc^2)^4}\right]^2E^4+\dots\,. \tag{19.88}$$

Such a term gives rise to a small amplitude for photon-photon scattering in the vacuum, a process which has been observed in the laboratory.

Another way of evaluating (19.83) is based on the representation (19.74) of the partition function in terms of the energy eigenvalues $\mathcal{E}_m = (m+\frac{1}{2})\hbar\omega_L^E$ of the Hamiltonian associated with (19.72). Then Z_1 is given by

$$Z_1 = \Delta x_4\, V \int_0^\infty d\beta \sqrt{\frac{M}{2\pi\hbar^2\beta}}^4 \omega_L^E \hbar \sum_{n=0}^\infty e^{-i(n+\frac{1}{2})\hbar\omega_L^E\beta}, \tag{19.89}$$

Then the integral representation (19.77) becomes (before the subtraction)

$$\Delta\mathcal{L}^{\rm eff} = \hbar c \left(\frac{Mc}{\hbar}\right)^4 \frac{1}{2(2\pi)^2} \int_0^\infty \frac{d\zeta}{\zeta^2} \frac{E}{E_c} \left[\sum_{m=0}^\infty e^{i(m+\frac{1}{2})2E\zeta/E_c}\right] e^{-\zeta}. \tag{19.90}$$

This can be rewritten as

$$\Delta\mathcal{L}^{\rm eff}_{\rm reg} = \hbar c \left(\frac{Mc}{\hbar}\right)^4 \frac{i}{2(2\pi)^2} \left[\sum_{k=4,6,\dots}^\infty \frac{(-1)^{k/2-1}}{(k-1)(k-2)} (2^{1-k}-1)\zeta(1-k)2^{k-1}\left(\frac{E}{E_c}\right)^k\right]_{\alpha=1}, \tag{19.91}$$

Expanding the terms in the sum in powers of E/E_c yields two divergent terms plus a regular series

$$\begin{aligned}\Delta\mathcal{L}^{\rm eff}_{\rm reg} &= \hbar c \left(\frac{Mc}{\hbar}\right)^4 \frac{i}{2(2\pi)^2} \left[\int d\alpha \int d\alpha \sum_{k=4,6,\dots}^\infty \right.\\ &\times \left. \sum_{m=0}^\infty \frac{(-1)^{k/2-1}}{\alpha^k} (-1)^{k/2-1}[(m+\tfrac{1}{2})]^{k-1} 2^{k-1}\left(\frac{E}{E_c}\right)^k\right]_{\alpha=1}. \end{aligned} \tag{19.92}$$

Performing the sum over m using Riemann's zetafuntions (2.521), this becomes

$$\Delta\mathcal{L}^{\rm eff}_{\rm reg} = \hbar c \left(\frac{Mc}{\hbar}\right)^4 \frac{i}{2(2\pi)^2} \left[\sum_{k=4,6,\dots}^\infty \frac{(-1)^{k/2-1}}{(k-1)(k-2)} (2^{1-k}-1)\zeta(1-k)2^{k-1}\left(\frac{E}{E_c}\right)^k\right]_{\alpha=1} \tag{19.93}$$

and coincides with the previous result (19.87).

Including Constant Magnetic Field parallel to Electric Field

Let us see how the decay rate (19.83) and the effective Lagrangian (19.77) are influenced by the presence of an additional constant **B**-field. This will at first be assumed to be parallel to the **E**-field, with both fields pointing in the z-direction. Then the action (19.68) for the motion in the $x^1 - x^2$ -plane becomes

$$\bar{\mathcal{A}}_{\rm e}^{(12)} = \int_0^{\hbar\beta} d\tau \frac{M}{2}\left[x_1'^2 + x_2'^2 + \frac{e}{c} iB(x_1 x_2' - x_2 x_1')\right]. \tag{19.94}$$

The partition function in the $x^1 - x^2$ -plane will therefore have the same form as (19.71), except with ω_L^E replaced by $i\omega_L^B$:

$$Z^{(12)}(B) = Z^{(34)}(iB). \tag{19.95}$$

Thus the B-field changes the partition function (19.89) of a single closed orbit to

$$Z_1 = \Delta x_4\, V \int_0^\infty \frac{d\beta}{\beta} \sqrt{\frac{M}{2\pi\hbar^2\beta}}^4 \frac{\omega_L^E\hbar\beta/2}{\sin\omega_L^E\hbar\beta/2} \frac{\omega_L^B\hbar\beta/2}{\sinh\omega_L^B\hbar\beta/2} e^{-\beta Mc^2/2}, \tag{19.96}$$

and the effective Lagrangian (19.77) becomes

$$\Delta\mathcal{L}^{\rm eff} = \hbar c\left(\frac{Mc}{\hbar}\right)^4 \frac{1}{4(2\pi)^2} \int_0^\infty \frac{d\zeta}{\zeta^3}\left[\frac{E\zeta/E_c}{\sin E\zeta/E_c}\frac{B\zeta/E_c}{\sinh B\zeta/E_c} - 1 - \frac{(E^2-B^2)\zeta^2}{6E_c^3}\right] e^{-\zeta}. \tag{19.97}$$

In the subtracted free-field term (19.78), the term E^2 is changed to the Lorentz-invariant combination $E^2 - B^2$. The decay rate (19.83) is modified to

$$\frac{\Gamma}{V} = \frac{2}{\hbar}{\rm Im}\,\Delta\mathcal{L}^{\rm eff} = c\left(\frac{Mc}{\hbar}\right)^4\left(\frac{E}{E_c}\right)^2 \frac{1}{4\pi^3}\,\frac{1}{2}\sum_{n=1}^\infty (-1)^{n-1}\frac{1}{n^2}\frac{n\pi B/E}{\sinh n\pi B/E} e^{-n\pi E_c/E}. \tag{19.98}$$

In Eqs. (7.521)–(7.525) we have shown that all results derived for parallel constant electric and magnetic fields remain valid if we replace $E \to \mathcal{E}$, $B \to \mathcal{B}$, where $\mathcal{E}$ and $\mathcal{B}$ are given in Eq. (7.524). After this we may expand the integrand in (19.97) in powers of ε and β using Eq. (19.85),

$$\begin{aligned}\frac{1}{\tau^3}\frac{e\beta\tau}{\sin e\beta\tau}\frac{e\varepsilon\tau}{\sinh e\varepsilon\tau} = \frac{1}{\tau^3} - \frac{e^2}{6\tau}\left(\varepsilon^2-\beta^2\right) + e^4\frac{\tau}{360}\left(7\varepsilon^4 - 10\varepsilon^2\beta^2 + 7\beta^4\right)\\ - e^6\frac{\tau^3}{1520}\left(31\varepsilon^6 - 49\varepsilon^4\beta^2 + 49\varepsilon^2\beta^4 - 31\beta^6\right) + \dots\,.\end{aligned} \tag{19.99}$$

and obtain the effective Lagrangian the fields generalizing (19.87):

$$\begin{aligned}\mathcal{L}^{\rm eff} = \frac{1}{2}\Bigg\{&(\mathbf{E}^2-\mathbf{B}^2) + \frac{7\alpha^2}{180}\frac{(\hbar c)^3}{(Mc^2)^4}(\mathbf{E}^2-\mathbf{B}^2)^2\\ &+\frac{31\pi\alpha^3}{315}\left[\frac{(\hbar c)^3}{(Mc^2)^4}\right]^2(\mathbf{E}^2-\mathbf{B}^2)[2(\mathbf{E}^2-\mathbf{B}^2)^2 - 4(\mathbf{EB})^2] + \dots\Bigg\},\end{aligned} \tag{19.100}$$

For strong fields, a saddle point approximation to the generalized version of the integral (19.97) yields the asymptotic form

$$\mathcal{L}^{\rm eff} \equiv -\frac{e^2}{192\pi^2}(\mathbf{E}^2-\mathbf{B}^2)\log\left[-4e^2\frac{(\hbar c)^3}{(Mc^2)^4}(\mathbf{E}^2-\mathbf{B}^2)\right] + \dots\,. \tag{19.101}$$

Spin-1/2 Particles

We anticipate the small modification which is necessary to obtain the analogous result for spin-1/2 fermions: The tools for this will be developed in Subsections 19.5.1–19.5.3. The relevant formula has actually been derived before in Eq. (7.520). Extending that equation to a constant field tensor that contains both magnetic and electric fields, and going to imaginary time $\tau = \lambda/c$ and antiperiodic boundary conditions in the interval $t_b - t_a = i\hbar\beta$, we find the functional determinant

$$4\,\mathrm{Det}^{1/2}\left(-g_{\mu\nu}\,\partial_\lambda + i\frac{e}{Mc^2}F_{\mu\nu}\right) = 4\det{}^{1/2}\left(\cosh\frac{e}{Mc}F_{\mu\nu}\frac{\hbar\beta}{2}\right), \quad (19.102)$$

with an ordinary determinant of the 4×4-dimensional matrix of the cosin on the right-hand side. According to Eqs. (7.520) and (7.525), the result is

$$4\,\mathrm{Det}^{1/2}\left(-g_{\mu\nu}\,\partial_\lambda + i\frac{e}{Mc^2}F_{\mu\nu}\right) = 4\cosh(\mu_B\mathcal{B}\beta/2)\cos(\mu_B\mathcal{E}\beta/2). \quad (19.103)$$

where $\mu_B = e\hbar/Mc$ is the Bohr magneton (2.649), and $\mathcal{B}$ and $\mathcal{E}$ are defined in Eq. (7.525)

For a purely electric field, the right-hand side becomes $4\cos(\mu_B E\beta/2) = 4\cos(\omega_L^E\hbar\beta/2)$ [recall (19.66)]. Multiplying the cos-factor into the expansion (19.81), this is modified to

$$z\,\frac{\cos z}{\sin z} = 1 + 2\sum_{n=1}^{\infty}\frac{z^2}{z^2 - n^2\pi^2} = 2\sum_{n=1}^{\infty}\frac{\zeta^2}{\zeta^2-\zeta_n^2}, \quad \zeta_n \equiv n\pi\frac{E_c}{E}. \quad (19.104)$$

Performing now the singular integral over ζ in (19.77), we obtain for the decay rate the same formula as in (19.83), except that the alternating signs are absent. The factor 4 in (19.103) reduces to a factor 2 in the effective action. The resulting effective Lagrangian density for spin-1/2 fermions is therefore

$$\Delta\mathcal{L}^{\mathrm{eff}}_{\mathrm{spin}\frac{1}{2}} = -\hbar c\left(\frac{Mc}{\hbar}\right)^4\frac{1}{2(2\pi)^2}\int_0^\infty\frac{d\zeta}{\zeta^3}\left(\frac{E\zeta/E_c}{\tan E\zeta/E_c} - 1 + \frac{E^2\zeta^2}{3E_c^3}\right)e^{-\zeta}, \quad (19.105)$$

a result first derived by Heisenberg and Euler in 1935 [8]. Its imaginary part yields the decay rate of the vacuum due to pair creation

$$\begin{aligned}\frac{\Gamma_{\mathrm{spin}\frac{1}{2}}}{V} &= \frac{2}{\hbar}\,\mathrm{Im}\,\Delta\mathcal{L}^{\mathrm{eff}}_{\mathrm{spin}\frac{1}{2}} = c\left(\frac{Mc}{\hbar}\right)^4\left(\frac{E}{E_c}\right)^2\frac{1}{4\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-n\pi E_c/E}\\ &= \frac{1}{4\pi^3}\frac{e^2}{\hbar c}E^2\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-n\pi E_c/E}. \end{aligned} \quad (19.106)$$

The reason for the reduction from 4 to 2 is that the sum over bosonic paths has to be first divided by a factor 2 to remove their orientation, before the fermionic factor 4 is applied. This procedure is not so obvious at this point but will be understood later in Subsection 19.5.2. The remaining factor 2 accounts for the two spin orientations of the charged particles.

The Taylor series

$$\frac{z}{\tan z} = 1 - \frac{z^2}{3} - \frac{1}{45}z^4 - \frac{2}{945}z^6 - \dots \quad (19.107)$$

in the integrand of (19.105) leads to the expansion

$$\Delta\mathcal{L}^{\rm eff}=\hbar c\left(\frac{Mc}{\hbar}\right)^4\frac{2}{16\pi^2}\left[\frac{1}{45}\left(\frac{E}{E_c}\right)^4+\frac{4}{315}\left(\frac{E}{E_c}\right)^6+\dots\right], \tag{19.108}$$

so that

$$\mathcal{L}^{\rm eff}=\frac{1}{2}\left\{E^2+\frac{4\alpha^2}{45}\frac{(\hbar c)^3}{(Mc^2)^4}E^4+\frac{64\pi\alpha^3}{315}\left[\frac{(\hbar c)^3}{(Mc^2)^4}\right]^2E^6+\dots\right\}. \tag{19.109}$$

The term proportional to α^2-term implies a small amplitude for photon-photon scattering which can be observed in the laboratory [9].

As in the boson result (19.100), each coefficient is exact to leading order in α, and the virtual creation and annihilation of fermion-antifermion pairs gives the physical vacuum a nontrivial E-dependent dielectric constant

$$\epsilon(E)=\frac{1}{E}\frac{\partial\mathcal{L}^{\rm eff}}{\partial E}=1+\frac{8\alpha^2}{45}\frac{(\hbar c)^3}{(Mc^2)^4}E^2+\frac{64\pi\alpha^3}{105}\left[\frac{(\hbar c)^3}{(Mc^2)^4}\right]^2E^4+\dots\,. \tag{19.110}$$

If we admit also a constant **B**-field parallel to **E**, the formulas (19.106) and (19.105) for the spin-$\frac{1}{2}$ -particles are modified in the same way as the bosonic formulas (19.83) and (19.84), except that the determinant (19.102) in spinor space introduces a further factor $\cosh(eB/Mc)$. Thus we obtain

$$\Delta\mathcal{L}^{\rm eff}_{{\rm spin}\frac{1}{2}}=-\hbar c\left(\frac{Mc}{\hbar}\right)^4\frac{1}{2(2\pi)^2}\int_0^\infty\frac{d\zeta}{\zeta^3}\left[\frac{E\zeta/E_c}{\tan E\zeta/E_c}\frac{B\zeta/E_c}{\tanh B\zeta/E_c}-1+\frac{(E^2-B^2)\zeta^2}{3E_c^3}\right]e^{-\zeta}. \tag{19.111}$$

For a general combination of constant electric and magnetic fields, we simply exchange E and B by the Lorentz invariants ε and β. From the imaginary part we obtain the decay rate

$$\frac{\Gamma_{{\rm spin}\frac{1}{2}}}{V}=\frac{2}{\hbar}\,{\rm Im}\,\Delta\mathcal{L}^{\rm eff}_{{\rm spin}\frac{1}{2}}=c\left(\frac{Mc}{\hbar}\right)^4\left(\frac{\varepsilon}{E_c}\right)^2\frac{1}{4\pi^3}\sum_{n=1}^\infty\frac{1}{n^2}\frac{n\pi\beta/\varepsilon}{\tanh n\pi\beta/\varepsilon}e^{-n\pi E_c/\varepsilon}. \tag{19.112}$$

For strong fields, a saddle point approximation to the generalized integral (19.105) yields the asymptotic form

$$\mathcal{L}^{\rm eff}\equiv-\frac{e^2}{48\pi^2}(\mathbf{E}^2-\mathbf{B}^2)\log\left[-4e^2\frac{(\hbar c)^3}{(Mc^2)^4}(\mathbf{E}^2-\mathbf{B}^2)\right]+\dots\,. \tag{19.113}$$

19.2.2 Birth of Universe

A similar tunneling phenomenon could explain the birth of the expanding universe [10].

As an idealization of the observed density of matter, the universe is usually assumed to be isotropic and homogeneous. Then it is convenient to describe it by a coordinate frame in which the metric is rotationally invariant. To account for the expansion, we have to allow for an explicit time dependence of the spatial part of the metric. In the spatial part, we choose coordinates which participate in the expansion. They can be imagined as being attached to the gas particles in a homogenized universe. Then the time passing at each coordinate point is the proper time. In this context it is the so-called *cosmic standard time* to be denoted by t. We imagine being an observer at a coordinate point with $dx^i/dt = 0$, and measure t by counting the number of orbits of an electron around a proton in a hydrogen atom, starting from the moment of the big bang (forgetting the fact that in the early time of the universe, the atom does not yet exist).

Geometry

With this time calibration, the component g_{00} of the metric tensor is identically equal to unity

$$g_{00}(x) \equiv 1, \tag{19.114}$$

such that at a fixed coordinate point, the proper time coincides with the coordinate time, $d\tau = dt$. Moreover, since all clocks in space follow the same prescription, there is no mixing between time and space coordinates, a property called *time orthogonality*, so that

$$g_{0i}(x) \equiv 0. \tag{19.115}$$

As a consequence, the Christoffel symbol $\bar{\Gamma}_{00}{}^{\mu}$ [recall (10.7)] vanishes identically:

$$\bar{\Gamma}_{00}{}^{\mu} \equiv \frac{1}{2} g^{\mu\nu}(\partial_0 g_{0\nu} + \partial_0 g_{0\nu} - g_{00}) \equiv 0. \tag{19.116}$$

This is the mathematical way of expressing the fact that a particle sitting at a coordinate point, which has $dx^i/dt = 0$, and thus $dx^\mu/dt = u^\mu = (c, 0, 0, 0)$, experiences no acceleration

$$\frac{du^\mu}{d\tau} = -\bar{\Gamma}_{00}{}^{\mu} c^2 = 0. \tag{19.117}$$

The coordinates themselves are trivially comoving.

