Acribus, ut ferme talia, initiis, incurioso fine

As is usual in such matters, keen in commencing, negligent at the end

TACITUS (58-120), Annales, Book 6, 17

14

Solution of Further Path Integrals by Duru-Kleinert Method

The combination of a path-dependent time reparametrization and a compensating coordinate transformation, used by Duru and Kleinert to transform the Coulomb path integral into a harmonic-oscillator path integral, can be generalized to relate a variety of path integrals to each other. In this way, many unknown path integrals can be solved by their relation to known path integrals. In this chapter, the method is explained for a typical sample of one-dimensional path integrals as well as for a more involved three-dimensional system. The latter describes a generalization of the Coulomb system consisting of two particles which carry both electric and magnetic charges. It is commonly referred to as the *dionium* atom (by analogy with the positronium atom, the bound state between electron and positron). We also discuss further possible generalizations of the solution method.

14.1 One-Dimensional Systems

In one space dimension, the general relation to be established is the following: Let $\hat{H}$ be a Hamiltonian operator

$$\hat{H} = \hat{T} + \hat{V}, \tag{14.1}$$

with the kinetic term $\hat{T} = \hat{p}^2/2M$, and define the auxiliary Hamiltonian operator

$$\hat{H}_E = \hat{H} - E, \tag{14.2}$$

with the associated time evolution amplitude

$$\langle x_b|\hat{U}_E(t)|x_a\rangle \equiv \langle x_b|e^{-it\hat{H}_E/\hbar}|x_a\rangle. \tag{14.3}$$

An integration of this amplitude over all $t > 0$ yields the fixed-energy amplitude

$$(x_b|x_a)_E = \int_{t_a}^{\infty} dt_b \langle x_b|\hat{U}_E(t_b - t_a)|x_a\rangle \tag{14.4}$$

[recall (12.8)]. This can formally be written as a path integral

$$(x_b|x_a)_E = \int_{t_a}^{\infty} dt_b \int \mathcal{D}x(t) e^{i\mathcal{A}_E[x]/\hbar}, \tag{14.5}$$

with an action

$$\mathcal{A}_E[x] = \int_{t_a}^{t_b} dt \left[\frac{M}{2}\dot{x}^2(t) - V(x(t)) + E \right]. \tag{14.6}$$

As in the Coulomb system, another path integral representation is found for the amplitude (14.5) by making use of the more general representation (12.21) of the resolvent operator. By choosing two arbitrary regulating functions $f_l(x)$, $f_r(x)$ whose product is $f(x)$, we introduce the modified auxiliary Hamiltonian operator

$$\hat{\mathcal{H}}_E = f_l(x)(\hat{H} - E)f_r(x). \tag{14.7}$$

The associated pseudotime evolution amplitude

$$\langle x_b|\hat{\mathcal{U}}_E(S)|x_a\rangle \equiv f_r(x_b)f_l(x_a)\langle x_b|e^{-iS\hat{\mathcal{H}}_E/\hbar}|x_a\rangle \tag{14.8}$$

yields upon integration over all $S > 0$ the same fixed-energy amplitude as (14.4):

$$(x_b|x_a)_E = \int_0^{\infty} dS \langle x_b|\hat{\mathcal{U}}_E(S)|x_a\rangle \tag{14.9}$$

[recall (12.30)]. The amplitude can therefore be calculated from the path integral

$$(x_b|x_a)_E = \int_0^{\infty} dS \left[f_r(x_b)f_l(x_a) \int \mathcal{D}x(s) e^{i\mathcal{A}_E^f[x]/\hbar} \right], \tag{14.10}$$

with the modified action

$$\mathcal{A}_E^f[x] = \int_0^S ds \left\{ \frac{M}{2f(x(s))} x'^2(s) - f(x(s))[V(x(s)) - E] \right\}. \tag{14.11}$$

As observed in (12.37), this action is obtained from (14.6) by a path-dependent time reparametrization satisfying

$$dt = ds\, f(x(s)). \tag{14.12}$$

The introduction of $f(x)$ has brought the kinetic term to an inconvenient form containing a space-dependent mass $M/f(x)$. This space dependence is removed by a coordinate transformation

$$x = h(q). \tag{14.13}$$

Since the coordinate differentials are related by

$$dx = h'(q)dq, \tag{14.14}$$

we require the function $h(q)$ to satisfy

$$h'^2(q) = f(h(q)). \tag{14.15}$$

Then the action (14.11) reads, in terms of the new coordinate q,

$$\mathcal{A}_E^{f,q} = \int_0^S ds \left\{ \frac{M}{2} q'^2(s) - f(q(s))[V(q(s)) - E] \right\}, \tag{14.16}$$

with the obvious notation

$$f(q) \equiv f(h(q)), \quad V(q) \equiv V(h(q)). \tag{14.17}$$

In the transformed action (14.16), the kinetic term has the usual form.

The important fact to be proved and exploited in the sequel is the following: The initial fixed-energy amplitude (14.5) can be related to the fixed-pseudoenergy amplitude associated with the transformed action (14.16), if this action is extended by an effective potential

$$V_{\text{eff}}(q) = -\frac{\hbar^2}{4M} \left[\frac{h'''}{h'} - \frac{3}{2} \left(\frac{h''}{h'} \right)^2 \right]. \tag{14.18}$$

The quantity in brackets is known as the Schwartz derivative $\{h, q\}$ of Eq. (4.19) encountered in the semiclassical expansion coefficient $q_2(x)$. The effective potential is caused by time slicing effects and will be derived in the next section. Thus, instead of the naively transformed action (14.16), the fixed-pseudoenergy amplitude $(q_b|q_a)_{\mathcal{E}}$ is obtained from the extended action

$$\mathcal{A}_{E,\mathcal{E}}^{\text{DK}}[q] = \int_0^S ds \left\{ \frac{M}{2} q'^2(s) - f(q(s))[V(q(s)) - E] - V_{\text{eff}}(q(s)) + \mathcal{E} \right\}, \tag{14.19}$$

by calculating the path integral

$$(q_b|q_a)_{\mathcal{E}} = \int_0^\infty dS \int \mathcal{D}q(s) e^{i\mathcal{A}_{E,\mathcal{E}}^{\text{DK}}[q]}. \tag{14.20}$$

The relation to be derived which leads to a solution of many nontrivial path integrals is

$$(x_b|x_a)_E = [f(x_b)f(x_a)]^{1/4} (q_b|q_a)_{\mathcal{E}=0}. \tag{14.21}$$

The procedure is an obvious generalization of the Duru-Kleinert transformation of the Coulomb path integral in Section 13.1, as indicated by the superscript DK on the transformed actions. Correspondingly, the actions $\mathcal{A}_E[x]$ and $\mathcal{A}_{E,\mathcal{E}}^{\text{DK}}[q]$, whose path integrals (14.5) and (14.20) producing the same fixed-energy amplitude $(x_b|x_a)_E$ via the relation (14.21), are called *DK-equivalent*.

The prefactor on the right-hand side has its origin in the normalization properties of the states. With $dx = dq\, h'(q) = dq\, f(h(q))^{1/2}$, the completeness relation

$$\int dx |x\rangle\langle x| = 1 \tag{14.22}$$

goes over into

$$\int dq \sqrt{f(q)} |h(q)\rangle\langle h(q)| = 1. \tag{14.23}$$

We want the transformed states $|q\rangle$ to satisfy the completeness relation

$$\int dq |q\rangle\langle q| \equiv 1. \tag{14.24}$$

This implies the relation between new and old states:

$$|x\rangle = f(q)^{-1/4} |q\rangle. \tag{14.25}$$

At first sight it appears as though the normalization factor in (14.21) should have the opposite power $-1/4$, but the sign is correct as it is. The reason lies, roughly speaking, in a factor $[f(x_b)f(x_a)]^{1/2}$ by which the pseudotimes dt and ds in the integrals (14.4) and (14.20) differ from each other. This causes the fixed-energy amplitude to be no longer proportional to the dimensions of the states, in which case Eq. (14.21) would have indeed carried a factor $[f(x_b)f(x_a)]^{-1/4}$. The extra factor $[f(x_b)f(x_a)]^{1/2}$ arising from the pseudotime integration *inverts* the naively expected prefactor.

In applications, the situation is usually as follows: There exists a solved path integral for a system with a singular potential. The time-sliced action is *not* the naively sliced classical action, but a more complicated regularized one which is free of path collapse problems. The most important example is the radial path integral (8.36) which involves a logarithm of a Bessel function rather than a centrifugal barrier. Further examples are the path integrals (8.174) and (8.207) of a particle near the surface of a sphere in $D = 3$ and $D = 4$ dimensions, where angular barriers are regulated by Bessel functions. In these examples, the explicit form of the time-sliced path integral without collapse as well as its solution are obtained from an angular momentum projection of a simple Euclidean path integral. In the first step of the solution procedure, the introduction of a path-dependent new time s via $dt = ds\, f(x(s))$ removes the dangerous singularities by an appropriate choice of the regulating function $f(x)$. The transformed system has a regular potential and possesses a time-sliced path integral, but it has an unconventional kinetic term. In the second step, the coordinate transformation brings the kinetic term to the conventional form. The final fixed-pseudoenergy amplitude $(q_b|q_a)_\mathcal{E}$ evaluated at $\mathcal{E} = 0$ coincides with the known amplitude of the initial system, apart from the above-discussed factor which is inversely related to the normalization of the states. Note that with (14.15), the relation (14.21) can also be written as

$$(x_b|x_a)_E = [h'(q_b)h'(q_a)]^{1/2} (q_b|q_a)_{\mathcal{E}=0}. \tag{14.26}$$

This transformation formula will be used to find a number of path integrals. First, however, we shall derive the effective potential (14.18) as promised.

14.2 Derivation of the Effective Potential

In order to derive the effective potential (14.18), we consider the pseudotime-sliced path integral associated with the regularized pseudotime evolution operator (14.8):

$$\langle x_b|\hat{\mathcal{U}}_E(S)|x_a\rangle \approx \frac{f_r(x_b)f_l(x_a)}{\sqrt{2\pi i\epsilon_s\hbar f_l(x_b)f_r(x_a)/M}}\prod_{n=1}^{N}\left[\int\frac{dx_n}{\sqrt{2\pi i\epsilon_s\hbar f_n/M}}\right]\exp\left(\frac{i}{\hbar}\mathcal{A}^N\right), \tag{14.27}$$

where

$$\mathcal{A}^N=\sum_{n=1}^{N+1}\left\{\frac{M}{2\epsilon_s}\frac{(\Delta x_n)^2}{f_l(x_n)f_r(x_{n-1})}+\epsilon_s[E-V(x_n)]f_l(x_n)f_r(x_{n-1})\right\}. \tag{14.28}$$

In the measure, we have used the abbreviation $f_n\equiv f(x_n)=f_l(x_n)f_r(x_n)$. From now on, the potential $V(x)$ is omitted as being inessential to the discussion. By shifting the product index and the subscripts of f_n by one unit, and by compensating for this with a prefactor, the integration measure in (14.27) acquires the postpoint form

$$\frac{[f(x_b)f(x_a)]^{1/4}}{\sqrt{2\pi i\epsilon_s\hbar/M}}\left[\frac{f_r(x_a)}{f_r(x_b)}\right]^{-5/4}\left[\frac{f_l(x_a)}{f_l(x_b)}\right]^{1/4}\prod_{n=2}^{N+1}\int\frac{d\Delta x_n}{\sqrt{2\pi i\epsilon_s\hbar f_n/M}}, \tag{14.29}$$

where the integrals over $\Delta x_n=x_n-x_{n-1}$ are done successively from high to low n, each at a fixed postpoint position x_n.

We now go over to the new coordinate q with a transformation function $x=h(q)$ satisfying (14.15) which makes the leading kinetic term simple:

$$\mathcal{A}_0^N=\sum_{n=1}^{N+1}\frac{M}{2\epsilon_s}(\Delta q_n)^2. \tag{14.30}$$

The postpoint expansion of Δx_n reads at each n (omitting the subscripts)

$$\Delta x=x(q)-x(q-\Delta q)=e_1\,\Delta q-\frac{1}{2}e_2\,(\Delta q)^2+\frac{1}{6}e_3\,(\Delta q)^3+\dots\,, \tag{14.31}$$

with the expansion coefficients

$$e_1\equiv h'=f^{1/2},\ e_2\equiv h'',\ e_3\equiv h''',\dots \tag{14.32}$$

evaluated at the postpoint q_n. The expansion (14.31) is the one-dimensional analog of the expansion (11.56), the coefficients corresponding to the basis triads ${e^i}_\mu$ in Eq. (10.12) and their derivatives ($e_1\hat{=}{e^i}_\mu$, $e_2\hat{=}{e^i}_{\mu,\nu},\dots$). Let us also introduce the analog of the reciprocal triad ${e_i}^\mu$ defined in (10.12):

$$\bar{e}\equiv 1/e_1=1/h'=1/f^{1/2}. \tag{14.33}$$

With it, we expand the kinetic term in (14.28) as

$$\frac{(\Delta x_n)^2}{2\epsilon_s f_l(x_n) f_r(x_{n-1})} = \frac{(\Delta q)^2}{2\epsilon_s}\left\{1 - \bar{e}e_2\,\Delta q + \left[\frac{1}{3}\bar{e}e_3 + \frac{1}{4}(\bar{e}e_2)^2\right](\Delta q)^2 + \dots\right\}$$
$$\times\left\{1 + \frac{f_r'}{f_r}\Delta q + \left[\left(\frac{f_r'}{f_r}\right)^2 - \frac{1}{2}\frac{f_r''}{f_r}\right](\Delta q)^2 + \dots\right\}, \tag{14.34}$$

where $f_r' \equiv df_r/dq$. From Eq. (14.31) we see that the transformation of the measure has the Jacobian

$$J = \frac{\partial \Delta x}{\partial \Delta q} = f^{1/2}\left[1 - \bar{e}e_2\Delta q + \frac{1}{2}\bar{e}e_3(\Delta q)^2 + \dots\ \right] \tag{14.35}$$

[this being a special case of (11.59)]. Since the subsequent algebra is tedious, we restrict the regulating functions $f_l(x)$ and $f_r(x)$ somewhat as in Eq. (13.3) by assuming them to be different powers $f_l(x) = f(x)^{1-\lambda}$ and $f_r(x) = f(x)^{\lambda}$ of a single function $f(x)$, where λ is an arbitrary splitting parameter. Then the measure (14.29) becomes

$$\frac{f_b^{3\lambda/2} f_a^{(1-3\lambda)/2}}{\sqrt{2\pi i\epsilon_s\hbar/M}}\prod_{n=2}^{N+1}\int\frac{d\Delta x_n}{\sqrt{2\pi i\epsilon_s\hbar f_n/M}}, \tag{14.36}$$

with the obvious notation $f_b \equiv f(x_b), f_a \equiv f(x_a)$. We now distribute the prefactor $f_b^{3\lambda/2} f_a^{(1-3\lambda)/2}$ evenly over the time interval by writing

$$f_b^{3\lambda/2} f_a^{(1-3\lambda)/2} = f_b^{1/4} f_a^{1/4}\prod_{n=1}^{N+1}\left(\frac{f_{n-1}}{f_n}\right)^{1/4-3\lambda/2}. \tag{14.37}$$

Then the path integral (14.27) becomes

$$\langle x_b|\hat{\mathcal{U}}_E(s)|x_a\rangle \approx \frac{f_b^{1/4} f_a^{1/4}}{\sqrt{2\pi i\epsilon_s\hbar/M}}\prod_{n=1}^{N}\left[\int\frac{d\Delta q_n}{\sqrt{2\pi i\epsilon_s\hbar/M}}\right] \tag{14.38}$$
$$\times\exp\left\{\frac{i}{\hbar}\left[\sum_{n=1}^{N+1}\frac{M}{2\epsilon_s}(\Delta q_n)^2 + \epsilon_s f(q_n)[E - V(q_n)] + \dots\right]\right\}[1 + C(q_n, \Delta q_n)],$$

where $1 + C$ is a correction factor arising from the three-step transformation

$$1 + C \equiv (1 + C_{\rm meas})(1 + C_f)(1 + C_{\rm act}). \tag{14.39}$$

Dropping irrelevant higher orders in Δq, the three contributions on the right-hand side have the following origins:

The transformation of the measure (14.35) gives rise to the time slicing correction

$$C_{\rm meas} = -\bar{e}e_2\ \Delta q + \frac{1}{2}\bar{e}e_3\ (\Delta q)^2 + \dots\ . \tag{14.40}$$

The rearrangement of the f-factors in (14.37) produces

$$C_f = \left(\frac{1}{4} - \frac{3\lambda}{2}\right)\left[-\frac{f'}{f}\,\Delta q + \frac{1}{2}\frac{f''}{f}\,(\Delta q)^2\right]$$
$$-\frac{1}{2}\left(\frac{3}{4} + \frac{3\lambda}{2}\right)\left(\frac{1}{4} - \frac{3\lambda}{2}\right)\left(\frac{f'}{f}\right)^2 (\Delta q)^2 + \dots\ . \tag{14.41}$$

The transformation of the pseudotime-sliced kinetic term (14.34) yields

$$C_{\rm act} = \frac{i}{\hbar}M\frac{(\Delta q)^2}{2\epsilon_s}\left\{-\left(\bar{e}e_2 - \lambda\frac{f'}{f}\right)\Delta q\right.$$
$$\left.+\left[\frac{1}{3}\bar{e}e_3 + \frac{1}{4}(\bar{e}e_2)^2 + \frac{1}{2}\left(-\lambda\frac{f''}{f} + \lambda(\lambda+1)\left(\frac{f'}{f}\right)^2\right) - \lambda\bar{e}e_2\frac{f'}{f}\right](\Delta q)^2\right\}$$
$$-\frac{M^2}{2\hbar^2}\frac{(\Delta q)^4}{4\epsilon_s^2}\left(\bar{e}e_2 - \lambda\frac{f'}{f}\right)^2(\Delta q)^2 + \dots\ . \tag{14.42}$$

We now calculate an equivalent kernel according to Section 11.2. The correction terms are evaluated perturbatively using the expectation values

$$\langle(\Delta q)^{2n}\rangle_0 = \left(\frac{i\hbar}{M}\right)^n (2n-1)!!. \tag{14.43}$$

First we find the expectation value (11.70). Listing only the relevant terms of order ϵ_s, we obtain

$$\langle C\Delta q\rangle_0 = i\hbar\epsilon_s\left[-\bar{e}e_2 - \left(\frac{1}{4} - \frac{3\lambda}{2}\right)\frac{f'}{f} + \frac{3}{2}\left(\bar{e}e_2 - \lambda\frac{f'}{f}\right)\right]. \tag{14.44}$$

The λ-terms cancel each other identically. The remainder vanishes upon using the relation (14.15), which reads in the present notation

$$e_1^2 = f, \tag{14.45}$$

implying that

$$2e_1e_2 = f', \qquad 2\bar{e}e_2 = f'/f, \tag{14.46}$$

and yielding indeed $\langle C\Delta q\rangle_0 \equiv 0$.

