Path Integrals

in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets

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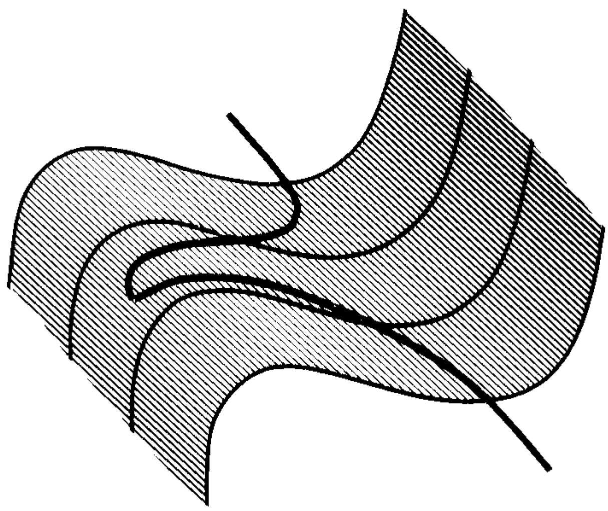
Hagen Kleinert
Professor of Physics
Freie Universität Berlin, Germany
ICRANet Pescara, Italy, and Nice, France



To Annemarie and Hagen II

Nature alone knows what she wants.
GOETHE

Preface to the Fifth Edition

The paperback version of the fourth edition of this book was sold out in Fall 2008. This gave me a chance to revise it at many places. In particular, I improved considerably Chapter 20 on financial markets and removed some technical sections of Chapter 5.

Among the many people who spotted printing errors and suggested changes of various text passages are Dr. A. Pelster, Dr. A. Redondo, and especially Dr. Annemarie Kleinert.

H. Kleinert
Berlin, January 2009

Nature alone knows what she wants.
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Preface to Fourth Edition

The third edition of this book appeared in 2004 and was reprinted in the same year without improvements. The present fourth edition contains several extensions. Chapter 4 includes now semiclassical expansions of higher order. Chapter 8 offers an additional path integral formulation of spinning particles whose action contains a vector field and a Wess-Zumino term. From this, the Landau-Lifshitz equation for spin precession is derived which governs the behavior of quantum spin liquids. The path integral demonstrates that fermions can be described by Bose fields—the basis of Skyrmion theories. A further new section introduces the Berry phase, a useful tool to explain many interesting physical phenomena. Chapter 10 gives more details on magnetic monopoles and multivalued fields. Another feature is new in this edition: sections of a more technical nature are printed in smaller font size. They can well be omitted in a first reading of the book.

Among the many people who spotted printing errors and helped me improve various text passages are Dr. A. Chervyakov, Dr. A. Pelster, Dr. F. Nogueira, Dr. M. Weyrauch, Dr. H. Baur, Dr. T. Iguchi, V. Bezerra, D. Jahn, S. Overesch, and especially Dr. Annemarie Kleinert.

H. Kleinert
Berlin, June 2006

Preface to Third Edition

This third edition of the book improves and extends considerably the second edition of 1995:

- Chapter 2 now contains a path integral representation of the scattering amplitude and new methods of calculating functional determinants for time-dependent second-order differential operators. Most importantly, it introduces the quantum field-theoretic definition of path integrals, based on perturbation expansions around the trivial harmonic theory.

- Chapter 3 presents more exactly solvable path integrals than in the previous editions. It also extends the Bender-Wu recursion relations for calculating perturbation expansions to more general types of potentials.

- Chapter 4 discusses now in detail the quasiclassical approximation to the scattering amplitude and Thomas-Fermi approximation to atoms.

- Chapter 5 proves the convergence of variational perturbation theory. It also discusses atoms in strong magnetic fields and the polaron problem.

- Chapter 6 shows how to obtain the spectrum of systems with infinitely high walls from perturbation expansions.

- Chapter 7 offers a many-path treatment of Bose-Einstein condensation and degenerate Fermi gases.

- Chapter 10 develops the quantum theory of a particle in curved space, treated before only in the time-sliced formalism, to perturbatively defined path integrals. Their reparametrization invariance imposes severe constraints upon integrals over products of distributions. We derive unique rules for evaluating these integrals, thus extending the linear space of distributions to a semigroup.

- Chapter 15 offers a closed expression for the end-to-end distribution of stiff polymers valid for all persistence lengths.

- Chapter 18 derives the operator Langevin equation and the Fokker-Planck equation from the forward–backward path integral. The derivation in the literature was incomplete, and the gap was closed only recently by an elegant calculation of the Jacobian functional determinant of a second-order differential operator with dissipation.

- Chapter 20 is completely new. It introduces the reader into the applications of path integrals to the fascinating new field of econophysics.