Under the above condition, the invariant distance has the general form

$$ds^2 = c^2 dt^2 - {}^{(3)}g_{ij}(x) dx^i dx^j. \tag{19.118}$$

We now impose the spatial isotropy upon the spatial metric g_{ij}. Denote the spatial length element by dl, so that

$$dl^2 = {}^{(3)}g_{ij}(x) dx^i dx^j. \tag{19.119}$$

The isotropy and homogeneity of space is most easily expressed by considering the spatial curvature ${}^{(3)}R_{ijk}{}^l$ calculated from the spatial metric ${}^{(3)}g_{ij}(x)$. The space corresponds to a spherical surface. If its radius is a, the curvature tensor is, according to Eq. (10.161),

$$ {}^{(3)}R_{ijkl}(x) = \frac{1}{a^2}\left[{}^{(3)}g_{il}(x){}^{(3)}g_{jk}(x) - {}^{(3)}g_{ik}(x){}^{(3)}g_{jl}(x)\right]. \tag{19.120} $$

The derivation of this expression in Section 10.4 was based on the assumption of a spherical space whose curvature $K \equiv 1/a^2$ is positive. If we allow also for hyperbolic and parabolic spaces with negative and vanishing curvature, and characterize these by a constant

$$ k = \left\{ \begin{array}{ll} 1 & \text{spherical} \\ 0 & \text{parabolic} \\ -1 & \text{hyperbolic} \end{array} \right\} \text{ universe,} \tag{19.121} $$

then the prefactor $1/a^2$ in (19.120) is replaced by $K \equiv k/a^2$. For $k = -1$ and 0, the space has an open topology and an infinite total volume.

The Ricci tensor and curvature scalar are in these three cases [compare (10.163) and (10.156)]

$$ {}^{(3)}R_{il} = k\,\frac{2}{a^2}\,g_{il}(x), \qquad {}^{(3)}R = k\,\frac{6}{a^2}. \tag{19.122} $$

By construction it is obvious that for $k = 1$, the three-dimensional space has a closed topology and a finite spatial volume which is equal to the surface of a sphere of radius a in four dimensions

$$ {}^4S^a = 2\pi^2 a^3. \tag{19.123} $$

A circle in this space has maximal radius a and a maximal circumference $2\pi a$. A sphere with radius $r_0 < a$ has a volume

$$ \begin{aligned} {}^{(3)}V^a_{r_0} &= \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \int_0^{r} dr \frac{r^2}{\sqrt{1-r^2/a^2}} \\ &= 4\pi\left(\frac{a^3}{2}\arcsin\frac{r_0}{a} - \frac{a^2 r_0}{2}\sqrt{1-\frac{{r_0}^2}{a^2}}\right). \end{aligned} \tag{19.124} $$

For small r_0, the curvature is irrelevant and the volume depends on r_0 like the usual volume of a sphere in three dimensions:

$$ {}^{(3)}V^a_{r_0} \approx V_{r_0} = \frac{4\pi}{3}{r_0}^2. \tag{19.125} $$

For $r_0 \to a$, however, ${}^{(3)}V^a_{r_0}$ approaches a saturation volume $2\pi a^3$.

The analogous expressions for negative and zero curvature are obvious.

Robertson-Walker Metric

In spherical coordinates, the four-dimensional invariant distance (19.118) defines the *Robertson-Walker metric.*

$$ds^2 = c^2dt^2 - dl^2 \tag{19.126}$$

$$dl^2 = \frac{dr^2}{1-kr^2/a^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{19.127}$$

It will be convenient to introduce, instead of r, the angle α on the surface of the four-sphere, such that

$$r = a\sin\alpha. \tag{19.128}$$

Then the metric has the four-dimensional angular form

$$ds^2 = c^2dt^2 - a^2(t)[d\alpha^2 + f^2(\alpha)(d\theta^2 + \sin^2\theta d\varphi^2)], \tag{19.129}$$

where for spherical, parabolic, and hyperbolic spaces, $f(\alpha)$ is equal to

$$f(\alpha) = \begin{cases} \sin\alpha & k=1, \\ \alpha & k=0, \\ \sinh\alpha & k=-1. \end{cases} \tag{19.130}$$

In order to have maximal symmetry, it is useful to absorb $a(t)$ into the time and define a new timelike variable η by

$$c\,dt = a(\eta)\,d\eta, \tag{19.131}$$

so that the invariant distance is measured by

$$ds^2 = a^2(\eta)[d\eta^2 - d\alpha^2 - f^2(\alpha)(d\theta^2 + \sin^2\theta d\varphi^2)]. \tag{19.132}$$

Then the metric is simply

$$g_{\mu\nu} = a^2(\eta)\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -f^2(\alpha) & \\ & & & -f^2(\alpha)\sin^2\theta, \end{pmatrix} \tag{19.133}$$

and the Christoffel symbols become

$$\Gamma_{00}{}^0 = \frac{a_\eta}{a},\quad \Gamma_{00}{}^i = 0,\quad \Gamma_{0i}{}^0 = 0,\quad \Gamma_{0i}{}^j = \frac{a_\eta}{a}\delta_i{}^j,\quad \Gamma_{ij}{}^0 = -\frac{a_\eta}{a^3}g_{ij},\quad \Gamma_{ij}{}^k = 0, \tag{19.134}$$

where the subscripts denote derivatives with respect to the corresponding variables:

$$a_\eta \equiv \frac{da}{d\eta} = \frac{a}{c}\frac{da}{dt} \equiv \frac{a}{c}a_t. \tag{19.135}$$

We now calculate the 00-component of the Ricci tensor:

$$R_{00} = \partial_\mu \Gamma_{00}{}^\mu - \partial_0 \Gamma_{\mu 0}{}^\mu - \Gamma_{\mu 0}{}^\nu \Gamma_{0\nu}{}^\mu + \Gamma_{00}{}^\mu \Gamma_{\nu\mu}{}^\nu. \tag{19.136}$$

Inserting the Christoffel symbols (19.134) we find

$$\partial_\mu \Gamma_{00}{}^\mu - \partial_0 \Gamma_{\mu 0}{}^\mu = -\partial_0 \Gamma_{i0}{}^i = -3 \frac{d}{d\eta} \frac{a_\eta}{a} = -3 \frac{1}{a^2} \left(a_{\eta\eta} a - a_\eta^2 \right), \tag{19.137}$$

$$\Gamma_{\mu 0}{}^\nu \Gamma_{0\nu}{}^\mu = \Gamma_{00}{}^0 \Gamma_{00}{}^0 + \Gamma_{00}{}^i \Gamma_{0i}{}^0 + \Gamma_{i0}{}^0 \Gamma_{00}{}^i + \Gamma_{i0}{}^j \Gamma_{0j}{}^i = \left(\frac{a_\eta}{a} \right)^2 + 3 \left(\frac{a_\eta}{a} \right)^2, \tag{19.138}$$

$$\Gamma_{00}{}^\mu \Gamma_{\nu\mu}{}^\nu = \Gamma_{00}{}^0 \Gamma_{00}{}^0 + \Gamma_{00}{}^0 \Gamma_{i0}{}^i + \Gamma_{00}{}^i \Gamma_{0i}{}^0 + \Gamma_{00}{}^i \Gamma_{ki}{}^k = \left(\frac{a_\eta}{a} \right)^2 + 3 \left(\frac{a_\eta}{a} \right)^2, \tag{19.139}$$

so that

$$R_{00} = -\frac{3}{a^2}(a a_{\eta\eta} - a_\eta^2), \qquad R_0{}^0 = g^{00} R_{00} = -\frac{3}{a^4}(a a_{\eta\eta} - a_\eta^2). \tag{19.140}$$

The other components can be determined by relating them to the three-dimensional curvature tensor ${}^{(3)}R_{ij}$ which has the simple form (19.120). So we calculate

$$\begin{aligned} R_{ij} = R_{\mu ij}{}^\mu &= R_{kij}{}^k + R_{0ij}{}^0 \\ &= {}^{(3)}R_{ij} - \Gamma_{kj}{}^0 \Gamma_{i0}{}^k + \Gamma_{ij}{}^0 \Gamma_{k0}{}^k + R_{0ij}{}^0. \end{aligned} \tag{19.141}$$

Inserting

$$R_{0ij}{}^0 = \partial_0 \Gamma_{ij}{}^0 - \partial_i \Gamma_{0j}{}^0 - \Gamma_{0j}{}^l \Gamma_{il}{}^0 - \Gamma_{0j}{}^0 \Gamma_{i0}{}^0 + \Gamma_{ij}{}^l \Gamma_{0l}{}^0 + \Gamma_{ij}{}^0 \Gamma_{00}{}^0, \tag{19.142}$$

$${}^{(3)}R_{ij} = k \frac{2}{a^2} g_{ij} \tag{19.143}$$

and the above Christoffel symbols (19.134) gives

$$R_{ij} = -\frac{1}{a^4}(2ka^2 + a_\eta^2 + a a_{\eta\eta}) g_{ij} \tag{19.144}$$

and thus a curvature scalar

$$\begin{aligned} R &= g^{00} R_{00} + g^{ij} R_{ij} = -\frac{1}{a^2} \left[\frac{3}{a^2}(a a_{\eta\eta} - a_\eta^2) \right] - \frac{3}{a^4}(2ka^2 + a_\eta^2 + a a_{\eta\eta}) \\ &= -\frac{6}{a^3}(a_{\eta\eta} + ka). \end{aligned} \tag{19.145}$$

Action and Field Equation

In the absence of matter, the *Einstein-Hilbert action* of the gravitational field is

$$\overset{f}{\mathcal{A}} = \int d^4x \sqrt{-g}\, \overset{f}{\mathcal{L}} = -\frac{1}{2\kappa} \int d^4x \sqrt{-g}(R + 2\lambda), \tag{19.146}$$

were κ is related to Newton's gravitational constant

$$G_N \approx 6.673 \cdot 10^{-8}\,\mathrm{cm}^3\mathrm{g}^{-1}\mathrm{s}^{-2} \tag{19.147}$$

by

$$\frac{1}{\kappa} = \frac{c^3}{8\pi G_N}. \tag{19.148}$$

A natural length scale of gravitational physics is the *Planck length*, which can be formed from a combination of Newton's gravitational constant (19.147), the light velocity $c \approx 3 \times 10^{10}$ cm/s, and Planck's constant $\hbar \approx 1.05459 \times 10^{-27}$:

$$l_{\mathrm{P}} = \left(\frac{c^3}{G_N \hbar}\right)^{-1/2} \approx 1.615 \times 10^{-33}\,\mathrm{cm}. \tag{19.149}$$

It is the Compton wavelength $l_{\mathrm{P}} \equiv \hbar/m_{\mathrm{P}}c$ associated with the Planck mass

$$m_{\mathrm{P}} = \left(\frac{c\hbar}{G_N}\right)^{1/2} \approx 2.177 \times 10^{-5}\,\mathrm{g} = 1.22 \times 10^{22}\,\mathrm{MeV/c^2}. \tag{19.150}$$

The constant $1/\kappa$ in the action (19.146) can be expressed in terms of the Planck length as

$$\frac{1}{\kappa} = \frac{\hbar}{8\pi l_P^2}. \tag{19.151}$$

If we add to (19.146) a matter action and vary to combined action with respect to the metric $g_{\mu\nu}$, we obtain the *Einstein equation*

$$\frac{1}{\kappa}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \lambda g_{\mu\nu}\right) = T_{\mu\nu}, \tag{19.152}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of matter. The constant λ is called the *cosmological constant*. It is believed to arise from the zero-point oscillations of all quantum fields in the universe.

A single field contributes to the Lagrangian density $\overset{f}{\mathcal{L}}$ in (19.146) a term $-\Lambda \equiv -\lambda/\kappa$ which is typically of the order of $\hbar/l_{\mathrm{P}}^4$. For bosons, the sign is positive, for fermions negative, reflecting the filling of all negative-energy in the vacuum. A constant of this size is much larger than the present experimental estimate. In the literature one usually finds estimates for the dimensionless quantity

$$\Omega_{\lambda 0} \equiv \frac{\lambda\, c^2}{3H_0^2}. \tag{19.153}$$

where H_0 is the *Hubble constant*, whose inverse is roughly the lifetime of the universe

$$H_0^{-1} \approx 14 \times 10^9\,\mathrm{years}. \tag{19.154}$$

Present fits to distant supernovae and other cosmological data yield the estimate [11]

$$\Omega_{\lambda 0} \approx 0.68 \pm 0.10. \tag{19.155}$$

The associated cosmological constant λ has the value

$$\lambda = \Omega_{\lambda 0} \frac{3H_0^2}{c^2} \approx \frac{\Omega_{\lambda 0}}{(6.55 \times 10^{27}\,\mathrm{cm})^2} \approx \frac{\Omega_{\lambda 0}}{(6.93 \times 10^9\,\mathrm{ly})^2} \approx \frac{\Omega_{\lambda 0}}{(2.14\,R_{\mathrm{universe}})^2}. \tag{19.156}$$

Note that in the presence of λ, the Schwarzschild solution around a mass M has the metric

$$ds^2 = B(r)c^2dt^2 - B^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2, \tag{19.157}$$

with

$$B(r) = 1 - \frac{2MG_N}{c^2r} - \frac{2}{3}\lambda\, r^2 = 1 - \frac{M}{m_{\mathrm{P}}}\frac{l_{\mathrm{P}}}{r} - \frac{2}{3}\Omega_{\lambda 0}\frac{r^2}{(2.14\,R_{\mathrm{universe}})^2}. \tag{19.158}$$

The λ-term adds a small repulsion to Newton's force between mass points. if the distances are the order of the radius of the universe.

The value of the constant Λ associated with (19.155) is

$$\Lambda = \frac{\lambda}{\kappa} = \Omega_{\lambda 0}\frac{3H_0^2}{c^2}\frac{l_{\mathrm{P}}^2}{8\pi} \approx 10^{-122}\frac{\hbar}{l_{\mathrm{P}}^4}. \tag{19.159}$$

Such a small prefactor can only arise from an almost perfect cancellation of the contributions of boson and fermion fields. This cancellation is the main reason for postulating a broken supersymmetry in the universe, in which every boson has a fermionic counterpart. So far, the known particle spectra show no trace of such a symmetry. Thus there is need to explain it by some other not yet understood mechanism.

The simplest model of the universe governed by the action (19.146) is called *Friedmann model* or *Friedmann universe*.

19.2.3 Friedmann Model

Inserting (19.140) and (19.145) into the 00-component of the Einstein equation (19.152), we obtain the equation for the energy

$$\frac{3}{a^4}\left(a_\eta^2 + ka^2\right) - \lambda = \kappa T_0{}^0. \tag{19.160}$$

In terms of the cosmic standard time t, the general equation reads

$$3\left[\left(\frac{a_t}{a}\right)^2 + k\frac{c^2}{a^2}\right] - \lambda c^2 = c^2\kappa T_0{}^0. \tag{19.161}$$

The simplest Friedmann model is based on an energy-momentum tensor ${T_0}^0$ of an ideal pressure-less gas of mass density ρ:

$$ {T_\mu}^\nu = c\rho u_\mu u^\nu, \tag{19.162} $$

where u^μ are the velocity four-vectors $u^\mu = (\gamma, \gamma \mathbf{v}/c)$ of the particles whose components are $u^\mu = (\gamma, \gamma \mathbf{v}/c)$ with $\gamma \equiv 1/\sqrt{1 - v^2/c^2}$ if the four components transform like $(dx^0 = cdt, d\mathbf{x})$. The gas is assumed to be at rest in our comoving coordinates. so that only the ${T_0}^0$-component is nonzero:

$$ {T_0}^0 = c\rho. \tag{19.163} $$

This component is invariant under the time transformation (19.131).

As a fortunate accident, this component of the Einstein equation has no $a_{\eta\eta}a$ term. Thus we may simply study the first-order differential equation

$$ \frac{3}{a^4}\left(a_\eta^2 + ka^2\right) - \lambda = c\kappa\rho. \tag{19.164} $$

Since the total volume of the universe is $2\pi a^3$, we can express ρ in terms of the total mass M as follows

$$ \rho = \frac{M}{2\pi^2 a^3}. \tag{19.165} $$

In this way we arrive at the differential equation

$$ \frac{3}{a^4}(a_\eta^2 + ka^2) - \lambda = \frac{\kappa M c}{2\pi^2 a^3} = \frac{4G_{\rm N}M}{\pi c^2 a^3}. \tag{19.166} $$

This equation of motion can also be obtained in another way. We express the action (19.146) in terms of $a(\eta)$ using the equation (19.180) for R. We use the volume (19.123) and the relation (19.131) to rewrite the integration measure as

$$ \int d^4x\,\sqrt{-g} = \int dt\,{}^{(4)}S^a = 2\pi^2 \int d\eta\, a^4(\eta), \tag{19.167} $$

so that

$$ \overset{f}{\mathcal{A}} = \frac{2\pi^2}{2\kappa}\int d\eta\left[6a(a_{\eta\eta} + ka) - 2\lambda a^4\right] \mathrel{\hat{=}} \frac{2\pi^2}{\kappa}\int d\eta\left[-3a_\eta^2 + 3ka^2 - \lambda a^4\right]. \tag{19.168} $$

The second expression arises from the first by a partial integration and ignoring the boundary terms which do not influence the equation of motion. The above matter is described by the action

$$ \overset{m}{\mathcal{A}} = -\int d^4x\,\sqrt{-g}c\rho = -2\pi^2 \int d\eta\,\frac{Mc}{2\pi^2}\,a(\eta). \tag{19.169} $$

Variation with respect to a yields the Euler-Lagrange equation

$$ 6(a_{\eta\eta} + ka) - 4\lambda a^3 - \frac{\kappa M c}{2\pi^2} = 0. \tag{19.170} $$

Note that in terms of the Robertson-Walker time t [recall (19.131)], the equation of motion reads

$$\ddot{a} = \frac{\lambda}{3}a - \frac{1}{6}\frac{\kappa Mc}{2\pi^2 a^2}. \tag{19.171}$$

As one should expect, the cosmological expansion is slowed down by matter, due to the gravitational attraction. A positive cosmological constant, on the other hand, accelerates the expansion. At the special value

$$\lambda = \lambda_{\text{Einstein}} \equiv \frac{\kappa Mc}{2\pi^2 a^3} = \frac{4G_{\text{N}}M}{\pi c^2 a^3} = \frac{4\pi G_{\text{N}}\rho}{c}, \tag{19.172}$$

the two effects cancel each other and there exist a time-independent solution at a radius a and a density ρ. This is the cosmological constant which Einstein chose before Hubble's discovery of the expanding universe to agree with Hoyle's steady-state model of the universe (a choice which he later called the biggest blunder of his life).