We now turn to the expectation $\langle C\rangle_0$ which determines the effective potential via Eq. (11.47). By differentiating the second equation in (14.46), we see that

$$f''/f = 2\left[(\bar{e}e_2)^2 + \bar{e}e_3\right]. \tag{14.47}$$

By expressing f and f'' in Eqs. (14.41) and (14.42) in terms of the e-functions, we obtain

$$C_f = \left(\frac{1}{4} - \frac{3\lambda}{2}\right)\left\{-2\bar{e}e_2\,\Delta q + [(\bar{e}e_2)^2 + \bar{e}e_3]\,(\Delta q)^2\right\}$$
$$-2\left(\frac{3}{4} + \frac{3\lambda}{2}\right)\left(\frac{1}{4} - \frac{3\lambda}{2}\right)(\bar{e}e_2)^2(\Delta q)^2 + \dots\ , \tag{14.48}$$

$$C_{\rm act} = i\frac{M}{\hbar}\frac{(\Delta q)^2}{2\epsilon_s}\Big\{ -(1-2\lambda)\bar{e}e_2\Delta q + \left[\frac{1}{3}\bar{e}e_3 + \frac{1}{4}(\bar{e}e_2)^2 - \lambda(\bar{e}e_2+\bar{e}e_3) + 2\lambda(\lambda+1)(\bar{e}e_2)^2 - 2\lambda(\bar{e}e_2)^2\right](\Delta q)^2\Big\} - \frac{M^2}{2\hbar^2}\frac{(\Delta q)^4}{4\epsilon_s^2}(1-2\lambda)^2(\bar{e}e_2)^2(\Delta q)^2 + \dots\,. \tag{14.49}$$

After forming the product (14.39), the total correction reads

$$C = \bar{e}e_2\left(\frac{1}{2}-\lambda\right)\Delta q\left[-\frac{iM}{\hbar\epsilon_s}(\Delta q)^2 - 3\right] +(\bar{e}e_2)^2\left[\frac{9}{2}\left(\lambda-\frac{1}{6}\right)\left(\lambda-\frac{1}{2}\right)(\Delta q)^2 + i\left(4\lambda^2-\frac{7}{2}\lambda+\frac{7}{8}\right)\frac{M}{\hbar\epsilon_s}(\Delta q)^4 - \frac{1}{2}\left(\lambda-\frac{1}{2}\right)^2\frac{M^2}{\hbar^2\epsilon_s^2}(\Delta q)^6\right] +\bar{e}e_3\left[-\frac{3}{2}\left(\lambda-\frac{1}{2}\right)(\Delta q)^2 - \frac{1}{2}\left(\lambda-\frac{1}{3}\right)i\frac{M}{\hbar\epsilon_s}(\Delta q)^4\right] + \dots\,. \tag{14.50}$$

Using (14.43), we find the expectation to the relevant order ϵ_s:

$$\langle C\rangle_0 = -\frac{\epsilon_s\hbar}{M}\left[\frac{1}{4}\bar{e}e_3 - \frac{3}{8}(\bar{e}e_2)^2\right]. \tag{14.51}$$

It amounts to an effective potential

$$V_{\rm eff} = -\frac{i\hbar^2}{M}\left[\frac{1}{4}\bar{e}e_3 - \frac{3}{8}(\bar{e}e_2)^2\right]. \tag{14.52}$$

By inserting (14.32) and (14.33), this turns into the expression (14.18) which we wanted to derive.

In summary we have shown that the kernel in (14.38)

$$K^{\epsilon_s}(\Delta q) = \frac{1}{\sqrt{2\pi i\epsilon_s\hbar/M}}\exp\left\{\frac{i}{\hbar}\sum_{n=1}^{N+1}\left[\frac{M}{2\epsilon_s}(\Delta q_n)^2 + \epsilon_s E f(q_n)\right]\right\}[1+C] \tag{14.53}$$

can be replaced by the simpler equivalent kernel

$$K^{\epsilon_s}(\Delta q) = \frac{1}{\sqrt{2\pi i\epsilon_s\hbar/M}}\exp\left\{\frac{i}{\hbar}\sum_{n=1}^{N+1}\left[\frac{M}{2\epsilon_s}(\Delta q_n)^2 + \epsilon_s E f(q_n) - \epsilon_s V_{\rm eff}\right]\right\}, \tag{14.54}$$

in which the correction factor $1+C$ is accounted for by the effective potential $V_{\rm eff}$ of Eq. (14.18). This result is independent of the splitting parameter λ [1]. The same result emerges, after a lengthier algebra, for a completely general splitting of the regulating function $f(x)$ into a product $f_l(x)f_r(x)$.

14.3 Comparison with Schrödinger Quantum Mechanics

The DK transformation of the action (14.11) into the action (14.19) has of course a correspondence in Schrödinger quantum mechanics. In analogy with the introduction of the pseudotime evolution amplitude (14.8), we multiply the Schrödinger equation

$$\left[-\frac{\hbar^2}{2M}\partial_x^2 - E\right]\psi(x,t) = i\hbar\partial_t\psi(x,t) \tag{14.55}$$

from the left by an arbitrary regulating function $f_l(x)$, and obtain

$$\left[-\frac{\hbar^2}{2M}f_l(x)\partial_x^2 f_r(x) - Ef(x)\right]\psi_f(x,t) = f(x)i\hbar\partial_t\psi_f(x,t), \tag{14.56}$$

with the transformed wave function $\psi_f(x,t) \equiv f_r(x)^{-1}\psi(x,t)$. After the coordinate transformation (14.14), we arrive at

$$\left[-\frac{\hbar}{2M}f_l(q)\left(\frac{1}{h'(q)}\partial_q\right)^2 f_r(q) - Ef(q)\right]\psi_f(q,t) = f(q)i\hbar\partial_t\psi_f(x,t), \tag{14.57}$$

having used the notation $f(q) \equiv f(h(q))$ as in (14.17). Inserting $h'^2(q) = f(h(q)) = f_l(h(q))f_r(h(q))$ from (14.15), the Schrödinger equation becomes

$$\left[-\frac{\hbar^2}{2M}f_r^{-1}(q)\left(\partial_q^2 - \frac{h''}{h'}\partial_q\right)f_r(q) - Ef(q)\right]\psi_f(q,t) = f(q)i\hbar\partial_t\psi_f(q,t). \tag{14.58}$$

After going from $\psi_f(q,t)$ to a new wave function

$$\phi(q,t) = f_r^{3/4}(q)f_l^{-1/4}(q)\psi_f(q,t)$$

related to the initial one by $\psi(x,t) \equiv f_r(q)\psi_f(q,t) = f^{1/4}(q)\phi(q,t)$, the Schrödinger equation takes the form

$$\begin{aligned}&\left[-\frac{\hbar^2}{2M}h'(q)^{-1/2}\left(\partial_q^2 - \frac{h''}{h'}\partial_q\right)h'(q)^{1/2} - Ef(q)\right]\phi(q,t)\\ &\qquad = \left[-\frac{1}{2M}\partial_q^2 + V_{\rm eff} - Ef(q)\right]\phi(q,t) = f(q)i\hbar\partial_t\phi(q,t),\end{aligned} \tag{14.59}$$

where $V_{\rm eff}$ is precisely the effective potential (14.18).

For the special coordinate transformation $r = h(q) = e^q$, we obtain

$$V_{\rm eff} = -\frac{\hbar^2}{4M}\left[\frac{h'''}{h'} - \frac{3}{2}\left(\frac{h''}{h'}\right)^2\right] = \frac{\hbar^2}{2M}\frac{1}{4}, \tag{14.60}$$

as first pointed out by Langer [2] when improving the WKB approximation of Schrödinger equations with a centrifugal barrier $\hbar^2 l(l+1)/2Mr^2$. This is too singular to allow for a semiclassical treatment. The transformation $r = h(q) = e^q$ leads to a smooth potential problem on the entire q-axis, in which the centrifugal barrier is replaced by $\hbar^2[l(l+1)+\frac{1}{4}]/2M$. Langer concluded from this that the original Schrödinger equation in r can be treated semiclassically if $l(l+1)$ is replaced by $l(l+1)+\frac{1}{4}$. The additional $\frac{1}{4}$ is known as the *Langer correction*.

The operator $f(q)\partial_t$ on the right-hand side of (14.59) plays the role of pseudotime derivative ∂_s.

14.4 Applications

We now present some typical solutions of path integrals via the DK method. The initial fixed-energy amplitudes will all have the generic action

$$\mathcal{A}_E = \int dt \left[\frac{M}{2}\dot{x}^2(t) - V(x) + E \right], \tag{14.61}$$

with different potentials $V(x)$ which usually do not allow for a naive time slicing. The associated path integrals are known from certain projections of Euclidean path integrals. In the sequel, we omit the subscript E for brevity (since we want to use its place for another subscript referring to the potential under consideration). The solution follows the general two-step procedure described in Section 14.4.

14.4.1 Radial Harmonic Oscillator and Morse System

Consider the action of a harmonic oscillator in D dimensions with an angular momentum $l_{\mathcal{O}}$ at a fixed energy $E_{\mathcal{O}}$:

$$\mathcal{A}_{\mathcal{O}} = \int dt \left[\frac{M}{2}\dot{r}^2 - \hbar^2 \frac{\mu_{\mathcal{O}}^2 - 1/4}{2Mr^2} - \frac{M}{2}\omega^2 r^2 + E_{\mathcal{O}} \right]. \tag{14.62}$$

Here $\mu_{\mathcal{O}}$ is an abbreviation for

$$\mu_{\mathcal{O}} = D_{\mathcal{O}}/2 - 1 + l_{\mathcal{O}} \tag{14.63}$$

[recall (8.138)], $D_{\mathcal{O}}$ denotes the dimension, and $l_{\mathcal{O}}$ the orbital angular momentum of the system. The subscript $\mathcal{O}$ indicates that we are dealing with the harmonic oscillator. A free particle is described by the $\omega \to 0$ -limit of this action.

Due to the centrifugal barrier, the time evolution amplitude possesses only a complicated time-sliced path integral involving Bessel functions. According to the rule (8.140), the centrifugal barrier requires the regularization

$$\epsilon\,\hbar^2 \frac{\mu_{\mathcal{O}}^2 - 1/4}{2Mr_n^2} \longrightarrow i\hbar \log \tilde{I}_{\mu_{\mathcal{O}}}\left(\frac{M}{i\hbar\epsilon} r_n r_{n-1} \right). \tag{14.64}$$

This smoothens the small-r fluctuations and prevents a path collapse in the Euclidean path integral with $\mu_{\mathcal{O}} = 0$. The time-sliced path integral can then be solved using the formula (8.14). The final amplitude is obtained most simply, however, by solving the harmonic oscillator in $D_{\mathcal{O}}$ Cartesian coordinates, and by projecting the result into a state of fixed angular momentum $l_{\mathcal{O}}$. The result was given in Eq. (9.32), and reads for $r_b > r_a$

$$(r_b|r_a)_{E_{\mathcal{O}},l_{\mathcal{O}}} = -i\frac{1}{\omega}\frac{1}{\sqrt{r_b r_a}}\frac{\Gamma((1+\mu)/2-\nu)}{\Gamma(\mu+1)}W_{\nu,\mu/2}\left(\frac{M\omega}{\hbar}r_b^2\right)M_{\nu,\mu/2}\left(\frac{M\omega}{\hbar}r_a^2\right), \quad (14.65)$$

where the parameters on the right-hand side are

$$\nu = \nu_{\mathcal{O}} \equiv \frac{E_{\mathcal{O}}}{2\omega\hbar}, \quad \mu = \mu_{\mathcal{O}}. \quad (14.66)$$

The poles determining the energy eigenvalues lie at $\nu = n/2 + D/4 = n_r + l/2 + D/4$, where $n = 0, 1, 2, 3, \ldots$.

A stable pseudotime evolution amplitude exists after a path-dependent time transformation with the regulating function

$$f(r) = r^2. \quad (14.67)$$

The time-transformed Hamiltonian

$$\mathcal{H}_{\mathcal{O}} = r^2\frac{p^2}{M} + \hbar^2\frac{\mu_{\mathcal{O}}^2 - 1/4}{2M} + \frac{M}{2}\omega^2 r^4 - E_{\mathcal{O}} r^2 \quad (14.68)$$

is free of the barrier singularity. Thus, when time-slicing the action

$$\mathcal{A}_{\mathcal{O}}^{f=r^2} = \int_0^S ds \left(\frac{M}{2}\frac{r'^2}{r^2} - \frac{\mu_{\mathcal{O}}^2 - 1/4}{2M} - \frac{M}{2}\omega^2 r^4 + E_{\mathcal{O}} r^2\right) \quad (14.69)$$

associated with $\mathcal{H}_{\mathcal{O}}$, no Bessel functions are needed.

Note that the factor $1/r^2$ accompanying $r'^2 = [dr(s)/ds]^2$ does not produce additional problems. It merely diminishes the fluctuations at small r. However, the r-dependence of the kinetic term is undesirable for an evaluation of the time-sliced path integral. We therefore go over to a new coordinate x via the transformation

$$r = h(x) \equiv e^x, \quad (14.70)$$

the transformation function $h(x)$ being related to the regulating function $f(x)$ by (14.15):

$$h'^2 = e^{2x} = f(r) = r^2. \quad (14.71)$$

The resulting effective potential (14.18) happens to be a constant:

$$V_{\text{eff}} = -\frac{\hbar^2}{M}\left[\frac{1}{4}\frac{h'''}{h'} - \frac{3}{8}\left(\frac{h''}{h'}\right)^2\right] = \frac{\hbar^2}{8M}. \quad (14.72)$$

Together with this constant, the DK-transformed radial oscillator action becomes

$$\mathcal{A}_{\mathcal{O}}^{\rm DK} = \int_0^S ds \left[\frac{M}{2}x'^2 - \frac{\mu_{\mathcal{O}}^2}{2M} - \frac{M\omega^2}{2}e^{4x} + E_{\mathcal{O}}e^{2x}\right]. \tag{14.73}$$

The effective potential (14.72) has changed the initial centrifugal barrier term from $(\mu_{\mathcal{O}}^2 - 1/4)/2M$ to $\mu_{\mathcal{O}}^2/2M$. We have omitted the pseudoenergy $\mathcal{E}$ since it is set equal to zero in the final DK relation (14.26). With the identifications

$$A = \frac{M}{2}\omega^2, \tag{14.74}$$

$$B = E_{\mathcal{O}}, \tag{14.75}$$

$$C = \frac{\hbar^2\mu_{\mathcal{O}}^2}{2M} + E_{\mathcal{M}}, \tag{14.76}$$

the action (14.73) goes over into

$$\mathcal{A}_{\mathcal{M}} = \int_0^S ds \left[\frac{M}{2}x'^2 - (V_{\mathcal{M}} - E_{\mathcal{M}})\right], \tag{14.77}$$

with the *Morse potential*

$$V_{\mathcal{M}}(x) = Ae^{4x} - Be^{2x} + C. \tag{14.78}$$

Its fixed-energy amplitude

$$(x_b|x_a)_{E_{\mathcal{M}}} = \int_0^\infty dS \int \mathcal{D}x(s)e^{i\mathcal{A}_{\mathcal{M}}/\hbar} \tag{14.79}$$

is therefore equivalent to the radial amplitude of the oscillator (14.65) via the DK relation (14.26), which now reads

$$(r_b|r_a)_{E_{\mathcal{O}},l} = e^{(x_b+x_a)/2}(x_b|x_a)_{E_{\mathcal{M}}}, \tag{14.80}$$

where $r = e^x$.

Let us use the relations (14.74)–(14.76) to extract the bound-state energy eigenvalues of the Morse potential. For this we use the relation (9.33) to rewrite B as

$$B = \hbar\omega(\mu_{\mathcal{O}} + n_r + 1/2), \quad n_r = 0, 1, 2, 3, \dots\ , \tag{14.81}$$

which becomes with (14.74)

$$B = \hbar\sqrt{2A/M}(\mu_{\mathcal{O}} + n_r + 1/2). \tag{14.82}$$

This is solved by $\mu_{\mathcal{O}} = B/\hbar\sqrt{2A/M} - (n_r + 1/2)$, and inserted into (14.76) to find

$$E_{\mathcal{M}} = C - \frac{B^2}{4A}\left[1 - \frac{\hbar}{B}\sqrt{\frac{2A}{M}}(n_r + 1/2)\right]^2, \quad (0 \le n_r \le \sqrt{MB^2/2A\hbar^2} - 1/2. \tag{14.83}$$

For studies of molecular vibrations one typically chooses the Morse potential $V_{\mathcal{M}} = V_0[e^{-4x} - 2e^{-2x}]$, in which case the energy spectrum becomes

$$E_{\mathcal{M}} = -V_0\left[1 - \frac{\hbar}{\sqrt{MV_0/2}}(n_r + 1/2)\right]^2, \quad (0 \le n_r \le \sqrt{MV_0/2\hbar^2} - 1/2. \tag{14.84}$$

14.4.2 Radial Coulomb System and Morse System

By a similar argument, the completely different path integral of the radial Coulomb system can be shown to be DK-equivalent to the path integral of the Morse potential. The action is

$$\mathcal{A}_{\mathcal{C}} = \int dt \left[\frac{M}{2}\dot{r}^2 - \hbar^2 \frac{\mu_{\mathcal{C}}^2 - 1/4}{2Mr^2} + \frac{e^2}{r} + E_{\mathcal{C}} \right], \tag{14.85}$$

where

$$\mu_{\mathcal{C}} = D_{\mathcal{C}}/2 - 1 + l_{\mathcal{C}}. \tag{14.86}$$

For $e^2 = 0$, the action describes a free particle moving in a centrifugal barrier potential. As in the previous example, the action (14.85) does not lead to a time-sliced amplitude of the Feynman type, but involves Bessel functions. We must again remove the barrier via a path-dependent time transformation with

$$f(r) = r^2 \tag{14.87}$$

by introducing the pseudotime s satisfying $dt = ds\, r^2(s)$. This leads to the time-transformed action

$$\mathcal{A}_{\mathcal{C}}^{f=r^2} = \int_0^S ds \left[\frac{M}{2}\frac{r'^2}{r^2} - \hbar^2 \frac{\mu_{\mathcal{C}}^2 - 1/4}{2M} + e^2 r + E_{\mathcal{C}} r^2 \right]. \tag{14.88}$$

To bring the kinetic term to the standard form, we change the variable r to x via

$$r = e^x. \tag{14.89}$$

This introduces the same effective potential as in (14.72),

$$V_{\text{eff}} = \frac{\hbar^2}{2M}\frac{1}{4}, \tag{14.90}$$

canceling the 1/4-term in the former centrifugal barrier [2]. Thus we arrive at the DK transform of the radial Coulomb action

$$\mathcal{A}_{\mathcal{C}}^{\text{DK}} = \int_0^S ds \left[\frac{M}{2}x'^2 - \hbar^2 \frac{\mu_{\mathcal{C}}^2}{2M} + e^2 e^x + E_{\mathcal{C}} e^{2x} \right]. \tag{14.91}$$

A trivial change of variables

$$\begin{aligned} x &= 2\bar{x}, \\ M &= \bar{M}/4, \\ \mu_{\mathcal{C}} &= 2\bar{\mu}, \end{aligned} \tag{14.92}$$

brings this to the form

$$\mathcal{A}_{\mathcal{C}}^{\rm DK} = \int_0^S ds \left[\frac{\bar{M}}{2} \bar{x}'^2 - \hbar^2 \frac{\bar{\mu}^2}{2\bar{M}} + e^2 e^{2\bar{x}} + E_{\mathcal{C}} e^{4\bar{x}} \right], \tag{14.93}$$

and establishes contact with the Morse action (14.77). Upon replacing $\bar{x}$ by x we see that

$$(r_b|r_a)_{E_{\mathcal{C}},l_{\mathcal{C}}} = \frac{1}{2} e^{(x_b+x_a)} (x_b|x_a)_{E_{\mathcal{M}}}, \tag{14.94}$$

with $r = e^{2x}$. The factor 1/2 accounts for the fact that the normalized states are related by $|x\rangle = |\bar{x}\rangle/2$. The identification of the parameters is now

$$A = -E_{\mathcal{C}}, \tag{14.95}$$
$$B = e^2, \tag{14.96}$$
$$C = \hbar^2 \frac{\mu_{\mathcal{C}}^2}{2M} + E_{\mathcal{M}}. \tag{14.97}$$

This can easily be checked inserting for $E_{\mathcal{M}}$ the previous result (14.84), with A and B expressed by (14.74) and (14.75). This yields

$$E_{\mathcal{C}} = -\frac{Me^4}{\hbar^2} \frac{1}{2(\mu_{\mathcal{C}} + n_r + \frac{1}{2})^2} = -Mc^2 \frac{\alpha^2}{2n^2}, \tag{14.98}$$

with $n = n_r + l_{\mathcal{C}} + (D_{\mathcal{C}} - 1)/2$, $n_r = 0, 1, 2, \ldots$. For $D_{\mathcal{C}} = 3$, this agrees with (13.212).