For a few years, the third edition has been freely available on the internet, and several readers have sent useful comments, for instance E. Babaev, H. Baur, B. Budnyj, Chen Li-ming, A.A. Drăgulescu, K. Glaum, I. Grigorenko, T.S. Hatamian, P. Hollister, P. Jizba, B. Kastening, M. Krämer, W.-F. Lu, S. Mukhin, A. Pelster, M.B. Pinto, C. Schubert, S. Schmidt, R. Scalettar, C. Tangui, and M. van Vugt. Reported errors are corrected in the internet edition.

When writing the new part of Chapter 2 on the path integral representation of the scattering amplitude I profited from discussions with R. Rosenfelder. In the new parts of Chapter 5 on polarons, many useful comments came from J.T. Devreese, F.M. Peeters, and F. Brosens. In the new Chapter 20, I profited from discussions with F. Nogueira, A.A. Drăgulescu, E. Eberlein, J. Kallsen, M. Schweizer, P. Bank, M. Tenney, and E.C. Chang.

As in all my books, many printing errors were detected by my secretary S. Endrias and many improvements are due to my wife Annemarie without whose permanent encouragement this book would never have been finished.

H. Kleinert
Berlin, August 2003

Preface to Second Edition

Since this book first appeared three years ago, a number of important developments have taken place calling for various extensions to the text.

Chapter 4 now contains a discussion of the features of the semiclassical quantization which are relevant for multidimensional chaotic systems.

Chapter 3 derives perturbation expansions in terms of Feynman graphs, whose use is customary in quantum field theory. Correspondence is established with Rayleigh-Schrödinger perturbation theory. Graphical expansions are used in Chapter 5 to extend the Feynman-Kleinert variational approach into a systematic *variational perturbation theory*. Analytically inaccessible path integrals can now be evaluated with arbitrary accuracy. In contrast to ordinary perturbation expansions which always diverge, the new expansions are convergent for all coupling strengths, including the strong-coupling limit.

Chapter 10 contains now a new action principle which is necessary to derive the correct classical equations of motion in spaces with curvature and a certain class of torsion (gradient torsion).

Chapter 19 is new. It deals with relativistic path integrals, which were previously discussed only briefly in two sections at the end of Chapter 15. As an application, the path integral of the relativistic hydrogen atom is solved.

Chapter 16 is extended by a theory of particles with fractional statistics (*anyons*), from which I develop a theory of polymer entanglement. For this I introduce nonabelian Chern-Simons fields and show their relationship with various knot polynomials (Jones, HOMFLY). The successful explanation of the fractional quantum Hall effect by anyon theory is discussed — also the failure to explain high-temperature superconductivity via a Chern-Simons interaction.

Chapter 17 offers a novel variational approach to tunneling amplitudes. It extends the semiclassical range of validity from high to low barriers. As an application, I increase the range of validity of the currently used large-order perturbation theory far into the regime of low orders. This suggests a possibility of greatly improving existing resummation procedures for divergent perturbation series of quantum field theories.

The Index now also contains the names of authors cited in the text. This may help the reader searching for topics associated with these names. Due to their great number, it was impossible to cite all the authors who have made important contributions. I apologize to all those who vainly search for their names.

In writing the new sections in Chapters 4 and 16, discussions with Dr. D. Wintgen and, in particular, Dr. A. Schakel have been extremely useful. I also thank Professors G. Gerlich, P. Hänggi, H. Grabert, M. Roncadelli, as well as Dr. A. Pelster, and Mr. R. Karrlein for many relevant comments. Printing errors were corrected by my secretary Ms. S. Endrias and by my editor Ms. Lim Feng Nee of World Scientific.

Many improvements are due to my wife Annemarie.

H. Kleinert
Berlin, December 1994

Preface to First Edition

These are extended lecture notes of a course on path integrals which I delivered at the Freie Universität Berlin during winter 1989/1990. My interest in this subject dates back to 1972 when the late R. P. Feynman drew my attention to the unsolved path integral of the hydrogen atom. I was then spending my sabbatical year at Caltech, where Feynman told me during a discussion how embarrassed he was, not being able to solve the path integral of this most fundamental quantum system. In fact, this had made him quit teaching this subject in his course on quantum mechanics as he had initially done.[1] Feynman challenged me: "Kleinert, you figured out all that group-theoretic stuff of the hydrogen atom, why don't you solve the path integral!" He was referring to my 1967 Ph.D. thesis[2] where I had demonstrated that all dynamical questions on the hydrogen atom could be answered using only operations within a *dynamical group* $O(4,2)$. Indeed, in that work, the four-dimensional oscillator played a crucial role and the missing steps to the solution of the path integral were later found to be very few. After returning to Berlin, I forgot about the problem since I was busy applying path integrals in another context, developing a field-theoretic passage from quark theories to a collective field theory of hadrons.[3] Later, I carried these techniques over into condensed matter (superconductors, superfluid ^{3}He) and nuclear physics. Path integrals have made it possible to build a unified field theory of collective phenomena in quite different physical systems.[4]