Multiplying (19.170) by a_η and integration over η yields the pseudo-energy conservation law

$$3(a_\eta^2 + ka^2) - \lambda a^4 - \frac{\kappa Mc}{2\pi^2}a = \text{const}, \tag{19.173}$$

in agreement with (19.166) for a vanishing pseudo-energy.

This equation may also be written as

$$a_\eta^2 + ka^2 - a_{\max}a - \frac{\lambda}{3}a^4 = 0, \tag{19.174}$$

where

$$a_{\max} \equiv \frac{\kappa Mc}{6\pi^2} = \frac{4G_{\text{N}}M}{3\pi c^2}. \tag{19.175}$$

This looks like the energy conservation law for a point particle of mass 2 in an effective potential of the universe

$$V^{\text{univ}}(a) = ka^2 - a_{\max}a - \frac{\lambda}{3}a^4, \tag{19.176}$$

at zero total energy. The potential is plotted for the spherical case $k = 1$ in Fig. 19.2.

Friedmann neglects the cosmological constant and considers the equation

$$a_\eta^2 + ka^2 - a_{\max}a = 0. \tag{19.177}$$

The solution of the differential equation for this trajectory is found by direct integration. Assuming $k = 1$ we obtain

$$\eta = \int \frac{da}{\sqrt{-V^{\text{univ}}(a)}} = \int \frac{da}{\sqrt{-(a - a_{\max}/2)^2 + a_{\max}^2/4}} = -\arccos\frac{2a}{a_{\max}}. \tag{19.178}$$

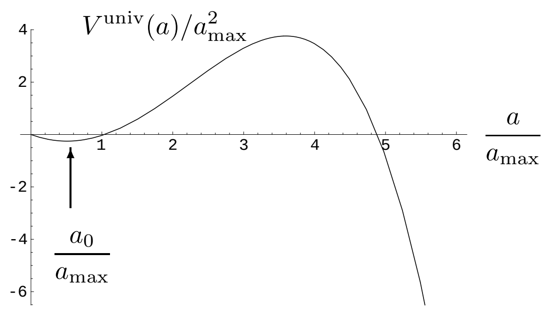

Figure 19.2 Potential of closed Friedman universe as a function of the reduced radius $a/a_{\rm max}$ for $\lambda a_{\rm max}^2 = 0.1$. Note the metastable minimum which leads to a possible solution $a \equiv a_0$. A tunneling process to the right leads to an expanding universe.

With the initial condition $a(0) = 0$, this implies

$$a(\eta) = \frac{a_{\rm max}}{2}\,(1 - \cos\eta). \tag{19.179}$$

Integrating Eq. (19.131), we find the relation between η and the physical (=proper) time

$$t = \frac{1}{c}\int d\eta\, a(\eta) = \frac{a_{\rm max}}{2c}\int d\eta\,(1 - \cos\eta) = \frac{a_{\rm max}}{2c}\,(\eta - \sin\eta). \tag{19.180}$$

The solution $a(t)$ is the cycloid pictured in Fig. 19.3. The radius of the universe bounces periodically with period $t_0 = \pi a_{\rm max}/c$ from zero to $a_{\rm max}$. Thus it emerges from a big bang, expands with a decreasing expansion velocity due to the gravitational attraction, and recontracts to a point.

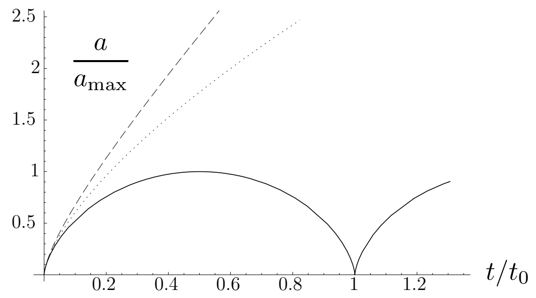

Figure 19.3 Radius of universe as a function of time in Friedman universe, measured in terms of the period $t_0 \equiv \pi a_{\rm max}/c$. (solid curve=closed, dashed curve=hyperbolic, dotted curve =parabolic). The curve for the closed universe is a cycloid.

Certainly, for high densities the solution is inapplicable since the pressure-less ideal gas approximation (19.163) breaks down.

Consider now the case of negative curvature with $k = -1$. Then the differential equation (19.177) reads

$$a_\eta^2 - a^2 - a_{\max} a - \frac{\lambda}{3} a^4 = 0. \tag{19.181}$$

In order to compare the curves we shall again introduce a mass parameter M and rewrite the density as in (19.165), although M has no longer the meaning of the total mass of the universe (which is now infinite). The solution for $\lambda = 0$ is now

$$a(\eta) = \frac{a_{\max}}{2}(\cosh\eta - 1), \tag{19.182}$$

$$t = \frac{a_{\max}}{2c}(\sinh\eta - \eta). \tag{19.183}$$

The solution is again depicted in Fig. 19.3. After a big bang, the universe expands forever, although with decreasing speed, due to the gravitational pull. The quantity $a_{\max}$ is no longer the maximal radius nor is t_0 the period.

Consider finally the parabolic case $k = 0$, where the equation of motion (19.177) reads

$$a_\eta^2 - a_{\max} a - \frac{\lambda}{3} a^4 = 0, \tag{19.184}$$

where M is a mass parameter defined as before in the negative curvature case. Now the solution for $\lambda = 0$ is simply

$$\eta = 2\sqrt{\frac{a}{a_{\max}}}, \tag{19.185}$$

which is inverted to

$$a(\eta) = a_{\max}\frac{\eta^2}{4}. \tag{19.186}$$

Now we solve (19.131) with

$$t = \frac{a_{\max}}{12c}\eta^3, \tag{19.187}$$

so that

$$a(t) = \left(\frac{9}{4}a_{\max}\right)^{1/3}(ct)^{2/3}. \tag{19.188}$$

This solution is simply the continuation of leading term in the previous two solutions to large t.

19.2.4 Tunneling of Expanding Universe

It is now interesting to observe that the potential for the spherical universe in Fig. 19.2 allows for a time-independent solution in which the radius lies at the metastable minimum which we may call a_0. The solution $a \equiv a_0$ is a time-independent universe. We may now imagine that the expanding universe arises from this by a tunneling process towards the abyss on the right of the potential [10]. Its rate can be calculated from the Euclidean action of the associated classical solution in imaginary time corresponding to the motion from a_0 towards the right in the inverted potential $-V^{\rm univ}(a)$.

Observe that this birth can only lead to universe of positive curvature. For a negative curvature, where the a^2-term in (19.176) has the opposite sign, the metastable minimum is absent.

19.3 Relativistic Coulomb System

An external time-independent potential $V(\mathbf{x})$ is introduced into the path integral (19.48) by substituting the energy E by $E - V(\mathbf{x})$. In the case of an attractive Coulomb potential, the second term in the action (19.49) becomes

$$\mathcal{A}^{\rm int} = -\int_{\lambda_a}^{\lambda_b} d\lambda\, h(\lambda) \frac{(E + e^2/r)^2}{2Mc^3}, \qquad (19.189)$$

where $r = |\mathbf{x}|$. The associated path integral is calculated via a Duru-Kleinert transformation as follows [12].

Consider the three-dimensional Coulomb system where the spacetime dimension is $D = 4$. Then we increase the three-dimensional space in a trivial way by a dummy fourth component x^4, just as in the nonrelativistic treatment in Section 13.4. The additional variable x^4 is eliminated at the end by an integral $\int dx_a^4/r_a = \int d\gamma_a$, as
in (13.120) and (13.127). Then we perform a Kustaanheimo-Stiefel transformation ref(13.120)
(13.106) $dx^\mu = 2A(u)^\mu{}_\nu du^\nu$. This changes $x'^{\mu 2}$ into $4\vec{u}^2\vec{u}'^{\,2}$, with the vector symbol lab(13.re)
indicating the four-vector nature. The transformed action reads: est(13.114)
ref(13.127)
lab(13.3d)
est(13.121)
ref(13.106)
lab(13.diffe)
est(13.101)

$$\tilde{\mathcal{A}}_{{\rm e},E} = \int_{\lambda_a}^{\lambda_b} d\lambda \left\{ \frac{4Mc\,\vec{u}^2}{2h(\lambda)}\, \vec{u}\,'^2(\lambda) + \frac{h(\lambda)}{2Mc^3\vec{u}^2} \left[(M^2c^4 - E^2)\vec{u}^2 - 2Ee^2 - \frac{e^4}{\vec{u}^2} \right] \right\}. \qquad (19.190)$$

We now choose the gauge $h(\lambda) = 1$, and go from λ to a new dimensionless parameter s via the path-dependent time transformation $d\lambda = f ds$ with $f = \vec{u}^2$. Result is the DK-transformed action

$$\tilde{\mathcal{A}}^{\rm DK}_{{\rm e},E} = \int_{s_a}^{s_b} \frac{ds}{c} \left\{ \frac{4Mc^2}{2} \vec{u}\,'^2(s) + \frac{1}{2Mc^2} \left[\left(M^2c^4 - E^2 \right) \vec{u}^2 - 2Ee^2 - \frac{e^4}{\vec{u}^2} \right] \right\}. \qquad (19.191)$$

It describes a particle of mass $\mu = 4M$ moving as a function of the "pseudotime" s in a harmonic oscillator potential of dimensionless "frequency"

$$\omega = \frac{1}{2Mc^2}\sqrt{M^2c^4 - E^2}. \qquad (19.192)$$

The oscillator possesses an additional attractive potential $-e^4/2Mc^2\vec{u}^2$, which is conveniently parametrized in the form of a centrifugal barrier

$$V_{\rm extra} = \hbar^2 \frac{l_{\rm extra}^2}{2\mu\vec{u}^2}, \tag{19.193}$$

whose squared angular momentum has the negative value

$$l_{\rm extra}^2 \equiv -4\alpha^2. \tag{19.194}$$

Here α denotes the fine-structure constant $\alpha \equiv e^2/\hbar c \approx 1/137$. In addition, there is also a trivial constant potential

$$V_{\rm const} = -\frac{E}{Mc^2}e^2. \tag{19.195}$$

If we ignore, for the moment, the centrifugal barrier $V_{\rm extra}$, the solution of the path
ref(13.127)integral can immediately be written down [see (13.127)]:
lab(13.3d)
est(13.121)

$$(\mathbf{x}_b|\mathbf{x}_a)_E = -i\frac{\hbar}{2Mc}\frac{1}{16}\int_0^\infty dS\; e^{e^2ES/Mc^2\hbar}\int_0^{4\pi} d\gamma_a\,(\vec{u}_bS|\vec{u}_a0)\,, \tag{19.196}$$

where $(\vec{u}_bS|\vec{u}_a0)$ is the pseudotime evolution amplitude of the four-dimensional harmonic oscillator.

There are no time slicing corrections for the same reason as in the three-dimensional case. This is ensured by the affine connection of the Kustaanheimo-Stiefel transformation satisfying

$$\Gamma_\mu{}^{\mu\lambda} = g^{\mu\nu}e_i{}^\lambda\partial_\mu e^i{}_\nu = 0 \tag{19.197}$$

(see the discussion in Section 13.6).

Performing the integral over γ_a in (19.196), we obtain

$$\begin{aligned}(\mathbf{x}_b|\mathbf{x}_a)_E &= -i\frac{\hbar}{2Mc}\frac{M\kappa}{\pi\hbar}\int_0^1 d\varrho\frac{\varrho^{-\nu}}{(1-\varrho)^2}I_0\left(2\kappa\frac{2\sqrt{\varrho}}{1-\varrho}\sqrt{(r_br_a+\mathbf{x}_b\mathbf{x}_a)/2}\right)\\ &\qquad\times\exp\left[-\kappa\frac{1+\varrho}{1-\varrho}(r_b+r_a)\right],\end{aligned} \tag{19.198}$$

with the variable

$$h \equiv e^{-2\omega S}, \tag{19.199}$$

and the parameters

$$\nu = \frac{e^2}{2\omega\hbar}\frac{E}{Mc^2} = \frac{\alpha}{\sqrt{M^2c^4/E^2-1}}, \tag{19.200}$$

$$\kappa = \frac{\mu\omega}{2\hbar} = \frac{1}{\hbar c}\sqrt{M^2c^4-E^2} = \frac{E}{\hbar c}\frac{\alpha}{\nu}. \tag{19.201}$$

ref(13.203) lab(13.n1) est(13.192) ref(13.207) lab(13.n5) est(13.197)

As in the further treatment of (13.203), the use of formula (13.207)

$$I_0(z\cos(\theta/2)) = \frac{2}{z}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)I_{2l+1}(z) \tag{19.202}$$

provides us with a partial wave decomposition

$$\begin{aligned}(\mathbf{x}_b|\mathbf{x}_a)_E &= \frac{1}{r_b r_a}\sum_{l=0}^{\infty}(r_b|r_a)_{E,l}\frac{2l+1}{4\pi}P_l(\cos\theta)\\ &= \frac{1}{r_b r_a}\sum_{l=0}^{\infty}(r_b|r_a)_{E,l}\sum_{m=-l}^{l}Y_{lm}(\hat{\mathbf{x}}_b)Y^*_{lm}(\hat{\mathbf{x}}_a).\end{aligned} \tag{19.203}$$

The radial amplitude is normalized slightly differently from (13.210): ref(13.210) lab(13.n9) est(13.201)

$$\begin{aligned}(r_b|r_a)_{E,l} &= -i\frac{\hbar}{2Mc}\sqrt{r_b r_a}\frac{2M}{\hbar}\int_0^\infty dy\frac{1}{\sinh y}e^{2\nu y}\\ &\quad\times\exp\left[-\kappa\coth y(r_b+r_a)\right]I_{2l+1}\left(2\kappa\sqrt{r_b r_a}\frac{1}{\sinh y}\right).\end{aligned} \tag{19.204}$$

At this place, we incorporate the additional centrifugal barrier via the replacement

$$2l+1 \to 2\tilde{l}+1 \equiv \sqrt{(2l+1)^2 + l_{\rm extra}^2}, \tag{19.205}$$

as in Eqs. (8.146) and (14.223). The integration over y according to (9.29) yields ref(8.146) lab(x8.146) est(19@8.146) ref(14.223) lab(14.239) est(14.239) ref(9.29) lab(intf3) est(9.52)

$$(r_b|r_a)_{E,l} = -i\frac{\hbar}{2Mc}\frac{M}{\hbar\kappa}\frac{\Gamma(-\nu+\tilde{l}+1)}{(2\tilde{l}+1)!}W_{\nu,\tilde{l}+1/2}(2\kappa r_b)\,M_{\nu,\tilde{l}+1/2}(2\kappa r_a). \tag{19.206}$$

This expression possesses poles in the Gamma function whose positions satisfy the equations $\nu - \tilde{l} - 1 = 0, 1, 2, \ldots$. These determine the bound states of the Coulomb system. To simplify subsequent expressions, we introduce the small positive l-dependent parameter

$$\delta_l \equiv l - \tilde{l} = l + 1/2 - \sqrt{(l+1/2)^2 - \alpha^2} \approx \frac{\alpha^2}{2l+1} + \mathcal{O}(\alpha^4). \tag{19.207}$$

Then the pole positions satisfy $\nu = \tilde{n}_l \equiv n - \delta_l$, with $n = l+1, l+2, l+3, \ldots$. Using the relation (19.200), we obtain the bound-state energies:

$$\begin{aligned}E_{nl} &= \pm Mc^2\left[1 + \frac{\alpha^2}{(n-\delta_l)^2}\right]^{-1/2}\\ &\approx \pm Mc^2\left[1 - \frac{\alpha^2}{2n^2} - \frac{\alpha^4}{n^3}\left(\frac{1}{2l+1} - \frac{3}{8n}\right) + \mathcal{O}(\alpha^6)\right].\end{aligned} \tag{19.208}$$

Note the appearance of the plus-minus sign as a characteristic property of energies in relativistic quantum mechanics. A correct interpretation of the negative energies as positive energies of antiparticles is straightforward only within quantum field theory, and will not be discussed here. Even if we ignore the negative energies, there is poor agreement with the experimental spectrum of the hydrogen atom. The spin of the electron must be included to get more satisfactory results.

To find the wave functions, we approximate near the poles $\nu \approx \tilde{n}_l$:

$$\begin{aligned}\Gamma(-\nu+\tilde{l}+1) &\approx -\frac{(-)^{n_r}}{n_r!}\frac{1}{\nu-\tilde{n}_l},\\ \frac{1}{\nu-\tilde{n}_l} &\approx \frac{2}{\tilde{n}_l}\frac{\hbar^2\kappa^2}{2M}\left(\frac{E}{Mc^2}\right)^2\frac{2Mc^2}{E^2-E_{nl}^2},\\ \kappa &\approx \frac{E}{Mc^2}\frac{1}{a_H}\frac{1}{\tilde{n}_l},\end{aligned} \tag{19.209}$$

with the radial quantum number $n_r = n-l-1$. By analogy with the nonrelativistic
ref(13.213) equation (13.213), the last equation can be rewritten as
lab(13.n11)
est(13.203)

$$\kappa = \frac{1}{\tilde{a}_H}\frac{1}{\nu}, \tag{19.210}$$

where

$$\tilde{a}_H \equiv a_H\frac{Mc^2}{E} \tag{19.211}$$

denotes a modified energy-dependent Bohr radius [compare (4.376)]. It sets the length scale of relativistic bound states in terms of the energy E. Instead of being $1/\alpha \approx 137$ times the Compton wavelength of the electron $\hbar/Mc$, the modified Bohr radius is equal to $1/\alpha$ times $\hbar c/E$.