14.4.3 Equivalence of Radial Coulomb System and Radial Oscillator

Since the radial oscillator and the radial Coulomb system are both DK-equivalent to a Morse system, they are DK-equivalent to each other. The relation between the parameters is

$$\begin{aligned} M_{\mathcal{O}} &= 4M_{\mathcal{C}}, \\ \mu_{\mathcal{O}} &= 2\mu_{\mathcal{C}}, \\ E_{\mathcal{O}} &= e^2, \\ -\frac{M_{\mathcal{O}}}{2}\omega^2 &= E_{\mathcal{C}}, \\ r_{\mathcal{O}} &= \sqrt{r_{\mathcal{C}}}. \end{aligned} \tag{14.99}$$

We have added subscripts $\mathcal{O}, \mathcal{C}$ also to the masses M to emphasize the systems to which they belong. The relation $\mu_{\mathcal{O}} = 2\mu_{\mathcal{C}}$ implies

$$D_{\mathcal{O}}/2 - 1 + l_{\mathcal{O}} = 2(D_{\mathcal{C}}/2 - 1 + l_{\mathcal{C}}) \tag{14.100}$$

for all dimensions and angular momenta of the two systems. Due to the square root relation $r_{\mathcal{O}} = \sqrt{r_{\mathcal{C}}}$, the orbital angular momenta satisfy

$$l_{\mathcal{O}} = 2l_{\mathcal{C}}. \tag{14.101}$$

For the dimensions, this implies

$$D_{\mathcal{O}} = 2D_{\mathcal{C}} - 2. \tag{14.102}$$

In the cases $D_{\mathcal{C}} = 2$ and 3, there is complete agreement with Chapter 13 where the dimensions of the DK-equivalent oscillators were 2 and 4, respectively.

To relate the amplitudes with each other we find it useful to keep the notation as close as possible to that of Chapter 13 and denote the radial coordinate of the radial oscillator by u. Then the DK relation for the pseudotime evolution amplitudes states that

$$(r_b|r_a)_{E_{\mathcal{C}},\mu_{\mathcal{C}}} = \frac{1}{2}\sqrt{u_b u_a}(u_b|u_a)_{E_{\mathcal{O}},\mu_{\mathcal{O}}}\ , \tag{14.103}$$

with the right-hand side given by (14.65) (after replacing r by u, $M_{\mathcal{O}}$ by $4M_{\mathcal{C}}$, and $M_{\mathcal{O}}\omega u^2/\hbar$ by $2\kappa r$).

Note once more that the prefactor on the right-hand side has a dimension opposite to what one might have expected from the quantum-mechanical completeness relation

$$\int_0^\infty dr|r\rangle\langle r| = 1, \tag{14.104}$$

whose u-space version reads

$$\int_0^\infty du\, 2u|r\rangle\langle r| = \int du|u\rangle\langle u| = 1. \tag{14.105}$$

As explained in Section 14.1, the reason lies in the different dimensions (by a factor r) of the pseudotimes over which the evolution amplitudes are integrated when going to the fixed-energy amplitudes. A further factor 1/4 contained in (14.103) is due to the mass relation $M_{\mathcal{O}} = 4M_{\mathcal{C}}$.

Let us check the relation (14.103) for $D_{\mathcal{C}} = 3$. The fixed-energy amplitude of the Coulomb system has the partial-wave expansion

$$(\mathbf{x}_b|\mathbf{x}_a)_{E_{\mathcal{C}}} = \sum_{l_{\mathcal{C}}=0}^{\infty}\sum_{m=-l_{\mathcal{C}}}^{l_{\mathcal{C}}} \frac{1}{r_b r_a}(r_b|r_a)_{E_{\mathcal{C}},l_{\mathcal{C}}} Y_{l_{\mathcal{C}}m}(\theta_b,\varphi_b)Y^*_{l_{\mathcal{C}}m}(\theta_a,\varphi_a). \tag{14.106}$$

The four-dimensional oscillator, on the other hand, has

$$\begin{aligned}(\vec{u}_b|\vec{u}_a)_{E_{\mathcal{O}}} \; &= \sum_{l_{\mathcal{O}}=0}^{\infty}(u_b|u_a)_{E_{\mathcal{O}},l_{\mathcal{O}}} \\ &\times \frac{l_{\mathcal{O}}+1}{2\pi^2}\sum_{m_1,m_2=-l_{\mathcal{O}}/2}^{l_{\mathcal{O}}/2} \mathcal{D}^{l_{\mathcal{O}}/2}_{m_1m_2}(\varphi_n,\theta_n,\gamma_n)\mathcal{D}^{l_{\mathcal{O}}/2\,*}_{m_1m_2}(\varphi_{n-1},\theta_{n-1},\gamma_{n-1}).\end{aligned} \tag{14.107}$$

We now take Eq. (13.127),

$$(\mathbf{x}_b|\mathbf{x}_a)_{E_\mathcal{C}} = \int_0^\infty dS e^{ie^2S/\hbar} \frac{1}{16} \int_0^{4\pi} d\gamma_a (\vec{u}_b S|\vec{u}_a 0), \tag{14.108}$$

and observe that the integral $\int_0^{4\pi} d\gamma_a$ over the sum of angular wave functions

$$\frac{l_\mathcal{O}+1}{2\pi^2} \sum_{m_1,m_2=-l_\mathcal{O}/2}^{l_\mathcal{O}/2} d^{l_\mathcal{O}/2}_{m_1 m_2}(\theta_b) d^{l_\mathcal{O}/2}_{m_1 m_2}(\theta_a) e^{im_1(\varphi_b-\varphi_a)+im_2(\gamma_b-\gamma_a)} \tag{14.109}$$

produces a sum

$$8 \sum_m^{l_\mathcal{O}/2} Y_{l_\mathcal{O}/2,m,0}(\theta_b,\phi_b) Y^*_{l_\mathcal{O}/2,m,0}(\theta_a,\phi_a), \tag{14.110}$$

with the spherical harmonics

$$Y_{l_\mathcal{O}/2,m}(\theta,\phi) = \sqrt{\frac{l_\mathcal{O}+1}{4\pi}} e^{im\phi} d^{l_\mathcal{O}/2}_{m,0}(\theta). \tag{14.111}$$

Only even $l_\mathcal{O}$-values survive the integration, and we identify $l_\mathcal{C} = l_\mathcal{O}/2$

Recalling the radial amplitude of the harmonic oscillator (9.32), we find from (14.103) the radial amplitude of the Coulomb system in any dimension $D_\mathcal{C}$ for $r_b > r_a$:

$$(r_b|r_a)_{E_\mathcal{C},l_\mathcal{C}} = -i\frac{M_\mathcal{C}}{\hbar\kappa} \frac{\Gamma(-\nu+l_\mathcal{C}+(D_\mathcal{C}-1)/2)}{(2l_\mathcal{C}+D_\mathcal{C}-2)!} W_{\nu,l_\mathcal{C}+D_\mathcal{C}/2-1}(2\kappa r_b) M_{\nu,l_\mathcal{C}+D_\mathcal{C}/2-1}(2\kappa r_a), \tag{14.112}$$

where $\kappa = \sqrt{-2ME/\hbar^2}$ and $\nu = \sqrt{-e^4 M_\mathcal{C}/2\hbar^2 E_\mathcal{C}}$ as in (13.39) and (13.40). For $D_\mathcal{C} = 3$, this agrees with (13.211).

The full $D_\mathcal{C}$-dimensional amplitude is given by the sum over partial waves

$$(\mathbf{x}_b|\mathbf{x}_a)_{E_\mathcal{C},l_\mathcal{C}} = \frac{1}{(r_b r_a)^{(D_\mathcal{C}-1)/2}} \sum_{l=0}^\infty (r_b|r_a)_{E_\mathcal{C},l_\mathcal{C}} \sum_\mathbf{m} Y_{l\mathbf{m}}(\hat{\mathbf{x}}_b) Y^*_{l\mathbf{m}}(\hat{\mathbf{x}}_a),$$

which becomes with (8.126)

$$(\mathbf{x}_b|\mathbf{x}_a)_{E_\mathcal{C},l_\mathcal{C}} = \frac{1}{(r_b r_a)^{(D_\mathcal{C}-1)/2}} \sum_{l_\mathcal{C}=0}^\infty (r_b|r_a)_{E_\mathcal{C},l_\mathcal{C}} \frac{2l_\mathcal{C}+D_\mathcal{C}-2}{D_\mathcal{C}-2} \frac{1}{S_{D_\mathcal{C}}} C^{(D_\mathcal{C}/2-1)}_{l_\mathcal{C}}(\cos\Delta\vartheta_n). \tag{14.113}$$

It is easy to perform the sum if we make use of an integral representation of the radial amplitude obtained by DK-transforming the integral representation (9.25) of the radial oscillator amplitude. Replacing the imaginary time by the new variable of integration $\varrho = e^{-2\omega(\tau_b-\tau_a)}$, the radial variables r by u, and the oscillator mass M

by $M_{\mathcal{O}}$ to match the notation of Chapter 13, the amplitude (9.25) can be rewritten as

$$(u_b|u_a)_{E_{\mathcal{O}},l_{\mathcal{O}}} = -i\,\frac{M_{\mathcal{O}}}{\hbar}\sqrt{u_bu_a}\int_0^1 \frac{d\varrho}{2\varrho}\varrho^{-\nu}e^{-\kappa(u_b^2+u_a^2)\frac{1+\varrho}{1-\varrho}}I_{l_{\mathcal{O}}+D_{\mathcal{O}}/2-1}\left(2\kappa u_bu_a\frac{2\sqrt{\varrho}}{1-\varrho}\right), \tag{14.114}$$

with

$$\kappa \equiv \frac{M_{\mathcal{O}}\omega}{2\hbar}, \qquad \nu \equiv E_{\mathcal{O}}/2\hbar\omega. \tag{14.115}$$

Recalling (9.35), the poles of this amplitude lie at $\nu = n/2 + D_{\mathcal{O}}/4 = n_r + l_{\mathcal{O}}/2 + D_{\mathcal{O}}/4$, where $n = 0, 1, 2, 3, \ldots$, so that the energy eigenvalues are

$$E_{\mathcal{O}\,n} = 2\hbar\omega\left(\frac{n}{2}+\frac{D_{\mathcal{O}}}{4}\right) = \hbar\omega\left(2n_r + l + \frac{D_{\mathcal{O}}}{2}\right). \tag{14.116}$$

From the DK relation (14.103) we obtain $(r_b|r_a)_{l_{\mathcal{C}},E_{\mathcal{C}}}$ and insert it into (14.113). Then we recall the summation formula

$$\left(\frac{1}{2}kz\right)^{D_{\mathcal{C}}/2-1/2} I_{D_{\mathcal{C}}/2-3/2}(kz) = k^{D_{\mathcal{C}}-2}\sum_{l=0}^{\infty}\frac{1}{l!}\frac{\Gamma(l+D_{\mathcal{C}}-2)}{\Gamma(D_{\mathcal{C}}/2-1/2)}(2l+D_{\mathcal{C}}-2)$$
$$\times F(-l, l+D_{\mathcal{C}}-2; D_{\mathcal{C}}/2-1/2; (1+k^2)/2)(-)^l I_{2l+D_{\mathcal{C}}-2}(z), \tag{14.117}$$

which follows from Eq. (13.205) for $\nu = D_{\mathcal{C}}/2-3/2$ and $\mu = D_{\mathcal{C}}-2$. After expressing the right-hand side in terms of the Gegenbauer polynomial $C_l^{D_{\mathcal{C}}/2-1}$ with the help of (8.106), this becomes

$$\frac{1}{2}\frac{1}{(2\pi)^{D_{\mathcal{C}}/2-1/2}}\left(\frac{z}{2}\right)^{D_{\mathcal{C}}-2} I_{D_{\mathcal{C}}/2-3/2}(kz)/(kz)^{D_{\mathcal{C}}/2-3/2}$$
$$= \sum_{l_{\mathcal{C}}=0}^{\infty}\frac{2l_{\mathcal{C}}+D_{\mathcal{C}}-2}{D_{\mathcal{C}}-2}\frac{1}{S_{D_{\mathcal{C}}}}C_{l_{\mathcal{C}}}^{(D_{\mathcal{C}}/2-1)}((1+k^2)/2)I_{2l_{\mathcal{C}}+D_{\mathcal{C}}-2}(z). \tag{14.118}$$

Setting

$$z \equiv 2\kappa u_bu_a\frac{2\sqrt{\varrho}}{1-\varrho}, \quad k \equiv \cos(\vartheta/2), \tag{14.119}$$

the sum over the partial waves in (14.113) is easily performed, and we obtain for the fixed-energy amplitude of the Coulomb system in $D_{\mathcal{C}}$ dimensions the generalization of the integral representations (13.43) and (13.133) in two and three dimensions:

$$(\mathbf{x}_b|\mathbf{x}_a)_E = -i\frac{M}{\hbar}\frac{\kappa^{D_{\mathcal{C}}-2}}{(2\pi)^{(D_{\mathcal{C}}-1)/2}}\int_0^1\frac{d\varrho}{(1-\varrho)^2}\varrho^{-\nu}$$
$$\times\left(\frac{2\sqrt{\varrho}}{1-\varrho}\right)^{(D_{\mathcal{C}}-3)/2} e^{-\kappa\frac{1+\varrho}{1-\varrho}(r_b+r_a)}I_{D_{\mathcal{C}}/2-3/2}(kz)/(kz)^{D_{\mathcal{C}}/2-3/2}, \tag{14.120}$$

where

$$kz = 2\kappa \frac{2\sqrt{\varrho}}{1-\varrho}\sqrt{(r_b r_a + \mathbf{x}_b\mathbf{x}_a)/2}, \tag{14.121}$$

and κ, ν are the Coulomb parameters (13.40).

By changing the integration variable to $\zeta = (1+\varrho)/(1-\rho)$ as in (13.49), the integral in (14.120) is transformed into a contour integral encircling the cut from $\zeta = 1$ to ∞ in the clockwise sense. Then the amplitude reads [3]

$$(\mathbf{x}_b|\mathbf{x}_a)_E = -i\frac{M}{2\hbar}\frac{\kappa^{D_\mathcal{C}-2}}{(2\pi)^{(D_\mathcal{C}-1)/2}}\frac{\pi e^{i\pi(\nu-D_\mathcal{C}/2+3/2)}}{\sin[\pi(\nu-D_\mathcal{C}/2+3/2)]} \tag{14.122}$$
$$\times\int_C \frac{d\zeta}{2\pi i}(\zeta-1)^{-\nu+D_\mathcal{C}/2-3/2}(\zeta+1)^{\nu+D_\mathcal{C}/2-3/2}e^{-\kappa\zeta(r_b+r_a)}I_{D_\mathcal{C}/2-3/2}(z)/z^{D_\mathcal{C}/2-3/2}.$$

This expression generalizes the integral representations (13.53) and (13.135) for $D_\mathcal{C} = 2$ and $D_\mathcal{C} = 3$, respectively. The poles of this amplitude lie at all integer $\nu = n_r + l_\mathcal{C} + (D_\mathcal{C}-1)/2$, corresponding to the energies

$$E_\mathcal{C} = -\frac{Me^4}{\hbar^2}\frac{1}{2n^2}, \quad \text{with} \quad n = n_r + l_\mathcal{C} + (D_\mathcal{C}-1)/2, \quad n_r = 0,1,2,\dots, \tag{14.123}$$

in agreement with (14.98) and, for $D_\mathcal{C} = 3$, with (13.212).

As a final remark let us emphasize that at the time-sliced level there is no way of going directly from the radial Coulomb problem to the radial oscillator via the DK relation, due to the catastrophic centrifugal barriers. This has been attempted in the literature [4]. An intermediate Morse potential is necessary to do this properly. See Appendix 14A.

14.4.4 Angular Barrier near Sphere, and Rosen-Morse Potential

For another application of the solution method, consider the path integral for a mass point near the surface of a sphere in three dimensions, projected into a state of fixed azimuthal angular momentum $m = 0, \pm1, \pm2, \dots$. The projection generates an angular barrier $\propto (m^2-1/4)/\sin^2\theta$ which is a potential of the Pöschl-Teller type. With $\mu = Mr^2$, the real-time action is

$$\mathcal{A}_{\mathcal{PT}} = \int dt\left[\frac{\mu}{2}\dot\theta^2 + \frac{\hbar^2}{8\mu} - \frac{\hbar^2}{2\mu}\text{“}\frac{m^2-1/4}{\sin^2\theta}\text{”} + E_{\mathcal{PT}}\right]. \tag{14.124}$$

The quotation marks are defined in analogy with those of the centrifugal barrier in Eq. (8.140). The precise meaning is given by the proper time-sliced expression in Eq. (8.175) whose limiting form for narrow time slices is (8.177). After an analytic continuation of the parameter m to arbitrary real numbers μ, the resulting amplitude was given in (8.187). In the sequel we refrain from using the symbol μ for the noninteger m-values to avoid confusion with the mass parameter μ.

