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Basic research is what I am doing
when I don't know what I am doing.
WERNHER VON BRAUN (1912–1977)

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Basics

A book on multivalued fields must necessarily review some basic concepts of classical mechanics and the theory of single-valued fields. This will be done in the first three chapters. Readers familiar with these subjects may move directly Chapter 4.

In his fundamental work on theoretical mechanics entitled *Principia*, Newton (1642–1727) assumed the existence of an absolute spacetime. Space is parametrized by vectors $\mathbf{x} = (x^1, x^2, x^3)$, and the movement of point particles is described by trajectories $\mathbf{x}(t)$ whose components $q^i(t)\,(i = 1, 2, 3)$ specify the coordinates $x^i = q^i(t)$ along which the particles move as a function of time t. In Newton's absolute spacetime, a single free particle moves without acceleration. Mathematically, this is expressed by the differential equation

$$\ddot{\mathbf{x}}(t) \equiv \frac{d^2}{dt^2}\mathbf{x}(t) = 0. \tag{1.1}$$

The dots denote derivatives with respect to the argument.

A set of N point particles $\mathbf{x}_n(t)$ $(n = 1, \ldots, N)$ with masses m_n is subject to gravitational forces which change the free equations of motion to

$$m_n\ddot{\mathbf{x}}_n(t) = G_{\rm N} \sum_{m\neq n} m_n m_m \frac{\mathbf{x}_m(t) - \mathbf{x}_n(t)}{|\mathbf{x}_m(t) - \mathbf{x}_n(t)|^3}, \tag{1.2}$$

where $G_{\rm N}$ is Newton's gravitational constant

$$G_{\rm N} \approx 6.67259(85) \times 10^{-8}\mathrm{cm}^3/\mathrm{g\,sec}^2. \tag{1.3}$$

1.1 Galilean Invariance of Newtonian Mechanics

The parametrization of absolute spacetime in which the above equations of motion hold is not unique. There is substantial freedom in choosing the coordinates.

1.1.1 Translations

The coordinates $\mathbf{x}$ may always be changed by translated coordinates

$$\mathbf{x}' = \mathbf{x} - \mathbf{x}_0. \tag{1.4}$$

It is obvious that the translated trajectories $\mathbf{x}'_n(t) = \mathbf{x}_n(t) - \mathbf{x}_0$ will again satisfy the equations of motion (1.2). The equations remain also true for a translated time

$$t' = t - t_0, \tag{1.5}$$

i.e., the trajectories

$$\mathbf{x}'(t) \equiv \mathbf{x}(t + t_0) \tag{1.6}$$

satisfy (1.2). This property of Newton's equations (1.2) is referred to as *translational symmetry* in spacetime.

An alternative way of formulating this invariance is by keeping the coordinate frame fixed and displacing the physical system in spacetime, moving all particles to new coordinates $\mathbf{x}' = \mathbf{x} + \mathbf{x}_0$ at a new time $t' = t + t_0$. The equations of motion are again invariant. The first procedure of reparametrizing the same physical system is called *passive symmetry transformation*, the second *active symmetry transformation*. One may use either procedure to discuss symmetries. In this book we shall use active or passive transformations, depending on the circumstance.

1.1.2 Rotations

The equations of motion are invariant under more transformations which mix different coordinates linearly with each other, for instance the *rotations*:

$$x'^i = R^i{}_j x^j, \tag{1.7}$$

where $R^i{}_j$ is the rotation matrix

$$R^i{}_j = \cos\theta\ \delta_{ij} + (1 - \cos\theta)\,\hat{\theta}_i\hat{\theta}_j + \sin\theta\ \epsilon_{ijk}\hat{\theta}_k\,, \tag{1.8}$$

in which $\hat{\theta}_i$ denotes the *directional* unit vector of the rotation axis. The matrices satisfy the *orthogonality relation*

$$R^T R = 1. \tag{1.9}$$

In Eq. (1.7) a sum from 1 to 3 is implied over the repeated spatial index j. This is called the *Einstein summation convention*, which will be followed throughout this text. As for the translations, the rotations can be applied in the passive or active sense.

The active rotations are obtained from the above passive ones by changing the sign of θ. For example, the active rotations around the z-axis with a rotation vector $\hat{\varphi} = (0, 0, 1)$ are given by the orthogonal matrices

$$R_3(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.10}$$

1.1.3 Galilei Boosts

A further set of transformations mixes space and time coordinates:

$$x'^i = x^i - v^i t, \tag{1.11}$$
$$t' = t. \tag{1.12}$$

These are called *pure Galilei transformations* of *Galilei boosts*. The coordinates x'^i, t' are positions and time of a particle observed in a frame of reference that moves uniformly through absolute spacetime with velocity $\mathbf{v} \equiv (v^1, v^2, v^3)$. In the active description, the transformation $x'^i = x^i + v^i t$ specifies the coordinates of a physical system moving past the observer with uniform velocity $\mathbf{v}$.

1.1.4 Galilei Group

The combined set of all transformations

$$x'^i = R^i{}_j x^j - v^i t - x_0^i, \tag{1.13}$$
$$t' = t - t_0, \tag{1.14}$$

forms a group. Group multiplication is defined by performing the transformations successively. This multiplication law is obviously associative, and each element has an inverse. The set of transformations (1.13) and (1.14) is referred to as the *Galilei group*.

Newton called all coordinate frames in which the equations of motion have the simple form (1.2) *inertial frames*.

1.2 Lorentz Invariance of Maxwell Equations

Problems with Newton's theory arose when J. C. Maxwell (1831–1879) formulated in 1864 his theory of electromagnetism. His equations for the *electric field* $\mathbf{E}(x)$ and the *magnetic flux density* or *magnetic induction* $\mathbf{B}(x)$ in empty space

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 0 \qquad \text{(Coulomb's law)}, \tag{1.15}$$
$$\boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = 0 \qquad \text{(Ampère's law)}, \tag{1.16}$$
$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \qquad \text{(absence of magnetic monopoles)}, \tag{1.17}$$
$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \text{(Faraday's law)}, \tag{1.18}$$

can be combined to obtain the second-order differential equations

$$\left(\frac{1}{c^2}\partial_t^2 - \boldsymbol{\nabla}^2\right) \mathbf{E}\,(\mathbf{x}, t) = 0, \tag{1.19}$$
$$\left(\frac{1}{c^2}\partial_t^2 - \boldsymbol{\nabla}^2\right) \mathbf{B}\,(\mathbf{x}, t) = 0. \tag{1.20}$$

The equations contain explicitly the light velocity

$$c \equiv 299\,792\,458\,\frac{\mathrm{m}}{\mathrm{sec}}, \tag{1.21}$$

and are not invariant under the Galilei group (1.14). Indeed, they contradict Newton's postulate of the existence of an absolute spacetime. If light propagates with the velocity c in absolute spacetime, it could not do so in other inertial frames which have a nonzero velocity with respect to the absolute frame. A precise measurement of the light velocity could therefore single out the absolute spacetime. However, experimental attempts to do this did not succeed. The experiment of Michelson (1852–1931) and Morley (1838–1923) in 1887 showed that light travels parallel and orthogonal to the earth's orbital motion with the same velocity up to ± 5 km/sec [1, 2]. This led Fitzgerald (1851–1901) [3], Lorentz (1855–1928) [4], Poincaré (1854–1912) [5], and Einstein (1879–1955) [6] to suggest that Newton's postulate of the existence of an absolute spacetime was unphysical [7].

1.2.1 Lorentz Boosts

The conflict was resolved by modifying the Galilei transformations (1.11) and (1.12) in such a way that Maxwell's equations remain invariant. This is achieved by the coordinate transformations

$$x'^i = x^i + (\gamma - 1)\frac{v^i v^j}{v^2} x^j - \gamma v^i t, \tag{1.22}$$

$$t' = \gamma t - \frac{1}{c^2}\gamma v^i x^i, \tag{1.23}$$

where γ is the velocity-dependent parameter

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{1.24}$$

The transformations (1.22) and (1.23) are referred to as *pure Lorentz transformations* or *Lorentz boosts*. The parameter γ has the effect that in different moving frames of reference, time elapses differently. This is necessary to make the light velocity the same in all frames.

Pure Lorentz transformations are conveniently written in a four-dimensional vector notation. Introducing the *four-vectors* x^a labeled by indices $a, b, c, \ldots$ running through the values $0, 1, 2, 3$,

$$x^a = \begin{pmatrix} ct \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \tag{1.25}$$

we rewrite (1.22) and (1.23) as

$$x'^a = \Lambda^a{}_b x^b, \tag{1.26}$$

where $\Lambda^a{}_b$ are the 4×4-matrices

$$\Lambda^a{}_b \equiv \left(\begin{array}{c|c} \gamma & -\gamma v^i/c \\ \hline -\gamma v^i/c & \delta_{ij} + (\gamma - 1) v_i v_j / v^2 \end{array} \right). \tag{1.27}$$

Note that we adopt Einstein's summation convention also for repeated labels $a, b, c, \ldots = 0, \ldots, 3$. The matrices $\Lambda^a{}_b$ satisfy the *pseudo-orthogonality relation* [compare (1.9)]:

$$\Lambda^T{}_a{}^c \, g_{cd} \, \Lambda^d{}_b = g_{ab}, \tag{1.28}$$

where g_{ab} is the *Minkowski metric* with the matrix elements

$$g_{ab} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}. \tag{1.29}$$

Equation (1.28) has the consequence that for any two four-vectors x^a and y^a, the scalar product formed with the help of the Minkowski metric

$$xy \equiv x^a g_{ab} y^b \tag{1.30}$$

is invariant under Lorentz transformation.

In order to verify the relation (1.28) it is convenient to introduce a dimensionless vector $\boldsymbol{\zeta}$ called *rapidity*, which points in the direction of the velocity $\mathbf{v}$ and has a length $\zeta \equiv |\boldsymbol{\zeta}|$ given by

$$\cosh \zeta = \gamma, \qquad \sinh \zeta = \gamma v / c. \tag{1.31}$$

We also define the unit vectors in three-space

$$\hat{\boldsymbol{\zeta}} \equiv \boldsymbol{\zeta}/\zeta = \hat{\mathbf{v}} \equiv \mathbf{v}/v, \tag{1.32}$$

so that

$$\boldsymbol{\zeta} = \zeta \, \hat{\boldsymbol{\zeta}} = \operatorname{atanh} \frac{v}{c} \, \hat{\mathbf{v}}. \tag{1.33}$$

Then the matrices $\Lambda^a{}_b$ of the pure Lorentz transformations (1.27) take the form

$$\Lambda^a{}_b = B^a{}_b(\boldsymbol{\zeta}) \equiv \left(\begin{array}{c|ccc} \cosh \zeta & -\sinh \zeta \, \hat{\zeta}_1 & -\sinh \zeta \, \hat{\zeta}_2 & -\sinh \zeta \, \hat{\zeta}_3 \\ \hline -\sinh \zeta \, \hat{\zeta}_1 & & & \\ -\sinh \zeta \, \hat{\zeta}_2 & & \delta_{ij} + (\cosh \zeta - 1) \, \hat{\zeta}_i \hat{\zeta}_j & \\ -\sinh \zeta \, \hat{\zeta}_3 & & & \end{array} \right). \tag{1.34}$$

The notation $B^a{}_b(\boldsymbol{\zeta})$ emphasizes that the transformations are boosts. The pseudo-orthogonality property (1.28) follows directly from the identities $\hat{\boldsymbol{\zeta}}^2 = 1$, $\cosh^2 \zeta - \sinh^2 \zeta = 1$.

For active transformations of a physical system, the above transformations have to be inverted. For instance, the active boosts with a rapidity $\boldsymbol{\zeta} = \zeta(0,0,1)$ pointing in the z-direction, have the pseudo-orthogonal matrix

$$\Lambda^a{}_b = B_3(\zeta) = \left(\begin{array}{c|ccc} \cosh\zeta & 0 & 0 & \sinh\zeta \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\zeta & 0 & 0 & \cosh\zeta \end{array}\right). \tag{1.35}$$

1.2.2 Lorentz Group

The set of Lorentz boosts (1.34) can be extended by rotations to form the *Lorentz group*. In 4×4 -matrix notation, the rotation matrices (1.8) have the block form

$$\Lambda^a{}_b(R) = R^a{}_b \equiv \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & R^i{}_j & \\ 0 & & & \end{array}\right). \tag{1.36}$$

It is easy to verify that these satisfy the relation (1.28), which becomes here an orthogonality relation (1.9).

The four-dimensional versions of the active rotations (1.10) around the z-axis with a rotation vector $\hat{\varphi} = (0,0,1)$ are given by the orthogonal matrices

$$\Lambda^b{}_a = R_3(\varphi) = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & \cos\varphi & -\sin\varphi & 0 \\ 0 & \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{array}\right). \tag{1.37}$$

The rotation matrix (1.37) differs from the boost matrix (1.35) mainly in the presence of trigonometric functions instead of hyperbolic functions. In addition, there is a sign change under transposition accounting for the opposite sign in the time- and space-like parts of the metric (1.29).

When combining all possible Lorentz boosts and rotations in succession, the resulting set of transformations forms a group called the *Lorentz group*.

1.3 Infinitesimal Lorentz Transformations

The transformation laws of continuous groups such as rotation and Lorentz group are conveniently expressed in an infinitesimal form. By combining successively many infinitesimal transformations it is always possible to reconstruct from these the finite transformation laws. This is a consequence of the fact that the exponential function e^x can always be obtained by a product of many small-x approximations $e^{\epsilon x} \approx 1+\epsilon x$:

$$e^x = \lim_{\epsilon \to 0} (1 + \epsilon x)^{1/\epsilon}. \tag{1.38}$$

1.3.1 Generators of Group Transformations

Let us illustrate this procedure for the active rotations (1.37). These can be written in the exponential form

$$R_3(\varphi) = \exp\left\{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\varphi\right\} \equiv e^{-iL_3\varphi}. \tag{1.39}$$

The matrix

$$L_3 = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.40}$$

is called the *generator* of this rotation in the Lorentz group. There are similar generators for rotations around x- and y-directions

$$L_1 = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{1.41}$$

$$L_2 = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{1.42}$$

The three generators may compactly be written as

$$L_i \equiv -i\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \epsilon_{ijk} \end{array}\right), \tag{1.43}$$

where ϵ_{ijk} is the completely antisymmetric *Levi-Civita tensor* with $\epsilon_{123} = 1$.

Introducing a vector notation for the three generators, $\mathbf{L} \equiv (L_1, L_2, L_2)$, the general pure rotation matrix (1.36) is given by the exponential

$$\Lambda(R(\boldsymbol{\varphi})) = e^{-i\boldsymbol{\varphi}\cdot\mathbf{L}}. \tag{1.44}$$

This follows from the fact that all orthogonal 3×3-matrices in the spatial block of (1.36) can be written as an exponential of i times all antisymmetric 3×3-matrices, and that these can all be reached by the linear combinations $\boldsymbol{\varphi} \cdot \mathbf{L}$.

Let us now find the generators of the active boosts, first in the z-direction. From Eq. (1.35) we see that the boost matrix can be written as an exponential

$$\begin{aligned} B_3(\zeta) &= \exp\left\{\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\zeta\right\} \\ &= e^{-iM_3\zeta}, \end{aligned} \tag{1.45}$$

with the generator

$$M_3 = i \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{1.46}$$

Similarly we find the generators for the x- and y-directions:

$$M_1 = i \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{1.47}$$

$$M_2 = i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1.48}$$

Introducing a vector notation for the three boost generators, $\mathbf{M} \equiv (M_1, M_2, M_2)$, the general Lorentz transformation matrix (1.34) is given by the exponential

$$\Lambda(B(\boldsymbol{\zeta})) = e^{-i\boldsymbol{\zeta}\cdot\mathbf{M}}. \tag{1.49}$$

The proof is analogous to the proof of the exponential form (1.44).