The hydrogen problem came up again in 1978 as I was teaching a course on quantum mechanics. To explain the concept of quantum fluctuations, I gave an introduction to path integrals. At the same time, a postdoc from Turkey, I. H. Duru, joined my group as a Humboldt fellow. Since he was familiar with quantum mechanics, I suggested that we should try solving the path integral of the hydrogen atom. He quickly acquired the basic techniques, and soon we found the most important ingredient to the solution: The transformation of time in the path integral to a new path-dependent pseudotime, combined with a transformation of the coordinates to

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[1]Quoting from the preface of the textbook by R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw-Hill, New York, 1965: "Over the succeeding years, ... Dr. Feynman's approach to teaching the subject of quantum mechanics evolved somewhat away from the initial path integral approach."

[2]H. Kleinert, Fortschr. Phys. *6*, 1, (1968), and *Group Dynamics of the Hydrogen Atom*, Lectures presented at the 1967 Boulder Summer School, published in *Lectures in Theoretical Physics*, Vol. X B, pp. 427–482, ed. by A.O. Barut and W.E. Brittin, Gordon and Breach, New York, 1968.

[3]See my 1976 Erice lectures, *Hadronization of Quark Theories*, published in *Understanding the Fundamental Constituents of Matter*, Plenum press, New York, 1978, p. 289, ed. by A. Zichichi.

[4]H. Kleinert, Phys. Lett. B *69*, 9 (1977); Fortschr. Phys. *26*, 565 (1978); *30*, 187, 351 (1982).

"square root coordinates" (to be explained in Chapters 13 and 14).[5] These transformations led to the correct result, however, only due to good fortune. In fact, our procedure was immediately criticized for its sloppy treatment of the time slicing.[6] A proper treatment could, in principle, have rendered unwanted extra terms which our treatment would have missed. Other authors went through the detailed time-slicing procedure,[7] but the correct result emerged only by transforming the measure of path integration inconsistently. When I calculated the extra terms according to the standard rules I found them to be zero only in two space dimensions.[8] The same treatment in three dimensions gave nonzero "corrections" which spoiled the beautiful result, leaving me puzzled.

Only recently I happened to locate the place where the three-dimensional treatment went wrong. I had just finished a book on the use of gauge fields in condensed matter physics.[9] The second volume deals with ensembles of defects which are defined and classified by means of operational cutting and pasting procedures on an ideal crystal. Mathematically, these procedures correspond to nonholonomic mappings. Geometrically, they lead from a flat space to a space with curvature and torsion. While proofreading that book, I realized that the transformation by which the path integral of the hydrogen atom is solved also produces a certain type of torsion (gradient torsion). Moreover, this happens only in three dimensions. In two dimensions, where the time-sliced path integral had been solved without problems, torsion is absent. Thus I realized that the transformation of the time-sliced measure had a hitherto unknown sensitivity to torsion.

It was therefore essential to find a correct path integral for a particle in a space with curvature and gradient torsion. This was a nontrivial task since the literature was ambiguous already for a purely curved space, offering several prescriptions to choose from. The corresponding equivalent Schrödinger equations differ by multiples of the curvature scalar.[10] The ambiguities are path integral analogs of the so-called *operator-ordering problem* in quantum mechanics. When trying to apply the existing prescriptions to spaces with torsion, I always ran into a disaster, some even yielding noncovariant answers. So, something had to be wrong with all of them. Guided by the idea that in spaces with constant curvature the path integral should produce the same result as an operator quantum mechanics based on a quantization of angular momenta, I was eventually able to find a consistent *quantum equivalence principle*

[5] I.H. Duru and H. Kleinert, Phys. Lett. B *84*, 30 (1979), Fortschr. Phys. *30*, 401 (1982).

[6] G.A. Ringwood and J.T. Devreese, J. Math. Phys. *21*, 1390 (1980).

[7] R. Ho and A. Inomata, Phys. Rev. Lett. *48*, 231 (1982); A. Inomata, Phys. Lett. A *87*, 387 (1981).

[8] H. Kleinert, Phys. Lett. B *189*, 187 (1987); contains also a criticism of Ref. 7.

[9] H. Kleinert, *Gauge Fields in Condensed Matter*, World Scientific, Singapore, 1989, Vol. I, pp. 1–744, *Superflow and Vortex Lines*, and Vol. II, pp. 745–1456, *Stresses and Defects*.

