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BILOCAL FORM FACTORS AND REGGE COUPLINGS

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Abstract: The infinitely many local observables defined by expanding the bilocal currents $j^{\mu}(x, y)$ on the light cone in powers of $(x-y)^{\mu}$ are used to study scattering processes where one or two external lines are Reggeons. The algebra of bilocal form factors of Fritzsch and Gell-Mann implies the existenc of an algebra of infinitely many form factors $F^{J\pm}(\mathbf{k})$ of any spin J, definite signature±, and arbitrary momentum transfer $\mathbf{k}$. These "signatured form factors" can be continued analytically in $\mathbf{k}$ as well as in J and superconvergence relations are obtained for the couplings of strongly interacting particles of arbitrarily high spin and for Regge couplings. Also the commutator of form factors with Regge residues vanishes, except at certain momentum transfers. In particular, the charges $F^{J\eta}(0)$ of spin J and signature $\eta = (-)^J$ act as "daughter lowering operators". The range of validity of these sum rules in momentum transfer is discussed by continuing the scattering amplitudes involving the spin J and spin J' currents analytically in J and J'.

When the sum rules break down one can truncate the intermediate sums at some value of the intermediate masses. In this way one obtains an algebraic form of finite-energy sum rules. They consist in commutation rules among Reggeons with the right-hand side being given by the Regge couplings that can be exchanged in the corresponding scattering process. The structure constants of this algebra are the corresponding triple-Regge couplings. As an example one may assume dominance of leading trajectories and finds that ρ, A_1, π and f residues commute like O(5). Including also the commutators with vector and axial vector charges, one obtains the "supergroup" SU(2)×SU(2)×O(5).

The Regge couplings we are dealing with here are all in the infinite-momentum frame. In order to give the connection with standard t-channel couplings the angular condition for bilocal form factors are derived and continued in angular momentum.

As another technical side results we point out that under very weak assumptions the bilocal form factors $F(\mathbf{k}, z)$ are analytic in z. $F(\mathbf{k}, z)$ does not, however, possess Regge behaviour for $|z| \to \infty$ as often stated. Only signatured bilocal form factors do, as shown in this work.

This work is dedicated to the memory of my friend Bruno Renner who never returned from an excursion in the Geneva mountains.

1. Introduction

The hypothesis that the leading light-cone singularity of current commutators be the same as in the free-quark model [1, 2] has brought to our attention the existence of an infinite string of new observable local currents of spin 1, 2, 3, ... In the free-quark model these currents are defined by expanding at $(x-y)^2 = 0$:

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$$j_a^\mu(x,y) = \bar\psi(x)\gamma^\mu \frac{\lambda_a}{2}\psi(y) \equiv \sum_{J=1}^{\infty} \frac{i^{J-1}}{(J-1)!}\, \zeta_{\mu_1}\dots\zeta_{\mu_{J-1}} j^{\mu\mu_1\dots\mu_{J-1}}(R), \tag{1.1}$$

$$j_{5a}^\mu(x,y) = \bar\psi(x)\gamma^\mu\gamma_5 \frac{\lambda_a}{2}\psi(y) = \sum_{J=1}^{\infty} \frac{i^{J-1}}{(J-1)!}\, \zeta_{\mu_1}\dots\zeta_{\mu_{J-1}} j^{\mu\mu_1\dots\mu_{J-1}}(R), \tag{1.2}$$

where* $\zeta = x-y$, $R = \frac{1}{2}(x+y)$.