The Lorentz group is therefore generated by the six matrices L_i, M_i, to be collectively denoted by $G_a (a = 1, \ldots, 6)$. Every element of the group can be written as

$$\Lambda = e^{-i(\boldsymbol{\varphi}\cdot\mathbf{L}+\boldsymbol{\zeta}\cdot\mathbf{M})} \equiv e^{-i\alpha_a G_a}. \tag{1.50}$$

There exists a Lorentz-covariant way of specifying the generators of the Lorentz group. We introduce the 4×4-matrices

$$(L^{ab})^{cd} = i(g^{ac}g^{bd} - g^{ad}g^{bc}), \tag{1.51}$$

labeled by the antisymmetric pair of indices ab, i.e.,

$$L^{ab} = -L^{ba}. \tag{1.52}$$

There are six independent matrices which coincide with the generators of rotations and boosts as follows:

$$L_i = \frac{1}{2}\epsilon_{ijk}L^{jk}, \qquad M_i = L^{0i}. \tag{1.53}$$

With the help of the generators (1.51), we can write every element (1.50) of the Lorentz group as follows

$$\Lambda = e^{-i\frac{1}{2}\omega_{ab}L^{ab}}, \tag{1.54}$$

where the antisymmetric angular matrix $\omega_{ab} = -\omega_{ba}$ collects both, rotation angles and rapidities:

$$\omega_{ij} = \epsilon_{ijk}\varphi^k, \tag{1.55}$$

$$\omega_{0i} = \zeta^i. \tag{1.56}$$

Summarizing the notation we have set up an exponential representation of all Lorentz transformations

$$\Lambda = e^{-i(\boldsymbol{\varphi}\cdot\mathbf{L}+\boldsymbol{\zeta}\cdot\mathbf{M})} = e^{-i(\frac{1}{2}\varphi^i\epsilon_{ijk}L^{jk}+\zeta^i L^{0i})} = e^{-i(\frac{1}{2}\omega_{ij}L^{ij}+\omega_{0i}L^{0i})} = e^{-i\frac{1}{2}\omega_{ab}L^{ab}}. \tag{1.57}$$

Note that for a Euclidean metric

$$g_{ab} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \tag{1.58}$$

the above representation are familiar from basic matrix theory. Then Eq. (1.28) implies that Λ comprises all real orthogonal matrices in four dimensions, which can be written as an exponential of all real antisymmetric 4×4-matrices. For the pseudo-orthogonal matrices satisfying (1.28) with the Minkowski metric (1.29), only the iL_i are antisymmetric while iM_i are symmetric.

1.3.2 Group Multiplication and Lie Algebra

The reason for expressing the group elements as exponentials of the six generators is that, in this way, the *multiplication rules* of infinitely many group elements can be completely reduced to the knowledge of the finite number of *commutation rules* among the six generators L_i, M_i. This is a consequence of the *Baker-Campbell-Hausdorff formula* [8]:

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A-B,[A,B]]-\frac{1}{24}[A,[B,[A,B]]]+\dots}. \tag{1.59}$$

According to this formula, the product of exponentials can be written as an exponential of commutators. Adapting the general notation $G_r = (L_i, M_i)$ for the six generators in Eqs. (1.53) and (1.57), the product of two group elements is

$$\begin{aligned} \Lambda_1\Lambda_2 &= e^{-i\alpha_r^1 G_r} e^{-i\alpha_s^2 G_s} \\ &= \exp\Big\{-i\alpha_r^1 G_r - i\alpha_s^2 G_s + \frac{1}{2}[-i\alpha_r^1 G_r, -i\alpha_s^2 G_s] \\ &\qquad + \frac{1}{12}[-i(\alpha_t^1 - \alpha_t^2)G_t, [-i\alpha_r^1 G_r, -i\alpha_s^2 G_s]] + \dots\Big\}. \end{aligned} \tag{1.60}$$

The exponent involves only commutators among G_r's. For the Lorentz group these can be calculated from the explicit 4×4 -matrices (1.40)–(1.42) and (1.46)–(1.48). The result is

$$
\begin{aligned}
[L_i, L_j] &= i\epsilon_{ijk} L_k, & (1.61)\\
[L_i, M_j] &= i\epsilon_{ijk} M_k, & (1.62)\\
[M_i, M_j] &= -i\epsilon_{ijk} L_k. & (1.63)
\end{aligned}
$$

This algebra of generators is called the *Lie algebra* of the group. In the general notation with generators G_r, the algebra reads

$$[G_r, G_s] = i f_{rst} G_t. \tag{1.64}$$

The number of linearly independent matrices G_r (here 6) is called the *rank* of the Lie algebra.

In any Lie algebra, the commutator of two generators is a linear combination of generators. The coefficients f_{abc} are called *structure constants*. They are completely antisymmetric in a, b, c, and satisfy the relation

$$f_{rsu} f_{utv} + f_{stu} f_{urv} + f_{tru} f_{usv} = 0. \tag{1.65}$$

This guarantees that the generators obey the *Jacobi identity*

$$[[G_r, G_s], G_t] + [[G_s, G_t], G_r] + [[G_t, G_r], G_s] = 0, \tag{1.66}$$

which ensures that multiplication of three exponentials $\Lambda_j = e^{-i\alpha_r^j G_r}$ $(i = 1, 2, 3)$ obeys the law of associativity $(\Lambda_1 \Lambda_2)\Lambda_3 = \Lambda_1(\Lambda_2 \Lambda_3)$ when evaluating the products via the expansion Eq. (1.60).

The relation (1.65) can easily be verified explicitly for the structure constants (1.61)–(1.63) of the Lorentz group using the identity for the ϵ-tensor

$$\epsilon_{ijl}\epsilon_{lkm} + \epsilon_{jkl}\epsilon_{lim} + \epsilon_{kil}\epsilon_{ljm} = 0. \tag{1.67}$$

The Jacobi identity implies that the r matrices with $r \times r$ elements

$$(F_r)_{st} \equiv -i f_{rst} \tag{1.68}$$

satisfy the commutation rules (1.64). They are the generators of the so-called *adjoint representation* of the Lie algebra. The matrix in the spatial block of Eq. (1.43) for L_i is precisely of this type.

In terms of the matrices F_r of the adjoint representation, the commutation rules can also be written as

$$[G_r, G_s] = -(F_t)_{rs} G_t. \tag{1.69}$$

Inserting for G_r the generators (1.68), we reobtain the relation (1.65).

Continuing the expansion in terms of commutators in the exponent of (1.60), all commutators can be evaluated successively and one remains at the end with an expression

$$\Lambda_{12} = e^{-i\alpha_r^{12}(\alpha^1, \alpha^2) G_r}, \tag{1.70}$$

in which the parameters of the product α_r^{12} are completely determined from those of the factor, α_r^1, α_r^2. The result depends only on the structure constants f_{abc}, not on the representation.

If we employ the tensor notation L^{ab} for L_i and M_i of Eqs. (1.53), (1.53), and perform multiplication covariantly, so that products $L^{ab}L^{cd}$ have the matrix elements $(L^{ab})_{\sigma\tau}(L^{cd})^{\tau}{}_{\delta}$, the commutators (1.61)–(1.63) can be written as

$$[L^{ab}, L^{cd}] = -i(g^{ac}L^{bd} - g^{ad}L^{bc} + g^{bd}L^{ac} - g^{bc}L^{ad}). \tag{1.71}$$

Due to the antisymmetry in $a \leftrightarrow b$ and $c \leftrightarrow d$ it is sufficient to specify only the simpler commutators

$$[L^{ab}, L^{ac}] = -ig^{aa}L^{bc}, \quad \text{no sum over } a. \tag{1.72}$$

This list of commutators omits only commutation rules of (1.71) which vanish since none of the indices ab is equal to one of the indices cd.

For infinitesimal transformations, the matrices (1.54) have the general form

$$\Lambda \equiv 1 - i\frac{1}{2}\omega_{ab}L^{ab}. \tag{1.73}$$

Inserting the e 4×4-generators (1.51), their matrix elements are

$$\Lambda^a{}_b = \delta^a{}_b + \omega^a{}_b, \qquad \left(\Lambda^{-1}\right)^a{}_b = \delta^a{}_b - \omega^a{}_b, \tag{1.74}$$

where $\omega^a{}_b$ and $\omega_a{}^b$ are related to the antisymmetric angular matrix ω_{ab} by

$$\omega^a{}_b = g^{aa'}\omega_{a'b}, \qquad \omega_a{}^b = g^{bb'}\omega_{ab'}. \tag{1.75}$$

1.4 Vector-, Tensor-, and Scalar Fields

We shall frequently consider four-component physical quantities v^a which, under Lorentz transformation, change in the same way as the coordinates x^a:

$$v'^a = \Lambda^a{}_b v^b. \tag{1.76}$$

This transformation property defines a *Lorentz vector*, or *four-vector*. In addition to such vectors, there are quantities with more indices $t^{ab}, t^{abc}, \dots$ which transform like products of vectors:

$$t'^{ab} = \Lambda^a{}_c\Lambda^b{}_d t^{cd}, \quad t'^{abc} = \Lambda^a{}_d\Lambda^b{}_e\Lambda^c{}_f t^{def}, \dots \ . \tag{1.77}$$

These are the transformation properties of *Lorentz tensors* of rank two, three, ...

Given any two four-vectors u^a and v^a, we define their scalar product in the same way as in (1.30) for two coordinate vectors x^a and y^a:

$$uv = u^a g_{ab} v^b. \tag{1.78}$$

Scalar products are, of course, invariant under Lorentz transformations due to their pseudo-orthogonality (1.28).

If $v^a, t^{ab}, t^{abc}, \ldots$ are functions of x, they are called *vector and tensor fields*. Derivatives with respect to x of such a field obey vector and tensor transformation laws. Indeed, since

$$x'^a = \Lambda^a{}_b x^b, \tag{1.79}$$

we see that the derivative $\partial/\partial x^b$ satisfies

$$\frac{\partial}{\partial x'^a} = \left(\Lambda^{T-1}\right)_a{}^b \frac{\partial}{\partial x^b}, \tag{1.80}$$

i.e., it transforms with the inverse of the transposed Lorentz matrix $\Lambda^a{}_b$. Using the pseudo-orthogonality relation (1.28),

$$\frac{\partial}{\partial x'^a} = \left(g\Lambda g^{-1}\right)_a{}^b \frac{\partial}{\partial x^b}. \tag{1.81}$$

It will be useful to define the matrix elements

$$\Lambda_a{}^b \equiv \left(g\Lambda g^{-1}\right)_a{}^b = g_{ac}\,\Lambda^c{}_d\, g_{db}. \tag{1.82}$$

The we can rewrite (1.81) as

$$\partial'_a = \Lambda_a{}^{b'} \partial_b. \tag{1.83}$$

In general, any four-component quantity v_a which transforms like the derivatives

$$v'_a = \Lambda_a{}^b v_b \tag{1.84}$$

is called a *covariant* four-vector or Lorentz vector, as opposed to the vector v^a transforming like the coordinates x^a, which is called *contravariant* vector.

A covariant vector v_a can be produced from a contravariant one v^b by multiplication with the metric tensor:

$$v_a = g_{ab} v^b. \tag{1.85}$$

This operation is called *lowering the index*. The operation can be inverted to what is called *raising the index*:

$$v^a = g^{ab} v_b, \tag{1.86}$$

where g^{ab} are the matrix elements of the *inverse metric*

$$g^{ab} \equiv \left(g^{-1}\right)_{ab}. \tag{1.87}$$

With Einstein's summation convention, the inverse metric $g^{ab} \equiv (g^{-1})^{ab}$ satisfies the equation

$$g^{ab} g_{bc} = \delta^a{}_c. \tag{1.88}$$

The sum over a common upper and lower index is called *contraction*.

Note that the notation (1.82) is perfectly compatible with the rules for raising and lowering indices.

In Minkowski spacetime, the matrices g and g^{-1} happen to be the same and so are the matrix elements g_{ab} and g^{ab}, both being equal to (1.29). This is no longer true in the general geometries of gravitational physics. For this reason it will be useful to keep separate symbols for the metric g and its inverse g^{-1}, and for their matrix elements g_{ab} and g^{ab}.

The contraction of a covariant vector with a contravariant vector is a scalar product, as is obvious if we rewrite the scalar product (1.78) as

$$uv = u^a g_{ab} v^b = u^a v_a = u_a v^a. \tag{1.89}$$

Of course, we can form also the scalar product of two covariant vectors with the help of the inverse metric g^{-1}:

$$uv = u_a g^{ab} v_b. \tag{1.90}$$

The invariance under Lorentz transformations (1.84) is easily verified using the pseudo-orthogonality property (1.28):

$$u'_a g^{ab} v'_b = u'^T g^{-1} v' \;\; = \;\; u^T g^{-1} \Lambda^T g\, g^{-1}\, g \Lambda g^{-1} v = u^T g^{-1} v = u_a g^{ab} v_b. \tag{1.91}$$

Since $\partial/\partial x^a$ transforms like a covariant vector, it is useful to emphasize this behavior by the notation

$$\partial_a \equiv \frac{\partial}{\partial x^a}. \tag{1.92}$$

Extending the definition of covariant vectors, one defines covariant tensors of rank two t_{ab}, three t_{abc}, etc. as quantities transforming like

$$t'_{ab} = \Lambda_a{}^c \Lambda_b{}^d\, t_{cd}, \quad t'_{abc} = \Lambda_a{}^c \Lambda_b{}^f \Lambda_c{}^g\, t_{efg}, \dots\ . \tag{1.93}$$

Co- and contravariant vectors and tensors can always be multiplied with each other to form new co- and contravariant quantities if the indices to be contracted are raised and lowered appropriately. If no uncontracted indices are left, one obtains an invariant, a *Lorentz scalar*.

It is useful to introduce a contravariant version of the covariant derivative vector

$$\partial^a \equiv g^{ab} \partial_b, \tag{1.94}$$

and covariant versions of the contravariant coordinate vector

$$x_a \equiv g_{ab} x^b. \tag{1.95}$$

The invariance of Maxwell's equations (1.20) is a direct consequence of these contraction rules since the differential operator on the left-hand side can be written covariantly as

$$\frac{1}{c^2}\partial_t^2 - \nabla^2 = \frac{\partial}{\partial x^a} g^{ab} \frac{\partial}{\partial x^b} = \partial_a g^{ab} \partial_b = \partial^a \partial_a = \partial^2. \tag{1.96}$$

The right-hand side is obviously a Lorentz scalar.

1.4.1 Discrete Lorentz Transformations

The Lorentz group can be extended to include space reflections in any of the four spacetime directions

$$x^a \to -x^a, \tag{1.97}$$

without destroying the defining property (1.28). The determinant of Λ, however, is then negative. If only x^0 is reversed, the reflection is also called *time reversal* and denoted by

$$T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. \tag{1.98}$$

The simultaneous reflection of the three spatial coordinates is called *parity transformation* and denoted by the 4×4 -matrix P:

$$P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}. \tag{1.99}$$

After this extension, the entire Lorentz group can no longer be obtained from the neighborhood of the identity by a product of infinitesimal transformations, i.e., by an exponential of the Lie algebra in Eq. (1.57). It consists of four topologically disjoint pieces which can be obtained by a product of infinitesimal transformations multiplied with 1, P, T, and PT. The four pieces of the group are

$$e^{-i\frac{1}{2}\omega_{ab}L^{ab}}, \quad e^{-i\frac{1}{2}\omega_{ab}L^{ab}}P, \quad e^{-i\frac{1}{2}\omega_{ab}L^{ab}}T, \quad e^{-i\frac{1}{2}\omega_{ab}L^{ab}}PT. \tag{1.100}$$

The Lorentz transformations Λ of the pieces associated with P and T have a negative determinant. This leads to the definition of *pseudotensors* which transform like a tensor, but with an additional determinantal factor $\det \Lambda$. A vector with this property is also called *axial vector*. In three dimensions, the angular momentum $\mathbf{L} = \mathbf{x} \times \mathbf{p}$ is an axial vector since it does not change sign under space reflections, as the vector $\mathbf{x}$ does, but remains invariant.

1.4.2 Poincaré group

Just as the Galilei transformations, the Lorentz transformations can be extended by the group of spacetime translations

$$x^a = x^a - a^a \tag{1.101}$$

to form the *inhomogeneous Lorentz group* or *Poincaré group*.

Inertial frames may be defined as all those frames in which Maxwell's equations are valid. They differ from each other by Poincaré transformations.

$$x'^a = \Lambda^a{}_b x^a - a^a. \tag{1.102}$$

1.5 Differential Operators for Lorentz Transformations

The physical laws in four-dimensional spacetime are formulated in terms of Lorentz-invariant field theories. The fields depend on the spacetime coordinates x^a. In order to perform transformations of the Lorentz group we need differential operators for the generators of this group.

For Lorentz transformations Λ with small rotation angles and rapidities, we can approximate the exponential in (1.57) as

$$\Lambda \equiv 1 - i\,\frac{1}{2}\omega_{ab}L^{ab}. \tag{1.103}$$

The Lorentz transformation of the coordinates

$$x \xrightarrow{\Lambda} x' = \Lambda x \tag{1.104}$$

is conveniently characterized by the infinitesimal change

$$\delta_\Lambda x = x' - x = -i\,\frac{1}{2}\omega_{ab}L^{ab}x. \tag{1.105}$$

Inserting the 4×4 -matrix generators (1.51), this becomes more explicitly [compare (1.74)]

$$\delta_\Lambda x^a = \omega^a{}_b x^b. \tag{1.106}$$

We now observe that (1.105) can be expressed in terms of the differential operators

$$\hat{L}^{ab} \equiv i(x^a\partial^b - x^b\partial^a) = -\hat{L}^{ba} \tag{1.107}$$

as a commutator

$$\delta_\Lambda x = i\,\frac{1}{2}\omega_{ab}[\hat{L}^{ab}, x]. \tag{1.108}$$

The differential operators (1.107) satisfy the same commutation relations (1.71), (1.72) as the 4×4 -generators L^{ab} of the Lorentz group. They form a representation of the Lie algebra (1.71), (1.72). By exponentiation we can thus form the operator representation of finite Lorentz transformations

$$\hat{D}(\Lambda) \equiv e^{-i\,\frac{1}{2}\omega_{ab}\hat{L}^{ab}}, \tag{1.109}$$

which satisfy the same group multiplication rules as the 4×4-matrices Λ.