Near the positive-energy poles, we now approximate

$$-i\Gamma(-\nu+\tilde{l}+1)\frac{M}{\hbar\kappa} \approx \frac{(-)^{n_r}}{\tilde{n}_l^2 n_r!}\frac{1}{\tilde{a}_H}\left(\frac{E}{Mc^2}\right)^2\frac{2Mc^2 i\hbar}{E^2-E_{nl}^2}. \tag{19.212}$$

ref(9.48) Using this behavior and formula (9.48) for the Whittaker functions [together with
lab(9.for) (9.50)] we write the contribution of the bound states to the spectral representation
est(9.73) of the fixed-energy amplitude as
ref(9.50)
lab(x9.75)
est(9.75)

$$(r_b|r_a)_{E,l} = \frac{\hbar}{Mc}\sum_{n=l+1}^{\infty}\left(\frac{E}{Mc^2}\right)^2\frac{2Mc^2 i\hbar}{E^2-E_{nl}^2}R_{nl}(r_b)R_{nl}(r_a)+\dots\ . \tag{19.213}$$

A comparison between the pole terms in (19.206) and (19.213) renders the radial wave functions

$$\begin{aligned}R_{nl}(r) &= \frac{1}{\tilde{a}_H^{1/2}\tilde{n}_l}\frac{1}{(2\tilde{l}+1)!}\sqrt{\frac{(\tilde{n}_l+\tilde{l})!}{(n-l-1)!}}\\ &\quad\times(2r/\tilde{n}_l\tilde{a}_H)^{\tilde{l}+1}e^{-r/\tilde{n}_l\tilde{a}_H}M(-n+l+1,2\tilde{l}+2,2r/\tilde{n}_l\tilde{a}_H) \qquad (19.214)\\ &= \frac{1}{\tilde{a}_H^{1/2}\tilde{n}_l}\sqrt{\frac{(n-l-1)!}{(\tilde{n}+\tilde{l})!}}e^{-r/\tilde{n}\tilde{a}_H}(2r/\tilde{n}_l\tilde{a}_H)^{\tilde{l}+1}L_{\tilde{n}_l-l-1}^{2\tilde{l}+1}(2r/\tilde{n}_l\tilde{a}_H).\end{aligned}$$

The properly normalized total wave functions are

$$\psi_{nlm}(\mathbf{x}) = \frac{1}{r} R_{nl}(r) Y_{lm}(\hat{\mathbf{x}}). \tag{19.215}$$

The continuous wave functions are obtained in the same way as from the non-relativistic amplitude in formulas (13.221)–(13.229). ref(13.221) lab(x13.210) est(13.210) ref(13.229) lab(x13.218) est(13.218)

19.4 Relativistic Particle in Electromagnetic Field

Consider now the relativistic particle in a general spacetime-dependent electromagnetic vector field $A^\mu(x)$.

19.4.1 Action and Partition Function

An electromagnetic field $A^\mu(x)$ is included into the canonical action (19.14) in the usual way by the minimal substitution (2.644):

$$\bar{\mathcal{A}}_{\rm e}[p,x] = \int_{\lambda_a}^{\lambda_b} d\lambda \left\{ -ip\dot{x} + \frac{h(\lambda)}{2Mc} \left[\left(p - \frac{e}{c} A \right)^2 + M^2 c^2 \right] \right\}, \tag{19.216}$$

and the amplitude (19.22):

$$(x_b|x_a) = \frac{\hbar}{2Mc} \int_0^\infty dS \int \mathcal{D}h\, \Phi[h] \int \mathcal{D}^D x\, e^{-\bar{\mathcal{A}}_{\rm e}/\hbar}, \tag{19.217}$$

with the minimally coupled action [compare (2.706)]

$$\bar{\mathcal{A}}_{\rm e} = \int_{\lambda_a}^{\lambda_b} d\lambda \left[\frac{Mc}{2h(\lambda)} \dot{x}^2(\lambda) + i\frac{e}{c} \dot{x}(\lambda) A(x(\lambda)) + h(\lambda) \frac{Mc}{2} \right], \tag{19.218}$$

which reduces in the simplest gauge (19.23) to the obvious extension of (19.26):

$$(x_b|x_a) = \frac{\hbar^2}{2M} \int_0^\infty d\beta\, e^{-\beta Mc^2/2} \int \mathcal{D}^4 x\, e^{-\mathcal{A}_{\rm e}}, \tag{19.219}$$

with the action

$$\mathcal{A}_{\rm e} = \mathcal{A}_{\rm e,0} + \mathcal{A}_{\rm e,int} \equiv \int_0^{\hbar\beta} d\tau \left[\frac{M}{2} \dot{x}^2(\tau) + i\frac{e}{c} \dot{x}(\tau) A(x(\tau)) \right]. \tag{19.220}$$

The partition function of a single closed particle loop of all shapes and lengths in an external electromagnetic field is from (19.37)

$$Z_1 = \int_0^\infty \frac{d\beta}{\beta} e^{-\beta Mc^2/2} \int \mathcal{D}^D x\, e^{-\mathcal{A}_{\rm e}/\hbar}. \tag{19.221}$$

As in (19.45) and (19.46) this yields, up to a factor $1/\hbar$ the effective action of an ensemble of closed particle loops in an external electromagnetic field.

19.4.2 Perturbation Expansion

Since the electromagnetic coupling is rather small, we can split the exponent $e^{-\mathcal{A}/\hbar}$ into $e^{-\mathcal{A}_0/\hbar}e^{-\mathcal{A}_{\rm int}/\hbar}$ and expand the second factor in powers of $\mathcal{A}_{\rm int}$:

$$e^{-\mathcal{A}_{\rm int}/\hbar} = \sum_{n=0}^{\infty}\frac{(-ie/\hbar c)^n}{n!}\prod_{i=1}^{n}\left[\int_0^{\hbar\beta}d\tau_i\,\dot{x}(\tau_i)A(x(\tau_i))\right]. \tag{19.222}$$

If the noninteracting effective action implied by Eqs. (19.39), (19.37), and (19.46) is denoted by

$$\frac{\Gamma_{\rm e,0}}{\hbar}\equiv -Z_1 = -\int_0^\infty\frac{d\beta}{\beta}\,e^{-\beta Mc^2/2}\frac{V_D}{\sqrt{2\pi\hbar^2\beta/M}^D} = -\frac{V_D}{{\lambda_M^{\rm C}}^D}\frac{1}{(4\pi)^{D/2}}\Gamma(1-D/2), \tag{19.223}$$

with the Compton wavelength $\lambda_M^{\rm C}$ of Eq. (19.31), we obtain the perturbation expansion

$$\frac{\Gamma_{\rm e}}{\hbar}=\frac{\Gamma_{\rm e,0}}{\hbar}-\int_0^\infty\frac{d\beta}{\beta}\,e^{-\beta Mc^2/2}\frac{V_D}{\sqrt{2\pi\hbar^2\beta/M}^D}\sum_{n=1}^{\infty}\frac{(-ie/c)^n}{n!}\left\langle\prod_{i=1}^{n}\left[\int_0^{\hbar\beta}d\tau_i\,\dot{x}(\tau_i)A(x(\tau_i))\right]\right\rangle_0, \tag{19.224}$$

where $\langle\dots\rangle_0$ denotes the free-particle expectation values [compare (3.483)–(3.486)] taken in the free path integral with periodic paths with a fixed β [compare (19.38) and (19.41)]:

$$\langle\mathcal{O}[x]\rangle_0\equiv\frac{\int\mathcal{D}^Dx\,\mathcal{O}[x]\,e^{-\mathcal{A}_{\rm e,0}/\hbar}}{\int\mathcal{D}^Dx\,e^{-\mathcal{A}_{\rm e,0}/\hbar}}. \tag{19.225}$$

The denominator is equal to $V_D/\sqrt{2\pi\hbar^2\beta/M}^D$.

The free effective action in the expansion (19.224) can be omitted by letting the sum start with $n=0$.

The evaluation of the cumulants proceeds by Fourier decomposing the vector fields as

$$A(x)=\int\frac{d^Dk}{(2\pi)^D}e^{ikx}A(k), \tag{19.226}$$

and rewriting (19.224) as

$$\frac{\Gamma_{\rm e}}{\hbar}=\frac{\Gamma_{\rm e,0}}{\hbar}-\int_0^\infty\frac{d\beta}{\beta}\,e^{-\beta Mc^2/2}\frac{V_D}{\sqrt{2\pi\hbar^2\beta/M}^D}$$
$$\times\sum_{n=1}^{\infty}\frac{(-ie/\hbar c)^n}{n!}\prod_{i=1}^{n}\left[\int\frac{d^Dk_i}{(2\pi)^D}A^{\mu_i}(k_i)\right]\left\langle\prod_{i=1}^{n}\left[\int_0^{\hbar\beta}d\tau_i\,\dot{x}^{\mu_i}(\tau_i)e^{ik_ix(\tau_i)}\right]\right\rangle_0. \tag{19.227}$$

First we evaluate the expectation values

$$\left\langle\dot{x}(\tau_1)e^{ik_1x(\tau_1)}\cdots\dot{x}(\tau_n)e^{ik_nx(\tau_n)}\right\rangle_0. \tag{19.228}$$

Due to the periodic boundary conditions, we separate, as in Section 3.25, the path average $x_0 = \bar{x}(\tau)$ [recall (3.807)], writing

$$x(\tau) = x_0 + \delta x(\tau), \tag{19.229}$$

and factorize (19.228) as

$$\left\langle e^{i(k_1+\ldots+k_n)x_0}\right\rangle_0 \left\langle \delta\dot{x}(\tau_1)e^{ik_1\delta x(\tau_1)}\cdots\delta\dot{x}(\tau_n)e^{ik_n\delta x(\tau_n)}\right\rangle_0. \tag{19.230}$$

The first average can be found as an average with respect to the x_0-part of the path integral whose measure was given in Eq. (3.811). It yields a δ function ensuring the conservation of the total energy and momenta of the n photons involved:

$$\left\langle e^{i(k_1+\ldots+k_n)x_0}\right\rangle_0 = \frac{1}{V_D}(2\pi)^D\delta^{(D)}(k_1+\ldots+k_n). \tag{19.231}$$

The denominator comes from the normalization of the expectation value which has an integral $\int d^Dx_0$ in the denominator.

The second average is obtained using Wick's theorem. The correlation function $\langle\delta x^\mu(\tau_1)\delta x^\nu(\tau_2)\rangle_0$, is obtained from Eq. (3.842) in the limit $\Omega\to 0$ for $\tau_1,\tau_2\in(0,\beta)$. It is the periodic propagator with subtracted zero mode:

$$\langle\delta x^\mu(\tau_1)\delta x^\nu(\tau_2)\rangle_0 = \delta^{\mu\nu}G(\tau_1,\tau_2) = \delta^{\mu\nu}\frac{\hbar}{M}\bar{\Delta}(\tau_1,\tau_2), \tag{19.232}$$

where $\bar{\Delta}(\tau_1-\tau_2)$ is the subtracted periodic Green function $G^{\rm a}_{\omega,{\rm e}}(\tau-\tau')$ of the differential operator $-\partial_\tau^2$ calculated in Eq. (3.254) in the short notation of Subsection 10.12.1 [see Eq. (10.565)]. In the presently used physical units it reads:

$$\bar{\Delta}(\tau,\tau') \equiv \bar{\Delta}(\tau-\tau') = \frac{(\tau-\tau')^2}{2\hbar\beta} - \frac{\tau-\tau'}{2} + \frac{\hbar\beta}{12}, \qquad \tau\in[0,\hbar\beta]. \tag{19.233}$$

The time derivatives of (19.233) are from (10.566):

$${}^{\cdot}\bar{\Delta}(\tau,\tau') = -\bar{\Delta}^{\cdot}(\tau,\tau') \equiv \frac{\tau-\tau'}{\hbar\beta} - \frac{\epsilon(\tau-\tau')}{2}, \quad \tau,\tau'\in[0,\hbar\beta]. \tag{19.234}$$

With these functions it is straightforward to calculate the expectation value using the Wick rule (3.310) for $j(\tau)=\sum_i^n k_i\delta(\tau-\tau_i)$:

$$\left\langle e^{ik_1\delta x(\tau_1)}\cdots e^{ik_n\delta x(\tau_n)}\right\rangle_0 = e^{-\frac{1}{2}\sum_{i,j=1}^n k_ik_jG(\tau_i,\tau_j)}. \tag{19.235}$$

By rewriting the right-hand side as

$$e^{-\frac{1}{2}\sum_{i,j=1}^n k_ik_jG(\tau_i,\tau_j)} = e^{-\frac{1}{2}\sum_{i,j=1}^n k_ik_j[G(\tau_i,\tau_j)-G(\tau_i,\tau_i)]-\frac{1}{2}\left(\sum_{i=1}^n k_i\right)^2G(\tau_i,\tau_i)}, \tag{19.236}$$

we see that if the momenta k_i add up to zero, $\sum_{i=1}^n k_i=0$, we can replace (19.235) by

$$\left\langle e^{ik_1\delta x(\tau_1)}\cdots e^{ik_n\delta x(\tau_n)}\right\rangle_0 = \exp\left\{-\sum_{i<j}^n k_ik_j[G(\tau_i,\tau_j)-G(\tau_i,\tau_i)]\right\}. \tag{19.237}$$

It is therefore useful to introduce the subtracted Green function

$$G'(\tau_i,\tau_j) \equiv G(\tau_i,\tau_j) - G(\tau_i,\tau_i). \tag{19.238}$$

Recall the similar situation in the evaluation (5.377).

An obvious extension of (19.235) is

$$\begin{aligned}&\left\langle e^{i[k_1\delta x(\tau_1)+q_1\dot{x}(\tau_1)]}\cdots e^{i[k_n\delta x(\tau_n)+q_n\dot{x}(\tau_n)]}\right\rangle_0 \qquad (19.239)\\ &\quad = e^{-\frac{1}{2}\sum_{i,j=1}^n k_ik_jG(\tau_i,\tau_j)-\frac{1}{2}\sum_{i,j=1}^n q_ik_j\,{}^{\cdot}G(\tau_i,\tau_j)-\frac{1}{2}\sum_{i,j=1}^n k_iq_j\,G^{\cdot}(\tau_i,\tau_j)-\frac{1}{2}\sum_{i,j=1}^n q_iq_j\,{}^{\cdot}G^{\cdot}(\tau_i,\tau_j)},\end{aligned}$$

where the dots have the same meaning as in (10.395).

19.4.3 Lowest-Order Vacuum Polarization

Consider the lowest nontrivial case $n=2$. By differentiating (19.237) with respect to iq_1 and iq_2, and setting $q_i=0$, we obtain for $k_2=-k_1=-k$:

$$\left\langle \dot{x}(\tau_1)e^{ik\delta x(\tau_1)}\dot{x}(\tau_2)e^{-ik\delta x(\tau_2)}\right\rangle_0 = \left[{}^{\cdot}G^{\cdot}(\tau_1,\tau_2)+k^2\,{}^{\cdot}G(\tau_1,\tau_2)G^{\cdot}(\tau_1,\tau_2)\right]e^{k^2[G(\tau_1,\tau_2)-G(\tau_1,\tau_1)]}. \tag{19.240}$$

Inserting this into (19.227) after factorization according to (19.230), we obtain the lowest correction to the effective action

$$\begin{aligned}\Delta\Gamma_{\rm e} = &-\frac{e^2}{2\hbar c^2}\int_0^\infty \frac{d\beta}{\beta}e^{-\beta Mc^2/2}\frac{1}{\sqrt{2\pi\hbar^2\beta/M}^D}\prod_{i=1}^2\left[\int\frac{d^Dk_i}{(2\pi)^D}\right](2\pi)^D\delta^{(D)}(k_1+k_2)A^\mu(k_1)A^\nu(k_2)\\ &\times\int_0^{\hbar\beta}d\tau_1\int_0^{\hbar\beta}d\tau_2\left[{}^{\cdot}G^{\cdot}(\tau_1,\tau_2)\delta^{\mu\nu}+k_1^\mu k_1^\nu\,{}^{\cdot}G(\tau_1,\tau_2)\,G^{\cdot}(\tau_1,\tau_2)\right]e^{k_1^2G'(\tau_1,\tau_2)},\end{aligned} \tag{19.241}$$

where we have displayed the proper vector indices. A partial integration over τ_1 brings the second line to the form

$$-\int_0^{\hbar\beta}d\tau_1\int_0^{\hbar\beta}d\tau_2\left(k_1^2\delta^{\mu\nu}-k_1^\mu k_1^\nu\right){}^{\cdot}G(\tau_1,\tau_2)\,G^{\cdot}(\tau_1,\tau_2)\,e^{k_1^2G'(\tau_1,\tau_2)}. \tag{19.242}$$

Expressing $G'(\tau_1,\tau_2)$ as $(\hbar/M)\left[\bar{\Delta}(\tau_1-\tau_2)-\bar{\Delta}(0)\right]$ and using the periodicity in τ_2, we calculate the integral

$$\frac{\hbar^2}{M^2}\left(k_1^2\delta^{\mu\nu}-k_1^\mu k_1^\nu\right)\hbar\beta\int_0^{\hbar\beta}d\tau\,\dot{\Delta}_{\rm p}'^{\,2}(\tau)e^{\hbar k_1^2M\left[\Delta_{\rm p}'(\tau)-\Delta_{\rm p}'(0)\right]}. \tag{19.243}$$

We now introduce reduced times $u\equiv\tau/\hbar\beta$ and rewrite the Green functions for $\tau\in(0,\hbar\beta)$, $u\in(0,1)$ as

$$\bar{\Delta}(\tau_1-\tau_2) = -\frac{\hbar\beta}{2}\left[u(1-u)-\frac{1}{6}\right], \tag{19.244}$$

$$\dot{\bar{\Delta}}(\tau_1-\tau_2) = u-\frac{1}{2}, \tag{19.245}$$

such that the integral in (19.243) becomes

$$\frac{1}{4}\int_0^1 du\,(2u-1)^2 e^{-\beta\hbar^2 k_1^2 u(1-u)/2M}. \tag{19.246}$$

Inserting this into (19.241) and dropping the irrelevant subscript of k_1 we arrive at

$$\begin{aligned}\Delta\Gamma_{\rm e}=&\frac{e^2}{2\hbar c^2}\int_0^\infty\frac{d\beta}{\beta}e^{-\beta Mc^2/2}\frac{1}{\sqrt{2\pi\hbar^2\beta/M}^D}\int\frac{d^Dk}{(2\pi)^D}A^\mu(k)A^\nu(-k)\\ &\times\left(k^2\delta^{\mu\nu}-k^\mu k^\nu\right)\frac{\hbar^4\beta^2}{4M^2}\int_0^1 du\,(2u-1)^2e^{-\beta\hbar^2k^2 2Mu(1-u)/2M}.\end{aligned} \tag{19.247}$$