The spectral representation of the associated fixed-energy amplitude is easily written down; it arises by simply integrating (8.187) over $-id\tau_b$ and reads

$$(\theta_b|\theta_a)_{m,E_{\mathcal{PT}}} = \sqrt{\sin\theta_b \sin\theta_a} \sum_{n=0}^{\infty} \frac{i\hbar}{E_{\mathcal{PT}} - \hbar^2 L_2/2\mu} \times \frac{2n+2m+1}{2} \frac{(n+2m)!}{n!} P_{n+m}^{-m}(\cos\theta_b) P_{n+m}^{-m}(\cos\theta_a), \tag{14.125}$$

where $L_2 = l(l+1)$ with $l = n+m$ [recall (8.225) for $D=3$]. The sum over n can be done using the so-called Sommerfeld-Watson transformation [5]. The sum is re-expressed as a contour integral in the complex n-plane and deformed in such a way that only the Regge poles at

$$n+m = l = l(E_{\mathcal{PT}}) \equiv -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\mu E_{\mathcal{PT}}}{\hbar^2}} \tag{14.126}$$

contribute, with both signs of the square root. The result for $\theta_b > \theta_a$ is [6]

$$(\theta_b|\theta_a)_{m,E_{\mathcal{PT}}} = \sqrt{\sin\theta_b \sin\theta_a} \frac{-i\mu}{\hbar} \Gamma(m - l(E_{\mathcal{PT}}))\Gamma(l(E_{\mathcal{PT}}) + m + 1) \times P_{l(E_{\mathcal{PT}})}^{-m}(-\cos\theta_b) P_{l(E_{\mathcal{PT}})}^{-m}(\cos\theta_a). \tag{14.127}$$

Here we shall consider m as a free parameter characterizing the interaction strength of the Pöschl-Teller potential [7]

$$V_{\mathcal{PT}}(\theta) = \frac{\hbar^2}{2\mu} \frac{m^2}{\sin^2\theta}. \tag{14.128}$$

The regulating function removing the angular barrier is

$$f(\theta) = \sin^2\theta, \tag{14.129}$$

and the time-transformed action reads with $dt = ds\ \sin^2\theta(s)$

$$\mathcal{A}_{\mathcal{PT}}^{f=\sin^2\theta} = \int_0^S ds \left[\frac{\mu}{2\sin^2\theta} \theta'^2 + \frac{\hbar^2}{8\mu} \sin^2\theta - \frac{\hbar^2}{2\mu}(m^2 - 1/4) + E_{\mathcal{PT}}\ \sin^2\theta \right]. \tag{14.130}$$

We now bring the kinetic term to the conventional form by the variable change

$$\sin\theta = \frac{1}{\cosh x}, \qquad \cos\theta = -\tanh x, \tag{14.131}$$

which maps the interval $\theta \in (0,\pi)$ into $x \in (-\infty,\infty)$. Then we have

$$h'(x) = \sin\theta = \frac{1}{\cosh x}. \tag{14.132}$$

Forming the higher derivatives

$$h''(x) = -\frac{\tanh x}{\cosh x}, \qquad h'''(x) = -\frac{1}{\cosh x}\left(1 - 2\tanh^2 x\right), \tag{14.133}$$

the effective potential is found to be

$$V_{\text{eff}} = \frac{\hbar^2}{8\mu}\left(1 + \frac{1}{\cosh^2 x}\right). \tag{14.134}$$

The DK-transformed action is therefore

$$\mathcal{A}^{\text{DK}}_{\mathcal{PT}} = \int_0^S ds \left[\frac{\mu}{2}x'^2 - \frac{\hbar^2 m^2}{2\mu} + E_{\mathcal{PT}}\frac{1}{\cosh^2 x}\right]. \tag{14.135}$$

It describes the motion of a mass point in a smooth potential well known as the *Rosen-Morse potential* (also called the *modified Pöschl-Teller potential*) [8]. The standard parametrization is[1]

$$V_{\mathcal{RM}}(x) = -\frac{\hbar^2}{2\mu}\frac{s(s+1)}{\cosh^2 x}, \tag{14.136}$$

This corresponds to $l(E_{\mathcal{PT}})$ in (14.126) having the value s. The energy of the Rosen-Morse potential determines the parameter m in the action (14.135), and we identify

$$m = m(E_{\mathcal{RM}}) = \sqrt{-2\mu E_{\mathcal{RM}}/\hbar^2}. \tag{14.137}$$

It is obvious that the time-sliced amplitude of the Rosen-Morse potential has no path collapse problems. Its fixed-energy amplitude is thus DK-equivalent to the Pöschl-Teller amplitude (14.127), with the precise relation being

$$(\theta_b|\theta_a)_{m,E_{\mathcal{PT}}} = \sqrt{\sin\theta_b \sin\theta_a}(x_b|x_a)_{m,E_{\mathcal{RM}}}, \tag{14.138}$$

where $\tanh x = -\cos\theta$, $\theta \in (0,\pi)$, $x \in (-\infty,\infty)$. Inserting (14.127), the amplitude of the Rosen-Morse system reads explicitly

$$\begin{aligned}(x_b|x_a)_{m(E_{\mathcal{RM}})} &= \frac{-i\mu}{\hbar}\Gamma(m(E_{\mathcal{RM}})-s)\Gamma(s+m(E_{\mathcal{RM}})+1) \\ &\times\; P_s^{-m(E_{\mathcal{RM}})}(\tanh x_b)P_s^{-m(E_{\mathcal{RM}})}(-\tanh x_a).\end{aligned} \tag{14.139}$$

The bound states lie at the poles of the first Gamma function where

$$m(E_{\mathcal{RM}}) = s - n, \qquad n = 0,1,2,\ldots,[s], \tag{14.140}$$

with $[s]$ denoting the largest integer number $\leq s$. From the residues we extract the normalized wave functions [6]

$$\psi_n(x) = \sqrt{\Gamma(2s-n+1)(s-n)/n}P_s^{n-s}(\tanh x). \tag{14.141}$$

For noninteger values of s, these are not polynomials. However, the identity between hypergeometric functions (1.453)

$$F(a,b;c;z) = (1-z)^{c-a-b}F(c-a,c-b;c;z) \tag{14.142}$$

[1] There is no danger of confusing this parameter s with the pseudotime s.

permits relating them to polynomials:

$$P_s^{n-s}(\tanh x) = \frac{2^{n-s}}{\Gamma(s-n+1)} \frac{1}{\cosh^{s-n} x} F\left(-n, 1+2s-n; s-n+1; (1-\tanh x)/2\right). \tag{14.143}$$

The continuum wave functions are obtained from (14.141) by an appropriate analytic continuation of m to $-ik$. This amounts to replacing n by $s+ik$.

14.4.5 Angular Barrier near Four-Dimensional Sphere and General Rosen-Morse Potential

Let us extend the previous path integral of a mass point moving near the surface of a sphere from $D=3$ to $D=4$ dimensions. By projecting the amplitude into a state of fixed azimuthal angular momenta m_1 and m_2, an angular barrier is generated in the Euler angle θ proportional to $(m_1^2+m_2^2-1/4-2m_1m_2\cos\theta)/\sin^2\theta$. This is again a potential of the Pöschl-Teller type, although of a more general form to be denoted by $\mathcal{PT}'$. The action (8.212) is, with $\mu = Mr^2/4$,

$$\mathcal{A}_{\mathcal{PT}'} = \int dt \left[\frac{\mu}{2}\dot\theta^2 + \frac{\hbar^2}{32\mu} - \frac{\hbar^2}{2\mu} \frac{\text{“}m_1^2+m_2^2-2m_1m_2\cos\theta-1/4\text{”}}{\sin^2\theta} + E_{\mathcal{PT}'}\right], \tag{14.144}$$

where the quotation marks indicate the need to regularize the angular barrier via Bessel functions as specified in (8.208). The projected amplitude was given in Eq. (8.203) and continued to arbitrary real values of $m_1 = \mu_1,\ m_2 = \mu_2$ with $\mu_1 \geq \mu_2 \geq 0$ in (8.213). As in subsection 14.4.4, we shall also use the parameters m_1, m_2 when they have noninteger values.

The most general Pöschl-Teller potential

$$V_{\mathcal{PT}'}(\theta) = \frac{\hbar^2}{2\mu}\left[\frac{s_1(s_1+1)}{\sin^2(\theta/2)} + \frac{s_2(s_2+1)}{\cos^2(\theta/2)}\right] \tag{14.145}$$

can easily be mapped onto the above angular barrier, where $s_1(s_1+1) = \frac{(m_1+m_2)^2}{4} - \frac{1}{16}$, $s_2(s_2+1) = \frac{(m_1-m_2)^2}{4} - \frac{1}{16}$ or $s_1 = -\frac{1}{2}\left(1-\sqrt{\frac{3}{4}+(m_1+m_2)^2}\right)$, $s_2 = -\frac{1}{2}\left(1-\sqrt{\frac{3}{4}+(m_1-m_2)^2}\right)$.

The fixed-energy amplitude is obtained directly from Eq. (8.213) by an integration over $-id\tau_b$. It reads for $m_1 \geq m_2$

$$(\theta_b|\theta_a)_{m_1,m_2,E_{\mathcal{PT}'}} = \sqrt{\sin\theta_b \sin\theta_a} \times \sum_{n=0}^{\infty} \frac{i\hbar}{E_{\mathcal{PT}'} - \hbar^2 L_2/8\mu} \frac{2n+2m_1+1}{2} d^{n+m_1}_{m_1,m_2}(\theta_b) d^{n+m_1}_{m_1,m_2}(\theta_a), \tag{14.146}$$

where L_2 is given by $L_2 = (l+1)^2 - 1/4$ with $l = 2n+2m_1$ [recall (8.220) with (8.225)].

As in Eq. (14.125), the sum over n can be performed with the help of a Sommerfeld-Watson transformation by rewriting the sum as a contour integral in the complex n-plane. After deforming the contour in such a way that only the Regge poles at

$$2n + 2m_1 = l = l(E_{\mathcal{PT}'}) \equiv -1 + 2\sqrt{\frac{1}{16} + \frac{2\mu E_{\mathcal{PT}'}}{\hbar^2}} \tag{14.147}$$

contribute, with both signs of the square root, we find for $\theta_b > \theta_a$:

$$\begin{aligned}(\theta_b|\theta_a)_{m_1,m_2,E_{\mathcal{PT}'}} &= \sqrt{\sin\theta_b \sin\theta_a}\,\frac{-2i\mu}{\hbar} \\ &\times \Gamma(m_1 - l(E_{\mathcal{PT}'})/2)\Gamma(l(E_{\mathcal{PT}'})/2 - m_1 + 1)\frac{1}{2} d^{l(E_{\mathcal{PT}'})/2}_{m_1,-m_2}(\theta_b - \pi) d^{l(E_{\mathcal{PT}'})/2}_{m_1,m_2}(\theta_a),\end{aligned} \tag{14.148}$$

with arbitrary real parameters m_1, m_2 characterizing the interaction strength. The eigenvalues of the discrete energies are given by the poles of the first Gamma function:

$$E_{\mathcal{PT}'} = \frac{\hbar^2}{2\mu}\left[(m_1 + n + \tfrac{1}{2})^2 - \tfrac{1}{16}\right], \qquad n = 0, 1, 2, \ldots\, . \tag{14.149}$$

The regulating function which removes the angular barrier is

$$f(\theta) = \sin^2\theta, \tag{14.150}$$

and the time-transformed action reads, with $dt = ds\, \sin^2\theta(s)$,

$$\begin{aligned}\mathcal{A}^{f=\sin^2\theta}_{\mathcal{PT}'} = \int_0^S ds \Bigg[&\frac{\mu}{2\sin^2\theta}\theta'^2 + \frac{\hbar^2}{32\mu}\sin^2\theta \\ &- \frac{\hbar^2}{2\mu}(m_1^2 + m_2^2 - 1/4 - 2m_1 m_2 \cos\theta) + E_{\mathcal{PT}'}\sin^2\theta\Bigg].\end{aligned} \tag{14.151}$$

We now bring the kinetic term to the conventional form by the variable change

$$\sin\theta = \pm\frac{1}{\cosh x}, \quad \cos\theta = -\tanh x. \tag{14.152}$$

As in the previous case, this leads to the effective potential

$$V_{\text{eff}} = \frac{\hbar^2}{8\mu}\left(1 + \frac{1}{\cosh^2 x}\right). \tag{14.153}$$

The DK-transformed action is then

$$\mathcal{A}^{\text{DK}}_{\mathcal{PT}'} = \int_0^S ds \left[\frac{\mu}{2}x'^2 - \frac{\hbar^2}{2\mu}(m_1^2 + m_2^2 + 2m_1 m_2 \tanh x) + \left(E_{\mathcal{PT}'} - \frac{3\hbar^2}{32\mu}\right)\frac{1}{\cosh^2 x}\right]. \tag{14.154}$$

It contains a smooth potential well near the origin known as the *general Rosen-Morse potential* [8]:

$$V_{\mathcal{RM}'}(x) = -\left(E_{\mathcal{PT}'} - \frac{3\hbar^2}{32\mu}\right)\frac{1}{\cosh^2 x} + \frac{\hbar^2}{2\mu}2m_1m_2\tanh x. \tag{14.155}$$

The standard parametrization for this is[2]

$$V_{\mathcal{RM}'}(x) = \frac{\hbar^2}{2\mu}\left[-\frac{s(s+1)}{\cosh^2 x} + 2c\tanh x\right], \tag{14.156}$$

which amounts to choosing

$$E_{\mathcal{PT}'} = \frac{\hbar^2}{2\mu}\left[s(s+1) + \frac{3}{16}\right], \qquad m_1m_2 = c, \tag{14.157}$$

in (14.154). Inserting this into (14.147) makes $l(E_{\mathcal{PT}'})/2$ equal to s.

The energy of the general Rosen-Morse potential is defined by the action in the fixed-energy path integral [compare (14.77)]

$$\mathcal{A}_{\mathcal{RM}'} = \int_0^S ds\left[\frac{\mu}{2}x'^2 - (V_{\mathcal{RM}'} - E_{\mathcal{RM}'})\right]. \tag{14.158}$$

Comparison with (14.154) shows that the bound-state energies have the eigenvalues

$$E_{\mathcal{RM}'} = -\frac{\hbar^2}{2\mu}(m_1^2 + m_2^2) = -\frac{\hbar^2}{2\mu}(m_1^2 + c^2/m_1^2). \tag{14.159}$$

The solution of this equation will be a function $m_1(E_{\mathcal{RM}'})$. Correspondingly, we may define $m_2(E_{\mathcal{RM}'}) \equiv c/m_1(E_{\mathcal{RM}'})$.

Feynman's time-sliced amplitude certainly exists for this potential, and the fixed-energy amplitude is determined in terms of the angular-projected amplitude (14.148) of a mass point near the surface of a sphere which describes the motion in a general Pöschl-Teller potential. The relation is [9]

$$(\theta_b|\theta_a)_{m_1,m_2,E_{\mathcal{PT}'}} = \sqrt{\sin\theta_b\sin\theta_a}(x_b|x_a)_{m_1,m_2,E_{\mathcal{RM}'}}, \tag{14.160}$$

with $\tanh x = -\cos\theta$, $\theta \in (0,\pi)$, $x \in (-\infty,\infty)$. Explicitly we have

$$\begin{aligned}(x_b|x_a)_{m_1,m_2,E_{\mathcal{RM}'}} &= \frac{-2i\mu}{\hbar}\Gamma(m_1(E_{\mathcal{RM}'}) - s)\Gamma(s - m_1(E_{\mathcal{RM}'}) + 1) \\ &\times \frac{1}{2}d^s_{m_1(E_{\mathcal{RM}'}),-m_2(E_{\mathcal{RM}'})}(\theta_b(x_b) - \pi)d^s_{m_1(E_{\mathcal{RM}'}),m_2(E_{\mathcal{RM}'})}(\theta_a(x_a)).\end{aligned} \tag{14.161}$$

Since $s - m_1 \geq 0$, the bound states lie at the poles of the first Gamma function. With the energy-dependent function $m_1(E_{\mathcal{RM}'})$ defined by (14.159), they are given by the solutions of the equation

$$m_1(E_{\mathcal{RM}'}) = s - n, \qquad n = 0, 1, \ldots, [s]. \tag{14.162}$$

[2]There is no danger of confusing this parameter s with the pseudotime s.

The residues in (14.161) render the normalized wave functions

$$\Psi_n(x) = \sqrt{\frac{m_1^2 - m_2^2}{m_1} \frac{\Gamma(s+1-m_1)n!}{\Gamma(s+1-m_2)\Gamma(s+1+m_2)}} \tag{14.163}$$
$$\times \left[\tfrac{1}{2}(1+\tanh x)\right]^{(m_1-m_2)/2} \left[\tfrac{1}{2}(1-\tanh x)\right]^{(m_1+m_2)/2} P_n^{(m_1-m_2, m_1+m_2)}(-\tanh x),$$

or, expressed in terms of hypergeometric functions,

$$\Psi_n(x) = \sqrt{\frac{m_1^2 - m_2^2}{m_1} \frac{\Gamma(s+1+m_1)\Gamma(s+1-m_2)}{n!\Gamma(1+m_1-m_2)^2\Gamma(s+1+m_2)}}$$
$$\times \left[\tfrac{1}{2}(1+\tanh x)\right]^{(m_1-m_2)/2} \left[\tfrac{1}{2}(1-\tanh x)\right]^{(m_1+m_2)/2}$$
$$\times F\left(2s-n+1, -n; 1+m_1-m_2; \tfrac{1}{2}(1+\tanh x)\right), \tag{14.164}$$

with $m_1 = s - n$ and $m_2 = c/m_1$ [10]. The continuum wave functions are obtained from these by an appropriate analytic continuation of m_1 to complex values $-ik$ satisfying the relation $k^2 = (m_1^2 + c^2/m_1^2)$ [compare (14.159)].

When inserting (14.161) into (14.160), we have to make sure to use for s the $E_{\mathcal{PT}'}$-dependent exression $s(E_{\mathcal{PT}'})$ obtained from solving (14.157).

14.4.6 Hulthén Potential and General Rosen-Morse Potential

For a further application of the solution method, consider the path integral of a particle moving along the positive r-axis with the singular Hulthén potential

$$V_{\mathcal{H}}(r) = g\frac{1}{e^{r/a} - 1}, \tag{14.165}$$

where g and a are energy and length parameters. Note that this potential contains the Coulomb system in the limit $a \to \infty$ at $ag = e^2 =$ fixed.

The fixed-energy amplitude is controlled by the action

$$\mathcal{A}_{\mathcal{H}} = \int dt \left[\frac{M}{2}\dot{r}^2 - V_{\mathcal{H}}(r) + E_{\mathcal{H}}\right]. \tag{14.166}$$

The potential is singular at $r = 0$, and for $g < 0$, the Euclidean time-sliced amplitude does not exist due to path collapse. A regulating function which stabilizes the fluctuations is

$$f(r) = 4(1 - e^{-r/a})^2. \tag{14.167}$$

The time-transformed action is therefore

$$\mathcal{A}_{\mathcal{H}}^f = \int_0^\infty ds \left[\frac{M}{2}\frac{r'^2}{4(1-e^{-r/a})^2} - g\,4e^{-r/a}(1-e^{-r/a}) + E_{\mathcal{H}}\;4(1-e^{-r/a})^2\right]. \tag{14.168}$$

The coordinate transformation leading to a conventional kinetic energy in terms of the new variable x is found by solving the differential equation

$$\frac{dr}{dx} = h'(x), \tag{14.169}$$

with

$$h' = \sqrt{f} = 2(1 - e^{-r/a}). \tag{14.170}$$

The solution is

$$\frac{r}{a} = x + a\log[2\cosh(x/a)] = \log(e^{2x/a} + 1), \tag{14.171}$$

so that

$$h'(x) = 2\frac{e^{2x/a}}{e^{2x/a} + 1} = \frac{e^{x/a}}{\cosh(x/a)}. \tag{14.172}$$

The semi-axis $r \in (0, \infty)$ is mapped into the entire x-axis.