The relation between the finite Lorentz transformations (1.104) and the operator version (1.109) is

$$x' = \Lambda x = e^{-i\,\frac{1}{2}\omega_{ab}L^{ab}}x = e^{i\,\frac{1}{2}\omega_{ab}\hat{L}^{ab}}\,x\,e^{-i\,\frac{1}{2}\omega_{ab}\hat{L}^{ab}} = \hat{D}^{-1}(\Lambda)\,x\,\hat{D}(\Lambda). \tag{1.110}$$

This is proved by expanding, on the left-hand side, $e^{-i\,\frac{1}{2}\omega_{ab}L^{ab}}x$ in powers of ω_{ab}, and doing the same on the right-hand expression $e^{i\,\frac{1}{2}\omega_{ab}\hat{L}^{ab}}x\,e^{-i\,\frac{1}{2}\omega_{ab}\hat{L}^{ab}}$ with the help of *Lie's expansion formula*

$$e^{-i\hat{A}}\,\hat{B}\,e^{iA} = 1 - i[\hat{A},\hat{B}] + \frac{i^2}{2!}[\hat{A},[\hat{A},\hat{B}]] + \dots\ . \tag{1.111}$$

This operator representation (1.109) can be used to generate Lorentz transformations on the spacetime argument of any function of x:

$$f'(x) \equiv f(\Lambda^{-1}x) = f\left(\hat{D}(\Lambda)x\,\hat{D}^{-1}(\Lambda)\right) = \hat{D}(\Lambda)f(x)\,\hat{D}^{-1}(\Lambda). \tag{1.112}$$

The latter step follows from a power series expansion of $f(x)$. Take for example an expansion term $f_{a,b}x^a x^b$ of $f(x)$. In the transformed function $f'(x)$, this becomes

$$f_{a,b}\hat{D}(\Lambda)x^a\,\hat{D}^{-1}(\Lambda)\hat{D}(\Lambda)x^b\hat{D}^{-1}(\Lambda) = \hat{D}(\Lambda)\left(f_{a,b}x^a x^b\right)\hat{D}^{-1}(\Lambda). \tag{1.113}$$

1.6 Vector and Tensor Operators

In working out the commutation rules among the differential operators $\hat{L}^{ab}$ one conveniently uses the commutation rules between $\hat{L}^{ab}$ and x^c, $\hat{p}^c$:

$$[\hat{L}^{ab}, x^c] = -i(g^{ac}x^b - g^{bc}x^a) = -(L^{ab})^c{}_d x^d, \tag{1.114}$$
$$[\hat{L}^{ab}, \hat{p}^c] = -i(g^{ac}\hat{p}^b - g^{bc}\hat{p}^a) = -(L^{ab})^c{}_d\hat{p}^d. \tag{1.115}$$

These commutation rules identify x^c and $\hat{p}^c$ as *vector operators*

In general, an operator $\hat{t}^{c_1,\cdots,c_n}$ is said to be a *tensor operator* of rank n if each of its tensor indices is transformed under commutation with L^{ab} like the index of x^a or $\hat{p}^a$ in (1.114) and (1.115):

$$\begin{aligned}[\hat{L}^{ab}, \hat{t}^{c_1,\dots,c_n}] &= -i[(g^{ac_1}\hat{t}^{b,\dots,c_n} - g^{bc_1}\hat{t}^{a,\dots,c_n}) + \ldots + (g^{ac_n}\hat{t}^{c_1,\dots,b} - g^{bc_n}\hat{t}^{c_1,\dots,a})] \\ &= -(L^{ab})^{c_1}{}_d\,\hat{t}^{dc_2,\dots,c_n} - (L^{ab})^{c_2}{}_d\,\hat{t}^{c_1d,\dots,c_n} - \ldots - (L^{ab})^{c_n}{}_d\,\hat{t}^{c_1c_2,\dots,d}.\end{aligned} \tag{1.116}$$

The commutators (1.71) between the generators imply that these are themselves tensor operators.

The simplest examples for such tensor operators are the direct products of vectors such as $\hat{t}^{c_1,\dots,c_n} = x^{c_1}\cdots x^{c_n}$ or $\hat{t}^{c_1,\dots,c_n} = \hat{p}^{c_1}\cdots\hat{p}^{c_n}$. In fact, the right-hand side can be found for such direct products using the commutation rules between products of operators

$$[\hat{a}, \hat{b}\hat{c}] = [\hat{a}, \hat{b}]\hat{c} + \hat{b}[\hat{a}, \hat{c}], \quad [\hat{a}\hat{b}, \hat{c}] = \hat{a}[\hat{b}, \hat{c}] + [\hat{a}, \hat{c}]\hat{b}. \tag{1.117}$$

These are the analogs of the *Leibnitz chain rule* for derivatives

$$\partial(fg) = (\partial f)g + f(\partial g). \tag{1.118}$$

1.7 Behavior of Vectors and Tensors under Finite Lorentz Transformations

Let us apply such a finite operator representation (1.109) to the vector x^c to form

$$\hat{D}(\Lambda)x^c\hat{D}^{-1}(\Lambda). \tag{1.119}$$

We shall do this separately for rotations and Lorentz transformations.

1.7.1 Rotations

An arbitrary three-vector (x^1, x^2, x^3) is rotated around the 3-axis by the operator $\hat{D}(R_3(\varphi)) = e^{-i\varphi\hat{L}_3}$ with $\hat{L}_3 = -i(x^1\partial_2 - x^2\partial_1)$ by the operation

$$\hat{D}(R_3(\varphi))x^i\hat{D}^{-1}(R_3(\varphi)) = e^{-i\varphi\hat{L}_3}x^i e^{i\varphi\hat{L}_3}. \tag{1.120}$$

Since $\hat{L}_3$ commutes with x^3, this component is invariant under the operation (1.120):

$$\hat{D}(R_3(\varphi))x^3\hat{D}^{-1}(R_3(\varphi)) = e^{-i\varphi\hat{L}_3}x^3 e^{i\varphi\hat{L}_3} = x^3. \tag{1.121}$$

For x^1 and x^2, the Lie expansion of (1.119) contains the commutators

$$-i[L_3, x^1] = x^2, \qquad -i[L_3, x^2] = -x^1. \tag{1.122}$$

Thus, the first-order expansion term on the right-hand side of (1.120) transforms the two-dimensional vector (x^1, x^2) into $(x^2, -x^1)$. The second-order term is obtained by commuting the operator $-i\hat{L}_3$ with $(x^2, -x^1)$, yielding $-(x^1, x^2)$. To third-order, this is again transformed into $-(x^2, -x^1)$, and so on. Obviously, all even orders reproduce the initial two-dimensional vector (x^1, x^2) with an alternating sign, while all odd powers are proportional to $(x^2, -x^1)$. Thus we obtain the expansion

$$\begin{aligned} e^{-i\varphi\hat{L}_3}(x^1, x^2)e^{i\varphi\hat{L}_3} &= \left(1 - \frac{1}{2!}\varphi^2 + \frac{1}{4!}\varphi^4 + \ldots\right)(x^1, x^2) \\ &+ \left(\varphi - \frac{1}{3!}\varphi^3 + \frac{1}{5!}\varphi^5 + \ldots\right)(x^2, -x^1). \end{aligned} \tag{1.123}$$

The even and odd powers can be summed up to a cosine and a sine, respectively, resulting in

$$e^{-i\varphi\hat{L}_3}(x^1, x^2)e^{i\varphi\hat{L}_3} = \cos\varphi\ (x^1, x^2) + \sin\varphi\ (x^2, -x^1). \tag{1.124}$$

Together with the invariant x^3 in (1.121), the right-hand side forms a vector arising from x^i by an *inverse* rotation (1.37). Thus

$$\hat{D}(R_3(\varphi))x^i\hat{D}^{-1}(R_3(\varphi)) = e^{-i\varphi\hat{L}_3}x^i e^{i\varphi\hat{L}_3} = \left(e^{i\varphi L_3}\right)^i{}_j x^j = R_3^{-1}(\varphi)^i{}_j x^j. \tag{1.125}$$

By performing successive rotations around the three axes we can generate in this way any inverse rotation:

$$\hat{D}(R(\boldsymbol{\varphi}))x^i\hat{D}^{-1}(R(\boldsymbol{\varphi})) = e^{-i\boldsymbol{\varphi}\cdot\hat{\mathbf{L}}}x^i e^{i\boldsymbol{\varphi}\cdot\hat{\mathbf{L}}} = \left(e^{i\boldsymbol{\varphi}\cdot\hat{\mathbf{L}}}\right)^i{}_j x^j = R^{-1}(\boldsymbol{\varphi})^i{}_j x^j. \tag{1.126}$$

This is the finite transformation law associated with the commutation relation

$$[\hat{L}_i, x_k] = x_j(L_i)_{jk}, \tag{1.127}$$

which characterizes the vector operator nature of x^i [compare (1.114)]. Thus also (1.126) holds for finite rotations of any vector operator $\hat{v}^i$.

The time component x^0 is obviously unchanged by rotations since $\hat{L}_3$ commutes with x^0. Hence we can extend (1.126) trivially to a four-vector, replacing $\hat{D}(R(\boldsymbol{\varphi}))$ by $\hat{D}(\Lambda(R(\boldsymbol{\varphi})))$ [recall (1.44)].

1.7.2 Lorentz Boosts

A similar calculation may be done for Lorentz boosts. Here we first consider a boost in the 3-direction $B_3(\zeta) = e^{-i\zeta\hat{M}_3}$ generated by $\hat{M}_3 = \hat{L}^{03} = -i(x^0\partial_3 + x^3\partial_0)$ [recall (1.57), (1.53), and (1.107)]. Note the positive relative sign of the two terms in the generator $\hat{L}^{03}$ is caused by the fact that $\partial_i = -\partial^i$, in contrast to $\partial_0 = \partial^0$. Thus we form

$$\hat{D}(B_3(\zeta))x^i\hat{D}^{-1}(B_3(\zeta)) = e^{-i\zeta\hat{M}_3}x^i e^{i\zeta\hat{M}_3}. \tag{1.128}$$

The Lie expansion of the right-hand side involves the commutators

$$-i[M_3, x^0] = -x^3, \quad -i[M_3, x^3] = -x^0, \quad -i[M_3, x^1] = 0, \quad -i[M_3, x^2] = 0. \tag{1.129}$$

Here the two-vector (x^1, x^2) is unchanged, while the two-vector (x^0, x^3) is transformed into $-(x^3, x^0)$. In the second expansion term, the latter becomes (x^0, x^3), and so on, yielding

$$\begin{aligned} e^{-i\zeta\hat{M}_3}(x^0, x^3)e^{i\zeta\hat{M}_3} &= \left(1 + \frac{1}{2!}\zeta^2 + \frac{1}{4!}\zeta^4 + \dots\right)(x^0, x^3) \\ &- \left(\zeta + \frac{1}{3!}\zeta^3 + \frac{1}{5!}\zeta^5 + \dots\right)(x^3, x^0). \end{aligned} \tag{1.130}$$

The right-hand sides can be summed up to hyperbolic cosines and sines:

$$e^{-i\zeta\hat{M}_3}(x^0, x^3)e^{i\zeta\hat{M}_3} = \cosh\zeta\ (x^0, x^3) - \sinh\zeta\ (x^3, x^0). \tag{1.131}$$

Together with the invariance of (x^1, x^2), this corresponds precisely to the inverse of the boost transformation (1.35):

$$\hat{D}(B_3(\zeta))x^a\hat{D}^{-1}(B_3(\zeta)) = e^{-i\zeta\hat{M}_3}x^a e^{i\zeta\hat{M}_3} = \left(e^{i\zeta M_3}\right)^a{}_b x^b = B_3^{-1}(\zeta)^a{}_b x^b. \tag{1.132}$$

1.7.3 Lorentz Group

By performing successive rotations and boosts in all directions we find all Lorentz transformations

$$\hat{D}(\Lambda)x^c\hat{D}^{-1}(\Lambda) = e^{-i\frac{1}{2}\omega_{ab}\hat{L}^{ab}}x^c e^{i\frac{1}{2}\omega_{ab}\hat{L}^{ab}} = (e^{i\frac{1}{2}\omega_{ab}L^{ab}})^c{}_{c'}x^{c'} = (\Lambda^{-1})^c{}_{c'}x^{c'}, \tag{1.133}$$

where ω_{ab} are the parameters (1.55) and (1.56). In the last term on the right-hand side we have expressed the 4×4 -matrix Λ as an exponential of its generators, to emphasize the one-to-one correspondence between the generators L^{ab} and their differential-operator representation $\hat{L}^{ab}$.

At first it may seem surprising that the group transformations appearing as a left-hand factor of the two sides of these equations are *inverse* to each other. However, we may easily convince ourselves that this is necessary to guarantee the correct group multiplication law. Indeed, if we perform two successive transformations they appear in opposite order on the right- and left-hand sides:

$$\begin{aligned} &\hat{D}(\Lambda_2\Lambda_1)x^c\hat{D}^{-1}(\Lambda_2\Lambda_1) = \hat{D}(\Lambda_2)\hat{D}(\Lambda_1)x^c\hat{D}^{-1}(\Lambda_1)\hat{D}^{-1}(\Lambda_2) \\ &\quad = (\Lambda_1^{-1})^c{}_{c'}\hat{D}(\Lambda_2)x^{c'}\hat{D}^{-1}(\Lambda_2) = (\Lambda_1^{-1})^c{}_{c'}(\Lambda_2^{-1})^{c'}{}_{c''}x^{c''} = [(\Lambda_2\Lambda_1)^{-1}]^c{}_{c'}x^{c'}. \end{aligned} \tag{1.134}$$

If the right-hand side of (1.133) would contain Λ instead of Λ^{-1}, the order of the factors in $\Lambda_2\Lambda_1$ on the right-hand side of (1.134) would be opposite to the order in $\hat{D}(\Lambda_2\Lambda_1)$ on the left-hand side.

A straightforward extension of the operation (1.133) yields the transformation law for a tensor $\hat{t}^{c_1,\dots,c_n} = x^{c_1}\cdots x^{c_n}$:

$$\begin{aligned}\hat{D}(\Lambda)\hat{t}^{c_1,\dots,c_n}\hat{D}^{-1}(\Lambda) &= e^{-i\frac{1}{2}\omega_{ab}\hat{L}^{ab}}\,\hat{t}^{c_1,\dots,c_n}\,e^{i\frac{1}{2}\omega_{ab}\hat{L}^{ab}} \\ &= (\Lambda^{-1})^{c_1}{}_{c_1'}\cdots(\Lambda^{-1})^{c_n}{}_{c_n'}\,\hat{t}^{c_1',\dots,c_n'} \\ &= (e^{i\frac{1}{2}\omega_{ab}L^{ab}})^{c_1}{}_{c_1'}\cdots(e^{i\frac{1}{2}\omega_{ab}L^{ab}})^{c_n}{}_{c_n'}\,\hat{t}^{c_1',\dots,c_n'}. \end{aligned} \tag{1.135}$$

This follows directly by inserting an auxiliary unit factor $1 = \hat{D}(\Lambda)\hat{D}^{-1}(\Lambda) = e^{-i\frac{1}{2}\omega_{ab}\hat{L}^{ab}}e^{i\frac{1}{2}\omega_{ab}\hat{L}^{ab}}$ into the product $x^{c_1}\cdots x^{c_n}$ between neighboring factors x^{c_i}, and performing the operation (1.135) on each of them. The last term in (1.135) can also be written as

$$\left[e^{i\frac{1}{2}\omega_{ab}(L^{ab}\times 1\times 1\cdots\times 1\;+\;\dots\;+\;1\times L^{ab}\times 1\cdots\times 1)}\right]^{c_1\dots c_n}{}_{c_1'\dots c_n'}\;t^{c_1'\dots c_n'}. \tag{1.136}$$

Since the commutation relations (1.116) determine the result completely, the transformation formula (1.135) is true for any tensor operator $\hat{t}^{c_1,\dots,c_n}$, not only for those composed from a product of vectors x^{c_i}.