In deep-inelastic scattering processes

$$\text{“}\gamma\text{”}(q) + \alpha(p) \to \text{“}\gamma\text{”}(q') + \beta(p'); \qquad q^2 \to q'^2 \to \infty;\ \text{at fixed } (q-q')^2 = -k^2,$$
$$\xi = -q^2/(p'+p)q\ , \tag{1.3}$$

in which “γ” can be a virtual photon or W meson, one is measuring structure functions symmetric and antisymmetric in ξ:

$$F^{\mathrm{S,A}}(k,\xi),\, F_5^{\mathrm{S,A}}(k,\xi). \tag{1.4}$$

If “γ”(q) is running with high mass and momentum in the $-z$ direction, these structure functions are simply the Fourier transforms

$$F^{\mathrm{S,A}}(k,z) = \int_{-\infty}^{\infty} \mathrm{e}^{i\xi z}\, F^{\mathrm{S,A}}(k,\xi)\,\mathrm{d}\xi \tag{1.5}$$

of the symmetrized and antisymmetrized “bilocal form factors”

$$(2\pi)^3(p'+p)^+\delta(p^{+\prime}-p^+)\delta^2(p^{\perp\prime}-p^\perp-k)F^{\mathrm{S,A}}(k,(p'+p)^+\zeta^-) \tag{1.6}$$
$$\equiv \tfrac{1}{2}\,\{\langle\beta p'|\textstyle\int \mathrm{d}R^-\mathrm{d}^2R^\perp\, \mathrm{e}^{ikR^\perp} j^+(R^+,R^\perp,R^-+\frac{1}{2}\zeta^-;R^+,R^\perp,R^--\frac{1}{2}\zeta^-)\pm(\zeta^-\to-\zeta^-)\}.$$

Expanding

$$F^{\mathrm{S,A}}(k,z) = \Sigma\, \frac{(iz)^{J-1}}{(J-1)!}\, F^J(k) \tag{1.7}$$

we see that all matrix elements of $j^{++\dots+}(R)$

$$(2\pi)^3\,(p'+p)^+\,\delta(p^{+\prime}-p^+)\,\delta^2(p^{\perp\prime}-p^\perp-k)F^J(k)$$
$$\equiv \langle\beta p'|\textstyle\int \mathrm{d}R^-\,\mathrm{d}^2R^{\perp R}\mathrm{e}^{ikR^\perp} j^{+\dots+}(R)\,|\alpha p\rangle. \tag{1.8}$$

are measurable in principle via the moments of the structure functions

$$F^J(k) = \int_{-\infty}^{\infty} \xi^{J-1}\, F^{\mathrm{S,A}}(k,\xi)\,\mathrm{d}\xi \qquad \text{for} \begin{cases} J = \text{odd} \\ J = \text{even} \end{cases}. \tag{1.9}$$

Equivalently $F^J(k)$ may be obtained as the infinite momentum matrix elements of $j^{00\dots0}(0)$:

* We use the standard notation $x^+ = x^0 + x^3$, $x^- = \frac{1}{2}(x^0-x^3)$, $x^\perp = (x^1, x^2)$.

$$F^J(p^{\perp\prime} - p^\perp) = \lim_{x\to\infty} \frac{1}{(p^{0\prime} + p^0)^J} \langle \beta, xe_3 + p^{\perp\prime} | j^{0\ldots 0}(O) | \alpha, xe_3 + p^\perp \rangle \tag{1.10}$$

(Since (1.7) is invariant with respect to Lorentz tranformations in the z-direction). Obviously, only the traceless and conserved part of the current survives the ∞ momentum limit. Conversely, the matrix elements (1.10) can easily be shown to be sufficient to describe the traceless and conserved part of the local current $j^{\mu_1 \cdots \mu_J}(R)$ completely. This is done by deriving the Lorentz transformation from the infinite-momentum form factors $F^J(R)$ to the standard helicity form factors. The resulting equations are commonly called the angular conditions [3]. Since the derivation is rather technical it will be presented in appendix A (there we shall also prove another pleasant property of $F^J(k)$, namely that if the states β and α in $F^J_{\beta\alpha}(k)$ differ in helicity by an amount Δh, then $F^J_{\beta\alpha}(k)/|k|^{\Delta h}$ is analytic in $t = -k^2$).