To find the effective potential we calculate the derivatives

$$\begin{aligned} h''(x) &= \frac{1}{a}\frac{1}{\cosh^2(x/a)} = \frac{1}{a}\frac{e^{x/a}}{\cosh(x/a)}[1 - \tanh(x/a)], \\ h'''(x) &= -\frac{2}{a^2}\frac{\sinh x}{\cosh^3 x} = -\frac{2}{a^2}\frac{e^{x/a}}{\cosh(x/a)}[\tanh(x/a) - \tan^2(x/a)], \end{aligned} \tag{14.173}$$

and obtain

$$\begin{aligned} \frac{h''}{h'} &= \frac{1}{a}\frac{e^{-x/a}}{\cosh(x/a)} = \frac{1}{a}[1 - \tanh(x/a)], \\ \frac{h'''}{h'} &= -\frac{2}{a^2}\frac{e^{-x/a}\sinh(x/a)}{\cosh^2(x/a)} = -\frac{2}{a^2}\tanh(x/a)[1 - \tanh(x/a)], \end{aligned} \tag{14.174}$$

so that the effective potential becomes

$$V_{\text{eff}} = \frac{\hbar^2}{8Ma^2}\left[2 - 2\tanh(x/a) - \frac{4}{\cosh^2(x/a)}\right]. \tag{14.175}$$

After adding this to the time-transformed potential, the DK-transformed action is found to be

$$\begin{aligned} \mathcal{A}_{\mathcal{H}}^{\text{DK}} = \int_0^S ds \left\{ \frac{M}{2}x'^2 - \left(g + E_{\mathcal{H}} - \frac{\hbar^2}{2Ma^2}\right)\frac{1}{\cosh^2(x/a)} \right. \\ \left. + \left(2E_{\mathcal{H}} + \frac{\hbar^2}{4Ma^2}\right)\tanh(x/a) + \left(2E_{\mathcal{H}} - \frac{\hbar^2}{4Ma^2}\right)\right\}. \end{aligned} \tag{14.176}$$

This is the action governing the fixed-energy amplitude of the general Rosen-Morse potential (14.156)

$$V_{\mathcal{RM}'}(x/a) = \frac{\hbar^2}{2Ma^2}\left[-\frac{s(s+1)}{\cosh^2(x/a)} + c\tanh(x/a)\right]. \tag{14.177}$$

Since this potential is smooth, there exists a time-sliced path integral of the Feynman type. The relation between the fixed-energy amplitudes is

$$(r_b|r_a)_{E_{\mathcal{H}}} = e^{(x_b+x_a)/2a}\left[\cosh(x_b/a)\cosh(x_a/a)\right]^{-1/2}(x_b|x_a)_{E_{\mathcal{RM}'}}, \tag{14.178}$$

with $r/a = \log(e^{2x/a}+1) \in (0,\infty),\ x \in (-\infty,\infty)$. The amplitude on the right-hand side is known from the last section; it is related to the amplitude for the motion of a mass point on the surface of a sphere in four dimensions, projected into a state of fixed azimuthal angular momenta m_1 and m_2. Only a simple rescaling of x/a to x is necessary to make the relation explicit.

From the energy spectrum (14.159), with (14.162), of the generalized Rosen-Morse potential and the above DK we derive the discrete spectrum of the Hulthén potential for $g<0$ [7]:

$$E_{\mathcal{H}n} = g\left(\frac{n_g^2-n^2}{2n_g n}\right)^2 = -\frac{\hbar^2}{2Ma^2}\frac{1}{4n^2}\left(n_g^2-n^2\right)^2, \qquad 1\le n<n_g, \tag{14.179}$$

where $n_g^2 = -2Mga^2/\hbar^2$.

In the literature, a solution of the time-sliced path integral with the action (14.166) has been attempted using a regulating function [11]

$$f = a^2(e^{r/a}-1). \tag{14.180}$$

This implies going to the new variables

$$\frac{r}{a} = -2\log\cos(\theta/2), \tag{14.181}$$

so that

$$f = a^2\tan^2(\theta/2) = a^2\left[\frac{1}{\cos^2(\theta/2)}-1\right]. \tag{14.182}$$

Note that this does not lead to a solution of the time-sliced path integral, since the transformed potential is still singular. Indeed, with $h' = a\tan(\theta/2),\quad h'' = a/2\cos^2(\theta/2),\quad h''' = a\sin(\theta/2)/2\cos^3(\theta/2)$, we would find the effective potential

$$V_{\text{eff}}(\theta) = \frac{\hbar^2}{8Ma^2}\frac{1}{\sin^2\theta}(1+2\cos\theta) = \frac{\hbar^2}{32Ma^2}\left[\frac{3}{\sin^2(\theta/2)} - \frac{1}{\cos^2(\theta/2)}\right], \tag{14.183}$$

and a transformed action

$$\tilde{\mathcal{A}}_{\mathcal{H}}^{\text{DK}} = \int_0^S (ds/a^2)\left\{\frac{Ma^4}{2}\theta'^2 - g + E_{\mathcal{H}}\left[\frac{1}{\cos^2(\theta/2)}-1\right] + V_{\text{eff}}(\theta)\right\}, \tag{14.184}$$

which is of the general Pöschl-Teller type (14.145). Due to the presence of the $1/\cos^2(\theta/2)$-term, the Euclidean time evolution amplitude cannot be time-sliced. Only by starting from the particle near the surface of a sphere with the particular Bessel function regularization of (8.208), can a well-defined time-sliced amplitude be written down whose action looks like (14.184) in the continuum limit. It would be impossible, however, to invent this regularization when starting from the continuum action (14.184).

14.4.7 Extended Hulthén Potential and General Rosen-Morse Potential

The alert reader will have noticed that the regulating function (14.165) overkills the ga/r singularity of the Hulthén potential (14.165). In fact, we may add to the potential a term

$$\Delta V_{\mathcal{H}} = \frac{g'}{(e^{r/a}-1)^2} \tag{14.185}$$

without loosing the stability of the path integral. In the limit $a \to \infty$, the extended potential contains the radial Coulomb system plus a centrifugal barrier, if we set $ga = -e^2 = \text{const}$ and $g'a^2 = \hbar^2 l(l+1)/2M$. The potential (14.185) adds to the time-transformed action (14.168) a term

$$\Delta\mathcal{A}^f_{\mathcal{H}} = -\int_0^S ds\, g' 4e^{-2r/a}, \tag{14.186}$$

which winds up in the final DK-transformed action as

$$\Delta\mathcal{A}^{\rm DK}_{\mathcal{H}} = -\int_0^S ds\, g' \left[2 - 2\tanh(x/a) - \frac{1}{\cosh^2(x/a)}\right]. \tag{14.187}$$

Therefore, the extended Hulthén potential is again DK-equivalent to the general Rosen-Morse potential with the same relation (14.178) between the amplitudes, but with different relations between the constants.

The discrete energy spectrum of the extended Hulthén potential is

$$E_{\mathcal{H}'\,n} = -\frac{\hbar^2}{2Ma^2}\left[\frac{n(n-1)+n_g^2}{2(n-s_2)} - n + \frac{1}{2}\right]^2, \qquad 1 \le n < \bar{n}, \tag{14.188}$$

where $n_{g'}^2 \equiv 2Mg'a^2/\hbar^2$, and $s_2 \equiv -\frac{1}{2}\left(1-\sqrt{1+4n_{g'}^2}\right)$ solves $s_2(s_2+1) = n_{g'}^2$. For $g' = 0$, this reduces to (14.179).

14.5 *D*-Dimensional Systems

Let us now perform the path-dependent time transformation in D dimensions. The fixed-energy amplitude is given by the integral

$$(\mathbf{x}_b|\mathbf{x}_a)_E = \int_0^\infty dS \langle \mathbf{x}_b|\hat{\mathcal{U}}_E(S)|\mathbf{x}_a\rangle, \tag{14.189}$$

with the pseudotime evolution amplitude

$$\langle \mathbf{x}_b|\hat{\mathcal{U}}_E(S)|\mathbf{x}_a\rangle = f_r(\mathbf{x}_b)f_l(\mathbf{x}_a)\langle \mathbf{x}_b| \exp\left[-\frac{i}{\hbar}S f_l(\mathbf{x})(\hat{H}-E)f_r(\mathbf{x})\right]|\mathbf{x}_a\rangle. \tag{14.190}$$

It has the time-sliced path integral

$$\langle \mathbf{x}_b|\hat{\mathcal{U}}_E(S)|\mathbf{x}_a\rangle \approx \frac{f_r(\mathbf{x}_b)f_l(\mathbf{x}_a)}{\sqrt{2\pi i\epsilon_s\hbar f_l(\mathbf{x}_b)f_r(\mathbf{x}_a)/M}^D}\prod_{n=1}^{N}\left[\int \frac{dx_n}{\sqrt{2\pi i\epsilon_s\hbar f_n/M}}\right]\exp\left\{\frac{i}{\hbar}\mathcal{A}^N\right\}, \tag{14.191}$$

with the action

$$\mathcal{A}^N = \sum_{n=1}^{N+1}\left\{\frac{M}{2\epsilon_s}\frac{(\Delta\mathbf{x}_n)^2}{f_l(\mathbf{x}_n)f_r(\mathbf{x}_{n-1})} + \epsilon_s[E-V(\mathbf{x}_n)]f_l(\mathbf{x}_n)f_r(\mathbf{x}_{n-1})\right\}, \tag{14.192}$$

where the integration measure contains the abbreviation $f_n \equiv f(r_n) = f_l(\mathbf{x}_n)f_r(\mathbf{x}_n)$. The time-transformed measure of path integration reads

$$\frac{f_r(\mathbf{x}_b)f_l(\mathbf{x}_a)}{\sqrt{2\pi i\epsilon_s\hbar f_l(\mathbf{x}_b)f_r(\mathbf{x}_a)/M}^D}\prod_{n=1}^{N}\int\frac{d^Dx_n}{\sqrt{2\pi i\epsilon_s\hbar f_n/M}^D}. \tag{14.193}$$

By shifting the product index and the subscripts of f_n by one unit, and by compensating for this with a prefactor, the integration measure in (14.27) acquires the postpoint form

$$\frac{f_r(\mathbf{x}_b)f_l(\mathbf{x}_a)}{\sqrt{2\pi i\epsilon_s\hbar f_l(\mathbf{x}_b)f_r(\mathbf{x}_a)/M}^D}\sqrt{\frac{f(\mathbf{x}_b)}{f(\mathbf{x}_a)}}^D\prod_{n=2}^{N+1}\int\frac{d^D\Delta x_n}{\sqrt{2\pi i\epsilon_s\hbar f_n/M}^D}. \tag{14.194}$$

The integrals over each coordinate difference $\Delta\mathbf{x}_n = \mathbf{x}_n - \mathbf{x}_{n-1}$ are done at fixed postpoint positions $\mathbf{x}_n$.

To simplify the subsequent discussion, it is preferable to work only with the postpoint regularization in which $f_l(\mathbf{x}) = f(\mathbf{x})$ and $f_r(\mathbf{x}) \equiv 1$. Then the measure becomes simply

$$\frac{f(\mathbf{x}_a)}{\sqrt{2\pi i\epsilon_s f(\mathbf{x}_a)\hbar/M}^D}\prod_{n=2}^{N+1}\int\frac{d^D\Delta x_n}{\sqrt{2\pi i\epsilon_s\hbar f_n/M}^D}. \tag{14.195}$$

We now introduce the coordinate transformation. In D dimensions it is given by

$$x^i = h^i(q). \tag{14.196}$$

The differential mapping may be written as in Chapter 10 as

$$dx^i = \partial_\mu h^i(q) = e^i{}_\mu(q)dq^\mu. \tag{14.197}$$

The transformation of a single time slice in the path integral can be done following the discussion in Sections 10.3 and 10.4. This leads to the path integral

$$(\mathbf{x}_b|\mathbf{x}_a)_E \approx \frac{f(q_a)}{\sqrt{2\pi i\epsilon_s f(q_a)\hbar/M}^D}\int_0^\infty dS \prod_{n=2}^{N+1}\left[\int \frac{d^D\Delta q_n g^{1/2}(q_n)}{\sqrt{2\pi i\epsilon_s \hbar f_n/M}^D}\right] e^{i\mathcal{A}_{\rm tot}/\hbar}, \tag{14.198}$$

with the total time-sliced action

$$\mathcal{A}_{\rm tot} = \sum_{n=1}^{N+1} \mathcal{A}^\epsilon_{\rm tot}. \tag{14.199}$$

Each slice contains three terms

$$\mathcal{A}^\epsilon_{\rm tot} = \mathcal{A}^\epsilon + \mathcal{A}^\epsilon_J + \mathcal{A}^\epsilon_{\rm pot}. \tag{14.200}$$

In the postpoint form, the first two terms were given in (13.163) and (13.164). They are equal to

$$\mathcal{A}^\epsilon + \mathcal{A}^\epsilon_J = \frac{M}{2\epsilon f} g_{\mu\nu}(q)\Delta q^\mu \Delta q^\nu - i\frac{\hbar}{2}\Gamma_\mu{}^\mu{}_\nu \Delta q^\nu - \epsilon_s f \frac{\hbar^2}{8M}(\Gamma_\mu{}^\mu{}_\nu)^2. \tag{14.201}$$

The third term contains the effect of a potential and a vector potential as derived in (10.183). After the DK transformation, it reads

$$\mathcal{A}^\epsilon_{\rm pot} = A_\mu \Delta q^\mu - i\epsilon_s f\frac{\hbar}{2M}(A_\nu \Gamma_\mu{}^{\mu\nu} + D_\mu A^\mu) - \epsilon_s f V(q). \tag{14.202}$$

14.6 Path Integral of the Dionium Atom

We now apply the generalized D-dimensional Duru-Kleinert transformation to the path integral of a dionium atom in three dimensions. This is a system of two particles with both electric and magnetic charges (e_1, g_1) and (e_2, g_2) [12]. Its Lagrangian for the relative motion reads

$$L = \frac{M}{2}\dot{\mathbf{x}}^2 + \mathbf{A}(\mathbf{x})\dot{\mathbf{x}} - V(\mathbf{x}), \tag{14.203}$$

where $\mathbf{x}$ is the distance vector pointing from the first to the second article, M the reduced mass, $V(\mathbf{x})$ a Coulomb potential

$$V(\mathbf{x}) = -\frac{e^2}{r}, \tag{14.204}$$

and $\mathbf{A}(\mathbf{x})$ the vector potential

$$\mathbf{A}(\mathbf{x}) = \hbar q \frac{\hat{\mathbf{z}} \times \mathbf{x}}{r}\left(\frac{1}{r-z} - \frac{1}{r+z}\right) = \hbar q \frac{(x\hat{\mathbf{y}} - y\hat{\mathbf{x}})z}{r(x^2+y^2)}. \tag{14.205}$$

The coupling constants are $q \equiv -(e_1 g_2 - e_2 g_1)/\hbar c$ and $e^2 \equiv -(e_1 e_2 + g_1 g_2)$. The vector potential (14.205) is a generalization of the magnetic monopole interaction (8.300) with an electric charge. If we take the coupling as and $e^2 \equiv -e_1 e_2 - g_1 g_2$ in (14.205) we allow for the two particles to carry both electric and magnetic charges of the two particles, if we take for $V(\mathbf{x})$ the potential

$$V(\mathbf{x}) = -\frac{e^2}{r}. \tag{14.206}$$

The hydrogen atom is a special case of the dionium atom with $e_1 = -e_2 = e$ and $q = 0, l_0 = 0$. An electron around a pure magnetic monopole has $e_1 = e$, $g_2 = g$, $e_2 = g_1 = 0$.

In the vector potential (14.205) we have made use of the gauge freedom $\mathbf{A} \to \mathbf{A}(\mathbf{x}) + \nabla\Lambda(\mathbf{x})$ to enforce the transverse gauge $\nabla\mathbf{A}(\mathbf{x}) = 0$. In addition, we have taken advantage of the extra *monopole gauge invariance* which allows us to choose the shape of the Dirac string that imports the magnetic flux to the monopoles. The field $\mathbf{A}(\mathbf{x})$ in (14.205) has two strings of equal strength importing the flux, one along the positive x^3-axis from minus infinity to the origin, the other along the negative x^3-axis from plus infinity to the origin. It is the average of the vector potentials (10A.59) and (10A.60).

For the sake of generality, we shall assume the potential $V(\mathbf{x})$ to contain an extra $1/r^2$-potential:

$$V(\mathbf{x}) = -\frac{e^2}{r} + \frac{\hbar^2 l_0^2}{2Mr^2}. \tag{14.207}$$

The extra potential is parametrized as a centrifugal barrier with an effective angular momentum $\hbar l_0$.

At the formal level, i.e., without worrying about path collapse and time slicing corrections, the amplitude has been derived in Ref. [13]. Here we reproduce the derivation and demonstrate, in addition, that the time slicing produces no corrections.