After replacing $\hbar^2\beta/2M\to\beta$, the integral over β can easily be performed using the formula (2.498) with the result

$$\begin{aligned}\Delta\Gamma_{\rm e}=&\frac{e^2\hbar}{c^2}\frac{1}{(4\pi)^{D/2}}\frac{1}{2c}\int\frac{d^Dk}{(2\pi)^D}A^\mu(k)A^\nu(-k)\left(k^2\delta^{\mu\nu}-k^\mu k^\nu\right)\\ &\times\Gamma\left(2-D/2\right)\int_0^1 du\,(2u-1)^2\left[u(1-u)k^2+M^2c^2/\hbar^2\right]^{D/2-2}.\end{aligned} \tag{19.248}$$

In the prefactor we recognize the fine structure constant $\alpha=e^2/\hbar c$ [recall (1.505)]. The momentum integral can be rewritten as

$$\frac{1}{2}\int\frac{d^Dk}{(2\pi)^D}A^\mu(k)A^\nu(-k)\left(k^2\delta^{\mu\nu}-k^\mu k^\nu\right)=\frac{1}{4}\int\frac{d^Dk}{(2\pi)^D}F_{\mu\nu}(-k)F_{\mu\nu}(k), \tag{19.249}$$

where

$$F_{\mu\nu}(x)=\partial_\mu A_\nu(x)-\partial_\nu A_\mu(x) \tag{19.250}$$

is the tensor of electromagnetic field strengths. We now abbreviate the integral over u as follows:

$$\Pi(k^2)\equiv\alpha\frac{4\pi}{(4\pi)^{D/2}}\Gamma\left(2-D/2\right)\int_0^1 du\,(2u-1)^2\left[u(1-u)k^2+M^2c^2/\hbar^2\right]^{D/2-2}. \tag{19.251}$$

This allows us to re-express (19.248) in configuration space:

$$\Delta\Gamma_{\rm e}=\frac{1}{16\pi c}\int d^4x F_{\mu\nu}(x)\Pi(-\partial^2)F_{\mu\nu}(x), \tag{19.252}$$

where $F_{\mu\nu}(x)$ is the Euclidean version of the gauge-invariant 4-dimensional curl of the vector potential:

$$F_{\mu\nu}(x)=\partial_\mu A_\nu(x)-\partial_\nu A_\mu(x). \tag{19.253}$$

In Minkowski space, the components of $F_{\mu\nu}$ are the electric and magnetic fields:

$$F_{0i}=-F^{0i}=-\partial^0A^i+\partial^iA^0=-\partial_0A^i-\partial_iA^0=-E^i, \tag{19.254}$$

$$F_{ij}=F^{ij}=\partial^iA^j+\partial^jA^i=-\partial_iA^j+\partial_jA^i=-\epsilon_{ijk}B^k. \tag{19.255}$$

This is in accordance with the electrodynamic definitions

$$\mathbf{E} \equiv -\frac{1}{c}\dot{\mathbf{A}} - \boldsymbol{\nabla}\phi, \qquad \mathbf{B} \equiv \boldsymbol{\nabla} \times \mathbf{A}, \tag{19.256}$$

where $A^0(x)$ is identified with the electric potential $\phi(x)$.

In terms of $F_{\mu\nu}(x)$, the Maxwell action in the presence of a charge density $\rho(x)$ and a electric current density $\mathbf{j}(x)$

$$\mathcal{A}^{\rm em} = \int dt\, d^3x \left\{ \frac{1}{4\pi}\left[\mathbf{E}^2(x) - \mathbf{B}^2(x)\right] - \left[\rho(x)\phi(x) - \frac{1}{c}\mathbf{j}(x)\cdot\mathbf{A}(x)\right]\right\}, \tag{19.257}$$

can be written covariantly as

$$\mathcal{A}^{\rm em} = -\int d^4x \left[\frac{1}{8\pi c}F^2_{\mu\nu}(x) + \frac{1}{c^2}j^\mu(x)A_\mu(x)\right], \tag{19.258}$$

where

$$j_\mu(x) = (c\rho(x), \mathbf{j}(x)) \tag{19.259}$$

is the four-vector formed by charge density and electric current.

By extremizing the action (19.258) in the vector field $A^\mu(x)$ we find the Maxwell equations in the covariant form

$$\partial_\nu F^{\nu\mu}(x) = \frac{1}{c}j^\mu(x), \tag{19.260}$$

whose zeroth and spatial components reduce to the time-honored laws of Gauss and Ampère:

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\rho \quad (\text{Gauss's law}), \tag{19.261}$$

$$\boldsymbol{\nabla}\times\mathbf{B} = \frac{4\pi}{c}\mathbf{j} \quad (\text{Ampère's law}). \tag{19.262}$$

Expressing $\mathbf{E}(x)$ in terms of the potential using Eq. (19.256) and inserting this into Gauss's law, we obtain for a static point charge e at the origin the Poisson equation

$$-\boldsymbol{\nabla}^2\phi(\mathbf{x}) = 4\pi e\delta^{(3)}(\mathbf{x}). \tag{19.263}$$

An electron of charge $-e$ experiences an attractive mechanical potential $V(\mathbf{x}) = -e\phi(\mathbf{x})$. In momentum space this satisfies the equation reads

$$\mathbf{k}^2 V(\mathbf{k}) = -4\pi e^2. \tag{19.264}$$

From this we find directly the Coulomb potential of a hydrogen atom

$$V(\mathbf{x}) = (\boldsymbol{\nabla}^2)^{-1}4\pi e\delta^{(3)}(\mathbf{x}) = -\int \frac{d^3k}{(2\pi)^3}\frac{4\pi e^2}{\mathbf{k}^2} = -\frac{e^2}{r}, \qquad r \equiv |\mathbf{x}|, \tag{19.265}$$

where e^2 can be expressed in terms of the fine-structure constant α as $e^2 = \hbar c\,\alpha$. (1.505).

The Euclidean result (19.252) implies that a fluctuating closed particle orbit changes the first term in the Maxwell action (19.258) to

$$\mathcal{A}_{\rm em}^{\rm eff} = -\int d^D x \, \frac{1}{16\pi c} F_{\mu\nu}(x) \left[1 + \Pi(-\partial^2)\right] F_{\mu\nu}(x). \tag{19.266}$$

The quantity $\Pi(-\partial^2)$ is the *self-energy* of the electromagnetic field caused by the fluctuating closed particle orbit.

The self-energy changes the Maxwell equations (19.261) and (19.262) into

$$\begin{aligned} \left[1 + \Pi(-\partial^2)\right] \ \boldsymbol{\nabla} \cdot \mathbf{E} \ &= \ 4\pi\rho, \\ \left[1 + \Pi(-\partial^2)\right] \boldsymbol{\nabla} \times \mathbf{B} \ &= \ \frac{4\pi}{c} \mathbf{j}. \end{aligned} \tag{19.267}$$

The static equation (19.264) for the atomic potential changes therefore into

$$\left[1 + \Pi(\mathbf{k}^2)\right] \mathbf{k}^2 V(\mathbf{k}) = -4\pi e^2. \tag{19.268}$$

Since $\Pi(\mathbf{k}^2)$ is of order $\alpha \approx 1/137$, this can be solved approximately by

$$V(\mathbf{k}) \equiv -4\pi e^2 \left[1 - \Pi(\mathbf{k}^2)\right] \frac{1}{\mathbf{k}^2}. \tag{19.269}$$

In real space, the attractive atomic potential is changed to lowest order in α as

$$-\frac{\alpha}{r} \to -\left[1 - \Pi(\boldsymbol{\nabla}^2)\right] \frac{\alpha}{r}. \tag{19.270}$$

Set us calculate this change explicitly. For small ϵ and k^2, we expand the self-energy (19.251) in $D = 4 - \epsilon$ dimensions as

$$\Pi(k^2) = \frac{\alpha}{24\pi} \left[-\frac{2}{\epsilon} + \log \frac{M^2c^2e^\gamma}{4\pi\hbar^2}\right] - \frac{\alpha\hbar^2 k^2}{160\pi M^2 c^2} + \mathcal{O}\left(\epsilon, \frac{k^2}{M^2c^2/\hbar^2}\right). \tag{19.271}$$

Inserting this into (19.269), or into (19.269) and using the Poisson equation $-\boldsymbol{\nabla}^2 \times 1/r = 4\pi\delta^{(3)}(\mathbf{x})$, we see that the self energy changes the Coulomb potential as follows:

$$-\frac{\alpha}{r} \to -\left[1 - \Pi(\boldsymbol{\nabla}^2)\right] \frac{\alpha}{r} \approx -\left\{1 - \frac{\alpha}{24\pi^2} \left[-\frac{2}{\epsilon} + \log \frac{M^2c^2e^\gamma}{4\pi\hbar^2}\right]\right\} \frac{\alpha}{r} - \frac{\alpha^2\hbar^2}{40 M^2 c^2} \delta^{(3)}(\mathbf{x}). \tag{19.272}$$

The first term amounts to a small renormalization of the electromagnetic coupling by the factor in curly brackets, which is close to unity for finite ϵ since α is small. We are, however, interested in the result in $D = 4$ spacetime dimensions where $\epsilon \to 0$ and (19.272) diverges. The physical resolution of this divergence problem is to assume the initial point charge e_0 in the electromagnetic interaction to be different from the experimentally observed e to precisely compensate the renormalization factor, i.e.,

$$e_0^2 = e^2 \left\{1 + \frac{\alpha}{24\pi^2}\left[-\frac{2}{\epsilon} + \log\frac{M^2c^2e^\gamma}{4\pi\hbar^2}\right]\right\}. \tag{19.273}$$

Thus, the result Eq. (19.272) is really obtained in terms of e_0, i.e., with α replaced by α_0. Then, using (19.273), we find that up to order α^2, the atomic potential is

$$V^{\rm eff}(\mathbf{x}) = -\frac{\alpha}{r} - \frac{\alpha^2\hbar^2}{40M^2c^2}\delta^{(3)}(\mathbf{x}). \tag{19.274}$$

The second is an additional attractive contact interaction. It shifts the energies of the s-wave bound states in Eq. (19.208) slightly downwards.

19.5 Path Integral for Spin-1/2 Particle

For particles of spin 1/2 the path integral formulation becomes algebraically more involved. Let us first recall a few facts from Dirac's theory of the electron.

19.5.1 Dirac Theory

In the Dirac theory, electrons are described by a four-component field $\psi_\alpha(x)$ in spacetime parametrized by $x^\mu = (ct, \mathbf{x})$ with $\mu = 0, 1, 2, 3$. The field satisfies the wave equation

$$(i\hbar\partial\!\!\!/ - Mc)\,\psi(x) = 0, \tag{19.275}$$

where $\partial\!\!\!/$ is a short notation for $\gamma^\mu\partial_\mu$ and γ^μ are 4×4 Dirac matrices satisfying the anticommutation rules

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \tag{19.276}$$

where $g_{\mu\nu}$ is now the Minkowski metric

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{19.277}$$

An explicit representation of these rules is most easily written in terms of the Pauli matrices (1.448):

$$\gamma^0 = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{19.278}$$

where σ^0 is a 2×2 unit matrix. The anticommutation rules (19.276) follow directly from the multiplication rules for the Pauli matrices:

$$\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k. \tag{19.279}$$

The action of the Dirac field is

$$\mathcal{A} = \int d^4x \, \bar{\psi}(x) \, (i\hbar\partial\!\!\!/ - Mc) \, \psi(x), \tag{19.280}$$

where the conjugate field $\bar{\psi}(x)$ is defined as

$$\bar{\psi}(x) \equiv \psi^\dagger(x)\gamma^0. \tag{19.281}$$

It can be shown that this makes $\bar{\psi}(x)\psi(x)$ a scalar field under Lorentz transformations, $\bar{\psi}(x)\gamma^\mu\psi(x)$ a vector field, and $\mathcal{A}$ an invariant. If we decompose $\psi(x)$ into its Fourier components

$$\psi(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{V}} e^{i\mathbf{k}\mathbf{x}} \psi_{\mathbf{k}}(t), \tag{19.282}$$

where V is the spatial volume, the action reads

$$\mathcal{A} = \int_{t_a}^{t_b} dt \sum_{\mathbf{k}} \psi_{\mathbf{k}}^\dagger(t) \, [i\hbar\partial_t - H(\hbar\mathbf{k})] \, \psi_{\mathbf{k}}(t), \tag{19.283}$$

with the 4×4 Hamiltonian matrix

$$H(\mathbf{p}) \equiv \gamma^0 \boldsymbol{\gamma} \, \mathbf{p} \, c + \gamma^0 Mc^2. \tag{19.284}$$

This can be rewritten in terms of 2×2-submatrices as

$$H(\mathbf{p}) = \begin{pmatrix} Mc & \mathbf{p}\boldsymbol{\sigma} \\ -\,\mathbf{p}\boldsymbol{\sigma} & -Mc \end{pmatrix} c. \tag{19.285}$$

Since the matrix is Hermitian, it can be diagonalized by a unitary transformation to

$$H^{\rm d}(\mathbf{p}) = \begin{pmatrix} \varepsilon_{\mathbf{k}} & 0 \\ 0 & -\varepsilon_{\mathbf{k}} \end{pmatrix}, \tag{19.286}$$

where

$$\varepsilon_{\mathbf{k}} \equiv c\sqrt{\mathbf{p}^2 + M^2c^2} \tag{19.287}$$

are energies of the relativistic particles of mass M and momentum $\mathbf{p}$. Each entry in (19.286) is a 2×2-submatrix.

This is achieved by the *Foldy-Wouthuysen transformation*

$$H^{\rm d} = e^{iS} H e^{-iS}, \tag{19.288}$$

where

$$S = -i\boldsymbol{\gamma} \cdot \boldsymbol{\zeta}/2, \qquad \boldsymbol{\zeta} \equiv \arctan{(\mathbf{v}/c)}, \qquad \mathbf{v} \equiv \mathbf{p}/M = \text{velocity}. \tag{19.289}$$

The vector $\boldsymbol{\zeta}$ points in the direction of the velocity $\mathbf{v}$ and has the length $\zeta = \arctan(v/c)$, such that

$$\cos\zeta = \frac{Mc}{\sqrt{\mathbf{p}^2 + M^2c^2}}, \qquad \sin\zeta = \frac{|\mathbf{p}|}{\sqrt{\mathbf{p}^2 + M^2c^2}}. \tag{19.290}$$

A function of a vector $\mathbf{v}$ is defined by its Taylor series where even powers of $\mathbf{v}$ are scalars $\mathbf{v}^{2n} = v^{2n}$ and odd powers are vectors $\mathbf{v}^{2n+1} = v^{2n}\mathbf{v}$. If $\hat{\mathbf{v}}$ denotes as usual the direction vector $\hat{\mathbf{v}} \equiv \mathbf{v}/|\mathbf{v}|$, the matrix $\boldsymbol{\gamma}\cdot\hat{\mathbf{v}}$ has the property that all even powers of it are equal to a 4×4 unity matrix up to an alternating sign: $(\boldsymbol{\gamma}\cdot\hat{\mathbf{v}})^{2n} = (-1)^n$. Thus $S = -i\boldsymbol{\gamma}\cdot\hat{\mathbf{v}}\zeta$ and the Taylor series of e^{iS} reads explictly

$$e^{iS} = \sum_{n=0,2,4,\dots} \frac{(-1)^n}{n!}\left(\frac{\zeta}{2}\right)^n + (\boldsymbol{\gamma}\cdot\hat{\mathbf{v}}) \sum_{n=1,3,5,\dots} \frac{(-1)^{n-1}}{n!}\left(\frac{\zeta}{2}\right)^n = \cos\frac{\zeta}{2} + \boldsymbol{\gamma}\cdot\hat{\mathbf{v}}\,\sin\frac{\zeta}{2}. \tag{19.291}$$

Now, S commutes trivially with $\boldsymbol{\gamma}\cdot\mathbf{p} = \boldsymbol{\gamma}\cdot\hat{\boldsymbol{\zeta}}\,|\mathbf{p}|$, while anticommuting with γ^0 due to the anticommutation rules (19.276). Hence we can move the right-hand transformation in (19.288) simply to the left-hand side with a sign change of S, and obtain

$$H^{\rm d} = e^{2iS}H. \tag{19.292}$$

It is easy to calculate e^{2iS}: we merely have to double the rapidity in (19.291) and obtain

$$e^{2iS} = \cos\zeta + \boldsymbol{\gamma}\cdot\hat{\mathbf{v}}\,\sin\zeta = \frac{Mc}{\sqrt{\mathbf{p}^2 + M^2c^2}}\left(1 + \boldsymbol{\gamma}\cdot\mathbf{p}/Mc\right). \tag{19.293}$$

Hence we obtain

$$H^{\rm d} = e^{2iS}H = \frac{Mc}{\sqrt{\mathbf{p}^2 + M^2c^2}}\left(1 + \boldsymbol{\gamma}\cdot\mathbf{p}/Mc\right)Mc^2\gamma^0\left(1 + \boldsymbol{\gamma}\cdot\mathbf{p}/Mc\right). \tag{19.294}$$

Taking the right-hand parentheses to the left of γ^0 changes the sign of $\boldsymbol{\gamma}$. The product $(1 + \boldsymbol{\gamma}\cdot\mathbf{p}/Mc)(1 - \boldsymbol{\gamma}\cdot\mathbf{p}/Mc)$ is simply $1 + \mathbf{p}^2/M^2c^2$, such that

$$H^{\rm d} = c\sqrt{\mathbf{p}^2 + M^2c^2}\,\gamma^0 = \varepsilon_{\mathbf{k}}\,\gamma^0 = \hbar\omega_{\mathbf{k}}\,\gamma^0. \tag{19.295}$$

Remembering γ^0 from Eq. (19.278) shows that $H^{\rm d}$ has indeed the diagonal form (19.286).