Therefore one may conclude that the form factors of the leading spin J of the currents $j^{\mu_1 \cdots \mu_J}$ appearing in the decomposition (1.1) can be uniquely measured by experiment. Notice that the integral in (1.5) can be constrained to the interval * $\xi \in [-1, 1]$ since the spectral condition makes $F(k, \xi)$ vanish for $\xi > 1$ and $F^{\mathrm{S,A}}(k, \xi)$ are symmetric and antisymmetric in ξ, respectively.

The new fundamental hypothesis put forward recently by Fritzsch and Gell-Mann [2] is that the bilocal form factors satisfy an algebra suggested by the bilocal operators of the quark model (with the quarks being held together by gluons):

$$\begin{aligned}
&[F_a(k, z), F_b(k', z')] = i f_{abc} F_c(k + k', z + z'),\\
&[F_a(k, z), F_{5b}(k', z')] = i f_{abc} F_{5c}(k + k', z + z'),\\
&[F_{5a}(k, z), F_{5b}(k', z')] = i f_{abc} F_c(k + k', z + z').
\end{aligned} \tag{1.11}$$

These sum rules are derived in the following way: First one sets $\zeta^+ = \zeta^\perp = 0$ in the algebra of bilocal currents. Then one obtains an algebra of the form (1.11) for the operators

$$\hat{F}_a(R^+, k, \zeta^-) = \int \mathrm{d}R^- \mathrm{d}^2 R^\perp j^+_a(R^+, R^\perp, R^- + \tfrac{1}{2}\zeta^-; R^+, R^\perp, R^- - \tfrac{1}{2}\zeta^-) \mathrm{e}^{ikR^\perp},$$

when commuted at equal R^+. Finally one assumes that the operator relation can be saturated by physical intermediate states of finite masses.

Expanding the bilocal form factors according to (1.7) we find that the form factors $F^J(k)$ satisfy the algebra

$$\begin{aligned}
&[F^J_a(k), F^{J'}_b(k')] = i f_{abc} F^{J+J'-1}_c(k + k'),\\
&[F^J_a(k), F^{J'}_{5b}(k')] = i f_{abc} F^{J+J'-1}_{5c}(k + k'),\\
&[F^J_{5a}(k), F^{J'}_{5b}(k')] = i f_{abc} F^{J+J'-1}_c(k + k').
\end{aligned} \tag{1.12}$$

* This fact follows basically from the dimension of the currents in the light-cone algebra being canonical. In the presence of anomalous dimensions the structure functions would not have this nice compact support. See ref. [4].

The subalgebra for $J = J' = 1$ coincides with the old algebra of form factors of vector currents proposed by Dashen and Gell-Mann [3] a long time ago.

There are several points we want to make in this paper.

(i) Upon introducting properly signature, $F^{J\pm}(k)$ can be continued analytically in the J plane.

(ii) $F^{J\pm}(k)$ coincides essentially with the t-channel partial wave amplitude of the scattering amplitude (1.3). If Regge poles survive the scaling limit, $F^{J\pm}(k)$ is expected to show the standard Regge pole structure in the J-plane.

(iii) $F(k, z)$ is analytic in z. Its properly signatured parts have Regge behaviour for $|z| \to \infty$ in the upper or lower half-plane. $F(k, z)$ itself is not, however, Regge behaved.

(iv) The algebra of form factors of Fritzsch and Gell-Mann gives rise to an algebra of signatured form factors. This algebra can be used to derive superconvergence relations for the scattering of hadrons of any spin $J \geqslant 1$, for Regge-particle scattering and, for the production of a reggeon by means of a spin-J current.

(v) In kinematical regions where the superconvergence relations break down, an algebra of finite-energy sum rules remains between form factors and Regge couplings. The charge of the spin-J current acts as a "daughter lowering operator" on Regge couplings.

(vi) The particular case of the algebra of isospin, axial charge and ρ, A_1, π, f trajectories is shown to yield the algebra SU(2)×SU(2)×O(5) and a solution is given for a system mesons consisting of π, A_1, ρ, σ.