14.6.1 Formal Solution

We extend the action of the type (14.11) by a dummy fourth coordinate as in the Coulomb system and go over to $\vec{u}$-coordinates depending on the radial coordinate $u = \sqrt{r}$ and the Euler angles θ, φ, γ as given in Eq. (13.102). Then the action reads

$$\mathcal{A} = \int dt \left\{ \frac{M}{2} 4u^2 \dot{u}^2 + \frac{M}{2} u^4 \left[\dot{\theta}^2 + \dot{\varphi}^2 + \dot{\gamma}^2 + 2\left(\dot{\gamma} + \frac{\hbar q}{Mu^4} \right) \dot{\varphi} \cos\theta \right] - \frac{e^2}{u^2} - \frac{\hbar^2 l_0^2}{2Mu^4} + E \right\}. \tag{14.208}$$

By performing the Duru-Kleinert time reparametrization $dt = ds\, r(s)$ and changing the mass to $\mu = 4M$, the action takes the form

$$\mathcal{A}^{\rm DK} = \int_0^S ds\, \frac{\mu}{2} \left\{ u'^2 + \frac{u^2}{4} \left[\theta'^2 + \varphi'^2 + \gamma'^2 + 2\left(\gamma' + \frac{4\hbar q}{\mu u^2} \right) \varphi' \cos\theta \right] - \frac{4\hbar^2 l_0^2}{2\mu u^2} + E u^2 \right\}. \tag{14.209}$$

This can be rewritten in a canonical form

$$\mathcal{A} = \int_0^S ds (p_u u' + p_\theta \theta + p_\varphi \varphi' + p_\gamma \gamma' - H), \tag{14.210}$$

with the Hamiltonian

$$\begin{aligned} H &= \frac{1}{2\mu} \left\{ p_u^2 + \frac{4}{u^2} \left[p_\theta^2 + \frac{1}{\sin^2\theta} \left(p_\varphi^2 + (p_\gamma + \hbar q)^2 - 2(p_\gamma + \hbar q) p_\varphi \cos\theta \right) \right] \right\} \\ &+ \frac{4}{2\mu u^2} \left[-2\hbar q p_\gamma + \hbar^2 (l_0^2 - q^2) \right]. \end{aligned} \tag{14.211}$$

In the canonical path integral, the momenta are dummy integration variables so that we can replace $p_\gamma + \hbar q$ by p_γ. Then the action becomes

$$\mathcal{A} = \int_0^S ds [p_u u' + p_\theta \theta + p_\varphi \varphi' + (p_\gamma - \hbar q)\gamma' - \bar{H}], \tag{14.212}$$

with the Hamiltonian

$$\begin{aligned} \bar{H} &= \frac{1}{2\mu} \left\{ p_u^2 + \frac{4}{u^2} \left[p_\theta^2 + \frac{1}{\sin^2\theta} \left(p_\varphi^2 + p_\gamma^2 - 2 p_\gamma p_\varphi \cos\theta \right) \right] \right\} \\ &+ \frac{4}{2\mu u^2} \left[-2\hbar q (p_\gamma - \hbar q) + \hbar^2 (l_0^2 - q^2) \right]. \end{aligned} \tag{14.213}$$

This differs from the pure Coulomb case in three ways:

First, the Hamiltonian has an extra centrifugal barrier proportional to the charge parameter $4q$:

$$V(r) = \frac{-8\hbar q (p_\gamma - \hbar q)}{2\mu u^2}. \tag{14.214}$$

Second, there is an extra centrifugal barrier

$$V(r) = \frac{\hbar^2 l_{\rm extra}^2}{2\mu u^2}, \tag{14.215}$$

whose effective quantum number of angular momentum is given by

$$l_{\rm extra}^2 \equiv 4(l_0^2 - q^2). \tag{14.216}$$

Third, the action (14.212) contains an additional term

$$\Delta\mathcal{A} = -\hbar q \int_0^S ds\gamma'. \tag{14.217}$$

Fortunately, this is a pure surface term

$$\Delta\mathcal{A} = -\hbar q(\gamma_b - \gamma_a). \tag{14.218}$$

In the case $q^2 = l_0^2$, the extra centrifugal barrier vanishes, making it straightforward to write down the fixed-energy amplitude $(\mathbf{x}_b|\mathbf{x}_a)_E$ of the system. It is given by a simple modification of the relation (13.127) that expresses the fixed-energy amplitude of the Coulomb system $(\mathbf{x}_b|\mathbf{x}_a)_E$ in terms of the four-dimensional harmonic oscillator amplitude $(\vec{u}_b S|\vec{u}_a 0)$. Due to (14.217), the modification consists of a simple extra phase factor $e^{-iq(\gamma_b-\gamma_a)}$ in the integral over γ_a so that

$$(\mathbf{x}_b|\mathbf{x}_a)_E = \int_0^\infty dS e^{ie^2S/\hbar} \frac{1}{16} \int_0^{4\pi} d\gamma_a e^{-iq(\gamma_a-\gamma_b)} (\vec{u}_b S|\vec{u}_a 0). \tag{14.219}$$

The integral over γ_a forces the momentum p_γ in the canonical action (14.212) to take the value $\hbar q$. This eliminates the term proportional to $p_\gamma - \hbar$ in (14.213).

In the general case $l_0 \neq q$, the amplitude becomes

$$(\mathbf{x}_b|\mathbf{x}_a)_E = \int_0^\infty dS e^{ie^2S/\hbar} \frac{1}{16} \int_0^{4\pi} d\gamma_a e^{-iq(\gamma_a-\gamma_b)} (\vec{u}_b S|\vec{u}_a 0)_{l_{\rm extra}}, \tag{14.220}$$

where the subscript $l_{\rm extra}$ indicates the presence of the extra centrifugal barrier potential in the harmonic oscillator amplitude. This amplitude was given for any dimension D in Eqs. (8.133) with (8.144). In the present case of $D = 4$, it has the partial-wave expansion [compare (8.162)]

$$(\vec{u}_b S|\vec{u}_a 0)_{l_{\rm extra}} = \frac{1}{(u_b u_a)^{3/2}} \sum_{l_\mathcal{O}=0}^{\infty} (u_b S|u_a 0)_{\tilde{l}_\mathcal{O}} \frac{l_\mathcal{O}+1}{2\pi^2} \tag{14.221}$$
$$\times \sum_{m_1,m_2=-l_\mathcal{O}/2}^{l_\mathcal{O}/2} d^{l_\mathcal{O}/2}_{m_1 m_2}(\theta_b) d^{l_\mathcal{O}/2}_{m_1 m_2}(\theta_a) e^{im_1(\varphi_b-\varphi_a)+im_2(\gamma_b-\gamma_a)},$$

with the radial amplitude

$$(u_b S|u_a 0)_{\tilde{l}_\mathcal{O}} = \frac{M_\mathcal{O}}{i\hbar} \frac{\omega\sqrt{u_b u_a}}{\sin\omega S} e^{i(M_\mathcal{O}\omega/2\hbar)(u_b^2+u_a^2)\cot\omega S} I_{\tilde{l}_\mathcal{O}+1}\left(\frac{M_\mathcal{O}\omega u_b u_a}{i\hbar\sin\omega S}\right). \tag{14.222}$$

This differs from the pure oscillator amplitude [compare (8.142) for $D = 4$] by having the index $l_\mathcal{O}+1$ of the Bessel function replaced by the square root of the "shifted square" as in (8.146):

$$\tilde{l}_\mathcal{O} + 1 \equiv \sqrt{(l_\mathcal{O}+1)^2 + l_{\rm extra}^2} = \sqrt{(l_\mathcal{O}+1)^2 + 4(l_0^2 - q^2)} = 2\sqrt{(j_\mathcal{D}+1/2)^2 + l_0^2 - q^2}. \tag{14.223}$$

The expansion (14.221) is inserted into (14.220) with the variables u_b, u_a replaced by $\sqrt{r_b}, \sqrt{r_a}$. Just as in the Coulomb case in (14.110) and (14.111), the integral $\int_0^{4\pi} d\gamma_a e^{-iq(\gamma_b-\gamma_a)}$ over the sum of angular wave functions

$$\frac{l_{\mathcal{O}}+1}{2\pi^2}\sum_{m_1,m_2=-l/2}^{l_{\mathcal{O}}/2} d^{l_{\mathcal{O}}/2}_{m_1m_2}(\theta_b)d^{l_{\mathcal{O}}/2}_{m_1m_2}(\theta_a)e^{im_1(\varphi_b-\varphi_a)+im_2(\gamma_b-\gamma_a)} \tag{14.224}$$

can immediately be done, resulting in

$$8\sum_{m}^{l_{\mathcal{O}}/2} Y^{l_{\mathcal{O}}/2}_{m,q}(\theta_b,\varphi_b)Y^{l_{\mathcal{O}}/2*}_{m,q}(\theta_a,\varphi_a), \tag{14.225}$$

where $Y^{l_{\mathcal{O}}/2}_{m,q}(\theta,\varphi)$ are the monopole spherical harmonics (8.278). They coincide with the wave functions of a spinning symmetric top which possesses a spin q along the body axis. Physically, this spin is caused by the field's momentum density $\boldsymbol{\pi} = (\mathbf{E}\times\mathbf{B})/4\pi c$ encircling the radial distance vector $\mathbf{x}$. The *Poynting* vector yielding the energy density is $\mathbf{S} = \mathbf{E}\times\mathbf{B}/4\pi$. If a magnetically charged particle lies at the origin and electrically charged particle orbits around it at $\mathbf{x}$, the total angular momentum carried by the fields is [14]

$$\mathbf{J} = \int d^3x'\,\mathbf{x}'\times\boldsymbol{\pi}(\mathbf{x}') = \frac{1}{4\pi c}\int d^3x'\,\mathbf{x}'\times\left[\frac{g\,\mathbf{x}'}{|\mathbf{x}'|^3}\times\frac{e(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^3}\right] = \frac{eg}{c}\hat{\mathbf{x}}. \tag{14.226}$$

The quantization of the angular momentum

$$\frac{eg}{c} = n\frac{\hbar}{2}, \qquad n = \text{integer} \tag{14.227}$$

is Dirac's famous charge quantization condition [16] [see also Eq. (8.304)].

Thus we arrive at the fixed-energy amplitude of the dionium atom, labeled by the subscript $\mathcal{D}$,

$$(\mathbf{x_b}|\mathbf{x_a})_{E_{\mathcal{D}}} = \frac{1}{r_br_a}\sum_{j_{\mathcal{D}}}(r_b|r_a)_{E_{\mathcal{D}},j_{\mathcal{D}}}\sum_{m=-j_{\mathcal{D}}}^{j_{\mathcal{D}}} Y^{j_{\mathcal{D}}}_{m,q}(\theta_b,\varphi_b)Y^{j_{\mathcal{D}}\,*}_{m,q}(\theta_a,\varphi_a), \tag{14.228}$$

where the sum over $j_{\mathcal{D}} = l_{\mathcal{O}}/2$ runs over integer or half-integer values depending on q, and with the radial amplitude given by the pseudotime integral over the radial oscillator amplitude of mass $M_{\mathcal{O}} = 4M$ and frequency $\omega = \sqrt{-E/2M_{\mathcal{C}}}$ [recall (13.123)]:

$$(r_b|r_a)_{E_{\mathcal{D}},j_{\mathcal{D}}} = \frac{1}{2}\int_0^\infty dS e^{ie^2S/\hbar}\frac{M_{\mathcal{O}}\,\omega\sqrt{r_br_a}}{i\hbar\sin\omega S}I_{\tilde{l}_{\mathcal{O}}+1}\left(\frac{M_{\mathcal{O}}\,\omega\sqrt{r_br_a}}{i\hbar\sin\omega S}\right) \times\exp\left[\frac{iM_{\mathcal{O}}\,\omega}{2\hbar}(r_b+r_a)\cot\omega S\right], \tag{14.229}$$

where $\kappa = \sqrt{-2ME_{\mathcal{D}}/\hbar^2}$ and $\nu = \sqrt{-e^4M/2\hbar^2E_{\mathcal{D}}}$ as in (13.39) and (13.40). Note that the dionium atom can be a fermion, even if the constituent particles are both bosons (or both fermions).

After the variable changes $e^2/\hbar = 2\omega\nu$, $\omega S = -iy$, we do the S-integral as in (9.29) and find for $r_b > r_a$ the radial amplitude of the dionium atom

$$(r_b|r_a)_{E_\mathcal{D},j_\mathcal{D}} = -i\frac{M}{\hbar\kappa}\frac{\Gamma(-\nu+\tilde{j}_\mathcal{D}+1)}{(2\tilde{j}_\mathcal{D}+1)!}W_{\nu,\tilde{j}_\mathcal{D}+1/2}(2\kappa r_b)M_{\nu,\tilde{j}_\mathcal{D}+1/2}(2\kappa r_a), \quad (14.230)$$

where $\tilde{j}_\mathcal{D} = \tilde{l}_\mathcal{O}/2 = \sqrt{(j_\mathcal{D}+\frac{1}{2})^2 + l_0^2 - q^2} - \frac{1}{2}$. For $q = 0$ and $l_0 = 0$, this reduces properly to the $D_\mathcal{C}$-dimensional Coulomb amplitude (14.112).

The energy eigenvalues are obtained from the poles of the Gamma function at

$$\nu = \nu_{n_r} \equiv \tilde{j}_\mathcal{D} + n_r + 1, \quad n_r = 0, 1, 2, 3, \dots\ , \quad (14.231)$$

which yield

$$E_n = -M\hbar^2 e^4 \frac{1}{2\left[n_r + \frac{1}{2} + \sqrt{(j_\mathcal{D}+\frac{1}{2})^2 + l_0^2 - q^2}\right]^2}. \quad (14.232)$$

From the residues at the poles and the discontinuity across the cut at $E_\mathcal{D} > 0$ in (14.230), we can extract the bound and continuum radial wave functions by the same method as in Section 13.8 from Eqs. (13.217)–(13.229).

14.6.2 Absence of Time Slicing Corrections

Let us now show that the above formal manipulations receive no correction in a proper time-sliced treatment [15]. Due to the presence of centrifugal and angular barriers, a path collapse can be avoided only after an appropriate regularization of both singularities. This is achieved by the path-dependent time transformation $dt = ds\, f(\mathbf{x}(s))$ with the postpoint regulating functions

$$f_l(\mathbf{x}) = f(\mathbf{x}) = r^2 \sin^2\theta, \quad f_r(\mathbf{x}) \equiv 1. \quad (14.233)$$

After the extension of the path integral by an extra dummy dimension x^4, the time-sliced time-transformed fixed-energy amplitude to be studied is [see (14.189)–(14.195)]

$$(\mathbf{x}_b|\mathbf{x}_a)_E \approx \int dx_a^4 \frac{1}{r_a^2 \sin^2\theta_a}\int_0^\infty dS \frac{1}{(2\pi i\hbar\epsilon_s/M)^2} \times \prod_{n=2}^{N+1}\left[\int \frac{d^4\Delta x_n}{(2\pi i\hbar\epsilon_s/M)^2 r_n^4 \sin^4\theta_n}\right] e^{i\mathcal{A}^N/\hbar}, \quad (14.234)$$

with the sliced postpoint action

$$\mathcal{A}^N = \sum_{n=1}^{N+1}\left\{\frac{M}{2\epsilon_s}\frac{(\Delta x_n^i)^2}{r_n^2\sin^2\theta_n} - \epsilon_s r_n^2 \sin^2\theta_n\left[V(\mathbf{x}_n) - E\right] + A_i(\mathbf{x}_n)\Delta x^i - \epsilon_s\frac{\hbar}{2M}A_{i,i}(\mathbf{x}_n)\right\}. \quad (14.235)$$

On this action, we now perform the coordinate transformation in two steps. First we go through the nonholonomic Kustaanheimo-Stiefel transformation as in Section 13.4 and express the four-dimensional $\vec{u}$-space in terms of $r = |\mathbf{x}|$ and the Euler angles θ, φ, γ. Explicitly,

$$\begin{aligned} x^1 &= r\sin\theta\cos\varphi, \\ x^2 &= r\sin\theta\sin\varphi, \\ x^3 &= r\cos\theta, \\ dx^4 &= r\cos\theta d\varphi + r d\gamma. \end{aligned} \quad (14.236)$$

Only the last equation is nonholonomic. If $q^\mu = 1,2,3,4$ denotes the components r,β,φ,γ, the transformation matrix reads

$$e^i{}_\mu = \frac{\partial x^i}{\partial q^\mu} = \begin{pmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi & 0 \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi & 0 \\ \cos\theta & -r\sin\theta & 0 & 0 \\ 0 & 0 & r\cos\theta & r \end{pmatrix}.$$

It has the metric

$$g_{\mu\nu} = e^i{}_\mu e_{i\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2 & r^2\cos\theta \\ 0 & 0 & r^2\cos\theta & r^2 \end{pmatrix}, \tag{14.237}$$

with an inverse

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/r^2 & 0 & 0 \\ 0 & 0 & 1/r^2\sin^2\theta & -\cos\theta/r^2\sin^2\theta \\ 0 & 0 & -\cos\theta/r^2\sin^2\theta & 1/r^2\sin^2\theta \end{pmatrix}. \tag{14.238}$$

The regulating function $f(\mathbf{x}) = r^2\sin^2\theta$ removes the singularities in $g^{\mu\nu}$ and thus in the free part of the Hamiltonian $(1/2M)g^{\mu\nu}p_\mu p_\nu$; there is no more danger of path collapse in the Euclidean amplitude.

In a second step we go to new coordinates

$$r = e^\xi, \quad \sin\theta = 1/\cosh\beta, \quad \cos\theta = -\tanh\beta, \tag{14.239}$$

as in the treatment of the angular barriers for $D = 3$ and $D = 4$ in Section 14.4. With $q^\mu = 1,2,3,4$ denoting the coordinates ξ,β,φ,γ, respectively, the combined transformation matrix reads

$$e^i{}_\mu = \begin{pmatrix} e^\xi\cosh^{-1}\beta\cos\varphi & -e^\xi\frac{\sinh\beta}{\cosh^2\beta}\cos\varphi & -e^\xi\cosh^{-1}\beta\sin\varphi & 0 \\ e^\xi\cosh^{-1}\beta\sin\varphi & -e^\xi\frac{\sinh\beta}{\cosh^2\beta}\sin\varphi & e^\xi\cosh^{-1}\beta\cos\varphi & 0 \\ -e^\xi\tanh\beta & -e^\xi\cosh^{-2}\beta & 0 & 0 \\ 0 & 0 & -e^\xi\tanh\beta & e^\xi \end{pmatrix}, \tag{14.240}$$

with the metric

$$g_{\mu\nu} = \begin{pmatrix} e^{2\xi} & 0 & 0 & 0 \\ 0 & e^{2\xi}\cosh^{-2}\beta & 0 & 0 \\ 0 & 0 & e^{2\xi} & -e^{2\xi}\tanh\beta \\ 0 & 0 & -e^{2\xi}\tanh\beta & e^{2\xi} \end{pmatrix}, \tag{14.241}$$

and the determinant

$$g = e^{8\xi}/\cosh^4\beta. \tag{14.242}$$

The inverse metric is completely regular:

$$g^{\mu\nu} = \begin{pmatrix} e^{-2\xi} & 0 & 0 & 0 \\ 0 & e^{-2\xi}\cosh^2\beta & 0 & 0 \\ 0 & 0 & e^{-2\xi}\cosh^2\beta & e^{-2\xi}\sinh\beta\cosh\beta \\ 0 & 0 & e^{-2\xi}\sinh\beta\cosh\beta & e^{-2\xi}\cosh^2\beta \end{pmatrix}. \tag{14.243}$$

We now calculate the transformed actions (14.201) and (14.202). The relevant quantities which could contain time slicing corrections are $D_\mu A^\mu$ and $\Gamma_\mu{}^{\mu\nu}$. The former quantity, being equal to

$\partial_i A_i$, vanishes in the transverse gauge under consideration. The calculation of $\Gamma_\mu{}^{\mu\nu}$ is somewhat tedious (see Appendix 14B) but yields a surprisingly simple result:

$$\Gamma_\mu{}^\mu{}_\nu = (-1, 0, 0, 0). \tag{14.244}$$

Because of this simplicity, the transformed sliced action is easily written down. It is split into two parts,

$$\mathcal{A}^\epsilon_{\rm tot} = \mathcal{A}^\epsilon_{\varphi\beta} + \mathcal{A}^\epsilon_{\varphi\gamma}, \tag{14.245}$$

one containing only the coordinates ξ, β,

$$\mathcal{A}^\epsilon_{\xi\beta} = \frac{M}{2\epsilon_s}[(\Delta\xi)^2_n \cosh^2\beta_n + (\Delta\beta_n)^2] + \frac{i\hbar}{2}\Delta\xi_n \\ - \epsilon_s \left[-\frac{e^2 e^{\xi_n}}{\cosh^2\beta_n} + \frac{\hbar^2(l^2_{\rm extra} + 1/4)}{2M\cosh^2\beta_n} - \frac{E e^{2\xi_n}}{\cosh^2\beta_n} \right], \tag{14.246}$$

the other dealing predominantly with φ, γ,

$$\mathcal{A}^\epsilon_{\varphi\gamma} = \frac{M\cosh^2\beta_n}{2\epsilon_s}[(\Delta\varphi_n)^2 + (\Delta\gamma_n)^2 - 2\Delta\gamma_n\Delta\varphi_n \tanh\beta_n] \\ + \hbar q \tanh\beta_n \Delta\varphi_n - \epsilon_s \frac{\hbar^2 q^2}{2M\cosh^2\beta_n}. \tag{14.247}$$

Hence the fixed-energy amplitude becomes

$$(\mathbf{x}_b|\mathbf{x}_a)_E \approx \int_0^\infty dS \int_0^{4\pi} d\gamma_a \frac{1}{2\pi i\hbar\epsilon_s r_a^2/M\cosh\beta_a} \prod_{n=1}^N \left[\int \frac{dr_n d\beta_n}{2\pi i\hbar\epsilon_s/M\cosh^2\beta_{n+1}} \right] \\ \times \exp\left(\frac{i}{\hbar}\sum_{n=1}^{N+1} \mathcal{A}^\epsilon_{\xi\beta} \right) (\varphi_b\,\gamma_b\, S|\varphi_a\,\gamma_a\, 0)_{[\beta]}. \tag{14.248}$$

The last factor is a pseudotime evolution amplitude in the angles φ, γ which is still a functional of $\beta(t)$, as indicated by the subscript $[\beta]$,

$$(\varphi_b\,\gamma_b\, S|\varphi_a\,\gamma_a\, 0)_{[\beta]} \approx \sum_{\varphi_{N+1}=\varphi_b+2\pi l_b^\varphi} \sum_{\gamma_{N+1}=\gamma_b+4\pi l_b^\gamma} \\ \times \frac{1}{2\pi i\hbar\epsilon_s/M\cosh\beta_b} \prod_{n=1}^N \left[\int \frac{d\varphi_n d\gamma_n}{2\pi i\hbar\epsilon_s/M\cosh\beta_n} \right] \exp\left(\frac{i}{\hbar}\sum_{n=1}^{N+1} \mathcal{A}^\epsilon_{\varphi\gamma} \right). \tag{14.249}$$

The sums over l_b^φ, l_b^γ account for the cyclic properties of the angles φ and γ with the periods 2π and 4π, respectively, at a fixed coordinate $\mathbf{x}_b$ (as in the examples in Section 6.1).