Going to the diagonal fields $\psi^{\rm d}_{\mathbf{k}}(t) = e^{iS}\psi_{\mathbf{k}}(t)$, the action becomes

$$\mathcal{A} = \int_{t_a}^{t_b} dt \sum_{\mathbf{k}} \psi^{{\rm d}\dagger}_{\mathbf{k}}(t)\left[i\hbar\partial_t - H^d(\hbar\mathbf{k})\right]\psi^{\rm d}_{\mathbf{k}}(t). \tag{19.296}$$

Thus a Dirac field is equivalent to a sum of infinitely many momentum states, each being associated with four harmonic oscillators of the Fermi type. The path integral is a product of independent harmonic path integrals of frequencies $\pm\omega_{\mathbf{k}}$.

It is then easy to calculate the quantum-mechanical partition function using the result (7.419) for each oscillator, continued to real times:

$$Z_{\rm QM} = \prod_{\mathbf{k}} \left\{ 2\cosh^4[\omega_{\mathbf{k}}(t_b - t_a)/2] \right\}. \tag{19.297}$$

This can also be written as

$$Z_{\rm QM} = \exp\left(4 \sum_{\mathbf{k}} \log\left\{ 2\cosh[\omega_{\mathbf{k}}(t_b - t_a)/2] \right\} \right), \tag{19.298}$$

or as

$$Z_{\rm QM} = \exp\left[4 \sum_{\mathbf{k}} {\rm Tr}\,\log\left(i\hbar\partial_t - \hbar\omega_{\mathbf{k}}\right) \right] = \exp\left\{ \sum_{\mathbf{k}} {\rm Tr}\,\log\left[i\hbar\partial_t - H^{\rm d}(\hbar\mathbf{k}) \right] \right\}. \tag{19.299}$$

Since the trace is invariant under unitary transformations, we can rewrite this as

$$Z_{\rm QM} = \exp\left\{ \sum_{\mathbf{k}} {\rm Tr}\,\log\left[i\hbar\partial_t - H(\hbar\mathbf{k}) \right] \right\}, \tag{19.300}$$

or, since the determinant of γ^0 is unity, as

$$Z_{\rm QM} = \exp\left\{ \sum_{\mathbf{k}} {\rm Tr}\,\log\left[i\hbar\gamma^0\partial_t - \gamma^0 H^{\rm d}(\hbar\mathbf{k}) \right] \right\} = \exp\left\{ \sum_{\mathbf{k}} {\rm Tr}\,\log\left[i\hbar\gamma^0\partial_t - \hbar c\boldsymbol{\gamma}\,\mathbf{k} - Mc^2 \right] \right\}.$$

If we include the spatial coordinates into the functional trace, this can also be written as

$$Z_{\rm QM} = \exp\left[{\rm Tr}\,\log\left(i\hbar\gamma^0\partial_t - i\hbar c\boldsymbol{\gamma}\,\boldsymbol{\nabla} - Mc^2 \right) \right] = \exp\left\{ {\rm Tr}\,\log\left[c\left(i\hbar\partial\!\!\!/ - Mc \right) \right] \right\}.$$

In analytic regularization of Section 2.15, the factor c in the tracelog can be dropped. Moreover, there exists a simple algebraic identity

$$(i\hbar\partial\!\!\!/ + Mc)(i\hbar\partial\!\!\!/ - Mc) = -\hbar^2\partial^2 - M^2c^2. \tag{19.301}$$

The factors on the left-hand side have the same functional determinant since [compare (7.338) and (7.420)]

$$\begin{aligned} {\rm Det}(i\hbar\partial\!\!\!/ - Mc) &= e^{{\rm Tr}\,\log(i\hbar\partial\!\!\!/ - Mc)} = e^{V_4\int\frac{d^4p}{(2\pi)^4}{\rm Tr}\,\log(\hbar p\!\!\!/ - Mc)} = e^{V_4\int\frac{d^4p}{(2\pi)^4}{\rm Tr}\,\log(-\hbar p\!\!\!/ - Mc)} \\ &= {\rm Det}(-i\hbar\partial\!\!\!/ - Mc). \end{aligned} \tag{19.302}$$

This allows us, as a generalization of (7.420), to write

$${\rm Det}(i\hbar\partial\!\!\!/ - Mc) = {\rm Det}(i\hbar\partial\!\!\!/ + Mc) = \sqrt{{\rm Det}(-\hbar^2\partial^2 - M^2c^2)1_{4\times 4}}, \tag{19.303}$$

where $1_{4\times4}$ is a 4×4 unit matrix. In this way we arrive at the quantum-mechanical partition function

$$Z_{\rm QM} = \exp\left[4\times\frac{1}{2}\,{\rm Tr}\,\log(-\hbar^2\partial^2 - M^2c^2) \right] \equiv e^{i\Gamma_0^{\rm f}/\hbar}. \tag{19.304}$$

The factor 4 comes from the trace in the 4×4 matrix space, whose indices have disappeared in the formula. The exponent determines the effective action Γ of the quantum system by analogy with the Euclidean relation Eq. (19.46).

The Green function of the Dirac equation (19.275) is a 4×4 -matrix defined by

$$(i\hbar\not\partial - Mc)_{\alpha\beta}(x|x_a)_{\beta\gamma} = i\hbar\delta^{(D)}(x - x_a)\delta_{\alpha\gamma}. \tag{19.305}$$

Suppressing the Dirac indices, it has the spectral representation

$$(x_b|x_a) = \int \frac{d^4p}{(2\pi\hbar)^4} \frac{i\hbar}{\not p - Mc + i\eta} e^{-ip(x_b - x_a)/\hbar}. \tag{19.306}$$

This can be written more formally as a functional matrix

$$(x_b|x_a) = \langle x_b|\frac{i\hbar}{i\hbar\not\partial - Mc}|x_a\rangle, \tag{19.307}$$

which obviously satisfies the differential equation (19.305).

19.5.2 Path Integral

It is straightforward to write down a path integral representation for the amplitude (19.307):

$$(x_b|x_a) = \int_0^\infty dS \int_{x_a = x(\lambda_a)}^{x_b = x(\lambda_b)} \mathcal{D}^4x \int \frac{\mathcal{D}^4p}{(2\pi\hbar)^4} \mathrm{e}^{i\mathcal{A}/\hbar}, \tag{19.308}$$

with the action

$$\mathcal{A}[x,p] = \int_0^S d\lambda \left[-p\dot{x} + (\not p - Mc)\right]. \tag{19.309}$$

As in Section 19.1, the parameter λ is a lengthalong the orbits, but in contrast to that section we shall work here with real time t, and λ is related t by $\lambda = ct$. and S is the total reparametrization-invariant length. The dot denotes the derivative $\dot{x} \equiv dx(\lambda)/d\lambda$. The path integral over $x(\lambda)$ ensures that the momentum is λ-independent, so that the path integral over $p(\lambda)$ reduces to an ordinary Fourier integral [compare Eq. (2.41)]:

$$\int_{x_a = x(\lambda_a)}^{x_b = x(\lambda_b)} \mathcal{D}^4x \int \frac{\mathcal{D}^4p}{(2\pi\hbar)^4} = \int \frac{d^4p}{(2\pi\hbar)^4} e^{ip(x_b - x_a) + iS(\not p - Mc)/\hbar} \tag{19.310}$$

Performing the integral over S in (19.308) leads to

$$(x_b|x_a) = \int \frac{d^4p}{(2\pi\hbar)^4} e^{ip(x_b - x_a)} \frac{i\hbar}{\not p - Mc}, \tag{19.311}$$

in agrrement with the amplitude (19.307). The minus sign in front of $p\dot{x}$ is necessary to have the positive sign for the spatial part $\mathbf{p}\dot{\mathbf{x}}$ in the Minkowski metric (19.277).

The action (19.309) can immediately be generalized to

$$\bar{\mathcal{A}}[x,p] \quad = \quad \int_0^S d\lambda \left[-p\dot{x} + h(\lambda)\left(\not{p} - Mc\right)\right], \tag{19.312}$$

with any function $h(\lambda) > 0$. This makes it invariant under the reparametrization

$$\lambda \to f(\lambda), \qquad h(\lambda) \to h(\lambda)/f(\lambda). \tag{19.313}$$

The path integral (19.308) contains then an extra functional integration over $h(\lambda)$ with some gauge-fixing functional $\Phi[h]$, as in (19.22), which has been chosen in (19.309) as $\Phi[h] = \delta[h-1]$.

The path integral alone yields an amplitude

$$\langle x|e^{iS(i\hbar\not{\partial} - Mc)/\hbar}|x_a\rangle, \tag{19.314}$$

and the integral over S in (19.308) produces indeed the propagator (19.307). In evaluating this we must assume, as usual, that the mass carries an infinitesimal negative imaginary part $i\eta$. This is also necessary to guarantee the convergence of the path integral (19.308).

Electromagnetism is introduced as usual by the minimal substitution (2.644). In the operator version, we have to substitute

$$\partial_\mu \longrightarrow \partial_\mu + i\frac{e}{\hbar c}A_\mu. \tag{19.315}$$

Thus we obtain the gauge-invariant action

$$\bar{\mathcal{A}}[x,p] = \int_0^S d\lambda \left[-p\dot{x} + h(\lambda)\left(\not{p} - \frac{e\hbar}{c}\not{A} - Mc\right)\right]. \tag{19.316}$$

Another path integral representation which is closer to the spinless case is obtained by rewriting (19.307) as

$$(x|x_a) = (i\hbar\not{\partial} + Mc)\,\langle x|\frac{i\hbar}{-\hbar^2\partial^2 - M^2c^2}|x_a\rangle, \tag{19.317}$$

where we have omitted the negative infinitesimal imaginary part $-i\hbar$ of the mass, for brevity, and used the fact that

$$(i\hbar\not{\partial} + Mc)(i\hbar\not{\partial} - Mc) = -\hbar^2\partial^2 - M^2c^2, \tag{19.318}$$

on account of the anticommutation relation (19.276). By rewriting (19.317) as a proper-time integral

$$(x|x_a) = \frac{1}{2Mc}\,(i\hbar\not{\partial} + Mc)\int_0^\infty dS\langle x|e^{iS\left(-\hbar^2\partial^2 - M^2c^2\right)/2Mc\hbar}|x_a\rangle, \tag{19.319}$$

we find immediately the canonical path integral

$$(x|x_a) = \frac{1}{2Mc}\,(i\hbar\not{\partial} + Mc)\int_0^\infty dS\int_{x_a=x(\lambda_a)}^{x=x(\lambda_b)}\mathcal{D}^4x\int\frac{\mathcal{D}^4p}{(2\pi\hbar)^4}e^{i\mathcal{A}/\hbar}, \tag{19.320}$$

with the action

$$\mathcal{A}[x,p]=\int_0^S d\lambda\left[-p\dot{x}+\frac{1}{2Mc}\left(p^2-M^2c^2\right)\right]. \tag{19.321}$$

The suppressed Dirac indices of the 4×4 -amplitude on the left-hand side, $(x|x_a)_{\alpha\beta}$, are entirely due to the prefactor $(i\hbar\partial\!\!\!/ + Mc)_{\alpha\beta}$ on the right-hand side.

As in the generalization of (19.309) to (19.309), this action can be generalized to

$$\bar{\mathcal{A}}[x,p] \;=\; \int_0^S d\lambda\left[-p\dot{x}+\frac{h(\lambda)}{2Mc}\left(p^2-M^2c^2\right)\right], \tag{19.322}$$

with any function $h(\lambda)>0$, thus becoming invariant under the reparametrization (19.313), and the path integral (19.308) contains then an extra functional integration $\int\mathcal{D}h(\lambda)\,\Phi[h]$. The action (19.322), is precisely the Minkowski version of the path integral of a spinless particle of the previous section [see Eq. (19.14)].

Introducing here electromagnetism by the minimal substitution (19.315) in the prefactor of (19.317) and on the left-hand side of (19.318), the latter becomes then

$$\left(i\hbar\partial\!\!\!/ - \frac{e}{c}A\!\!\!/ + Mc\right)\left(i\hbar\partial\!\!\!/ - \frac{e}{c}A\!\!\!/ - Mc\right)=\hbar^2\left[\left(i\partial-\frac{e}{\hbar c}A\right)^2-\frac{e}{\hbar c}\Sigma^{\mu\nu}F_{\mu\nu}\right]-M^2c^2, \tag{19.323}$$

where

$$\Sigma^{\mu\nu}\equiv\frac{i}{4}[\gamma^\mu,\gamma^\nu]=-\Sigma^{\nu\mu} \tag{19.324}$$

are the generators of Lorentz transformations in the space of Dirac spinors. For any fixed index μ, they satisfy the commutation rules:

$$[\Sigma^{\mu\nu},\Sigma^{\mu\kappa}]=ig^{\mu\mu}\Sigma^{\nu\kappa}. \tag{19.325}$$

Due to the antisymmetry in the two indices, this determines all nonzero commutators of the Lorentz group.

Using Eqs. (19.254), we can write the last interaction term in (19.323) as

$$\Sigma^{\mu\nu}F_{\mu\nu}=-2\Sigma^iB^i+2\Sigma^{0i}E^i, \tag{19.326}$$

where Σ^i are the generators of rotation

$$\Sigma^i\equiv\frac{1}{2}\epsilon_{ijk}\Sigma^{jk}=\frac{1}{2}\begin{pmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix}, \tag{19.327}$$

and

$$\Sigma^{0i}\equiv i\alpha^i\equiv i\gamma^0\gamma^i=i\begin{pmatrix}-\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix} \tag{19.328}$$

are the generators of rotation-free Lorentz transformations. Thus

$$\Sigma^{\mu\nu}F_{\mu\nu}=-\begin{pmatrix}\boldsymbol{\sigma}(\mathbf{B}+i\mathbf{E}) & 0\\ 0 & \boldsymbol{\sigma}(\mathbf{B}-i\mathbf{E})\end{pmatrix}. \tag{19.329}$$

19.5.3 Amplitude with Electromagnetic Interaction

The obvious generalization of the path integral (19.320) which includes minimal electromagnetic interactions is then

$$(x|x_a)=\frac{1}{2M}\left[\left(i\hbar\partial\!\!\!/ - \frac{e}{c}A\!\!\!/\right)+Mc\right]\int_0^\infty dS\int\mathcal{D}h(\lambda)\,\Phi[h]\int_{x_a=x(\lambda_a)}^{x=x(\lambda_b)}\mathcal{D}^4x\int\frac{\mathcal{D}^4p}{(2\pi\hbar)^4}\hat{T}e^{i\mathcal{A}/\hbar}, \tag{19.330}$$

with the action

$$\bar{\mathcal{A}}[x,p]=\int_0^S d\lambda\left\{-p\dot{x}+\frac{h(\lambda)}{2Mc}\left[\left(p-\frac{e}{c}A\right)^2-\frac{\hbar e}{c}\Sigma^{\mu\nu}F_{\mu\nu}-M^2c^2\right]\right\}. \tag{19.331}$$

The symbol $\hat{T}$ is the time-ordering operator defined in (1.241), now with respect to the proper time λ, which has to be present to account for the possible noncommutativity of $F_{\mu\nu}\Sigma^{\mu\nu}/2$ at different λ. Integrating out the momentum variables yields the configuration-space path integral

$$(x|x_a)=\frac{1}{2M}\left[\left(i\hbar\partial\!\!\!/ -\frac{e}{c}A\!\!\!/\right)+Mc\right]\int_0^\infty dS\int\mathcal{D}h(\lambda)\,\Phi[h]\int_{x_a=x(\lambda_a)}^{x=x(\lambda_b)}\mathcal{D}^4x\,\hat{T}e^{i\mathcal{A}/\hbar}, \tag{19.332}$$

with the action

$$\bar{\mathcal{A}}[x]=\int_0^S d\lambda\left[-\frac{Mc}{2h(\lambda)}\dot{x}^2-\frac{e}{c}\,\dot{x}A-h(\lambda)\frac{\hbar e}{2Mc^2}\Sigma^{\mu\nu}F_{\mu\nu}-h(\lambda)\frac{Mc}{2}\right]. \tag{19.333}$$

The coupling to the magnetic field adds to the rest energy Mc^2 an interaction energy

$$H_{\rm int}=-\frac{\hbar e}{Mc}\boldsymbol{\sigma}\cdot\mathbf{B}. \tag{19.334}$$

From this we extract the magnetic moment of the electron. We compare (19.334) with the general interaction energy (8.316), and identify the magnetic moment as

$$\boldsymbol{\mu}=\frac{\hbar e}{Mc}\boldsymbol{\sigma}. \tag{19.335}$$

Recall that in 1926, Uhlenbeck and Goudsmit explained the observed Zeeman splitting of atomic levels by attributing to an electron a half-integer spin. However, the magnetic moment of the electron turned out to be roughly twice as large as what one would expect from a charged rotating sphere of angular momentum $\mathbf{L}$, whose magnetic moment is

$$\boldsymbol{\mu}=\mu_B\,\frac{\mathbf{L}}{\hbar}, \tag{19.336}$$

where $\mu_B\equiv\hbar e/Mc$ is the Bohr magneton (2.649). On account of this relation, it is customary to parametrize the magnetic moment of an elementary particle of spin $\mathbf{S}$ as follows:

$$\boldsymbol{\mu}=g\mu_B\frac{\mathbf{S}}{\hbar}. \tag{19.337}$$

The dimensionless ratio g with respect to (19.336) is called the *gyromagnetic ratio* or *Landé factor*. For a spin-1/2 particle, $\mathbf{S}$ is equal to $\boldsymbol{\sigma}/2$, and comparison with (19.335) yields the gyromagnetic ratio

$$g = 2, \tag{19.338}$$

the famous result found first by Dirac, predicting the intrinsic magnetic moment μ of an electron to be equal to the Bohr magneton μ_B, thus being twice as large as expected from the relation (19.336), if we insert there the spin 1/2 for the orbital angular momentum.