The result can easily be extended to an exponential function e^{-ipx}, and further to any function $f(x)$ which possesses a Fourier representation

$$\hat{D}(\Lambda)f(x)\hat{D}^{-1}(\Lambda) \;=\; f(\Lambda^{-1}x) = e^{-i\frac{1}{2}\omega_{ab}\hat{L}^{ab}}\,f(x)\,e^{i\frac{1}{2}\omega_{ab}\hat{L}^{ab}}. \tag{1.137}$$

Since the last differential operator has nothing to act on, it can also be omitted and we can also write

$$\hat{D}(\Lambda)f(x)\hat{D}^{-1}(\Lambda) \;=\; f(\Lambda^{-1}x) = e^{-i\frac{1}{2}\omega_{ab}\hat{L}^{ab}}\,f(x). \tag{1.138}$$

1.8 Relativistic Point Mechanics

The Lorentz invariance of the Maxwell equations explains the observed invariance of the light velocity in different inertial frames. It is, however, incompatible with Newton's mechanics. There exists a modification of Newton's laws which makes them Lorentz-invariant as well, while differing very little from Newton's equations in their description of slow macroscopic bodies, for which Newton's equations were originally designed. Let us introduce the Poincaré-invariant distance measure in spacetime

$$ds \equiv \sqrt{dx^2} = \left(g_{ab}dx^a dx^b\right)^{1/2}. \tag{1.139}$$

At a fixed coordinate point of an inertial frame, ds is equal to c times the elapsed time:

$$ds = \sqrt{g_{00}dx^0dx^0} = dx^0 = cdt. \tag{1.140}$$

Einstein called the quantity

$$\tau \equiv s/c \tag{1.141}$$

the *proper time*.

When going from one inertial frame to another, two simultaneous events at different points in the first frame will take place at *different* times in the other frame. Their invariant distance, however, remains the same, due to the pseudo-orthogonality relation (1.28) which ensures that

$$ds' = \left(g_{ab}dx'^a dx'^b\right)^{1/2} = \left(g_{ab}dx^a dx^b\right)^{1/2} = ds. \tag{1.142}$$

A particle moving with a constant velocity along a trajectory $\mathbf{x}(t)$ in one Minkowski frame remains at rest in another frame moving with velocity $\mathbf{v} = \dot{\mathbf{x}}(t)$ relative to the first. Its proper time is then related to the coordinate time in the first frame by the Lorentz transformation

$$cd\tau = ds = \sqrt{c^2dt^2 - d\mathbf{x}^2} = cdt\sqrt{1 - \frac{1}{c^2}\left(\frac{d\mathbf{x}}{dt}\right)^2} = cdt\sqrt{1 - \frac{\mathbf{v}^2}{c^2}} = \frac{cdt}{\gamma}. \tag{1.143}$$

This is the famous Einstein relation implying that a moving particle lives longer by a factor γ. There exists direct experimental evidence for this phenomenon. For example, the meson π^+ lives on the average $\tau_a = 2.60 \times 10^{-8}$ sec, after which it decays into a muon and a neutrino. If the pion is observed in a bubble chamber with a velocity equal to 10% of the light velocity $c \equiv 299\,792\,458$ m/sec, it leaves trace of an average length $l \approx \tau_a \times c \times 0.1/\sqrt{1 - 0.1^2} \approx 0.78$ cm. A very fast muon moving with 90% of the light velocity, however, leaves a trace which is longer by a factor $(0.9/0.1) \times \sqrt{1 - 0.10^2}/\sqrt{1 - 0.9^2} \equiv 20.6$. Massless particles move with light velocity and have $d\tau = 0$, i.e., the proper time stands still along their paths. This implies that massless particles can never decay — they are necessarily stable particles.

Another way to see the time dilation is by observing the spectral lines of a moving atom, say a hydrogen atom. If the atom is at rest, the frequency of the line is given by

$$\nu = -\mathrm{Ry}\left(\frac{1}{n^2} - \frac{1}{m^2}\right) \tag{1.144}$$

where $\mathrm{Ry} = m_e c^2 \alpha^2/2 \approx 13.6$ eV, is the *Rydberg constant*,

$$\alpha \equiv \frac{e^2}{4\pi\hbar c} \approx 1/137.035\,989 \tag{1.145}$$

is the *fine-structure constant*, and n and m are the principal quantum numbers of initial and final electron orbits. If the atom emits a light quantum while moving with velocity $\mathbf{v}$ through the laboratory orthogonal to the direction of observation, this frequency is *lowered* by a factor $1/\gamma$:

$$\frac{\nu_{\text{obs}}}{\nu} = \frac{1}{\gamma} = \sqrt{1 - \frac{v^2}{c^2}}. \tag{1.146}$$

If the atom runs away from the observer or towards him, the frequency is further changed by the *Doppler shift*. Due to the growing or decreasing distance, the wave trains arrive with a smaller or higher frequency given by

$$\frac{\nu_{\text{obs}}}{\nu} = \left(1 \pm \frac{v}{c}\right)^{-1} \frac{1}{\gamma} = \sqrt{\frac{1 \mp v/c}{1 \pm v/c}}. \tag{1.147}$$

In the first case the observer sees an additional *red shift*, in the second a *violet shift* of the spectral lines.

Without external forces, the trajectories of free particles are straight lines in four-dimensional spacetime. If the particle positions are parametrized by the proper time τ, they satisfy the equation of motion

$$\frac{d^2}{d\tau^2} x^a(\tau) = \frac{d}{d\tau} p^a(\tau) = 0. \tag{1.148}$$

The first derivative of $x^a(\tau)$ is the *relativistic four-vector of momentum* $p^a(\tau)$, briefly called *four-momentum*:

$$p^a(\tau) \equiv m \frac{d}{d\tau} x^a(\tau) \equiv m\, u^a(\tau). \tag{1.149}$$

On the right-hand side we have introduced the *relativistic four-vector of velocity* $u^a(\tau)$, or *four-velocity*. Inserting (1.143) into (1.149) we identify the components of $u^a(\tau)$ as

$$u^a = \begin{pmatrix} \gamma c \\ \gamma v^a \end{pmatrix}, \tag{1.150}$$

and see that $u^a(\tau)$ is normalized to the light-velocity:

$$u^a(\tau) u_a(\tau) = c^2. \tag{1.151}$$

The time and space components of (1.149) are

$$p^0 = m\gamma c = m u^0, \qquad p^i = m\gamma v^i = m u^i. \tag{1.152}$$

This shows that the time dilation factor γ is equal to p^0/mc, and the same factor increases the spatial momentum with respect to the nonrelativistic momentum mv^i. This correction becomes important for particles moving near the velocity of light, which are called *relativistic*. The light particle has $m = 0$ and $v = c$. It is ultra-relativistic.

Note that by Eq. (1.152), the hyperbolic functions of the rapidity in Eq. (1.31) are related to the four velocity and to energy and momentum by

$$\cosh\zeta = u^0/c = p^0/mc, \qquad \sinh\zeta = |\mathbf{u}|/c = |\mathbf{p}|/mc. \tag{1.153}$$

Under a Lorentz transformation of space and time, the four-momenta p^a transform in exactly the same way as the coordinate four-vectors x^a. This is, of course, due to the Lorentz invariance of the proper time τ in Eq. (1.149). Indeed, from Eq. (1.152) we derive the important relation

$${p^0}^2 - \mathbf{p}^2 = m^2c^2, \tag{1.154}$$

which shows that the square of the four-momentum taken with the Minkowski metric is an invariant:

$$p^2 \equiv p^a g_{ab} p^b = m^2c^2. \tag{1.155}$$

Since both x^a and p^a are Lorentz vectors, the scalar product of them,

$$xp \equiv g_{ab}x^a p^b, \tag{1.156}$$

is an invariant. In the canonical formalism, the momentum p^i is the conjugate variable to the space coordinate x^i. Equation (1.156) suggests that the quantity cp^0 is conjugate to $x^0/c = t$. As such it must be the energy of the particle:

$$E = cp^0. \tag{1.157}$$

From relation (1.154), we calculate the energy as a function of the momentum of a relativistic particle:

$$E = c\sqrt{\mathbf{p}^2 + m^2c^2}. \tag{1.158}$$

For small velocities, this can be expanded as

$$E = mc^2 + \frac{m}{2}\mathbf{v}^2 + \dots\ . \tag{1.159}$$

The first term gives a nonvanishing *rest energy* which is unobservable in nonrelativistic physics. The second term is Newton's kinetic energy.

The first term has dramatic observable effects. Particles can be produced and disappear in collision processes. In the latter case, their rest energy mc^2 can be transformed into kinetic energy of other particles. The large factor c makes unstable particles a source of immense energy, with disastrous consequences for Hiroshima and Nagasaki in 1945.

1.9 Quantum Mechanics

In quantum mechanics, free spinless particles of momentum p are described by plane waves of the form

$$\phi_p(x) = \mathcal{N}\ e^{-ipx/\hbar}, \tag{1.160}$$

where $\mathcal{N}$ is some normalization factor. The momentum components are the eigenvalue of the differential operators

$$\hat{p}_a = i\hbar\frac{\partial}{\partial x^a}, \tag{1.161}$$

which satisfy with x^b the commutation rules

$$[\hat{p}_a, x^b] = i\hbar\delta_a{}^b. \tag{1.162}$$

In terms of these, the generators (1.107) can be rewritten as

$$\hat{L}^{ab} \equiv \frac{1}{\hbar}(x^a\hat{p}^b - x^b\hat{p}^a). \tag{1.163}$$

Apart from the factor $1/\hbar$, this is the tensor version of the four-dimensional angular momentum.

It is worth observing that the differential operators (1.163) can also be expressed as a sandwich of the 4×4 -matrix generators (1.51) between x^c and $\hat{p}^d$:

$$\hat{L}^{ab} = -\frac{i}{\hbar}(L^{ab})_{cd}x^c\hat{p}^d = -\frac{i}{\hbar}x^TL^{ab}\hat{p} = i\hat{p}^TL^{ab}x. \tag{1.164}$$

This way of forming operator representations of the 4×4 -Lie algebra (1.71) is a special application of a general construction technique of higher representations of a defining matrix representations. In fact, the procedure of second quantization is based on this construction, which extends the single-particle Schrödinger operators to the Fock space of many-particle states.

In general, one may always introduce vectors of creation and annihilation operators $\hat{a}_c^\dagger$ and $\hat{a}^d$ with the commutation rules

$$[\hat{a}^c, \hat{a}^d] = [\hat{a}_c^\dagger, \hat{a}_d^\dagger] = 0; \quad [\hat{a}^c, \hat{a}_d^\dagger] = \delta^c{}_d, \tag{1.165}$$

and form sandwich operators

$$\hat{L}^{ab} = \hat{a}_c^\dagger(L^{ab})^c{}_d\hat{a}^d. \tag{1.166}$$

These satisfy the same commutation rules as the sandwiched matrices due to the Leibnitz chain rule (1.117). Since $-i\hat{p}_a/\hbar$ and x^a commute in the same way as $\hat{a}$ and $\hat{a}^\dagger$, the commutation rules of the matrices go directly over to the sandwich operators (1.164). The higher representations generated by them lie in the Hilbert space of square-integrable functions.

Under a Lorentz transformation, the momentum of the particle described by the wave function (1.160) goes over into $p' = \Lambda p$, so that the wave function transforms as follows:

$$\phi_p(x) \xrightarrow{\Lambda} \phi'_p(x) \equiv \phi_{p'}(x) = \mathcal{N}e^{-i(\Lambda p)x} = \mathcal{N}e^{-ip\Lambda^{-1}x} = \phi_p(\Lambda^{-1}x). \tag{1.167}$$

This can also be written as $\phi'_{p'}(x') = \phi_p(x)$. An arbitrary superposition of such waves transforms like

$$\phi(x) \xrightarrow{\Lambda} \phi'(x) = \phi(\Lambda^{-1}x), \tag{1.168}$$

which is the defining relation for a *scalar field*.

The transformation (1.168) may be generated by the differential-operator representation of the Lorentz group (1.138) as follows:

$$\phi(x) \xrightarrow{\Lambda} \phi'(x) = \hat{D}(\Lambda)\phi(x). \tag{1.169}$$

1.10 Relativistic Particles in Electromagnetic Field

Lorentz and Einstein formulated a theory of relativistic massive particles with electromagnetic interactions referred to as *Maxwell-Lorentz theory*. It is invariant under the Poincaré group and describes the dynamical properties of charged particles such as electrons moving with nonrelativistic and relativistic speeds.

The motion for a particle of charge e and mass m in an electromagnetic field is governed by the *Lorentz equations*

$$\frac{dp^a(\tau)}{d\tau} = m\frac{d^2x^a(\tau)}{d\tau^2} = f^a(\tau), \tag{1.170}$$

where f^a is the four-vector associated with the *Lorentz force*

$$f^a = \frac{e}{c}F^a{}_b\frac{dx^b}{d\tau} = \frac{e}{mc}F^a{}_b(x(\tau))\, p^b(\tau), \tag{1.171}$$

and $F^a{}_b(x)$ is a 4×4 -combination of electric and magnetic fields with the components

$$F^i{}_j = \epsilon^{ijk}B^k, \quad F^0{}_i = E^i. \tag{1.172}$$

By raising the second index of $F^a{}_b$ one obtains the tensor

$$F^{ac} = g^{cb}F^a{}_b \tag{1.173}$$

associated with the antisymmetric matrix of the six electromagnetic fields

$$F^{ab} = \left(\begin{array}{c|ccc} 0 & -E^1 & -E^2 & -E^3 \\ \hline E^1 & 0 & B^3 & -B^2 \\ E^2 & -B^3 & 0 & B^1 \\ E^3 & B^2 & -B^1 & 0 \end{array}\right). \tag{1.174}$$

This tensor notation is useful since F^{ab} transforms under the Lorentz group in the same way as the direct product x^ax^b, which goes over into $x'^a x'^b = \Lambda^a{}_c\Lambda^b{}_d\, x^cx^d$. In

$F^{ab}(x)$, also the arguments must be transformed as in the scalar field in Eq. (1.168), so that we find the generic transformation behavior of a *tensor field*:

$$F^{ab}(x) \xrightarrow{\Lambda} F'^{ab}(x) = \Lambda^a{}_c \Lambda^b{}_d \, F^{cd}(\Lambda^{-1}x). \tag{1.175}$$

Recalling the exponential representation (1.136) of the direct product of the Lorentz transformations and the differential operator generation (1.138) of the transformation of the argument x, this can also be written as

$$F^{ab}(x) \xrightarrow{\Lambda} F'^{ab}(x) = [e^{-i\frac{1}{2}\omega_{ab}\hat{J}^{ab}} \, F]^{ab}(\Lambda^{-1}x), \tag{1.176}$$

where

$$\hat{J}^{cd} \equiv L^{cd} \times 1 + 1 \times L^{cd} \tag{1.177}$$

are the generators of the total four-dimensional angular momentum of the tensor field. The factors in the direct products apply successively to the representation spaces associated with the two Lorentz indices and the spacetime coordinates. The generators $\hat{J}^{ab}$ obey the same commutation rules (1.71) and (1.72) as L_{ab} and $\hat{L}_{ab}$.

In order to verify the transformation law (1.175), we recall the basic result of electromagnetism that, under a change to a coordinate frame $x \to x' = \Lambda x$ moving with a velocity $\mathbf{v}$, the electric and magnetic fields change as follows

$$\mathbf{E}'_{\|}(x') \;=\; \mathbf{E}_{\|}(x), \quad \mathbf{E}'_{\perp}(x') \;=\; \gamma\left[\mathbf{E}_{\perp}(x) + \frac{1}{c}\mathbf{v}\times\mathbf{B}(x)\right], \tag{1.178}$$

$$\mathbf{B}'_{\|}(x') \;=\; \mathbf{B}_{\|}(x), \quad \mathbf{B}'_{\perp}(x') \;=\; \gamma\left[\mathbf{B}_{\perp}(x) - \frac{1}{c}\mathbf{v}\times\mathbf{E}(x)\right], \tag{1.179}$$

where the subscripts $\|$ and $\perp$ denote the components parallel and orthogonal to $\mathbf{v}$. Recalling the matrices (1.27) we see that (1.178) and (1.179) correspond precisely to the transformation law (1.175) of a tensor field.

The field tensor in the electromagnetic force of the equation of motion (1.170) transforms accordingly:

$$F^a{}_b(x(\tau)) \xrightarrow{\Lambda} F'^a{}_b(x(\tau)) = \Lambda^a{}_c \Lambda^T{}_b{}^d F'^c{}_d(\Lambda^{-1}x(\tau)). \tag{1.180}$$

This can be verified by rewriting $F^a{}_b(x(\tau))$ as

$$F^a{}_b(x(\tau)) = \int d^4x \, F^a{}_b(x) \, \delta^{(4)}(x - x(\tau)), \tag{1.181}$$

and applying the transformation (1.175).