We now introduce auxiliary momentum variables p_n^φ, p_n^γ and go over to the canonical form of the amplitude (14.249):

$$(\varphi_b\gamma_b\, S|\varphi_a\gamma_a\, 0)_{[\beta]} \approx \prod_{n=1}^N \left[\int d\varphi_n \right] \prod_{n=1}^{N+1} \left[\int \frac{dp_n^\varphi}{2\pi\hbar} \right] \prod_{n=1}^N \left[\int d\gamma_n \right] \prod_{n=1}^{N+1} \left[\int \frac{dp_n^\gamma}{2\pi\hbar} \right] \\ \times \exp\left[\frac{i}{\hbar}\sum_{n=1}^{N+1} \left(p_n^\varphi \Delta\varphi_n + p_n^\gamma \Delta\gamma_n \right.\right. \\ \left.\left. - \frac{\epsilon_s}{2M}[(p_n^\varphi)^2 + (p_n^\gamma + \hbar q)^2 + 2p_n^\varphi(p_n^\gamma + \hbar q)\tanh\beta_n] + \epsilon_s \frac{p_n^\gamma \hbar q}{M\cosh^2\beta_n} \right) \right]. \tag{14.250}$$

The momenta p_n^γ are dummy integration variables and can be replaced by $p_n^\gamma - \hbar q$. The $d\varphi_n, d\gamma_n$-integrals run over the full extended zone schemes $\varphi_n, \gamma_n \in (-\infty, \infty)$ and enforce the equality of all p_n^γ. At the end, only the integrals over a common single momentum p^φ, p^γ remain and we arrive at

$$
\begin{aligned}
&(\varphi_b\gamma_b\, S|\varphi_a\gamma_a\, 0)_{[\beta]} \qquad (14.251)\\
&\approx e^{-iq(\gamma_b-\gamma_a)} \sum_{l_b^\varphi=-\infty}^{\infty} \sum_{l_b^\gamma=-\infty}^{\infty} \int \frac{dp_\varphi}{2\pi\hbar} \int \frac{dp_\gamma}{2\pi\hbar} e^{ip_\varphi(\varphi_b+2\pi l_b^\varphi-\varphi_a)/\hbar} e^{ip_\gamma(\gamma_b+4\pi l_b^\gamma-\gamma_a)/\hbar}\\
&\times \exp\left\{-\frac{i}{\hbar}\sum_{n=1}^{N+1}\left[\frac{1}{2M\epsilon_s}(p_\varphi^2+p_\gamma^2+2p_\varphi p_\gamma \tanh\beta_n) - \epsilon_s\frac{(p_\gamma-\hbar q)\hbar q}{M\cosh^2\beta_n}\right]\right\}.
\end{aligned}
$$

We can now do the sums over l_b^φ, l_b^γ which force the momenta p_φ to integer values and p_γ to half-integer values by Poisson's formula, so that

$$
\begin{aligned}
(\varphi_b\gamma_b\, S|\varphi_a\gamma_a\, 0)_{[\beta]} &= e^{-iq(\gamma_b-\gamma_a)} \sum_{m_1,m_2} \frac{1}{2\pi} e^{im_1(\varphi_b-\varphi_a)} \frac{1}{4\pi} e^{im_2(\gamma_b-\gamma_a)} \qquad (14.252)\\
&\times \exp\left\{-\frac{i}{\hbar}\sum_{n=1}^{N+1}\left[\frac{\hbar^2}{2M\epsilon_s}(m_1^2+m_2^2+2m_1m_2\tanh\beta_n) - \frac{\epsilon_s\hbar^2(m_2-q)q}{M\cosh^2\beta_n}\right]\right\}.
\end{aligned}
$$

With this, the expression for the fixed-energy amplitude (14.248) of the dionium atom contains the magnetic charge only at three places: the extra centrifugal barrier in $\mathcal{A}_{\xi\beta}^\epsilon$, the phase factor of the remaining integral over γ_a, and the last term in (14.252). This last term, however, can be dropped since the integral over γ_a forces the half-integer number m_2 to become equal to $\hbar q$. The γ_b-integral over the remaining functional of $\beta(t)$ gives, incidentally,

$$
\begin{aligned}
\int_0^{4\pi} d\gamma_a (\varphi_b\gamma_b\, S|\varphi_a\gamma_a\, 0)_{[\beta]} &= \sum_{m_1} \frac{1}{2\pi} e^{im_1(\varphi_b-\varphi_a)}\\
&\times \exp\left\{-\frac{i}{\hbar}\sum_{n=1}^{N+1}\left[\frac{\hbar^2}{2M\epsilon_s}[m_1^2+q^2+2m_1 q\tanh\beta_n]\right]\right\}. \qquad (14.253)
\end{aligned}
$$

The time-sliced expression has the parameter q at precisely the same places as the previous formal one. This proves that formula (14.220) with (14.221) is unchanged by time slicing corrections, thus completing the solution of the path integral of the dionium atom.

Note that after inserting (14.253), the time-sliced path integral (14.248) is a combination of a general Rosen-Morse system in β and a Morse system in ξ.

Let us end this discussion by the remark that like the Coulomb system, the dionium atom can be treated in a purely group-theoretic way, using only operations within the dynamical group O(4,2). This is explained in Appendix 14C.

14.7 Time-Dependent Duru-Kleinert Transformation

By generalizing the above transformation method to time-dependent regulating functions, we can derive further relations between amplitudes of different physical systems. In the path-dependent time transformation $dt = ds\, f_l(\mathbf{x})f_r(\mathbf{x})$ regularizing the path integrals, we may allow for functions $f_l(\mathbf{p},\mathbf{x},t)$ and $f_r(\mathbf{p},\mathbf{x},t)$ depending on positions, momenta, and time. Such functions complicate the subsequent transformation to new coordinates $\mathbf{q}$, in which the kinetic term of the amplitude (12.47) with respect to the pseudotime s has the standard form $(M/2)\mathbf{q}'^2(s)$. In particular, a momentum dependence of f_l and f_r leads to involved formulas, which is the reason why this case has not yet been investigated (just like the even more general case where the right-hand side of the transformation $dt = ds\,f$ contains terms proportional to $d\mathbf{x}$). If one restricts the transformation to depend only on time and uses the special splitting with the regulating functions $f_l = f$ and $f_r = 1$, or $f_l = 1$ and $f_r = f$, the result in one spatial dimension is relatively simple. On the basis of Section 12.3, the following relation is found [instead of (14.26)] between the time evolution amplitude of an initial system and a fixed-energy amplitude of the transformed systems at $\mathcal{E} = 0$:

$$(x_b t_b|x_a t_a) = g(q_b, t_b)g(q_a, t_a)\{q_b t_b|q_a t_a\}_{\mathcal{E}=0}, \tag{14.254}$$

where $\{q_b t_b|q_a t_a\}_{\mathcal{E}=0}$ denotes the spacetime extension of the fixed-pseudoenergy amplitude. It is calculated by time-slicing the expression

$$\int_0^\infty dS\{x_b t_b|\hat{\mathcal{U}}_E(S)|x_a t_a\} \tag{14.255}$$

on the right-hand side of (12.55), transforming the coordinates x to q, and adapting the normalization to the completeness relation of the states

$$\int dx \int dt|q\,t\}\{q\,t| = 1. \tag{14.256}$$

This leads to the path integral representation

$$\{q_b t_b|q_a t_a\}_{\mathcal{E}} = \int_0^\infty dS \int dE e^{-iE(t_b-t_a)/\hbar} \int \mathcal{D}q(s) e^{i\mathcal{A}^{\rm DK}_{E,\mathcal{E}}/\hbar}, \tag{14.257}$$

with the DK-transformed action

$$\mathcal{A}^{\rm DK}_{E,\mathcal{E}} = \int_0^S ds\Big\{\frac{M}{2}q'^2(s) - f(q(s),t(s))[V(q(s),t) - E] + \mathcal{E} \\ -V_{\rm eff}(q(s),t(s)) - \Delta V_{\rm eff}(q(s),t(s))\Big\}. \tag{14.258}$$

Note that the initial potential may depend explicitly on time. The function $t(s)$ is now given by the time-dependent differential equation

$$\frac{dt}{ds} = f(x,t). \tag{14.259}$$

The coordinate transformation also depends on time,

$$x = h(q,t), \tag{14.260}$$

and satisfies the equation

$$h'^2(q,t) = f(h(q,t),t), \tag{14.261}$$

where $h'(q,t) \equiv \partial_q h(q,t)$ [compare (14.15)]. The function $f(q(s),t(s))$ used in (14.258) is an abbreviation for $f(h(q,t),t)$ evaluated at the time $t = t(s)$.

In addition to the effective potential $V_{\rm eff}$ determined by Eq. (14.18), there is now a further contribution which is due to the time dependence of $h(q,t)$ [17]:

$$\Delta V_{\rm eff} = Mh'^2 \int dq\, h'\ddot{h} \mp i\hbar\dot{h}'h'. \tag{14.262}$$

The upper sign must be used if the relation between t and s is calculated from the time-sliced postpoint recursion relation

$$t_{n+1} - t_n = \epsilon_s f(q_{n+1}, t_{n+1}). \tag{14.263}$$

The lower sign holds when solving the prepoint relation

$$t_{n+1} - t_n = \epsilon_s f(q_n, t_n). \tag{14.264}$$

Note that the first term of $\Delta V_{\rm eff}$ contributes even at the classical level. If a function $h(q,t)$ is found satisfying Newton's equation of motion

$$M\ddot{h} = -\frac{\partial V(h,t)}{\partial h}, \tag{14.265}$$

with $V(h) \equiv V(x)|_{x=h}$, then the first term eliminates the potential in the action (14.258), and the transformed system is classically free. This happens if the new coordinate $q(t)$ associated with x, t is identified with the initial value, at some time t_0, of the classical orbit running through x, t. These are trivially time-independent and therefore behave like the coordinates of a free particle (see the subsequent example).

The normalization factor $g(q,t)$ is determined by the differential equation

$$\frac{g'}{g} = \frac{1}{2}\frac{h''}{h'} + i\frac{M}{\hbar}h'\dot{h}. \tag{14.266}$$

The solution reads

$$g(q,t) = e^{i\Lambda(q,t)}\sqrt{h'(q,t)}, \tag{14.267}$$

with

$$\Lambda(q,t) = \pm\frac{M}{\hbar}\int^q dq\, h'\dot{h}. \tag{14.268}$$

Thus, in addition to the normalization factor in (14.26), the time-dependent DK relation (14.254) also contains a phase factor.

As an example [18], we transform the amplitude of a harmonic oscillator to that of a free particle. The classical orbits are given by $x(t) = x_0 \cos\omega t$, so that the transformation $x(t) = h(q,t) = q\cos\omega t$ leads to a coordinate $q(t)$ which moves without acceleration. For brevity, we write $\cos\omega t$ as $c(t)$. Obviously, $f(q,t) = c^2(t)$ is a pure function of the time [19], and the differential relation between the time t and the pseudotime s is integrated to

$$S = \frac{1}{\omega}\frac{\sin\omega(t_b - t_a)}{c(t_b)c(t_a)}. \tag{14.269}$$

This equation can be solved for $t_a(S)$ at fixed t_b, or for $t_b(S)$ at fixed t_a. The solution $t_a(S)$ is obtained from the time-sliced postpoint recursion relation (14.263), while $t_b(S)$ arises from the prepoint recursion (14.264). The DK action (14.258) is simplified in these two cases to

$$\mathcal{A}^{\rm DK}_{E,\mathcal{E}} = \frac{M}{2}\frac{(q_b-q_a)^2}{S} \pm i\hbar \log\frac{c(t_b)}{c(t_a)} + E\begin{Bmatrix}[t_b-t_a(S)]\\ [t_b(S)-t_a]\end{Bmatrix} + \mathcal{E}S. \tag{14.270}$$

The E-integration in (14.257) yields in the first case

$$\{q_bt_b|q_at_a\}_{\mathcal{E}} = \int_0^\infty dS\,\delta(t_b-t_a(S))\,\frac{c(t_a)}{c(t_b)}\frac{e^{(i/\hbar)M(q_b-q_a)^2/2S}}{\sqrt{2\pi\hbar iS/M}}, \tag{14.271}$$

and the integration over S using $-dt_a(S)/dS = c^2(t_a)$ results in

$$\{q_bt_b|q_at_a\}_{\mathcal{E}} = \frac{1}{c(t_b)c(t_a)}\frac{e^{(i/\hbar)M(q_b-q_a)^2/2S}}{\sqrt{2\pi\hbar iS/M}}. \tag{14.272}$$

The same amplitude is obtained for the lower sign in (14.270) with $dt_b(S)/dS = c^2(t_b)$. After inserting this together with (18.671) and (14.268) into (14.254) [the integration there gives $\Lambda(q,t) = (M/\hbar)q^2c(t)\dot{c}(t)/2$], we obtain

$$(x_bt_b|x_at_a) = e^{(i/\hbar)M[q_b^2c(t_b)\dot{c}(t_b)-q_a^2c(t_a)\dot{c}(t_a)]/2}\frac{1}{\sqrt{c(t_b)c(t_a)}}\frac{e^{(i/\hbar)M(q_b-q_a)^2/2S}}{\sqrt{2\pi\hbar iS/M}}. \tag{14.273}$$

Since q_b and q_a are equal to $x_b/c(t_b)$ and $x_a/c(t_a)$, respectively, a few trigonometric identities lead to the well-known expression (2.175) for the amplitude of the harmonic oscillator.

It is obvious that a combination of this transformation with a time-independent Duru-Kleinert transformation makes it possible to reduce also the path integral of the Coulomb system to that of a free particle.

It will be interesting to find out which hitherto unsolved path integrals can be integrated by means of such generalized DK transformations. Applications to statistical problems have been given in Ref. [21].

Appendix 14A Remarks on the DK Transformation of Time-Sliced Radial Coulomb Problem

Here we show that due to the catastrophic centrifugal barriers in the multiple integrals of a time-sliced formulation, the DK relation has to be used twice to transform the radial amplitude of the Coulomb problem to the radial harmonic oscillator. An intermediate Morse potential is needed to have no divergent integrals. The use of the DK transformation with the regulating function $f(r) = r$ and a pseudotime s satisfying $dt = ds\,r(s)$ (which were successful in two and three dimensions), removes the Coulomb singularity, but it weakens the centrifugal barrier insufficiently to a still catastrophic $1/r$-singularity.

Let us exhibit the failure of this direct transformation. The starting point is the pseudotime-sliced amplitude (13.8),

$$\langle \mathbf{x}_b|\hat{\mathcal{U}}_E(s)|\mathbf{x}_a\rangle \approx \frac{r_b^{\lambda} r_a^{1-\lambda}}{\sqrt{2\pi\epsilon_s\hbar r^{1-\lambda}r^{\lambda}/M}^{D_{\mathcal{C}}/2}} \prod_{n=1}^{N}\left[\int \frac{d^{D_{\mathcal{C}}}\Delta x_n}{\sqrt{2\pi\epsilon_s\hbar r_n/M}^{D_{\mathcal{C}}}}\right] e^{-\mathcal{A}_E^N/\hbar}, \tag{14A.1}$$

with the action

$$\mathcal{A}_E^N = -(N+1)\epsilon_s e^2 + \sum_{n=1}^{N+1}\left[\frac{M}{2\epsilon_s}\frac{(x_n-x_{n-1})^2}{r_n^{1-\lambda}r_{n-1}^{\lambda}} + \epsilon_s E r_n^{1-\lambda} r_{n-1}^{\lambda}\right]. \tag{14A.2}$$

In contrast to Chapter 13, we work here conveniently with an imaginary-time. In any dimension $D_{\mathcal{C}}$, the amplitude has the angular decomposition

$$\langle \mathbf{x}_b|\hat{\mathcal{U}}_E(s)|\mathbf{x}_a\rangle = \frac{1}{(r_b r_a)^{D_{\mathcal{C}}-1/2}} \sum \langle r_b|\hat{\mathcal{U}}_E(s)|r_a\rangle_l Y_{lm}(\hat{\mathbf{x}}_b) Y_{lm}^*(\hat{\mathbf{x}}_a). \tag{14A.3}$$

The action for the radial amplitude is obtained by decomposing

$$\mathcal{A}_E^N = -(N+1)\epsilon_s e^2 + \sum_{n=1}^{N+1}\left[\frac{M}{2\epsilon_s r_n^{1-\lambda}r_{n-1}^{\lambda}}(r_n^2 + r_{n-1}^2 - 2r_n r_{n-1}\cos\vartheta_n) + \epsilon_s E r_n\right], \tag{14A.4}$$

where ϑ_n is the angle between $\mathbf{x}_n$ and $\mathbf{x}_{n-1}$. We have replaced $Er_n^{1-\lambda}r_{n-1}^{\lambda}$ by Er_n since the difference is of order ϵ_s^2 and thus negligible.