In quantum electrodynamics one can calculate further corrections to this Dirac result as a perturbation expansion in powers of the fine-structure constant α [recall (1.505)]. The first correction to g due to one-loop Feynman diagrams was found by Schwinger:

$$g = 2 \times \left(1 + \frac{\alpha}{2\pi}\right) \approx 2 \times 1.001161, \tag{19.339}$$

where α is the fine-structure constant (1.505). Experimentally, the gyromagnetic ratio has been measured to an incredible accuracy:

$$g = 2 \times 1.001\,159\,652\,193(10), \tag{19.340}$$

in excellent agreement with (19.339). If the perturbation expansion is carried to higher orders, one is able to reach agreement up to the last experimentally known digits [14].

In the literature, there exist other representations of path integrals for Dirac particles involving Grassmann variables. For this we recall the discussion in Subsection 7.11.3 that a path integral over four real Grassmann fields θ^μ, $\mu = 0, 1, 2, 3$

$$\int \mathcal{D}^4\theta \, \exp\left[\frac{i}{\hbar}\int dt \left(-\frac{i\hbar}{4}\theta_\mu\dot{\theta}^\mu\right)\right], \tag{19.341}$$

generates a matrix space corresponding to operators $\hat{\theta}^\mu$ with the anticommutation rules

$$\left\{\hat{\theta}^\mu, \hat{\theta}^\nu\right\} = 2g^{\mu\nu}, \tag{19.342}$$

and the matrix elements

$$\langle\beta|\hat{\theta}^\mu|\alpha\rangle = (\gamma_5\gamma^\mu)_{\beta\alpha}\,, \qquad \beta, \alpha = 1, 2, 3, 4. \tag{19.343}$$

It is then possible to replace path integral (19.332) by

$$\begin{aligned}(x|x_a) \;&=\; \frac{1}{2M}\int_0^\infty dS \int \mathcal{D}h\, \Phi[h] \\ &\times \int \mathcal{D}\chi\Phi[\chi] \int \mathcal{D}^4\theta \int \mathcal{D}x \int \frac{\mathcal{D}p}{(2\pi\hbar)^4}\, e^{i\left(\bar{\mathcal{A}}[x]+\mathcal{A}_{\rm G}[\theta^\mu,A]\right)/\hbar},\end{aligned} \tag{19.344}$$

with the action of a relativistic spinless particle [the action (19.333) without the spin coupling]

$$\bar{\mathcal{A}}[x,p] = \int_0^S d\lambda \left\{ -p\dot{x} + \frac{h(\lambda)}{2Mc}\left[\left(p - \frac{e}{c}A\right)^2 - M^2c^2\right]\right\}, \tag{19.345}$$

and an action involving the Grassmann fields θ^μ:

$$\mathcal{A}_{\rm G}[\theta^\mu, A] = \int_0^S d\lambda \left\{ -\frac{i\hbar}{4}\theta_\mu(\lambda)\dot{\theta}^\mu(\lambda) + h(\lambda)\frac{i\hbar e}{4Mc^2}F_{\mu\nu}(x(\lambda))\theta^\mu(\lambda)\theta^\nu(\lambda)\right\}. \tag{19.346}$$

This follows directly from Eq. (7.513). The function $h(\lambda)$ is the same as in the bosonic actions (19.14) and the path integral (19.22) guaranteeing the reparametrization invariance (19.13).

After integrating out the momentum variables in the path integral (19.344), the canonical action is of course replaced by the configuration space action (19.218). In the simplest gauge (19.23), the total action reads

$$\bar{\mathcal{A}}[x,\theta^\mu] = \int_0^S d\lambda \left[-\frac{Mc}{2}\dot{x}^2 - \frac{e}{c}\left(\dot{x}A - i\frac{\hbar}{4Mc}F_{\mu\nu}\theta^\mu\theta^\nu\right) - \frac{Mc}{2} + \frac{i\hbar}{4}\theta_\mu(\tau)\dot{\theta}^\mu(\tau)\right]. \tag{19.347}$$

The Grassmann variables can be integrated out using formula (19.103), which reads here [compare (7.501)]:

$$\int \mathcal{D}^4\theta\, e^{\frac{i}{4}\int dt\left[-i\theta_\mu(t)\dot{\theta}^\mu(t) + (ie/4Mc)F_{\mu\nu}\theta^\mu\theta^\nu\right]} = 4\,{\rm Det}^{1/2}\left[-i\delta_{\mu\nu}\,\partial_t + \frac{ie}{Mc}F_{\mu\nu}(x(\lambda))\right]. \tag{19.348}$$

For a constant field tensor $F_{\mu\nu}$ and with the usual antiperiodic boundary conditions in the interval $ct = \lambda \in (0, S)$, the right-hand side has been given before in Eq. (19.103). In the present case it reads:

$$4\,{\rm Det}^{1/2}\left(-ig_{\mu\nu}\,\partial_\lambda + i\frac{e}{Mc^2}F_{\mu\nu}\right) = 4\cos\left(\frac{e}{Mc^2}\mathcal{B}\frac{S}{2}\right)\cosh\left(\frac{e}{Mc^2}\mathcal{E}\frac{S}{2}\right). \tag{19.349}$$

19.5.4 Effective Action in Electromagnetic Field

In the absence of electromagnetism, the effective action of the fermion orbits is given by (19.304). Its Euclidean version differs from the Klein-Gordon expression in (19.38) only by a factor -2:

$$\frac{\Gamma^{\rm f}_{\rm e,0}}{\hbar} = -2\,{\rm Tr}\,\log\left[-\hbar^2\partial^2 + M^2c^2\right]. \tag{19.350}$$

Explicitly we have from (19.39), (19.41), and (19.46):

$$\frac{\Gamma^{\rm f}_{\rm e,0}}{\hbar} = 2V_D\int_0^\infty \frac{d\beta}{\beta}\, e^{-\beta Mc^2/2}\frac{1}{\sqrt{2\pi\hbar^2\beta/M}^D} = 2\frac{V_D}{{\lambda^{\rm C}_M}^D}\frac{1}{(4\pi)^{D/2}}\Gamma(1-D/2). \tag{19.351}$$

The factor 2 may be thought of as $4 \times 1/2$ where the factor 4 comes from the free path integral over the Grassmann field,

$$\int \mathcal{D}^D\theta\, e^{-\mathcal{A}_{\rm e,0}[\theta]/\hbar} = 4. \tag{19.352}$$

counts the four components of the Dirac field. Recall that by (19.286), the Dirac field carries four modes, one of energy $\hbar\omega_{\mathbf{k}}$, with two spin degrees of freedom, the other of energy $-\hbar\omega_{\mathbf{k}}$ with two spin degrees. The latter are shown in quantum field theory to correspond to an antiparticle with spin 1/2. The path integral over $x(\lambda)$ which counts paths in opposite directions with the ground state energy (19.39) describes particles and antiparticles [recall the remarks after Eq. 19.45]. This explains why only the spin factor 2 remains in (19.351).

By including the vector potential via the minimal substitution $\hat{p} \to \hat{p} - (e/c)A$, we obtain the Euclidean effective action from Eq. (19.221), and thus obtain immediately the path integral representation

$$\frac{\tilde{\Gamma}^{\rm f}_{\rm e}}{\hbar} = 2\int_0^\infty \frac{d\beta}{\beta}\, e^{-\beta Mc^2/2} \int \mathcal{D}^D x\, e^{-\mathcal{A}_{\rm e}/\hbar}, \tag{19.353}$$

with the Euclidean action (19.220).

This is not yet the true partition function $\Gamma_{\rm e}$ of the spin-1/2 particle, since the proper path integral contains the additional Grassmann terms of the action (19.347). In the Euclidean version, the full interaction is

$$\mathcal{A}_{\rm e,int}[x,\theta] = \int_0^{\hbar\beta} d\lambda\, \frac{e}{c}\left[i\,\dot{x}_\mu(\lambda)A_\mu(x(\lambda)) - \frac{i}{4M}F_{\mu\nu}(x(\lambda))\theta^\mu(\lambda)\theta^\nu(\lambda)\right]. \tag{19.354}$$

Thus we obtain the path integral representation

$$\frac{\Gamma^{\rm f}_{\rm e}}{\hbar} = 2\int_0^\infty \frac{d\beta}{\beta}\, e^{-\beta Mc^2/2} \int \mathcal{D}^D x \int \mathcal{D}^D\theta\, e^{-\{\mathcal{A}_{\rm e,0}[x,\theta]+\mathcal{A}_{\rm e,int}[x,\theta]\}/\hbar}, \tag{19.355}$$

where the free part of the Euclidean action is

$$\mathcal{A}_{\rm e,0}[x,\theta] = \mathcal{A}_{\rm e,0}[x] + \mathcal{A}_{\rm e,0}[\theta] \equiv \int_0^{\hbar\beta} d\tau \frac{M}{2}\dot{x}^2(\tau) + \int_0^{\hbar\beta} d\tau\, \frac{\hbar}{4}\theta^\mu(\tau)\dot{\theta}^\mu(\tau). \tag{19.356}$$

19.5.5 Perturbation Expansion

The perturbation expansion is a straightforward generalization of the expansion (19.224):

$$\frac{\Gamma^{\rm f}_{\rm e}}{\hbar} = \frac{\Gamma^{\rm f}_{\rm e,0}}{\hbar} + \int_0^\infty \frac{d\beta}{\beta}\, e^{-\beta Mc^2/2} \frac{2V_D}{\sqrt{2\pi\hbar^2\beta/M}^D} \sum_{n=1}^\infty \frac{(-ie/c)^n}{n!} \tag{19.357}$$
$$\times \left\langle \prod_{i=1}^n \left\{ \int_0^{\hbar\beta} d\tau_i \left[\dot{x}_\mu(\tau_i)A_\mu(x(\tau_i)) - \frac{\hbar}{4Mc}F_{\mu\nu}(x(\tau_i))\theta^\mu(\tau_i)\theta^\nu(\tau_i)\right]\right\}\Bigg]\right\rangle_0 .$$

The leading free effective action coincides, of course, with the $n = 0$ -term of the sum [compare (19.351)].

The expectation values are now defined by the Grassmann extension of the Gaussian path integral (19.225):

$$\langle \mathcal{O}[x,\theta]\rangle_0 \equiv \frac{\int \mathcal{D}^D x \int \mathcal{D}^D\theta\, \mathcal{O}[x,\theta]\, e^{-\mathcal{A}_{\rm e,0}[x,\theta]/\hbar}}{\int \mathcal{D}^D x\, e^{-\mathcal{A}_{\rm e,0}[x]/\hbar} \int \mathcal{D}^D\theta\, e^{-\mathcal{A}_{\rm e,0}[\theta]/\hbar}}. \tag{19.358}$$

where the denominator is equal to $(1/2)V_D/\sqrt{2\pi\hbar^2\beta/M}^D \times 4$.

There exists also an expansion analogous to (19.227), where the vector potentials have been Fourier decomposed according to (19.226). Then we obtain an expansion just like (19.227), except for a factor -2 and with the expectation values replaced as follows:

$$\left\langle \prod_{i=1}^{n} \left[\int_0^{\hbar\beta} d\tau_i\, \dot{x}^{\mu_i}(\tau_i) e^{ik_i x(\tau_i)} \right] \right\rangle_0$$
$$\to \left\langle \prod_{i=1}^{n} \left\{ \int_0^{\hbar\beta} d\tau_i \left[\dot{x}^{\mu_i}(\tau_i) + \frac{i\hbar}{2Mc} k_i^{\nu_i} \theta^{\nu_i}(\tau_i)\theta^{\mu_i}(\tau_i) \right] e^{ik_i x(\tau_i)} \right] \right\rangle_0 . \tag{19.359}$$

The evaluation of these expectation values proceeds as in Eqs. (19.228)–(19.239), except that we also have to form Wick contractions of Grassmann variables which have the free correlation functions

$$\langle \theta^\mu(\tau)\theta^\nu(\tau')\rangle = 2\delta^{\mu\nu} G^{\rm a}_{\omega,\rm e}(\tau-\tau'), \tag{19.360}$$

where

$$G^{\rm a}_{\omega,\rm e}(\tau-\tau') = \frac{1}{2}\epsilon(\tau), \qquad \tau \in [-\hbar\beta, \hbar\beta) \tag{19.361}$$

is the Euclidean version of the antiperiodic Green function (3.109) solving the inhomogeneous equation

$$\partial_\tau G^{\rm a}_{\omega,\rm e}(\tau) = \delta(\tau). \tag{19.362}$$

Outside the basic interval $[-\hbar\beta, \hbar\beta)$ the function is to be continued antiperiodically. in accordance with the fermionic nature of the Grassmann variables.

In operator language, the correlation function (19.360) is the time-ordered expectation value $\langle \hat{T}\hat{\theta}(\tau)\hat{\theta}(\tau')\rangle_0$ [recall (3.299)]. By letting $\tau \to \tau'$ once from above and once from below, the correlation function shows agreement with the anticommutation rule (19.342). In verifying this we must use the fact that the time ordered product of fermion operators is defined by the following modification of the bosonic definition in Eq. (1.241):

$$\hat{T}(\hat{O}_n(t_n)\cdots\hat{O}_1(t_1)) \equiv \epsilon_P \hat{O}_{i_n}(t_{i_n})\cdots\hat{O}_{i_1}(t_{i_1}), \tag{19.363}$$

where $t_{i_n}, \ldots, t_{i_1}$ are the times $t_n, \ldots, t_1$ relabeled in the causal order, so that

$$t_{i_n} > t_{i_{n-1}} > \ldots > t_{i_1}. \tag{19.364}$$

The difference lies in the sign factor ϵ_P which is equal to 1 for an even and -1 for an odd number of permutations of fermion variables.

19.5.6 Vacuum Polarization

Let us see how the fluctuations of an electron loop change the electromagnetic field action. To lowest order, we must form the expectation value (19.359) for $n=0$ and $k_1=-k_2\equiv k$:

$$\left\langle\left[\dot x^{\mu_1}(\tau_1)+\frac{i\hbar}{2Mc}k^{\nu_1}\theta^{\nu_1}(\tau_1)\theta^{\mu_1}(\tau_1)\right]e^{ik\delta x(\tau_1)}\left[\dot x^{\mu_2}(\tau_2)-\frac{i\hbar}{2Mc}k^{\nu_2}\theta^{\nu_2}(\tau_2)\theta^{\mu_2}(\tau_2)\right]e^{-ik\delta x(\tau_2)}\right\rangle_0. \tag{19.365}$$

From the contraction of the velocities $\dot x^{\mu_1}(\tau_1)$ and $\dot x^{\mu_2}(\tau_2)$ we obtain again the spinless result (19.240) leading in (19.242) to the integrand

$$\left(k_1^2\delta^{\mu_1\nu_2}-k_1^{\mu_1}k_1^{\mu_2}\right)\dot G^{\,2}(\tau_1,\tau_2)=\left(k_1^2\delta^{\mu_1\nu_2}-k_1^{\mu_1}k_1^{\mu_2}\right)\frac{\hbar^2}{M^2c^2}(u-1/2)^2. \tag{19.366}$$

In addition, there are the Wick contractions of the Grassmann variables:

$$\left\langle\left[\frac{\hbar}{2Mc}k^{\nu_1}\theta^{\nu_1}(\tau_1)\theta^{\mu_1}(\tau_1)\right]e^{ik\delta x(\tau_1)}\left[\frac{i\hbar}{2Mc}k^{\nu_2}\theta^{\nu_2}(\tau_2)\theta^{\mu_2}(\tau_2)\right]e^{-ik\delta x(\tau_2)}\right\rangle_0$$
$$=-\left(k_1^2\delta^{\mu_1\nu_2}-k_1^{\mu_1}k_1^{\mu_2}\right)\frac{\hbar^2}{M^2c^2}\frac{1}{4}\,\epsilon^2(\tau_1-\tau_2). \tag{19.367}$$

Since $\epsilon^2(\tau_1-\tau_2)=1$, this changes the spinless result (19.366) to

$$\left(k_1^2\delta^{\mu_1\nu_2}-k_1^{\mu_1}k_1^{\mu_2}\right)\dot G^{\,2}(\tau_1,\tau_2)=\left(k_1^2\delta^{\mu_1\nu_2}-k_1^{\mu_1}k_1^{\mu_2}\right)\frac{\hbar^2}{M^2c^2}[(u-1/2)^2-1/4]. \tag{19.368}$$

Remembering the factor -2 in the expansion (19.358) with respect to the spinless one, we find that the vacuum polarization due to fluctuating spin-1/2 orbits is obtained from the spinless result (19.251) by changing the factor $4(u-1/2)^2=(2u-1)^2$ in the integrand to $-2\times 4u(u-1)=8u(1-u)$. The resulting function $\Pi(k^2)$ has the expansion

$$\Pi(k^2)=\frac{1}{3\pi}\left[\frac{2}{\epsilon}-\log\frac{M^2c^2e^\gamma}{4\pi\hbar^2}\right]-\frac{\hbar^2k^2}{15\pi M^2c^2}+\mathcal{O}\left(\epsilon,\frac{k^2}{M^2c^2/\hbar^2}\right). \tag{19.369}$$

The first term produces a renormalization of the charge which is treated as in the bosonic case [recall (19.271)–(19.274)], which causes an additional contact interaction

$$-\frac{\alpha}{r}\to-\frac{\alpha}{r}-\frac{4\alpha^2\hbar^2}{15M^2c^2}\,\delta^{(3)}(\mathbf{x}). \tag{19.370}$$

There, the vacuum polarization has the effect of lowering the state $2S_{1/2}$, which is the s-state of principal quantum number $n=2$, against the p-state $2P_{1/2}$ by 27.3 MHz. The experimental frequency shift is positive ≈ 1057 MHz [recall Eq. (18.600)], and is mainly due to the effect of the electron moving through a bath of photons as calculated in Eq. (18.599).

The effect of vacuum polarization was first calculated by Uehling [13], who assumed it to be the main cause for the Lamb shift. He was disappointed to find only 3% of the experimental result, and a wrong sign.

The situation in muonic atoms is different. There the vacuum polarization *does* produce the dominant contribution to the Lamb shift for a simple reason: The other effects contain in a factor M/M_μ^2, where M_μ^2 is the mass of the muon, whereas the vacuum polarization still involves an electron loop containing only the electron mass M, thus being enhanced by a factor $(M_\mu/M)^2 \approx 210^2$ over the others.