Separating time and space components of the four-vector of the Lorentz force (1.171) we find

$$\frac{d}{d\tau}p^0 \;=\; f^0 = \frac{e}{Mc}\mathbf{E}\cdot\mathbf{p}, \tag{1.182}$$

$$\frac{d}{d\tau}\mathbf{p} \;=\; \mathbf{f} \;=\; \frac{e}{Mc}\Big(\mathbf{E}\,p^0 + \mathbf{p}\times\mathbf{B}\Big). \tag{1.183}$$

The Lorentz force can also be stated in terms of velocity as

$$f^a = \frac{e}{c} F^a{}_b \frac{dx^b}{d\tau} = \gamma \begin{pmatrix} \frac{e}{c} \mathbf{v} \cdot \mathbf{E} \\ e\mathbf{E}^i + \frac{1}{c} (\mathbf{v} \times \mathbf{B})^i \end{pmatrix}. \tag{1.184}$$

It should be noted that if we do not use the proper time τ to describe the particle orbits but the coordinate time $dt = \gamma d\tau$, the equation of motion reads

$$\frac{dp^a}{dt} = \frac{1}{\gamma} f^a, \tag{1.185}$$

so that the acceleration is governed by the three-vector of the Lorentz force

$$\mathbf{f}^{\rm em} = e \left[\mathbf{E}(x) + \frac{\mathbf{v}}{c} \mathbf{B}(x) \right]. \tag{1.186}$$

The above equations rule the movement of charged point particles in a given external field. The moving particles will, however, also give rise to additional electromagnetic fields. These are calculated by solving the *Maxwell equations* in the presence of charge and current densities ρ and $\mathbf{j}$, respectively:

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \rho \quad \text{(Coulomb's law)}, \tag{1.187}$$

$$\boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{c}\mathbf{j} \quad \text{(Ampère's law)}, \tag{1.188}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \quad \text{(absence of magnetic monopoles)}, \tag{1.189}$$

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \quad \text{(Faraday's law)}. \tag{1.190}$$

In a dielectric and paramagnetic medium with dielectric constant ϵ and magnetic permeability μ one defines the displacement field $\mathbf{D}(x)$ and the magnetic field $\mathbf{H}(x)$ by the relations

$$\mathbf{D}(x) = \epsilon \mathbf{E}(x), \qquad \mathbf{B}(x) = \epsilon \mathbf{B}(x), \tag{1.191}$$

and the Maxwell equations become

$$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho \quad \text{(Coulomb's law)}, \tag{1.192}$$

$$\boldsymbol{\nabla} \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = \frac{1}{c}\mathbf{j} \quad \text{(Ampère's law)}, \tag{1.193}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \quad \text{(absence of magnetic monopoles)}, \tag{1.194}$$

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \quad \text{(Faraday's law)}. \tag{1.195}$$

On the right-hand sides of (1.187), (1.188) and (1.192), (1.193) we have omitted factors 4π, for convenience. This makes the charge of the electron equal to $-e = -\sqrt{4\pi\alpha\hbar c}$.

In the vacuum, the two inhomogeneous Maxwell equations (1.187) and (1.188) can be combined to a single equation

$$\partial_b F^{ab} = -\frac{1}{c} j^a, \tag{1.196}$$

where j^a is the *four-vector of current density*

$$j^a(x) = \begin{pmatrix} c\rho(\mathbf{x}, t) \\ \mathbf{j}(\mathbf{x}, t) \end{pmatrix}. \tag{1.197}$$

Indeed, the zeroth component of (1.196) is equal to (1.187):

$$\partial_i F^{0i} = -\boldsymbol{\nabla} \cdot \mathbf{E} = -\rho, \tag{1.198}$$

whereas the spatial components with $a = i$ reduce to Eq. (1.188):

$$\partial_0 F^{i0} + \partial_j F^{ij} = \partial_j \epsilon^{ijk} B^k + \frac{1}{c}\frac{\partial}{\partial t} E^i = -\left(\boldsymbol{\nabla} \times \mathbf{B}\right)^i + \frac{1}{c}\frac{\partial}{\partial t} E^i = -\frac{1}{c} j^i. \tag{1.199}$$

The remaining homogeneous Maxwell equations (1.189) and (1.190) can also be rephrased in tensor form as

$$\partial_b \tilde{F}^{ab} = 0. \tag{1.200}$$

Here $\tilde{F}^{ab}$ is the so-called *dual field tensor* defined by

$$\tilde{F}^{ab} = \frac{1}{2} \epsilon^{abcd} F_{cd}, \tag{1.201}$$

where ϵ^{abcd} is the totally antisymmetric unit tensor with $\epsilon^{0123} = 1$. Its properties are summarized in Appendix A.

The antisymmetry of F^{ab} in (1.196) implies the vanishing of the four-divergence of the current density:

$$\partial_a j^a(x) = 0. \tag{1.202}$$

This is the four-dimensional way of expressing the *local conservation law* of charges. Written out in space and time components it reads

$$\partial_t \rho(\mathbf{x}, t) + \boldsymbol{\nabla} \cdot \mathbf{j}(\mathbf{x}, t) = 0. \tag{1.203}$$

Integrating this over a finite volume gives

$$\partial_t \left[\int d^3x \, \rho(\mathbf{x}, t) \right] = -\int d^3x \, \boldsymbol{\nabla} \cdot \mathbf{j}(\mathbf{x}, t) = 0. \tag{1.204}$$

The right-hand side vanishes by the Gauss divergence theorem, according to which the volume integral over the divergence of a current density is equal to the surface integral over the flux through the boundary of the volume. This vanishes if currents do not leave a finite spatial volume, which is usually true for an infinite system. Thus we find that, as a consequence of local conservation law (1.202), the charge of the system

$$Q(t) \equiv \int d^3\, \rho(\mathbf{x},t) \equiv \frac{1}{c}\int d^3x\, j^0(x) \tag{1.205}$$

satisfies the *global conservation law* according to which charge is time-independent

$$Q(t) \equiv Q. \tag{1.206}$$

For a set of point particles of charges e_n, the charge and current densities are

$$\rho(\mathbf{x},t) \;=\; \sum_n e_n \delta^{(3)}\left(\mathbf{x}-\mathbf{x}_n(t)\right), \tag{1.207}$$

$$\mathbf{j}(\mathbf{x},t) \;=\; \sum_n e_n \dot{\mathbf{x}}_n(t)\delta^{(3)}\left(\mathbf{x}-\mathbf{x}_n(t)\right). \tag{1.208}$$

Combining these expressions to a four-component current density (1.197), we can easily verify that $j^a(x)$ transforms like a *vector field* [compare with the behaviors (1.168) of scalar field and (1.175) of tensor fields]:

$$j^a(x) \xrightarrow{\Lambda} j'^a(x) = \Lambda^a{}_b\, j^b(\Lambda^{-1}x). \tag{1.209}$$

To verify this we note that $\delta^{(3)}\left(\mathbf{x}-\mathbf{x}(t)\right)$ can also be written as an integral along the path of the particle parametrized with the help of the proper time τ. This is done with the help of the identity

$$\begin{aligned}\int_{-\infty}^{\infty} d\tau\, \delta^{(4)}(x-x(\tau)) \;&=\; \int_{-\infty}^{\infty} d\tau\, \delta(x^0-x^0(\tau))\delta^{(3)}\left(\mathbf{x}-\mathbf{x}(\tau)\right)\\ &=\; \frac{d\tau}{dx^0}\delta^{(3)}\left(\mathbf{x}-\mathbf{x}(t)\right) = \frac{1}{c\gamma}\delta^{(3)}\left(\mathbf{x}-\mathbf{x}(t)\right).\end{aligned} \tag{1.210}$$

This allows us to rewrite (1.207) and (1.208) as

$$c\rho\left(\mathbf{x},t\right) \;=\; c\sum_n \int_{-\infty}^{\infty} d\tau_n e_n \gamma_n c\, \delta^{(4)}\left(x-x_n(\tau)\right), \tag{1.211}$$

$$\mathbf{j}\left(\mathbf{x},t\right) \;=\; c\sum_n \int_{-\infty}^{\infty} d\tau_n e_n \gamma_n \mathbf{v}_n \delta^{(4)}\left(x-x_n(\tau)\right). \tag{1.212}$$

These equations can be combined in a single four-vector equation

$$j^a(x) = c\sum_n \int_{-\infty}^{\infty} d\tau_n e_n \dot{x}_n^a(\tau)\delta^{(4)}\left(x-x_n(\tau)\right), \tag{1.213}$$

which makes the transformation behavior (1.209) an obvious consequence of the vector nature of $\dot{x}_n^a(\tau)$.

In terms of the four-dimensional current density, the inhomogeneous Maxwell equation (1.196) becomes the *Maxwell-Lorentz equation*

$$\partial_b F^{ab} = -\frac{1}{c}\, j^a = -\sum_n \int_{-\infty}^{\infty} d\tau_n e_n \dot{x}_n^a(\tau)\delta^{(4)}\left(x-x_n(\tau)\right). \tag{1.214}$$

It is instructive to verify the conservation law (1.202) for the current density (1.213). Applying the derivative ∂_a to the δ-function gives $\partial_a \delta^{(4)}(x-x_n(\tau)) = -\partial_{x_n^a}\delta^{(4)}(x-x_n(\tau))$, and therefore

$$\begin{aligned}\partial_a j^a(x) &= -c\sum_n \int_{-\infty}^{\infty} d\tau_n e_n \frac{dx_n^a(\tau)}{d\tau}\frac{\partial}{\partial x_n^a}\delta^{(4)}(x-x_n(\tau)) \\ &= -c\sum_n \int_{-\infty}^{\infty} d\tau_n e_n \partial_\tau \delta^{(4)}(x-x_n(\tau))\,. \end{aligned} \tag{1.215}$$

If the particle orbits $x(\tau)$ are stable, they are either closed in spacetime, or they come from negative infinite x_0 and run to positive infinite x_0. Then the right-hand side vanishes in any finite volume so that the current density is indeed conserved.

We end this section by remarking that the vector transformation law (1.209) can also be written by analogy with the tensor law (1.175) as

$$j^a(x) \xrightarrow{\Lambda} j'^a(x) = [e^{-i\frac{1}{2}\omega_{ab}\hat{J}^{ab}}\, j]^a(\Lambda^{-1}x), \tag{1.216}$$

where

$$\hat{J}^{cd} \equiv L^{cd}\times\hat{1} + 1\times\hat{L}^{cd} \tag{1.217}$$

are the generators of the total four-dimensional angular momentum of the vector field. As in (1.177), the factors in the direct products apply separately to the representation spaces associated with the Lorentz index and the spacetime coordinates, and the generators $\hat{J}^{ab}$ obey the same commutation rules (1.71) and (1.72) as L_{ab} and $\hat{L}_{ab}$.

1.11 Dirac Particles and Fields

The observable matter of the universe consists mainly of electrons and nucleons, the latter being predominantly bound states of three quarks. Electrons and quarks are spin-1/2 particles which may be described by four-component Dirac fields $\psi(x)$. These obey the Dirac equation

$$(i\gamma^a\partial_a - m)\,\psi(x) = 0, \tag{1.218}$$

where γ^a are the 4×4 -*Dirac matrices*

$$\gamma^a = \begin{pmatrix} 0 & \sigma^a \\ \tilde{\sigma}^a & 0 \end{pmatrix}, \tag{1.219}$$

in which the 2×2 -submatrices σ^a and $\tilde{\sigma}^a$ with $a = 0,\dots,3$ form the *four-vectors of Pauli matrices*

$$\sigma^a \equiv (\sigma^0, \sigma^i), \quad \tilde{\sigma}^a \equiv (\sigma^0, -\sigma^i). \tag{1.220}$$

The spatial components σ^i are the ordinary *Pauli matrices*

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.221}$$

while the zeroth component σ^0 is defined as the 2×2 -unit matrix:

$$\sigma^0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.222}$$

From the algebraic properties of these matrices

$$(\sigma^a)^2 = \sigma^0 = 1, \quad \sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k, \quad \sigma^a \tilde{\sigma}^b + \sigma^b \tilde{\sigma}^a = 2g^{ab}, \tag{1.223}$$

we deduce that the Dirac matrices γ^a satisfy the anticommutation rules

$$\left\{\gamma^a, \gamma^b\right\} = 2g^{ab}. \tag{1.224}$$

Under Lorentz transformations, the Dirac field transforms according to the spinor representation of the Lorentz group

$$\psi_A(x) \xrightarrow{\Lambda} \psi'_A(x) = D_A{}^B(\Lambda)\psi_B(\Lambda^{-1}x), \tag{1.225}$$

by analogy with the transformation law (1.209) of a vector field. The 4×4 -matrices Λ of the defining representation of the Lorentz group in (1.209) are replaced by the 4×4 -matrices $D(\Lambda)$ representing the Lorentz group in spinor space.

It is easy to find these matrices. If we denote the spinor representation of the Lie algebra (1.72) by 4×4 -matrices Σ^{ab}, these have to satisfy the commutation rules

$$[\Sigma^{ab}, \Sigma^{ac}] = -ig^{aa}\Sigma^{bc}, \quad \text{no sum over } a. \tag{1.226}$$

These can be solved by the matrices

$$\Sigma^{ab} \equiv \frac{1}{2}\sigma^{ab}, \tag{1.227}$$

where σ^{ab} is the antisymmetric tensor of matrices

$$\sigma^{ab} \equiv \frac{i}{2}[\gamma^a, \gamma^b]. \tag{1.228}$$

The representation matrices of finite Lorentz transformations may now be expressed as exponentials of the form (1.54):

$$D(\Lambda) = e^{-i\frac{1}{2}\omega_{ab}\Sigma^{ab}}, \tag{1.229}$$

where ω_{ab} is the same antisymmetric matrix as in (1.54), containing the rotation and boost parameters as specified in (1.55) and (1.56). Comparison with (1.57) shows that pure rotations and pure Lorentz transformations are generated by the spinor representations of L^{ab} in (1.57):

$$\Sigma^{ij} = \epsilon_{ijk}\,\frac{1}{2}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}, \qquad \Sigma^{0i} = \frac{i}{2}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}. \tag{1.230}$$

The generators of the rotation group $\Sigma^i = \frac{1}{2}\epsilon_{ijk}\Sigma^{jk}$ corresponding to L_i in (1.53) consist of a direct sum of two Pauli matrices, the 4×4 -spin matrix:

$$\boldsymbol{\Sigma} \equiv \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}. \tag{1.231}$$

The generators Σ^{0i} of the pure Lorentz transformations corresponding to M_i in (1.53) can also be expressed as $\Sigma^{0i} = i\alpha^i/2$ with the vector of 4×4 -matrices

$$\boldsymbol{\alpha} = \begin{pmatrix} -\boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}. \tag{1.232}$$

In terms of $\boldsymbol{\Sigma}$ and $\boldsymbol{\alpha}$, the representation matrices (1.229) for pure rotations and pure Lorentz transformations are seen to have the explicit form

$$D(R) = e^{-i\boldsymbol{\varphi}\cdot\boldsymbol{\Sigma}} = \begin{pmatrix} e^{-i\boldsymbol{\varphi}\cdot\boldsymbol{\sigma}/2} & 0 \\ 0 & e^{-i\boldsymbol{\varphi}\cdot\boldsymbol{\sigma}/2} \end{pmatrix}, \quad D(B) = e^{\boldsymbol{\zeta}\cdot\boldsymbol{\alpha}} = \begin{pmatrix} e^{-\boldsymbol{\zeta}\cdot\boldsymbol{\sigma}/2} & 0 \\ 0 & e^{\boldsymbol{\zeta}\cdot\boldsymbol{\sigma}/2} \end{pmatrix}. \tag{1.233}$$

The commutation relations (1.226) are a direct consequence of the commutation relations of the generators Σ^{ab} with the gamma matrices:

$$[\Sigma^{ab}, \gamma^c] = -(L^{ab})^c{}_d\,\gamma^d = -i(g^{ac}\gamma^b - g^{bc}\gamma^a). \tag{1.234}$$

Comparison with (1.114) and (1.115) shows that the matrices γ^a transform like x^a, i.e., they form a vector operator. The commutation rules (1.226) follow directly from (1.234) upon using the Leibnitz chain rule (1.117).

For global transformations, the vector property (1.234) implies that γ^a behaves like the vector x^a in Eq. (1.133):

$$D(\Lambda)\gamma^c D^{-1}(\Lambda) = e^{-i\frac{1}{2}\omega_{ab}\Sigma^{ab}}\gamma^c e^{i\frac{1}{2}\omega_{ab}\Sigma^{ab}} = (e^{i\frac{1}{2}\omega_{ab}L^{ab}})^c{}_{c'}\,\gamma^{c'} = (\Lambda^{-1})^c{}_{c'}\,\gamma^{c'}. \tag{1.235}$$

In terms of the generators Σ^{ab}, we can write the field transformation law (1.225) more explicitly as

$$\psi(x) \xrightarrow{\Lambda} \psi'_\Lambda(x) = D(\Lambda)\psi(\Lambda^{-1}x) = e^{-i\frac{1}{2}\omega_{ab}\Sigma^{ab}}\psi(\Lambda^{-1}x), \tag{1.236}$$

in perfect analogy with the transformation laws of scalar, tensor, and vector fields in Eqs. (1.168), (1.175), and (1.209).

It is useful to re-express the transformation of the spacetime argument on the right-hand side in terms of the differential operator of four-dimensional angular momentum and rewrite (1.236) as in (1.177) and (1.217) as

$$\psi(x) \xrightarrow{\Lambda} \psi'_\Lambda(x) = \hat{D}(\Lambda)\times D(\Lambda)\psi(x) = e^{-i\frac{1}{2}\omega_{ab}\hat{J}^{ab}}\psi(x), \tag{1.237}$$

where

$$\hat{J}^{cd} \equiv \Sigma^{cd}\times\hat{1} + 1\times\hat{L}^{cd} \tag{1.238}$$

are the generators of the total four-dimensional angular momentum of the Dirac field.