We now go through the same steps as in Section 8.5. For an individual time slice, the ϑ_n-part of the exponential is expanded as

$$\exp\left(\frac{M}{\epsilon_s} {r_n}^{\lambda} r_{n-1}^{1-\lambda} \cos\vartheta_n\right) = e^h \sum_{l_{\mathcal{C}}=0}^{\infty} \tilde{a}_{l_{\mathcal{C}}}(h) \sum_{\mathbf{m}} Y_{l_{\mathcal{C}}\,\mathbf{m}}(\hat{\mathbf{x}}_b) Y_{l_{\mathcal{C}}\,\mathbf{m}}^*(\hat{\mathbf{x}}_a), \tag{14A.5}$$

with

$$a_{l_{\mathcal{C}}}(h) = \left(\frac{2\pi}{h}\right)^{(D_{\mathcal{C}}-1)/2} \tilde{I}_{D_{\mathcal{C}}/2-1+l_{\mathcal{C}}}(h), \quad h = \frac{M}{\hbar\epsilon_s} {r_n}^{\lambda} r_{n-1}^{1-\lambda} \tag{14A.6}$$

[recall (8.130) and (8.101)]: The radial part of the propagator is then

$$\langle r_b|\hat{\mathcal{U}}_E(s)|r_a\rangle_{l_{\mathcal{C}}} \approx \frac{{r_b}^{\lambda}{r_a}^{1-\lambda}}{\sqrt{2\pi\hbar\epsilon_s {r_b}^{1-\lambda}{r_a}^{\lambda}/M}} \prod_{n=2}^{N+1}\left[\int \frac{d\Delta r_n\ r_{n-1}^{-1/2}}{\sqrt{2\pi\hbar\epsilon_s/M}}\right] e^{-\mathcal{A}_E^N/\hbar}, \tag{14A.7}$$

with the radial action

$$\mathcal{A}_E^N = -(N+1)\epsilon_s e^2 + \sum_{n=1}^{N+1}\left[\frac{M}{2\epsilon_s}\frac{(r_n - r_{n-1})^2}{r_n^{1-\lambda}r_{n-1}^{\lambda}} - \hbar\log\tilde{I}_{D_{\mathcal{C}}/2-1+l_{\mathcal{C}}}\left(\frac{M}{\hbar\epsilon_s}{r_n}^{\lambda}r_{n-1}^{1-\lambda}\right) - \epsilon_s E r_n\right]. \tag{14A.8}$$

At this place we simplify the calculation by choosing the symmetric splitting parameter $\lambda = 1/2$. Going over to square root coordinates

$$u_n = \sqrt{r_n}, \tag{14A.9}$$

we calculate

$$\begin{aligned} \Delta r_n &= (u_n + u_{n-1})\Delta u_n, \\ &= 2u_n(1 - \Delta u_n/2u_n)\Delta u_n, \\ \frac{\partial \Delta r_n}{\partial \Delta u_n} &= 2u_n(1 - \Delta u_n/u_n), \\ r_{n-1}^{-1/2} &= u_n^{-1}(1 - \Delta u_n/u_n)^{-1}, \end{aligned} \tag{14A.10}$$

transforming the measure of integration into

$$\frac{\sqrt{u_b u_a}}{\sqrt{2\pi\hbar\epsilon_s/M}} \prod_{n=1}^{N} \int \frac{d\Delta u_n}{\sqrt{2\pi\hbar\epsilon_s/M}}. \tag{14A.11}$$

Note that there are no higher Δu_n correction terms. The kinetic energy is

$$\frac{4\bar{u}_n{}^2(\Delta u_n)^2}{2\epsilon_s u_n u_{n-1}} = \frac{4}{2\epsilon_s}\left[(\Delta u_n)^2 + \frac{1}{4}\frac{(\Delta u_n)^4}{u_n{}^2} + \dots\right]. \tag{14A.12}$$

The $(\Delta u_n)^4$-term can be replaced right away by its expectation value and renders an effective potential

$$V_{\rm eff}(u_n{}^2) = \epsilon_s \hbar^2 \frac{1}{2\cdot 4M}\frac{3}{4u_n{}^2}. \tag{14A.13}$$

The radial amplitude becomes simply

$$\langle r_b|\hat{\mathcal{U}}_E(s)|r_a\rangle_{l_\mathcal{C}} \approx \frac{\sqrt{u_b u_a}}{\sqrt{2\pi\hbar\epsilon_s/M}} \prod_{n=2}^{N+1}\left[\int_0^\infty \frac{2du_n}{\sqrt{2\pi\hbar\epsilon_s/M}}\right] e^{-\mathcal{A}_E^N/\hbar}, \tag{14A.14}$$

with

$$\mathcal{A}_E^N = -(N+1)\epsilon_s + \sum_{n=1}^{N+1}\left[\frac{4M}{2}\frac{(\Delta u_n)^2}{2\epsilon_s} + V_{\rm eff}(u_n{}^2) - \hbar\log \tilde{I}_{D_\mathcal{C}/2-1+l_\mathcal{C}}\left(\frac{M}{\hbar\epsilon_s}u_n u_{n-1}\right)\right]. \tag{14A.15}$$

Due to the $1/u_n{}^2$-singularity in $V_{\rm eff}(u_n{}^2)$, the time-sliced path integral does not exist. Apart from the $\epsilon_s/u_n{}^2$-term, there should be infinitely many terms of increasing order of the type $(\epsilon_s/u_n{}^2)^2$, ... , whose resummation is needed to obtain the correct threshold small-u_n behavior of the amplitude as discussed in Section 8.2. To have the usual kinetic term of the harmonic oscillator $M_\mathcal{O}(\Delta u_n)^2/2\epsilon_s$, we must identify $4M$ with the oscillator mass $M_\mathcal{O}$ [called μ in (13.27); see also (14.99) with $M_\mathcal{C} \equiv M$],

$$M_\mathcal{O} = 4M. \tag{14A.16}$$

The centrifugal barrier in (14A.15) resides in

$$-\hbar\log \tilde{I}_{D_\mathcal{C}/2-1+l_\mathcal{C}}\left(\frac{M_\mathcal{O}/4}{\hbar\epsilon_s}u_n u_{n-1}\right) + \epsilon_s\hbar^2\frac{3}{8M_\mathcal{O}u_n^2} + \dots\ , \tag{14A.17}$$

and is given by

$$\epsilon_s\frac{\hbar^2}{2M_\mathcal{O}}\frac{4}{u_n u_{n-1}}\left[\left(\frac{D_\mathcal{C}}{2} - 1 + l_\mathcal{C}\right)^2 - \frac{1}{4}\right] + \epsilon_s\hbar^2\frac{3}{8M_\mathcal{O}u_n^2} + \dots\ . \tag{14A.18}$$

This can be rewritten more explicitly as

$$\epsilon_s\frac{\hbar}{2M_\mathcal{O}}\frac{1}{u_n u_{n-1}}\left[(D_\mathcal{C} - 2 + 2l_\mathcal{C})^2 - \frac{1}{4}\right]. \tag{14A.19}$$

The expression in parentheses is identified with the parameter $\mu_\mathcal{O}$ of the harmonic oscillator, which appears in the subscript of the Bessel function in (8.140). This implies

$$\mu_\mathcal{O} = 2\mu_\mathcal{C}, \tag{14A.20}$$

in agreement with the relation (14.100). Indeed, the higher terms in the expansion (14A.18) must all conspire to sum up to the Bessel-regulated centrifugal barrier in the time-sliced radial amplitude of the harmonic oscillator

$$-\hbar \log \tilde{I}_{D_\mathcal{C}-2+2l_\mathcal{C}}\left(\frac{M_\mathcal{O}}{\hbar\epsilon_s}u_n u_{n-1}\right). \tag{14A.21}$$

This is quite hard to verify term by term, although it must happen.

These difficulties are avoided by using the stronger regulating function $f = r^2$, which transform the radial Coulomb problem to the Morse problem. Instead of the pseudotime evolution amplitude (14A.1), we have

$$\langle \mathbf{x}_E|\hat{\mathcal{U}}_e(s)|\mathbf{x}_a\rangle \approx \frac{{r_b}^2{r_a}^{2-2\lambda}}{\sqrt{2\pi\epsilon_s\hbar{r_b}^{2-2\lambda}{r_a}^{2\lambda}/M}^{D_\mathcal{C}/2}}\prod_{n=1}^{N}\left[\int\frac{d^{D_\mathcal{C}}\Delta x_n}{\sqrt{2\pi\epsilon_s\hbar r_n^2/M}^{D_\mathcal{C}}}\right]e^{-\mathcal{A}_E^{f\,N}/\hbar}, \tag{14A.22}$$

with the time-sliced transformed action

$$\mathcal{A}_E^{f\,N} = -(N+1)\epsilon_s e^2 + \sum_{n=1}^{N+1}\left[\frac{M}{2\epsilon_s {r_n}^{2-2\lambda}r_{n-1}^{2\lambda}}(r_n^2+r_{n-1}^2-2r_nr_{n-1}\cos\vartheta_n)-\epsilon_s E r_n^2\right]. \tag{14A.23}$$

For $\lambda = 1/2$, the $\cos\Delta\vartheta_n$-term is now free of the radial variables $r_n, r_{n-1}, r_n, r_{n-1}$, and the angular decomposition of the amplitude as in (14A.3)–(14A.8) gives the radial amplitude with a time-sliced action

$$\mathcal{A}_E^{f\,N} = -(N+1)\epsilon_s e^2 + \sum_{n=1}^{N+1}\left[\frac{M}{2\epsilon_s}\frac{(r_n-r_{n-1})^2}{r_nr_{n-1}} - \hbar\log\tilde{I}_{D_\mathcal{C}/2-1+l_\mathcal{C}}\left(\frac{M}{\hbar\epsilon_s}\right)-\epsilon_s E{r_n}^2\right]. \tag{14A.24}$$

Since r_n, r_{n-1} are absent in the Bessel function, the limit of small ϵ_s is now uniform in the integration variables r_n and the logarithmic term in the energy can directly be replaced by

$$\epsilon_s\frac{\hbar}{2M_\mathcal{C}}\left[(D_\mathcal{C}/2-1+l_\mathcal{C})^2-1/4\right], \tag{14A.25}$$

where we have added the subscripts $\mathcal{C}$, for clarity. To perform the integration over the r_n variables, one goes over to new coordinates x with

$$r = h(x) = e^x. \tag{14A.26}$$

The measure of integration is

$$\frac{\sqrt{r_br_a}}{\sqrt{2\pi\hbar\epsilon_s/M}^{N+1}}\prod_{n=2}^{N+1}\int\frac{d\Delta r_n}{r_{n-1}}. \tag{14A.27}$$

Expanding $1/r_{n-1}$ around the postpoint r_n gives

$$\frac{1}{r_{n-1}} = \frac{1}{r_n}\left(\frac{r_n}{r_{n-1}}\right) = \frac{e^{\Delta x_n}}{e^{x_n}}. \tag{14A.28}$$

We now write (dropping subscripts n)

$$\Delta r = e^x - e^{x-\Delta x} = e^x(1-e^{-\Delta x}), \tag{14A.29}$$

and find the Jacobian

$$\frac{\partial\Delta r}{\partial\Delta x} = e^x e^{-\Delta x}. \tag{14A.30}$$

In the x-coordinates, the measure becomes simply

$$\frac{e^{(x_b+x_a)/2}}{\sqrt{2\pi\hbar\epsilon_s/M}^{N+1}}\prod_{n=2}^{N+1}\int d\Delta x_n. \tag{14A.31}$$

The kinetic term in the action turns into

$$\mathcal{A}_E^N=\sum_{n=1}^{N+1}\frac{M}{\epsilon_s}(1-\cos\Delta x_n), \tag{14A.32}$$

and has the expansions

$$\mathcal{A}_E^N=\sum_{n=1}^{N+1}\frac{M}{2\epsilon_s}\left[(\Delta x)^2-\frac{1}{12}(\Delta x_n)^4+\ldots\right]. \tag{14A.33}$$

The higher-order terms contribute with higher powers of ϵ_s *uniformly* in x. They can be treated as usual. This is why the path-dependent time transformation of the radial Coulomb system to a radial oscillator with the regulating function $f=r^2$ is free of problems.

Appendix 14B Affine Connection of Dionium Atom

From the transformation matrices (14.240), we calculate the derivatives [setting $q^\mu=(\xi,\beta,\varphi,\gamma)$ and using the abbreviation $f_{,\xi}\equiv\partial_\xi f$]

$$e^i{}_{\mu,\xi}=e^i{}_\mu, \tag{14B.1}$$

$$e^i{}_{\mu,\beta} = \begin{pmatrix} -e^\xi\frac{\sinh\beta}{\cosh^2\beta}\cos\phi & -e^\xi\frac{1-\sinh^2\beta}{\cosh^3\beta}\cos\phi & e^\xi\frac{\sinh\beta}{\cosh^2\beta}\sin\phi & 0\\ -e^\xi\frac{\sinh\beta}{\cosh^2\beta}\sin\phi & -e^\xi\frac{1-\sinh^2\beta}{\cosh^3\beta}\sin\phi & -e^\xi\frac{\sinh\beta}{\cosh^2\beta}\cos\phi & 0\\ -e^\xi\cosh^{-2}\beta & 2e^\xi\frac{\sinh\beta}{\cosh^3\beta} & 0 & 0\\ 0 & 0 & -e^\xi\cosh^{-2}\beta & 0 \end{pmatrix}, \tag{14B.2}$$

$$e^i{}_{\mu,\phi} = \begin{pmatrix} -e^\xi\cosh^{-1}\beta\sin\phi & e^\xi\frac{\sinh\beta}{\cosh^2\beta}\sin\phi & -e^\xi\cosh^{-1}\beta\cos\phi & 0\\ e^\xi\cosh^{-1}\beta\cos\phi & -e^\xi\frac{\sinh\beta}{\cosh^2\beta}\cos\phi & -e^\xi\cosh^{-1}\beta\sin\phi & 0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \tag{14B.3}$$

$$e^i{}_{\mu,\alpha}=0. \tag{14B.4}$$

From these we find $\Gamma_{\mu\nu\lambda}=e^i{}_\lambda e^i{}_{\nu,\mu}$ by contraction with $e^i{}_\lambda$:

$$\Gamma_{\xi\mu\nu}=g_{\mu\nu}, \tag{14B.5}$$

$$\Gamma_{\beta\mu\nu} = \begin{pmatrix} 0 & e^{2\xi}\cosh^{-2}\beta & 0 & 0\\ -e^{2\xi}\cosh^{-2} & -e^{2\xi}\frac{\sinh\beta}{\cosh^3\beta} & 0 & 0\\ 0 & 0 & 0 & -e^{2\xi}\cosh^{-2}\beta\\ 0&0&0&0 \end{pmatrix}, \tag{14B.6}$$

$$\Gamma_{\phi\mu\nu} = \begin{pmatrix} 0 & 0 & e^{2\xi}\cosh^{-2}\beta & 0\\ 0 & 0 & -e^{2\xi}\frac{\sinh\beta}{\cosh^3\beta} & 0\\ -e^{2\xi}\cosh^{-2} & e^{2\xi}\frac{\sinh\beta}{\cosh^3\beta} & 0 & 0\\ 0&0&0&0 \end{pmatrix}, \tag{14B.7}$$

$$\Gamma_{\alpha\mu\nu}=0. \tag{14B.8}$$

A contraction with the inverse metric $g^{\mu\nu}$ yields $\Gamma_\mu{}^\mu{}_\nu=(-1,0,0,0)$, as stated in Eq. (14.244).

Appendix 14C Algebraic Aspects of Dionium States

In Appendix 13A we have shown that certain combinations of x^μ and ∂_μ operators satisfy the commutation rules of the Lie algebra of the group O(4,2) [see Eqs. (13A.11)]. This permits solving all dynamical problems via group operations. In the case $l_0 = q$ (i.e., $l_{\rm extra} = 0$), the group-theoretic approach can be extended to include the dionium atom. In fact, it is easy to see [20] that the Lie algebra of O(4,2) remains the same if the generators L_{AB} $(A, B = 1, \dots, 6)$ of Eq. (13A.11) are extended to $(x^i \equiv x_i)$

$$
\begin{aligned}
\hat{L}_{ij} &= -\frac{i}{2}(x_i\partial_{x^j} - x_j\partial_{x^i}) + q\frac{r}{\mathbf{x}_\perp^2}x_\perp^k, \\
\hat{L}_{i4} &= \frac{1}{2}\left[-x^i\partial_{\mathbf{x}}^2 - x^i + 2\partial_{x^i}\mathbf{x}\partial_{\mathbf{x}} + 2iq\frac{r}{\mathbf{x}_\perp^2}(\mathbf{x}_\perp\times\boldsymbol{\nabla})_i - (-)^{\delta_{i3}}q^2\frac{x_i}{\mathbf{x}_\perp^2}\right], \\
\hat{L}_{i5} &= \frac{1}{2}\left[-x^i\partial_{\mathbf{x}}^2 + x^i + 2\partial_{x^i}\mathbf{x}\partial_{\mathbf{x}} + 2iq\frac{r}{\mathbf{x}_\perp^2}(\mathbf{x}_\perp\times\boldsymbol{\nabla})_i - (-)^{\delta_{i3}}q^2\frac{x_i}{\mathbf{x}_\perp^2}\right], \\
\hat{L}_{i6} &= -ir\partial_{x^i} - \frac{q}{\mathbf{x}_\perp^2}(\mathbf{x}\times\mathbf{x}_\perp)_i, \\
\hat{L}_{45} &= -i(\mathbf{x}\partial_{\mathbf{x}} + 1), \\
\hat{L}_{46} &= \frac{1}{2}\left[-r\partial_{\mathbf{x}}^2 - r + 2iq\frac{z}{\mathbf{x}_\perp^2}(\mathbf{x}\times\boldsymbol{\nabla})_3 + q^2\frac{r}{\mathbf{x}_\perp^2}\right], \\
\hat{L}_{56} &= \frac{1}{2}\left[-r\partial_{\mathbf{x}}^2 + r + 2iq\frac{z}{\mathbf{x}_\perp^2}(\mathbf{x}\times\boldsymbol{\nabla})_3 + q^2\frac{r}{\mathbf{x}_\perp^2}\right].
\end{aligned}
\tag{14C.1}
$$

The representation space is now characterized by the eigenvalue of the operator $\hat{L}_{05} = -ir\partial_{x^4} = -i\partial_\gamma = -\frac{1}{2}(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})$ being equal to q. The wave functions are generalizations of the $q = 0$ -wave functions of the Coulomb system.

Notes and References

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Watch out for mistakes. For instance, in Eq. (3.28) of the first paper, the authors claim to have calculated the fixed-energy amplitude, but give only its imaginary part restricted to the bound-state poles.Their result (3.33) lacks the continuum states. Further errors in their Section V have been pointed out in Footnote 20 of Chapter 8.

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