The calculations for the electron in an atom have been performed to quite high orders [14] within quantum electrodynamics. We have gone through the above calculation only to show that it is possible to re-obtain quantum field-theoretic result within the path integral formalism. More details are given in the review article [5],

As mentioned in the beginning, the above calculations are greatly simplified version of analogous calculations within superstring theory, which so far have not produced any physical results. If this ever happens, one should expect that also in this field a second-quantized field theory would be extremely useful to extract efficiently observable consequences. Such a theory still need development [15].

19.6 Supersymmetry

It is noteworthy that the various actions for a spin-1/2 particle is invariant under certain supersymmetry transformations.

19.6.1 Global Invariance

Consider first the fixed-gauge action (19.347). Its appearance can be made somewhat more symmetric by absorbing a factor $\sqrt{\hbar/2M}$ into the Grassmann variables $\theta^\mu(\tau)$, so that it reads

$$\bar{\mathcal{A}}[x,\theta^\mu]=\int d\tau\left[-\frac{M}{2}\dot{x}^2-\frac{e}{c}\left(\dot{x}A+\frac{i}{2}F_{\mu\nu}\theta^\mu\theta^\nu\right)-\frac{Mc^2}{2}+\frac{M}{2}i\theta_\mu(\tau)\dot{\theta}^\mu(\tau)\right]. \quad (19.371)$$

The correlation functions (19.360) of the θ-variables are now

$$\langle\theta^\mu(\tau)\theta^\nu(\tau')\rangle=\delta^{\mu\nu}G^{\rm f}(\tau,\tau')\equiv\delta^{\mu\nu}\frac{\hbar}{M}\Delta_0^{\rm f}(\tau-\tau'), \quad (19.372)$$

with $\Delta_0^{\rm f}(\tau-\tau')=\epsilon(\tau-\tau')/2$. In this normalization, $G^{\rm f}(\tau,\tau')$ coincides, up to a sign, with the first term in the derivative $\dot{G}(\tau,\tau')$ of the bosonic correlation function [recall (19.232) and the first term in (19.234)].

Let us apply to the variables the infinitesimal transformations

$$\delta x^\mu(\tau)=i\alpha\theta^\mu(\tau), \qquad \delta\theta^\mu(\tau)=\alpha\dot{x}^\mu(\tau). \quad (19.373)$$

where α is an arbitrary Grassmann parameter. For the free terms this is obvious. The interacting terms change by

$$-i\alpha \int d^4x \, \frac{e}{c} \left(\dot{\theta}^\mu A_\mu + F_{\mu\nu} \dot{x}^\mu \theta^\nu \right). \tag{19.374}$$

Inserting $F_{\mu\nu}\dot{x}^\mu(\tau) = dA_\nu(x(\tau))/d\tau - \partial_\nu[A^\mu(x(\tau))\dot{x}^\mu(\tau)]$, the first term cancels and the second is a pure surface term, such that the action is indeed invariant.

Supersymmetric theories have a compact representation in an extended space called *superspace*. This space is formed by pairs (τ, ζ), where ζ is a Grassmann variable playing the role of a supersymmetric partner of the time parameter τ. The coordinates $x^\mu(\tau)$ are extended likewise by defining

$$X^\mu(\tau) \equiv x^\mu(\tau) + i\zeta\theta^\mu(\tau). \tag{19.375}$$

A supersymmetric derivative is defined by

$$DX^\mu(\tau) \equiv \left(\frac{\partial}{\partial\zeta} + i\zeta \frac{\partial}{\partial\tau} \right) X^\mu(\tau) = i\theta^\mu(\tau) + i\zeta\dot{x}^\mu(\tau). \tag{19.376}$$

If we now form the integral, using the Grassmann formula (7.379),

$$\int d\tau \frac{\mathrm{d}\zeta}{2\pi} \, i\dot{X}_\mu(\tau) DX^\mu(\tau) = \int d\tau \frac{\mathrm{d}\zeta}{2\pi} \, i \left[\dot{x}(\tau) + i\zeta\dot{\theta}^\mu(\tau) \right] \left[i\theta^\mu(\tau) + i\zeta\dot{x}^\mu(\tau) \right], \tag{19.377}$$

we find

$$\int d\tau \left(-\dot{x}^2 + i\theta_\mu \dot{\theta}^\mu \right), \tag{19.378}$$

which proportional to the free part of the action (19.347). As a curious property of differentiations in superspace we note that

$$D^2X^\mu(\tau) = i\dot{x}^\mu(\tau) - \zeta\dot{\theta}^\mu(\tau), \qquad D^3X^\mu(\tau) = -\dot{\theta}^\mu(\tau) - \zeta\ddot{x}(\tau), \tag{19.379}$$

such that the kinetic term (19.377) can also be written as

$$-\int d\tau \frac{\mathrm{d}\zeta}{2\pi} X_\mu(\tau) D^3 X^\mu(\tau). \tag{19.380}$$

The interaction is found from the integral in superspace

$$\begin{aligned} i \int d\tau \frac{\mathrm{d}\zeta}{2\pi} \, & A^\mu(X(\tau)) DX(\tau) \\ &= i \int d\tau \frac{\mathrm{d}\zeta}{2\pi} \left[A^\mu(x(\tau)) + i\partial_\nu A^\mu(x(\tau))\theta^\nu(\tau) \right] \left[i\theta^\mu(\tau) + i\zeta\dot{x}^\mu(\tau) \right], \end{aligned} \tag{19.381}$$

which is equal to

$$-\int d\tau \left[A^\mu(\tau) \, \dot{x}(\tau) + \frac{i}{2} F_{\mu\nu} \theta^\mu(\tau) \theta^\nu(\tau) \right],$$

thus reproducing the interaction in (19.347). The action in superspace can therefore be written in the simple form

$$\mathcal{A}[X] = i \int d\tau \, \frac{\mathrm{d}\zeta}{2\pi} \left[-\frac{M}{2} X_\mu(\tau) D^3 X^\mu(\tau) + \frac{e}{c} A^\mu(X(\tau)) DX(\tau) \right]. \tag{19.382}$$

19.6.2 Local Invariance

A larger class of supersymmetry transformations exists for the action without gauge fixing which is the sum of the free part (19.345) and the interacting part (19.346). Absorbing again the factor $\sqrt{\hbar/2M}$ into the Grassmann variable $\theta^\mu(\tau)$, and rescaling in addition $h(\tau)$ by a factor $1/c$, the reparametrization-invariant action reads

$$\bar{\mathcal{A}}[x,p,\theta,h] = \int d\tau \left\{ -p\dot{x} + \frac{h(\tau)}{2M}\left[\left(p - \frac{e}{c}A\right)^2 - M^2c^2\right] \right.$$
$$\left. + \frac{M}{2} i\theta_\mu(\tau)\dot{\theta}^\mu(\tau) - ih(\tau)\frac{e}{c}F_{\mu\nu}(x(\tau))\theta^\mu(\tau)\theta^\nu(\tau) \right\}. \quad (19.383)$$

Let us now compose the action from invariant building blocks. For simplicity, we ignore the electromagnetic interaction. In a first step we also omit the mass term. The extra variable $h(\tau)$ requires an extra Grassmann partner $\chi(\tau)$ for symmetry, and we form the action

$$\bar{\mathcal{A}}_1[x,p,\theta,h,\chi] = \int d\tau \left\{ -p\dot{x} + \frac{h(\tau)}{2M}p^2 + \frac{M}{2} i\theta_\mu(\tau)\dot{\theta}^\mu(\tau) + \frac{i}{2}\chi(\tau)\theta^\mu(\tau)p_\mu(\tau) \right\}. \quad (19.384)$$

This action possesses a *local supersymmetry.* If we now perform τ-dependent versions of the supersymmetry transformations (19.373)

$$\delta x^\mu = i\alpha(\tau)\theta^\mu, \qquad \delta\theta^\mu = \alpha(\tau)p, \qquad \delta p = 0,$$
$$\delta h = i\alpha(\tau)\chi, \qquad \delta\chi = 2\dot{\alpha}(\tau). \quad (19.385)$$

If we integrate out the momenta in the path integral, the action (19.384) goes over into

$$\bar{\mathcal{A}}_1[x,\theta,h,\chi] = \int d\tau \left\{ -\frac{\dot{x}^2}{2h(\tau)} + \frac{M}{2} i\theta_\mu(\tau)\dot{\theta}^\mu(\tau) + \frac{i}{2h(\tau)}\chi(\tau)\theta^\mu(\tau)\dot{x}_\mu(\tau) \right\}, \quad (19.386)$$

where a term proportional to $\chi^2(\tau)$ has been omitted since it vanishes due to the nilpotency (7.375). This action is locally supersymmetric under the transformations

$$\delta x^\mu = i\alpha(\tau)\theta^\mu, \qquad \delta\theta^\mu = \frac{\alpha(\tau)}{h(\tau)}\left[\dot{x} - \frac{i}{2}\chi\,\theta^\mu\right],$$
$$\delta h = i\alpha(\tau)\chi, \qquad \delta\chi = 2\dot{\alpha}(\tau). \quad (19.387)$$

We now add the mass term

$$\mathcal{A}_M = -\frac{1}{2}\int d\tau\, h(\tau)Mc^2. \quad (19.388)$$

This needs a supersymmetric partner to compensate the variation of $\mathcal{A}_m$ under (19.387).

$$\mathcal{A}_5 = \frac{i}{2}\int d\tau \left[\theta_5(\tau)\dot{\theta}_5(\tau) + Mc\chi(\tau)\theta_5(\tau)\right]. \quad (19.389)$$

Indeed, add to (19.387) the transformation

$$\delta\theta_5 = Mc\,\alpha(\tau), \tag{19.390}$$

we see that the sum $\mathcal{A}_M + \mathcal{A}_5$ is invariant. Adding this to (19.384), we obtain the locally invariant canonical action

$$\bar{\mathcal{A}}[x,p,\theta,\theta_5,h,\chi] = \int d\tau \left\{ -p\dot{x} + \frac{h(\tau)}{2M}p^2 - \frac{h(\tau)}{2}Mc + \frac{M}{2}i\left[\theta_\mu(\tau)\dot{\theta}^\mu(\tau) + \theta_5(\tau)\dot{\theta}_5(\tau)\right] \right.$$
$$\left. + \frac{i}{2}\chi(\tau)\left[\theta^\mu(\tau)p_\mu(\tau) + Mc\theta_5(\tau)\right]\right\}. \tag{19.391}$$

Appendix 19A Proof of Same Quantum Physics of Modified Action

Consider the sliced path integral for a relativistic point particle associated with the original action (19.12). If we set the initial and final paramters λ_a and λ_b equal to λ_0 and λ_{N+1}, and slice the λ-axis at the places λ_n $(n = 1, 2, \dots, N)$, the action becomes

$$\mathcal{A}_{\rm cl,e} = Mc \sum_{n=1}^{N+1} |x_n - x_{n-1}|, \tag{19A.1}$$

where $|x_n - x_{n-1}| = \sqrt{(x_n - x_{n-1})^2}$ are the Euclidean distances [recall (19.2)]. The Euclidean amplitude for the particle to run from $x_a = x_0$ to $x_b = x_{N+1}$ is therefore given by the product of integrals in D spacetime dimensions

$$(x_b|x_a) = \mathcal{N} \prod_{n=1}^{N} \left[\int d^D x_n\right] e^{-Mc\sum_{n=1}^{N+1}|x_n - x_{n-1}|/\hbar}. \tag{19A.2}$$

As a consequence of the reparamterization invariance of (19.12), this expression is independent of the thickness $\lambda_n - \lambda_{n-1}$ of the slices, which we shall denote by rename as $\equiv h_n\epsilon$, where ϵ is some fixed small number.

We now factorize the exponential of the sum into a product of $N+1$ exponentials and represent each factor as an integral [using Formulas (1.347) and (1.349)]

$$e^{-Mc|x_n - x_{n-1}|/\hbar} = \sqrt{\frac{\epsilon Mc}{2\pi\hbar}} \int_0^\infty dh_n h_n^{-1/2} e^{-\epsilon h_n Mc/2\hbar - Mc(x_n - x_{n-1})^2/2h_n\epsilon\hbar}. \tag{19A.3}$$

Absorbing constants in the normalization factor $\mathcal{N}$, we arrive at

$$(x_b|x_a) = \mathcal{N} \prod_{n=1}^{N+1} \left[\int_0^\infty dh_n h_n^{(D-1)/2}\right] e^{-Mc\epsilon\Sigma_{n=1}^{N+1} h_n/2}$$
$$\times \frac{1}{\sqrt{2\pi\epsilon h_{N+1}/Mc}^D} \prod_{n=1}^{N} \left[\int \frac{d^D x_n}{\sqrt{2\pi\epsilon h_n/Mc}^D}\right] e^{-Mc\Sigma_{n=1}^{N+1}(x_n - x_{n-1})^2/2h_n\epsilon}, \tag{19A.4}$$

The second line contains only harmonic integrals over x_n, which can all be done with the help of the formulas of Appendix 2B, with the result

$$(x_b S|x_a 0) = \frac{1}{\sqrt{2\pi S\hbar/Mc}^D} e^{-Mc(x_b - x_a)^2/2S\hbar}, \tag{19A.5}$$

where S is the total parameter length

$$S \equiv \sum_{n=1}^{N+1} \epsilon h_n, \tag{19A.6}$$

Replacing (19A.5) by its Fourier representation [compare with (1.333) and (1.341)], we can rewrite (19A.4) as

$$(x_b|x_a) = \mathcal{N} \prod_{n=1}^{N+1} \left[\int_0^\infty dh_n h_n^{(D-1)/2}\right] e^{-McS/2\hbar} \int \frac{d^Dp}{(2\pi\hbar)^D} e^{-Sp^2/2Mc+ip(x_b-x_a)/\hbar}. \tag{19A.7}$$

Thus we are left with the product of integrals over h_n. Before we can perform these, we must make sure to respect the sum (19A.6). This is done by inserting an auxiliary unit integral that separates out an integral over the total length S:

$$1 = \int_0^\infty dS\, \delta(\Sigma_{n=1}^{N+1}\epsilon h_n - S) = \int_0^\infty \frac{dS}{2\lambda_C} \int_{-i\infty}^\infty \frac{d\sigma}{2\pi i} e^{-\sigma(\Sigma_{n=1}^{N+1}\epsilon h_n - S)/2\lambda_C}, \tag{19A.8}$$

where λ_C is the Compton wavelength (19.31). Then the product of integrals over h_n in the brackets of (19A.7) can be rewritten as follows:

$$\int_0^\infty \frac{dS}{2\lambda_C} \left\{ \int_{-i\infty}^{i\infty} \frac{d\sigma}{2\pi i} e^{\sigma S/2\lambda_C} \prod_{n=1}^{N+1} \left[\int_0^\infty dh_n h_n^{(D-1)/2} e^{-\sigma\epsilon h_n/2\lambda_C}\right]\right\} e^{-S/2\lambda_C} \tag{19A.9}$$

Setting $h_n = r_n^2$, we treat the curly backets as

$$\int_{-i\infty}^{i\infty} \frac{d\sigma}{2\pi i} e^{\sigma S/2\lambda_C} \prod_{n=1}^{N+1} \left[\int_{-\infty}^\infty dr_n r_n^D e^{-\sigma\epsilon r_n^2/2\lambda_C}\right]$$
$$= \left[\frac{2\pi\lambda_C}{\Gamma(D+1)\epsilon}\right]^{N+1} \int_{-i\infty}^{i\infty} \frac{d\sigma}{2\pi i} e^{\sigma S/2\lambda_C} \sigma^{-(N+1)(D+1)/2}. \tag{19A.10}$$

For large N, the integral over σ can be approximated by the Gaussian integral around the neighborhood of the saddle point at $\sigma = (N+1)(D+1)\lambda_C/S$:

$$\int_{-i\infty}^{i\infty} \frac{d\sigma}{2\pi i} e^{\sigma S/2\lambda_C} \sigma^{-(N+1)(D+1)/2} \underset{\text{large } N}{\approx} \frac{1}{\sqrt{2\pi}} \left[\frac{S}{(N+1)(D+1)\lambda_C}\right]^{(N+1)(D+1)/2}. \tag{19A.11}$$

While letting N tend to infinity, we keep $S/(N+1) \equiv \bar\epsilon$ fixed. Then we may write the right-hand side as an exponential

$$\frac{1}{\sqrt{2\pi}} \left[\frac{(D+1)\lambda_C}{\bar\epsilon}\right]^{-S(D+1)/2\bar\epsilon} = \frac{1}{\sqrt{2\pi}} e^{-zS/2\lambda_C}, \tag{19A.12}$$

where

$$z \equiv \nu \log \nu \quad \text{with} \quad \nu \equiv (D+1)\lambda_C/\bar\epsilon \tag{19A.13}$$

is a large number. Inserting (20.246) back into the curly brackets of (19A.9), the constant z can be absorbed into the mass of the particle by replacing M by the renormalized quantity $M_1 = M(1+z)$. With this, (19A.9) becomes

$$\int_0^\infty \frac{dS}{2\lambda_C} e^{-SM_1c/2\hbar}, \tag{19A.14}$$

and the path integral (19A.4) reduces to

$$(x_b|x_a) = \mathcal{N}'' \int_0^\infty \frac{dS}{2\lambda_C}\, e^{-M_1 cS/2\hbar} \int \frac{d^D p}{(2\pi\hbar)^D} e^{-Sp^2/2Mc+ip(x_b-x_a)/\hbar}, \tag{19A.15}$$

where all irrelevant factors are contained in the normalization factor $\mathcal{N}''$. The remaining integral over S can now be done and yields

$$(x_b|x_a) = \mathcal{N}'' \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + M_R^2 c^2/\hbar^2} e^{ik(x_b-x_a)},$$

where M_R is the renomalized mass

$$M_R = M(1+z)^{1/2}. \tag{19A.16}$$

If we choose the normalization factor $\mathcal{N}'' = 1$, this is exactly the same result as that obtained before in Eq. (19.30) from the modified action (19.10), if we use in the original action (19.12) the mass $M/(1+z)^{1/2}$ rather than M for the calculation.

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