1.12 Energy-Momentum Tensor

The four-dimensional current density $j^a(x)$ contains all information on the electric properties of relativistic particle orbits. It is possible to collect also the mechanical properties in a tensor, the *energy-momentum tensor*.

1.12.1 Point Particles

The *energy density* of the particles can be written as

$$\overset{\text{part}}{\mathcal{E}}(\mathbf{x},t)=\sum_n m_n\gamma c^2\delta^{(3)}\left(\mathbf{x}-\mathbf{x}_n(t)\right). \tag{1.239}$$

We have previously seen that the energy transforms like a zeroth component of a four-vector [recall (1.152)]. The energy density measures the energy per spatial volume element. An infinitesimal four-volume d^4x is invariant under Lorentz transformations, due to the unit determinant $|\Lambda^a{}_b|=1$ implied by the pseudo-orthogonality relation (1.28), so that indeed

$$d^4x'=\left|\frac{\partial x'^a}{\partial x^b}\right|d^4x=|\Lambda^a{}_b|d^4x=d^4x. \tag{1.240}$$

This shows that $\delta^{(3)}(\mathbf{x})$ which transforms like an inverse spatial volume

$$\frac{1}{d^3x}=\frac{dx^0}{d^4x} \tag{1.241}$$

behaves like the zeroth component of a four-vector. The energy density (1.239) can therefore be viewed as a 00-component of a Lorentz tensor called the symmetric energy-momentum tensor. By convention, this is chosen to have the dimension of momentum density, so that we must identify the energy density with $c\overset{\text{part}}{T}{}^{ab}$. In fact, using the identity (1.210), we may rewrite (1.239) as

$$\overset{\text{part}}{\mathcal{E}}(\mathbf{x},t)=c\sum_n\int_{-\infty}^{\infty}d\tau_n\frac{1}{m_n}p_n^0(\tau)p_n^0(\tau)\delta^{(4)}(x-x(\tau)), \tag{1.242}$$

which is equal to c times the 00-component of the energy-momentum tensor

$$\overset{\text{part}}{T}{}^{ab}(\mathbf{x},t)=\sum_n\int_{-\infty}^{\infty}d\tau_n\frac{1}{m_n}p_n^a(\tau)p_n^b(\tau)\delta^{(4)}(x-x(\tau)). \tag{1.243}$$

The spatial momenta of the particles

$$\overset{\text{part}}{\mathcal{P}}{}^i(\mathbf{x},t)=\sum_n m_n\gamma_n\dot{x}_n^i(\tau)\delta^{(3)}\left(\mathbf{x}-\mathbf{x}(\tau)\right) \tag{1.244}$$

are three-vectors. Their densities transform therefore like $0i$-components of a Lorentz tensor. Indeed, using once more the identity (1.210), we may rewrite (1.244) as

$$\overset{\text{part}}{\mathcal{P}}{}^i(\mathbf{x},t)=\overset{\text{part}}{T}{}^{0i}(\mathbf{x},t)=\sum_n\int_{-\infty}^{\infty}d\tau_n\frac{1}{m_n}p_n^0(\tau)p_n^i(\tau)\delta^{(4)}(x-x(\tau)), \tag{1.245}$$

which shows precisely the tensor character. The four-vector of the total energy-momentum of the many-particle system is given by the integrals over the $0a$-components

$$\overset{\text{part}}{P}{}^{a}(t) \equiv \int d^3x \, \overset{\text{part}}{T}{}^{0a}(\mathbf{x}, t). \tag{1.246}$$

Inserting here (1.242) and (1.245), we obtain the sum over all four-momenta

$$\overset{\text{part}}{P}{}^{a}(t) = \sum_n p_n^a(\tau). \tag{1.247}$$

By analogy with the four-dimensional current density $j^a(\mathbf{x})$, let us calculate the four-divergence $\partial_b \overset{\text{part}}{T}{}^{ab}$. A partial integration yields

$$\begin{aligned}
\sum_n \int_{-\infty}^{\infty} d\tau_n \, p_n^a(\tau)\dot{x}_n^b(\tau)\partial_b\delta^{(4)}(x - x(\tau)) &= -\sum_n \int_{-\infty}^{\infty} d\tau_n \, p_n^a(\tau)\partial_\tau\delta^{(4)}(x - x(\tau)) \\
= -\sum_n \int_{-\infty}^{\infty} d\tau_n \partial_\tau \left[p_n^a(\tau)\delta^{(4)}(x - x(\tau)\right] &+ \sum_n \int_{-\infty}^{\infty} d\tau_n \dot{p}_n^a(\tau)\delta^{(4)}\left(x - x(\tau)\right).
\end{aligned} \tag{1.248}$$

The first term on the right-hand side disappears if the particles are stable, i.e., if the orbits are closed or come from negative infinite x^0 and disappear into positive infinite x^0. The derivative $\dot{p}_n^a(\tau)$ in the second term can be made more explicit if only electromagnetic forces act on the particles. Then it is equal to the Lorentz force, i.e., the four-vector $f^a(\tau)$ of Eq. (1.184), and we obtain

$$\begin{aligned}
\partial_b \overset{\text{part}}{T}{}^{ab} &= \sum_n \int_{-\infty}^{\infty} d\tau_n f_n(\tau)\delta^{(4)}\left(x - x(\tau)\right) \qquad (1.249) \\
&= \frac{1}{c}\sum_n \int_{-\infty}^{\infty} d\tau_n e_n F^a{}_b\left(x_n(\tau)\right)\dot{x}_n{}^b(\tau)\delta^{(4)}\left(x - x(\tau)\right).
\end{aligned}$$

Expressed in terms of the current four-vector (1.213), this reads

$$\partial_b \overset{\text{part}}{T}{}^{ab}(x) = \frac{1}{c^2}F^a{}_b(x)j^b(x). \tag{1.250}$$

In the absence of electromagnetic fields, the energy-momentum tensor of the particles is conserved.

Integrating (1.246) over the spatial coordinates gives the time change of the total four-momentum

$$\begin{aligned}
\partial_t \overset{\text{part}}{P}{}^{a}(t) &= c\,\partial_0\left[\int d^3x \, \overset{\text{part}}{T}{}^{a0}\right] = c\int d^3x \, \partial_b \overset{\text{part}}{T}{}^{ab} - c\int d^3x \, \partial_i \overset{\text{part}}{T}{}^{0i} \\
&= \frac{e}{c}\sum_n F^a{}_b\left(x_n(\tau)\right)\dot{x}_n^b(\tau)\gamma_n(\tau).
\end{aligned} \tag{1.251}$$

This agrees, of course, with the Lorentz equations (1.170) since by (1.247)

$$\partial_t \overset{\text{part}}{P}{}^{a}(t) = \partial_t \sum_n p_n^a(\tau) = \sum_n \dot{p}_n^a(\tau)\,\gamma_n. \tag{1.252}$$

If there are no electromagnetic forces, then $\overset{\text{part}}{P}{}^{a}$ is time-independent.

1.12.2 Perfect Fluid

A perfect fluid is defined as an idealized uniform material medium moving with velocity $\mathbf{v}(\mathbf{x},t)$. The uniformity is an acceptable approximation as long as the microscopic mean free paths are short with respect to the length scale recognizable by the observer. Consider such a fluid at rest. Then the energy-momentum tensor has no momentum density:

$$\overset{\text{fluid}}{T}{}^{0i} = 0. \tag{1.253}$$

The energy density is given by

$$c\overset{\text{fluid}}{T}{}^{00} = c^2\rho, \tag{1.254}$$

where ρ is the mass density.

Due to the isotropy, the *purely spatial* part of the energy-momentum tensor must be diagonal:

$$\overset{\text{fluid}}{T}{}^{ij} = \frac{p}{c}\,\delta_{ij}, \tag{1.255}$$

where p is the *pressure* of the fluid. We can now calculate the energy-momentum tensor of a moving perfect fluid by performing a Lorentz transformation on the energy-momentum tensor at rest:

$$\overset{\text{fluid}}{T}{}^{ab} \to \Lambda^a{}_c\Lambda^b{}_d\overset{\text{fluid}}{T}{}^{cd}. \tag{1.256}$$

Applying to this the Lorentz boosts from rest to momentum $\mathbf{p}$ of Eq. (1.34), and expressing the hyperbolic functions in terms of energy and momentum according to Eq. (1.153), we obtain

$$\overset{\text{fluid}}{T}{}_{ab} = \frac{1}{c}\left[\left(\frac{p}{c^2}+\rho\right)u^a u^b - pg^{ab}\right], \tag{1.257}$$

where u^a is the four-velocity (1.150) of the fluid with $u^a u_a = c^2$.

1.12.3 Electromagnetic Field

The energy density of an electromagnetic field is well-known:

$$\mathcal{E}(x) = \frac{1}{2}\left[\mathbf{E}^2(x) + \mathbf{B}^2(x)\right]. \tag{1.258}$$

The associated energy current density is given by the *Poynting vector*:

$$\mathbf{S}(x) = c\,\mathbf{E}(x)\times\mathbf{B}(x). \tag{1.259}$$

From these we find four components of the energy-momentum tensor:

$$\overset{\text{em}}{T}{}^{00}(x) \equiv \frac{1}{c}\mathcal{E}(x), \qquad \overset{\text{em}}{T}{}^{0i} = \overset{\text{em}}{T}{}^{i0} \equiv \frac{1}{c^2}\mathbf{S}^i(x). \tag{1.260}$$

The remaining components are determined by the tensor

$$\overset{\text{em}}{T}{}^{ab}(x) = \frac{1}{c}\left[- F^{a}{}_{c}F^{bc} + \frac{1}{4}g^{ab}F^{cd}F_{cd}\right]. \tag{1.261}$$

The four-divergence of this is

$$\partial_b \overset{\text{em}}{T}{}^{ab} = \frac{1}{c}\left[- F^{a}{}_{c}\partial_b F^{ab} - \left(\partial_b F^{a}{}_{c}\right) F^{bc} + \frac{1}{4}\partial^a \left(F^{cd}F_{cd}\right)\right]. \tag{1.262}$$

The second and third terms cancel each other, due to the homogeneous Maxwell equations (1.189) and (1.190). In order to see this, take the trivial identity $\partial_b\epsilon^{abcd}F_{cd} = 2\epsilon^{abcd}\partial_b\partial_c A_d = 0$, and multiply this by $\epsilon_{aefg}F_{fg}$. Using the identity (1A.23):

$$\epsilon^{abcd}\epsilon_{aefg} \quad = \quad -\left(\delta^b_e\delta^c_f\delta^d_g + \delta^c_e\delta^d_f\delta^b_g + \delta^d_e\delta^b_f\delta^c_g - \delta^b_e\delta^d_f\delta^c_g - \delta^d_e\delta^c_f\delta^b_g - \delta^c_e\delta^b_f\delta^d_g\right), \tag{1.263}$$

we find

$$-F^{cd}\partial_e F_{cd} - F^{db}\partial_b F_{ed} - F^{bc}\partial_b F_{ce} + F^{dc}\partial_e F_{cd} + F^{cb}\partial_b F_{ce} + F^{bd}\partial_b F_{ed} = 0. \tag{1.264}$$

Due to the antisymmetry of F_{ab}, this gives

$$-\partial_e \left(F^{cd}F_{cd}\right) + 4F^{bd}\partial_b F_{bd} = 0, \tag{1.265}$$

so that we obtain the conservation law

$$\partial_b \overset{\text{em}}{T}{}^{ab}(x) = -\frac{1}{c}\left[F^{a}{}_{c}(x)\partial_b F^{bc}(x)\right] = 0. \tag{1.266}$$

In the last step we have used Maxwell's equation Eq. (1.196) with zero currents.

The timelike component of the conservation law (1.266) reads

$$\partial_t \overset{\text{em}}{T}{}^{00}(x) + c\,\partial_i \overset{\text{em}}{T}{}^{0i}(x) = 0, \tag{1.267}$$

which can be rewritten with (1.258) and (1.260) as the well-known *Poynting law* of energy flow:

$$\partial_t\,\mathcal{E}(x) + \boldsymbol{\nabla}\cdot\mathbf{S}(x) = 0. \tag{1.268}$$

If currents are present, the Maxwell equation (1.196) changes the conservation law (1.266) to

$$c\,\partial_b \overset{\text{em}}{T}{}^{ab}(x) = -\frac{1}{c}F^{a}{}_{c}(x)j^c(x) = 0, \tag{1.269}$$

which modifies (1.268) to

$$\partial_t\,\mathcal{E}(x) + \boldsymbol{\nabla}\cdot\mathbf{S}(x) = -\mathbf{j}(x)\cdot\mathbf{E}(x). \tag{1.270}$$

A current parallel to the electric field reduces the field energy.

In a medium, the energy density and Poynting vector become

$$\mathcal{E}(x) \equiv \frac{1}{2}\left[\mathbf{E}(x)\cdot\mathbf{D}(x)+\mathbf{B}(x)\cdot\mathbf{H}(x)\right], \qquad \mathbf{S}(x) \equiv c\mathbf{E}(x)\times\mathbf{H}(x), \tag{1.271}$$

and the conservation law can easily be verified using the Maxwell equations (1.193) and (1.195):

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{S}(x) &= c\boldsymbol{\nabla}\cdot[\mathbf{E}(x)\times\mathbf{H}(x)] = c[\boldsymbol{\nabla}\times\mathbf{E}(x)]\cdot\mathbf{H}(x) - c\mathbf{E}(x)\cdot[\boldsymbol{\nabla}\times\mathbf{B}(x)]\\ &= \{\partial_t\mathbf{B}(x)\cdot\mathbf{H}(x)+\mathbf{E}(x)\cdot[\partial_t\mathbf{D}(x)+\mathbf{j}(x)]\} = \partial_t\mathcal{E}(x)+\mathbf{j}(x)\cdot\mathbf{E}(x).\end{aligned} \tag{1.272}$$

We now observe that the force on the right-hand side of (1.269) is precisely the opposite of the right-hand side of (1.250), as required by Newton's third axiom of actio = reactio. Thus, the total energy-momentum tensor of the combined system of particles and electromagnetic fields

$$T^{ab}(x) = \overset{\text{part}}{T}{}^{ab}(x) + \overset{\text{em}}{T}{}^{ab}(x) \tag{1.273}$$

has a vanishing four-divergence,

$$\partial_b T^{ab}(x) = 0 \tag{1.274}$$

implying that the total four-momentum $P^a \equiv \int d^3x\, T^{0a}$ is a conserved quantity

$$\partial_t P^a(t) = 0. \tag{1.275}$$

1.13 Angular Momentum and Spin

Similar considerations apply to the total angular momentum of particles and fields. Since $T^{i0}(x)$ is a momentum density, we may calculate the spatial tensor of total angular momentum from the integral

$$J^{ij}(t) = \int d^3x \left[x^i T^{j0}(x) - x^j T^{i0}(x)\right]. \tag{1.276}$$

In three space dimensions one describes the angular momentum by a vector $J^i = \frac{1}{2}\epsilon^{ijk}J^{jk}$. The angular momentum (1.276) may be viewed as the integral

$$J^{ij}(t) = \int d^3x\, J^{ij,0}(x) \tag{1.277}$$

over the $i, j, 0$-component of the Lorentz tensor

$$J^{ab,c}(x) = x^a T^{bc}(x) - x^b T^{ac}(x). \tag{1.278}$$

It is easy to see that due to (1.274) and the symmetry of the energy-momentum tensor, the Lorentz tensor $J^{ab,c}(x)$ is divergenceless in the index c

$$\partial_c J^{ab,c}(x) = 0. \tag{1.279}$$

As a consequence, the spatial integral

$$J^{ab}(t) = \int d^3x \, J^{ab,0}(x) \tag{1.280}$$

is a conserved quantity. This is the four-dimensional extension of the conserved total angular momentum. The conservation of the components J^{0i} is the center-of-mass theorem.

A set of point particles with the energy-momentum tensor (1.243) possesses a four-dimensional angular momentum

$$\overset{\text{part}}{J}{}^{ab}(\tau) = \sum_n \left[x_n^a(\tau)p_n^b(\tau) - x_n^b(\tau)p_n^a(\tau)\right]. \tag{1.281}$$

In the absence of electromagnetic fields, this is conserved, otherwise the τ-dependence is important.

The spin of a particle is defined by its total angular momentum in its rest frame. It is the *intrinsic angular momentum* of the particle. Electrons, protons, neutrons, and neutrinos have spin 1/2. For nuclei and atoms, the spin can take much larger values.