[10] B.S. DeWitt, Rev. Mod. Phys. *29*, 377 (1957); K.S. Cheng, J. Math. Phys. *13*, 1723 (1972), H. Kamo and T. Kawai, Prog. Theor. Phys. *50*, 680, (1973); T. Kawai, Found. Phys. *5*, 143 (1975), H. Dekker, Physica A *103*, 586 (1980), G.M. Gavazzi, Nuovo Cimento *101*A, 241 (1981); M.S. Marinov, Physics Reports *60*, 1 (1980).

for path integrals in spaces with curvature and gradient torsion,[11] thus offering also a unique solution to the operator-ordering problem. This was the key to the leftover problem in the Coulomb path integral in three dimensions — the proof of the absence of the extra time slicing contributions presented in Chapter 13.

Chapter 14 solves a variety of one-dimensional systems by the new techniques.

Special emphasis is given in Chapter 8 to instability (*path collapse*) problems in the Euclidean version of Feynman's time-sliced path integral. These arise for actions containing bottomless potentials. A general stabilization procedure is developed in Chapter 12. It must be applied whenever centrifugal barriers, angular barriers, or Coulomb potentials are present.[12]

Another project suggested to me by Feynman, the improvement of a variational approach to path integrals explained in his book on statistical mechanics[13], found a faster solution. We started work during my sabbatical stay at the University of California at Santa Barbara in 1982. After a few meetings and discussions, the problem was solved and the preprint drafted. Unfortunately, Feynman's illness prevented him from reading the final proof of the paper. He was able to do this only three years later when I came to the University of California at San Diego for another sabbatical leave. Only then could the paper be submitted.[14]

Due to recent interest in lattice theories, I have found it useful to exhibit the solution of several path integrals for a finite number of time slices, without going immediately to the continuum limit. This should help identify typical lattice effects seen in the Monte Carlo simulation data of various systems.

The path integral description of polymers is introduced in Chapter 15 where stiffness as well as the famous excluded-volume problem are discussed. Parallels are drawn to path integrals of relativistic particle orbits. This chapter is a preparation for ongoing research in the theory of fluctuating surfaces with extrinsic curvature stiffness, and their application to world sheets of strings in particle physics.[15] I have also introduced the field-theoretic description of a polymer to account for its increasing relevance to the understanding of various phase transitions driven by fluctuating line-like excitations (vortex lines in superfluids and superconductors, defect lines in crystals and liquid crystals).[16] Special attention has been devoted in Chapter 16 to simple topological questions of polymers and particle orbits, the latter arising by the presence of magnetic flux tubes (Aharonov-Bohm effect). Their relationship to Bose and Fermi statistics of particles is pointed out and the recently popular topic of fractional statistics is introduced. A survey of entanglement phenomena of single orbits and pairs of them (ribbons) is given and their application to biophysics is indicated.

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[11] H. Kleinert, Mod. Phys. Lett. A *4*, 2329 (1989); Phys. Lett. B *236*, 315 (1990).
[12] H. Kleinert, Phys. Lett. B *224*, 313 (1989).
[13] R.P. Feynman, *Statistical Mechanics*, Benjamin, Reading, 1972, Section 3.5.
[14] R.P. Feynman and H. Kleinert, Phys. Rev. A *34*, 5080, (1986).
[15] A.M. Polyakov, Nucl. Phys. B *268*, 406 (1986), H. Kleinert, Phys. Lett. B *174*, 335 (1986).
[16] See Ref. 9.

Finally, Chapter 18 contains a brief introduction to the path integral approach of nonequilibrium quantum-statistical mechanics, deriving from it the standard Langevin and Fokker-Planck equations.

I want to thank several students in my class, my graduate students, and my post-docs for many useful discussions. In particular, T. Eris, F. Langhammer, B. Meller, I. Mustapic, T. Sauer, L. Semig, J. Zaun, and Drs. G. Germán, C. Holm, D. Johnston, and P. Kornilovitch have all contributed with constructive criticism. Dr. U. Eckern from Karlsruhe University clarified some points in the path integral derivation of the Fokker-Planck equation in Chapter 18. Useful comments are due to Dr. P.A. Horvathy, Dr. J. Whitenton, and to my colleague Prof. W. Theis. Their careful reading uncovered many shortcomings in the first draft of the manuscript. Special thanks go to Dr. W. Janke with whom I had a fertile collaboration over the years and many discussions on various aspects of path integration.

Thanks go also to my secretary S. Endrias for her help in preparing the manuscript in LaTeX, thus making it readable at an early stage, and to U. Grimm for drawing the figures.

Finally, and most importantly, I am grateful to my wife Dr. Annemarie Kleinert for her inexhaustible patience and constant encouragement.

H. Kleinert
Berlin, January 1990

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