There exists a four-vector $S^a(\tau)$ along the orbit of a particle whose spatial part reduces to the angular momentum in the rest frame. It is defined by a combination of the angular momentum (1.281) and the four-velocity $u^d(\tau)$ [recall (1.150)]

$$S^a(\tau) \equiv \frac{1}{2c} \epsilon^{abcd} \overset{\text{part}}{J}{}_{bc}(\tau) u_d(\tau). \tag{1.282}$$

In the rest frame where

$$u_R^a = (c, 0, 0, 0), \tag{1.283}$$

this reduces indeed to the three-vector of total angular momentum

$$S_R^a(\tau) = (0, \overset{\text{part}}{J}{}_{23}(\tau), \overset{\text{part}}{J}{}_{31}(\tau), \overset{\text{part}}{J}{}_{12}(\tau)) = (0, \overset{\text{part}}{\mathbf{J}}(\tau)). \tag{1.284}$$

For a free particle we find, due to conservation of momentum and total angular momentum

$$\frac{d}{d\tau} u_d(\tau) = 0, \qquad \frac{d}{d\tau} \overset{\text{part}}{J}{}_{bc}(\tau) = 0, \tag{1.285}$$

that also the spin vector $S^a(\tau)$ is conserved:

$$\frac{d}{d\tau} S^a(\tau) = 0. \tag{1.286}$$

The spin four-vector is useful to understand an important phenomenon in atomic physics called the *Thomas precession* of the electron spin in an atom. It explains why the observed fine structure of atomic physics determines the gyromagnetic ratio g_e of the electron to be close to 2.

The relation between spin (1.284) and its four-vector is exhibited clearly by applying the pure Lorentz transformation matrix (1.27) to (1.284) yielding

$$S^i = S_R^i + \frac{\gamma^2}{\gamma+1}\frac{v^i v^j}{c^2} S_R{}^j, \quad S^0 = \gamma \frac{v^i}{c} S_R^i. \tag{1.287}$$

Note that S^0 and S^i satisfy $S^0 = v^i S^i/c$, which can be rewritten covariantly as

$$u^a S_a = 0. \tag{1.288}$$

The inverse of the transformation (1.287) is found with the help of the identity $v^2/c^2 = (\gamma^2 - 1)/\gamma^2$ as follows:

$$S_R^i = S^i - \frac{\gamma}{\gamma+1}\frac{v^i v^j}{c^2} S^j = S^i - \frac{\gamma-1}{\gamma}\frac{v^i v^j}{v^2} S^j. \tag{1.289}$$

If external forces act on the system, the spin vector starts moving. This movement is called *precession*. If the point particle moves in an orbit under the influence of a *central force* (for example, an electron around a nucleus in an atom), there is no torque on the particle so that the total angular momentum in its rest frame is conserved. Hence $dS_R^i(\tau)/d\tau = 0$, which is expressed covariantly as $dS^a(\tau)/d\tau \propto u^a(\tau)$. In the rest frame of the atom, however, the spin shows precession. Let us calculate its rate. From the definition (1.282) we have

$$\frac{dS_a}{d\tau} = \frac{1}{2}\epsilon_{abcd} \overset{\text{part}}{J}{}^{bc} \frac{du^d}{d\tau}. \tag{1.290}$$

There is no contribution from

$$\frac{d}{d\tau} \overset{\text{part}}{J}{}^{bc} = x^a(\tau)\dot{p}^b(\tau) - x^b(\tau)\dot{p}^a(\tau), \tag{1.291}$$

since $\dot{p} = m\dot{u}$, and the ϵ-tensor is antisymmetric.

The right-hand side of (1.290) can be simplified by multiplying it with the trivial expression

$$g_{st} u^s u^t = c^2, \tag{1.292}$$

and using the identity for the ϵ-tensor

$$\epsilon^{abcd} g^{st} \;\; = \;\; \epsilon^{abcs} g^{dt} + \epsilon^{absd} g^{ct} + \epsilon^{ascd} g^{bt} + \epsilon^{sbcd} g^{at}. \tag{1.293}$$

This can easily be proved by taking advantage of the antisymmetry of ϵ^{abcd} and choosing a, b, c, d to be equal to $0, 1, 2, 3$, respectively. After this, the right-hand side of (1.290) becomes a sum of the four terms

$$\frac{1}{2}\left(\epsilon_{abcs} \overset{\text{part}}{J}{}^{bc} u^s u^d u'^d + \epsilon_{absd} \overset{\text{part}}{J}{}^{bc} u_c u^s u'^d + \epsilon_{ascd} \overset{\text{part}}{J}{}^{bc} u_b u^a u'^d + \epsilon_{sbcd} \overset{\text{part}}{J}{}^{bc} u^s u^a \dot{u}^d\right).$$

The first term vanishes, since $u^d \dot{u}_d = (1/2)du^2/d\tau = (1/2)dc^2/d\tau = 0$. The last term is equal to $-S_d \, \dot{u}^d u_a/c^2$. Inserting the identity (1.293) into the second and third terms, we obtain twice the left-hand side of (1.290). Taking this to the left-hand side, we find the equation of motion

$$\frac{dS_a}{d\tau} = \frac{1}{c^2} S_c \frac{du^c}{d\tau} u_a. \tag{1.294}$$

Note that on account of this equation, the time derivative $dS_a/d\tau$ points in the direction of u^a, in accordance with the initial assumption of a torque-free force.

We are now prepared to calculate the rate of the Thomas precession. Denoting in the final part of this section the derivatives with respect to the physical time $t = \gamma\tau$ by a dot, we can rewrite (1.294) as

$$\dot{\mathbf{S}} \equiv \frac{d\mathbf{S}}{dt} = \frac{1}{\gamma}\frac{d\mathbf{S}}{d\tau} = -\frac{1}{c^2}\left(S^0 \dot{u}^0 + \mathbf{S}\cdot\dot{\mathbf{u}}\right)\mathbf{u} = \frac{\gamma^2}{c^2}\left(\mathbf{S}\cdot\dot{\mathbf{v}}\right)\mathbf{v}, \tag{1.295}$$

$$\dot{S}_0 \equiv \frac{dS_0}{dt} = \frac{1}{c}\frac{d}{dt}\left(\mathbf{S}\cdot\mathbf{v}\right) = \frac{\gamma^2}{c^2}\left(\mathbf{S}\cdot\dot{\mathbf{v}}\right). \tag{1.296}$$

We now differentiate Eq. (1.289) with respect to the time using the relation $\dot{\gamma} = \gamma^3 \dot{\mathbf{v}}\mathbf{v}/c^2$, and find

$$\dot{\mathbf{S}}_R = \dot{\mathbf{S}} - \frac{\gamma}{\gamma+1}\frac{1}{c^2}\dot{S}^0\mathbf{v} - \frac{\gamma}{\gamma+1}\frac{1}{c^2}S^0\dot{\mathbf{v}} - \frac{\gamma^3}{(\gamma+1)^2}\frac{1}{c^4}\left(\mathbf{v}\cdot\dot{\mathbf{v}}\right)S^0\,\mathbf{v}. \tag{1.297}$$

Inserting here Eqs. (1.295) and (1.296), we obtain

$$\dot{\mathbf{S}}_R = \frac{\gamma^2}{\gamma+1}\frac{1}{c^2}(\mathbf{S}\cdot\dot{\mathbf{v}})\mathbf{v} - \frac{\gamma}{\gamma+1}\frac{1}{c^2}S^0\,\dot{\mathbf{v}} - \frac{\gamma^3}{(\gamma+1)^2}\left(\mathbf{v}\cdot\dot{\mathbf{v}}\right)S^0\mathbf{v}. \tag{1.298}$$

On the right-hand side we return to the spin vector $\mathbf{S}_R$ using Eqs. (1.287), and find

$$\dot{\mathbf{S}}_R = \frac{\gamma^2}{\gamma+1}\frac{1}{c^2}\left[(\mathbf{S}_R\cdot\dot{\mathbf{v}})\mathbf{v} - (\mathbf{S}_R\cdot\mathbf{v})\dot{\mathbf{v}}\right] = \boldsymbol{\Omega}_{\mathrm{T}}\times\mathbf{S}_R, \tag{1.299}$$

with the Thomas precession frequency

$$\boldsymbol{\Omega}_{\mathrm{T}} = -\frac{\gamma^2}{(\gamma+1)}\frac{1}{c^2}\mathbf{v}\times\dot{\mathbf{v}}. \tag{1.300}$$

This is a purely kinematic effect. If an electromagnetic field is present, there will be an additional dynamic precession. For slow particles, it is given by

$$\dot{\mathbf{S}} \equiv -\mathbf{S}\times\boldsymbol{\Omega}_{\mathrm{em}} \approx \boldsymbol{\mu}\times\left(\mathbf{B} - \frac{\mathbf{v}}{c}\times\mathbf{E}\right), \tag{1.301}$$

where $\boldsymbol{\mu}$ is the magnetic moment

$$\boldsymbol{\mu} = g\mu_B\frac{\mathbf{S}}{\hbar} = \frac{eg}{2Mc}\mathbf{S}, \tag{1.302}$$

and g the dimensionless *gyromagnetic ratio*, also called *Landé factor*. Recall the value of the *Bohr magneton*

$$\mu_B \equiv \frac{e\hbar}{2Mc} \approx 3.094 \times 10^{-30}\,\mathrm{C\,cm} \approx\ 0.927 \times 10^{-20}\ \frac{\mathrm{erg}}{\mathrm{gauss}} \approx\ 5.788 \times 10^{-8}\ \frac{\mathrm{eV}}{\mathrm{gauss}}. \tag{1.303}$$

If the electron moves fast, we transform the electromagnetic field to the electron rest frame by a Lorentz transformation (1.178), (1.179), and obtain an equation of motion for the spin

$$\dot{\mathbf{S}}_R = \boldsymbol{\mu} \times \mathbf{B}' = \boldsymbol{\mu} \times \left[\gamma\left(\mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E}\right) - \frac{\gamma^2}{\gamma+1}\frac{\mathbf{v}}{c}\left(\frac{\mathbf{v}}{c}\cdot\mathbf{B}\right)\right]. \tag{1.304}$$

Expressing $\boldsymbol{\mu}$ via Eq. (1.302), this becomes

$$\dot{\mathbf{S}}_R \equiv -\mathbf{S}_R \times \boldsymbol{\Omega}_{\mathrm{em}} = \frac{eg}{2mc}\mathbf{S}_R \times \left[\left(\mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E}\right) - \frac{\gamma}{\gamma+1}\frac{\mathbf{v}}{c}\left(\frac{\mathbf{v}}{c}\cdot\mathbf{B}\right)\right], \tag{1.305}$$

which is the relativistic generalization of Eq. (1.301). It is easy to see that the associated fully covariant equation is

$$S^{a\prime} = \frac{g}{2mc}\left[eF^{ab}S_b + \frac{1}{mc}p^a S_c \frac{d}{d\tau}p^c\right] = \frac{eg}{2mc}\left[F^{ab}S_b + \frac{1}{m^2c^2}p^a S_c F^{c\kappa} p_\kappa\right]. \tag{1.306}$$

On the right-hand side we have inserted the relativistic equation of motion (1.170) of a point particle in an external electromagnetic field.

If we add to this the torque-free Thomas precession rate (1.294), we obtain the covariant *Bargmann-Michel-Telegdi equation* [9]

$$S^{a\prime} = \frac{1}{2mc}\left[egF^{ab}S_b + \frac{g-2}{mc}p^a S_c \frac{d}{d\tau}p^c\right] = \frac{e}{2mc}\left[gF^{ab}S_b + \frac{g-2}{m^2c^2}p^a S_c F^{c\kappa} p_\kappa\right]. \tag{1.307}$$

For the spin vector $\mathbf{S}_R$ in the electron rest frame this implies a change in the electromagnetic precession rate in Eq. (1.305) to [10]

$$\frac{d\mathbf{S}}{dt} = \boldsymbol{\Omega}_{\mathrm{em\,T}} \times \mathbf{S} \equiv (\boldsymbol{\Omega}_{\mathrm{em}} + \boldsymbol{\Omega}_{\mathrm{T}}) \times \mathbf{S} \tag{1.308}$$

with a frequency given by the *Thomas equation*

$$\boldsymbol{\Omega}_{\mathrm{em\,T}} = -\frac{e}{mc}\left[\left(\frac{g}{2} - 1 + \frac{1}{\gamma}\right)\mathbf{B} - \left(\frac{g}{2} - 1\right)\frac{\gamma}{\gamma+1}\left(\frac{\mathbf{v}}{c}\cdot\mathbf{B}\right)\frac{\mathbf{v}}{c} - \left(\frac{g}{2} - \frac{\gamma}{\gamma+1}\right)\frac{\mathbf{v}}{c} \times \mathbf{E}\right]. \tag{1.309}$$

The contribution of the Thomas precession is the part of the right-hand side without the gyromagnetic factor g:

$$\boldsymbol{\Omega}_{\mathrm{T}} = -\frac{e}{mc}\left[-\left(1 - \frac{1}{\gamma}\right)\mathbf{B} + \frac{\gamma}{\gamma+1}\frac{1}{c^2}\left(\mathbf{v}\cdot\mathbf{B}\right)\mathbf{v} + \frac{\gamma}{\gamma+1}\frac{1}{c}\mathbf{v} \times \mathbf{E}\right]. \tag{1.310}$$

This agrees with the Thomas frequency in Eq. (1.300) if we insert the acceleration

$$\dot{\mathbf{v}}(t) = c\frac{d}{dt}\frac{\mathbf{p}}{p^0} = \frac{e}{\gamma m}\left[\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B} - \frac{\mathbf{v}}{c}\left(\frac{\mathbf{v}}{c}\cdot\mathbf{E}\right)\right], \tag{1.311}$$

which follows directly from (1.182) and (1.183).

The Thomas equation (1.309) can be used to calculate the time dependence of the helicity $h \equiv \mathbf{S}_R\cdot\hat{\mathbf{v}}$ of an electron, i.e., its component of the spin in the direction of motion. Using the chain rule of differentiation,

$$\frac{d}{dt}\left(\mathbf{S}_R\cdot\hat{\mathbf{v}}\right) = \dot{\mathbf{S}}_R\cdot\hat{\mathbf{v}} + \frac{1}{v}\left[\mathbf{S}_R - (\hat{\mathbf{v}}\cdot\mathbf{S}_R)\hat{\mathbf{v}}\right]\frac{d}{dt}\mathbf{v} \tag{1.312}$$

and inserting (1.308) as well as the equation for the acceleration (1.311), we obtain

$$\frac{dh}{dt} = -\frac{e}{mc}\mathbf{S}_{R\perp}\cdot\left[\left(\frac{g}{2}-1\right)\hat{\mathbf{v}}\times\mathbf{B} + \left(\frac{gv}{2c}-\frac{c}{v}\right)\mathbf{E}\right], \tag{1.313}$$

where $\mathbf{S}_{R\perp}$ is the component of the spin vector orthogonal to $\mathbf{v}$. This equation shows that for a Dirac electron which has $g=2$ the helicity remains constant in a purely magnetic field. Moreover, if the electron moves ultra-relativistically ($v\approx c$), the value $g=2$ makes the last term extremely small, $\approx (e/mc)\gamma^{-2}\mathbf{S}_{R\perp}\cdot\mathbf{E}$, so that the helicity is almost unaffected by an electric field. The anomalous magnetic moment of the electron $a\equiv(g-2)/2$, however, changes this to a finite value $\approx -(e/mc)a\mathbf{S}_{R\perp}\cdot\mathbf{E}$. This drastic effect was used to measure the experimental values of a for electrons, positrons, and muons:

$$a(e^-) = (115\,965.77\pm 0.35)\times 10^{-8}, \tag{1.314}$$
$$a(e^+) = (116\,030\pm 120)\times 10^{-8}, \tag{1.315}$$
$$a(\mu^\pm) = (116\,616\pm 31)\times 10^{-8}. \tag{1.316}$$

1.14 Spacetime-Dependent Lorentz Transformations

The theory of gravitation to be developed in this book will not only be Lorentz-invariant, but also invariant under local Lorentz transformations

$$x'^a = \Lambda^a{}_b(x)x^b. \tag{1.317}$$

As a preparation for dealing with such theories let us derive a group-theoretic formula which is useful for many purposes.

1.14.1 Angular Velocities

Consider a time-dependent 3×3 -rotation matrix $R(\boldsymbol{\varphi}(t)) = e^{-i\boldsymbol{\varphi}(t)\cdot\mathbf{L}}$ with the generators $(L_i)_{jk} = -i\epsilon_{ijk}$ [compare (1.43)]. As time proceeds, the rotation angles change with an *angular velocity* $\boldsymbol{\omega}(t)$ defined by the relation

$$R^{-1}(\boldsymbol{\varphi}(t))\,\dot{R}(\boldsymbol{\varphi}(t)) = -i\boldsymbol{\omega}(t)\cdot\mathbf{L}. \tag{1.318}$$

The components of $\boldsymbol{\omega}(t)$ can be specified more explicitly by parametrizing the rotations in terms of *Euler angles* α, β, γ:

$$R(\alpha,\beta,\gamma) = R_3(\alpha)R_2(\beta)R_3(\gamma), \tag{1.319}$$

where $R_3(\alpha)$, $R_3(\gamma)$ are rotations around the z-axis by angles α, γ, respectively, and $R_2(\beta)$ is a rotation around the y-axis by β, i.e.,

$$R(\alpha,\beta,\gamma) \equiv e^{-i\alpha\hat{L}_3} e^{-i\beta\hat{L}_2} e^{-i\gamma\hat{L}_3}. \tag{1.320}$$

The relations between the vector $\boldsymbol{\varphi}$ of rotation angles in (1.57) and the Euler angles α, β, γ can be found by purely geometric considerations. Most easily, we equate the 2×2 -representation of the rotations $R(\boldsymbol{\varphi})$,

$$R(\boldsymbol{\varphi}) = \cos\frac{\varphi}{2} - i\,\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\varphi}}\,\sin\frac{\varphi}{2}, \tag{1.321}$$

with the 2×2 -representation of the Euler decomposition (1.320):

$$R(\alpha,\beta,\gamma) = \left(\cos\frac{\alpha}{2} - i\sigma_3\,\sin\frac{\alpha}{2}\right)\left(\cos\frac{\beta}{2} - i\sigma_2\,\sin\frac{\beta}{2}\right)\left(\cos\frac{\gamma}{2} - i\sigma_3\,\sin\frac{\gamma}{2}\right). \tag{1.322}$$

The desired relations follow directly from the multiplication rules for the Pauli matrices (1.223).

In the Euler decomposition, we may calculate the derivatives:

$$i\hbar\partial_\alpha R = R\left[\cos\beta\, L_3 - \sin\beta(\cos\gamma\, L_1 - \sin\gamma\, L_2)\right], \tag{1.323}$$

$$i\hbar\partial_\beta R = R\,(\cos\gamma\, L_2 + \sin\gamma\, L_1), \tag{1.324}$$

$$i\hbar\partial_\gamma R = R\, L_3. \tag{1.325}$$

The third equation is trivial, the second follows from the rotation of the generator

$$e^{i\gamma L_3/\hbar} L_2 e^{-i\gamma L_3/\hbar} = \cos\alpha\, L_2 + \sin\gamma\, L_1, \tag{1.326}$$

which is a consequence of *Lie's expansion formula*

$$e^{iA}Be^{-iA} = 1 + i[A,B] + \frac{i^2}{2!}[A,[A,B]] + \dots\ , \tag{1.327}$$

and the commutation rules (1.61) of the 3×3 -matrices L_i. The derivation of the first equation (1.323) requires, in addition, the rotation

$$e^{i\beta L_2/\hbar} L_3 e^{-i\beta L_2/\hbar} = \cos\beta L_3 - \sin\beta L_1. \tag{1.328}$$

We may now calculate the time derivative of $R(\alpha,\beta,\gamma)$ using Eqs. (1.323)–(1.325) and the chain rule of differentiation, and find the right-hand side of (1.318) with the angular velocities

$$\omega_1 = \dot{\beta}\sin\gamma - \dot{\alpha}\sin\beta\cos\gamma, \tag{1.329}$$

$$\omega_2 = \dot{\beta}\cos\gamma + \dot{\alpha}\sin\beta\sin\gamma, \tag{1.330}$$

$$\omega_3 = \dot{\alpha}\cos\beta + \dot{\gamma}. \tag{1.331}$$

Only commutation relations have been used to derive (1.323)–(1.325), so that the formulas (1.329)–(1.331) hold for *all* representations of the rotation group.

1.14.2 Angular Gradients

The concept of angular velocities can be generalized to spacetime-dependent Euler angles $\alpha(x)$, $\beta(x)$, $\gamma(x)$, replacing (1.318) by *angular gradients*

$$R^{-1}(\boldsymbol{\varphi}(x))\,\partial_a R(\boldsymbol{\varphi}(x)) = -i\boldsymbol{\omega}_a(x)\cdot\mathbf{L}, \tag{1.332}$$

with the generalization of the vector of angular velocity

$$\omega_{a;1} = \partial_a\beta\sin\gamma - \partial_a\alpha\sin\beta\cos\gamma, \tag{1.333}$$

$$\omega_{a;2} = \partial_a\beta\cos\gamma + \partial_a\alpha\sin\beta\sin\gamma, \tag{1.334}$$

$$\omega_{a;3} = \partial_a\alpha\cos\beta + \partial_a\gamma. \tag{1.335}$$

The derivatives ∂_a act only upon the functions right after it. These equations are again valid if $R(\boldsymbol{\varphi}(x))$ and $\mathbf{L}$ in (1.332) are replaced by any representation of the rotation group and its generators.

A relation of type (1.332) exists also for the Lorentz group where $\Lambda(\omega_{ab}(x)) = e^{-i\frac{1}{2}\omega_{ab}(x)L^{ab}}$ [recall (1.57)], and the generalized angular velocities are defined by

$$\Lambda^{-1}(\omega_{ab}(x))\,\partial_c\Lambda(\omega_{ab}(x)) = -i\,\frac{1}{2}\omega_{c;ab}(x)L^{ab}. \tag{1.336}$$

Inserting the explicit 4×4 -generators (1.51) on the right-hand side, we find for the matrix elements the relation

$$[\Lambda^{-1}(\omega_{ab}(x))\,\partial_c\Lambda(\omega_{ab}(x))]_{ef} = \omega_{c;ef}(x). \tag{1.337}$$

As before, the matrices $\Lambda(\omega_{ab}(x))$ and L^{ab} in (1.336) can be replaced by any representations of the Lorentz group and its generators, in particular in the spinor representation (1.229) where

$$D^{-1}(\Lambda(\omega_{ab}(x)))\,\partial_c D(\Lambda(\omega_{ab}(x)) = -i\,\frac{1}{2}\omega_{c;ab}(x)\Sigma^{ab}. \tag{1.338}$$

Appendix 1A Tensor Identities

In the tensor calculus of Euclidean as well as Minkowski space in d spacetime dimensions, a special role is played by the contravariant *Levi-Civita tensor*

$$\epsilon^{a_1a_2\dots a_d}, \quad a_i = 0,1,\dots,d-1. \tag{1A.1}$$

This is a totally antisymmetric unit tensor with the normalization

$$\epsilon^{012\dots(d-1)} = 1. \tag{1A.2}$$

It vanishes if any two indices coincide, and is equal to ± 1 if they differ from the natural ordering $0,1,\dots,(d-1)$ by an even or odd perturbation. The Levi-Civita tensor serves to calculate a determinant of a tensor t_{ab} as follows

$$\det(t_{ab}) = \frac{1}{d!}\epsilon^{a_1,a_2\dots a_d}\epsilon^{b_1b_2\dots b_d}\,t_{a_1b_1}\cdots t_{a_db_d}. \tag{1A.3}$$

In order to see this it is useful to introduce also the covariant version of $\epsilon^{a_1\dots a_d}$ defined by

$$\epsilon_{a_1a_2\dots a_d} \equiv g_{a_1b_1}g_{a_2b_2}\dots g_{a_db_d}\,\epsilon^{b_1b_2\dots b_d}. \tag{1A.4}$$

This is again a totally antisymmetric unit tensor with

$$\epsilon_{012\dots(d-1)} = (-1)^{d-1}. \tag{1A.5}$$

The contraction of the two is easily seen to be

$$\epsilon_{a_1\dots a_d}\epsilon^{a_1\dots a_d} = -d!. \tag{1A.6}$$

Now, by definition, a determinant is a totally antisymmetric sum

$$\det(t_{ab}) = \epsilon^{a_1\dots a_d}t_{a_10}\cdots t_{a_d(d-1)}. \tag{1A.7}$$

We may also write

$$\det(t_{ab})\,\epsilon_{b_1\dots b_d} = -\epsilon^{a_1\dots a_d}t_{a_1b_1}\cdots t_{a_db_d}. \tag{1A.8}$$

By contracting with $\epsilon^{b_1\dots b_d}$ and using (1A.6) we find

$$\det(t_{ab}) = -\frac{1}{d!}\epsilon^{a_1\dots a_d}\epsilon^{b_1\dots b_d}t_{a_1b_1}\cdots t_{a_db_d}, \tag{1A.9}$$

which agrees with (1A.7).

In the same way we can derive the formula

$$\det\left(t_a{}^b\right) = -\frac{1}{d!}\epsilon^{a_1\dots a_d}\epsilon_{b_1\dots b_d}t_{a_1}{}^{b_1}\cdots t_{a_d}{}^{b_d}. \tag{1A.10}$$

Under mirror reflection, the Levi-Civita tensor behaves like a pseudotensor. Indeed, if we subject it to a Lorentz transformation $\Lambda^a{}_b$, we obtain

$$\epsilon'^{\,a_1\dots a_d} = \Lambda^{a_1}{}_{b_1}\cdots\Lambda^{a_d}{}_{b_d}\,\epsilon^{b_1\dots b_d} = \det(\Lambda)\,\epsilon^{a_1\dots a_d}. \tag{1A.11}$$

As long as $\det\Lambda = 1$, the tensor $\epsilon^{a_1\dots a_d}$ is covariant under Lorentz transformations. If space or time inversion are included, then $\det\Lambda = -1$, and (1A.11) exhibits the pseudotensor nature of $\epsilon^{a_1\dots a_d}$.

We now collect a set of useful identities of the Levi-Civita tensor which will be needed in this text.

1A.1 Product Formulas

a) $d = 2$ Euclidean space with $g_{ij} = \delta_{ij}$.

The antisymmetric Levi-Civita tensor ϵ_{ij} with the normalization $\epsilon_{12} = 1$ satisfies the identities

$$
\begin{aligned}
\epsilon_{ij}\epsilon_{kl} &= \delta_{ik}\delta_{il} - \delta_{il}\delta_{jk}, && (1A.12)\\
\epsilon_{ij}\epsilon_{ik} &= \delta_{jk}, && (1A.13)\\
\epsilon_{ij}\epsilon_{ij} &= 2, && (1A.14)\\
\epsilon_{ij}\delta_{kl} &= \epsilon_{ik}\delta_{jl} + \epsilon_{kj}\delta_{il}. && (1A.15)
\end{aligned}
$$

b) $d = 3$ Euclidean space with $g_{ij} = \delta_{ij}$.
The antisymmetric Levi-Civita tensor ϵ_{ijk} with the normalization $\epsilon_{123} = 1$ satisfies the identities

$$
\begin{aligned}
\epsilon_{ijk}\epsilon_{lmn} &= \delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km}, && \\
&- \delta_{il}\delta_{jn}\delta_{km} - \delta_{in}\delta_{jm}\delta_{kl} - \delta_{im}\delta_{jl}\delta_{kn}, && (1A.16)\\
\epsilon_{ijk}\epsilon_{imn} &= \delta_{jm}\delta_{kn} - \delta_{in}\delta_{km}, && (1A.17)\\
\epsilon_{ijk}\epsilon_{ijn} &= 2\delta_{kn}, && (1A.18)\\
\epsilon_{ijk}\epsilon_{ijk} &= 6, && (1A.19)\\
\epsilon_{ijk}\delta_{lm} &= \epsilon_{ijl}\delta_{km} + \epsilon_{ilk}\delta_{jm} + \epsilon_{ljk}\delta_{im}, && (1A.20)
\end{aligned}
$$

c) $d = 4$ Minkowski space with metric

$$
g_{ab} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}. \qquad (1A.21)
$$

The antisymmetric Levi-Civita tensor with the normalization $\epsilon^{0123} = -\epsilon_{0123} = 1$ satisfies the product identities

$$
\begin{aligned}
\epsilon_{abcd}\epsilon^{efgh} &= -\Big(\delta^e_a\delta^f_b\delta^g_c\delta^h_d + \delta^f_a\delta^g_b\delta^h_c\delta^c_d + \delta^g_a\delta^k_b\delta^e_c\delta^f_d + \delta^h_a\delta^e_b\delta^f_c\delta^g_d \\
&+ \delta^f_a\delta^e_b\delta^h_c\delta^g_d + \delta^e_a\delta^h_b\delta^g_c\delta^f_d + \delta^h_a\delta^g_b\delta^f_c\delta^e_d + \delta^g_a\delta^f_b\delta^e_c\delta^h_d \\
&+ \delta^h_a\delta^g_b\delta^f_c\delta^e_d + \delta^g_a\delta^f_b\delta^e_c\delta^h_d + \delta^f_a\delta^e_b\delta^h_c\delta^g_d + \delta^c_a\delta^h_b\delta^g_c\delta^f_d \\
&- \delta^e_a\delta^f_b\delta^h_c\delta^g_d - \delta^f_a\delta^h_b\delta^g_c\delta^e_d - \delta^h_a\delta^g_b\delta^e_c\delta^f_d - \delta^g_a\delta^c_b\delta^f_c\delta^h_d \\
&- \delta^f_a\delta^e_b\delta^g_c\delta^h_d - \delta^e_a\delta^g_b\delta^h_c\delta^f_d - \delta^g_a\delta^h_b\delta^f_c\delta^e_d - \delta^h_a\delta^f_b\delta^e_c\delta^g_d \\
&- \delta^g_a\delta^h_b\delta^f_c\delta^e_d - \delta^h_a\delta^f_b\delta^e_c\delta^g_d - \delta^f_a\delta^e_b\delta^g_c\delta^h_d - \delta^e_a\delta^g_b\delta^h_c\delta^f_d\Big), && (1A.22)
\end{aligned}
$$

$$
\begin{aligned}
\epsilon_{abcd}\epsilon^{afgh} &= -\left(\delta^f_b\delta^g_c\delta^h_d + \delta^g_b\delta^h_c\delta^f_d + \delta^h_b\delta^f_c\delta^g_d - \delta^f_b\delta^h_c\delta^g_d - \delta^h_b\delta^g_c\delta^f_d - \delta^g_b\delta^f_c\delta^h_d\right), && (1A.23)\\
\epsilon_{abcd}\epsilon^{abgh} &= -2\left(\delta^g_c\delta^h_d - \delta^h_c\delta^g_d\right), && (1A.24)\\
\epsilon_{abcd}\epsilon^{abch} &= -6\delta^h_d, && (1A.25)\\
\epsilon_{abcd}\epsilon^{abcd} &= -24, && (1A.26)\\
\epsilon_{abcd}g_{ef} &= \epsilon_{abce}g_{dt} + \epsilon_{abcd}g_{cf} + \epsilon_{aecd}g_{bf} + \epsilon_{ebcd}g_{af}. && (1A.27)
\end{aligned}
$$

1A.2 Determinants

a) $d=2$ Euclidean:

$$\begin{aligned} g &= \det(g_{ij}) = \frac{1}{2!}\epsilon_{ik}\epsilon_{il}g_{ij}g_{kl} \equiv \frac{1}{2}g_{ij}C^{ij}, \\ C^{ij} &= \epsilon_{ik}\epsilon_{jl}g_{kl} = \text{cofactor}, \\ g^{ij} &= \frac{1}{g}C_{ij} = \text{inverse of } g_{ij}. \end{aligned} \tag{1A.28}$$

b) $d=3$ Euclidean:

$$\begin{aligned} g &= \det(g_{ij}) = \frac{1}{3!}\epsilon_{ikl}\epsilon_{jmn}g_{ij}g_{km}g_{ln} = g_{ij}C^{ij}, \\ C^{ij} &= \frac{1}{2!}\epsilon_{ikl}\epsilon_{jmn}g_{km}g_{ln} = \text{cofactor}, \\ g^{ij} &= \frac{1}{g}C^{ij} = \text{inverse of } g_{ij}. \end{aligned} \tag{1A.29}$$

c) $d=4$ Minkowski:

$$\begin{aligned} g = \det(g_{ab}) &= -\frac{1}{4!}\epsilon^{abcd}\epsilon^{efgh}g_{ac}g_{bf}g_{cg}g_{dh} = \frac{1}{4}g^{ae}C^{ae}, \\ C^{ae} &= -\frac{1}{3!}\epsilon^{abcd}\epsilon^{efgh}g_{bf}g_{cg}g_{dh} = \text{cofactor}, \\ g^{ab} &= \frac{1}{g}C^{ab} = \text{inverse of } g_{ab}. \end{aligned} \tag{1A.30}$$

1A.3 Expansion of Determinants

From Formulas (1A.28)–(1A.30) together with (1A.12), (1A.16), (1A.22), we find

$$\begin{aligned} &d{=}2: \ \det(g_{ij}) = \frac{1}{2!}\left[(\text{tr}g)^2 - \text{tr}(g^2)\right], \\ &d{=}3: \ \det(g_{ij}) = \frac{1}{3!}\left[(\text{tr}g)^3 + 2\,\text{tr}(g^3) - 3\,\text{tr}g\,\text{tr}(g^2)\right], \\ &d{=}4: \ \det(g_{ab}) = \frac{1}{24}\left[(\text{tr}g)^4 - 6(\text{tr}g)^2\,\text{tr}(g^2) + 3[\text{tr}(g^2)]^2 + 8\,\text{tr}(g)\,\text{tr}(g^3) - 6\,\text{tr}(g^4)\right]. \end{aligned} \tag{1A.31}$$

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