(vii) The form factors $F^J(k)$ are analytic in k and obey a simple angular condition.

2. Analytic continuations of $F^J(k)$ in J

Let us recall that the structure functions $F(k, \xi)$ are defined as the absorptive part of some Compton amplitudes $T(k, q^2, q'^2, \nu)$ in the scaling limit (1.3):

$$F(k, \xi) \equiv \frac{1}{\pi} \operatorname{Im} T(k, \xi). \tag{2.1}$$

As an example, consider the structure functions measured in deep inelastic scattering of electrons or neutrinos on nucleons:

$$F^{\mathrm{A}}(\xi) \equiv \frac{1}{\xi} F_2(\xi), \tag{2.2}$$

$$F^{\mathrm{S}}(\xi) \equiv \quad F_3(\xi), \tag{2.3}$$

where the argument k has been dropped since it vanishes. Here the corresponding amplitudes $T(\xi)$ are given by the scaling limits

$$T^{\mathrm{A}}(\xi) = \lim_{\mathrm{S.L.}} \frac{2\nu^2}{-q^2} T_2(q^2, \nu), \tag{2.4}$$

$$T^{\mathrm{S}}(\xi) = \lim_{\mathrm{S.L.}} \frac{\nu}{m} T_3(q^2, \nu). \tag{2.5}$$

Using (2.1) one can write for $T(k, \xi)$ a dispersion relation which may in general contain $N(k)$ subtractions

$$T(k, \xi) = -\frac{1}{\xi^{N-1}} \int_{-1}^{1} \xi'^{N-1} \frac{F(k, \xi')\mathrm{d}\xi'}{\xi'-\xi + i\epsilon} + P_{N-1}(k, \xi^{-1}). \tag{2.6}$$

For $|\xi| > 1$, the denominator can be expanded in a power series giving

$$T(k, \xi) = \sum_{n=0}^{\infty} \xi^{-N-n} \int_{-1}^{1} \xi'^{N+n-1} F(k, \xi')\mathrm{d}\xi' + P_{N-1}(k, \xi^{-1}). \tag{2.7}$$

If we denote the expansion coefficients of $T(k, \xi)$ with respect to powers of ξ^{-1} by $T^J(\xi)$:

$$T(k, \xi) \equiv \sum_{J=0}^{\infty} \xi^{-J}\, T^J(k), \tag{2.8}$$

we see that for $J \geqslant N(k)$ the coefficients $T^J(k)$ coincide exactly with the previously defined form factors $F^J(k)$ [see eq. (1.9)]:

$$T^J(k) \equiv F^J(k), \quad J \geqslant N(k)\,. \tag{2.9}$$

Notice that in the important particular examples (2.4) and (2.5) this equality holds even for all $n = 1, 2, 3, \ldots$. The reason is that $T_2(q^2, \nu)$ and $T_3(q^2, \nu)$ are expected to behave for fixed q^2 and large ν like $\nu^{\alpha-2}$ and $\nu^{\alpha-1}$, respectively where α is the leading Regge trajectory exchanged in the t-channel. If we assume this Regge behaviour to govern the amplitude also in the scaling region and use the experimental fact * that $\alpha \leqslant 1$ we see that $T^A(\xi)$ and $T^S(\xi)$ cannot diverge faster than ξ^{-1} for $\xi \to 0$. This excludes more than two subtractions. On the other hand, $T^S(\xi)$ has a single zero, $T^A(\xi)$ a double zero at $\xi = \infty$. Due to the symmetry properties of $T^S(\xi)$, $T^A(\xi)$ this forces the subtraction polynomials to vanish identically.

With the standard argument [5] one can now show that upon introducing signatured amplitudes

$$\begin{aligned} T^{\pm}(k, \xi) &\equiv -\frac{1}{\xi^{N-1}} \int_{0}^{1} \xi'^{N-1} \frac{F^{A,S}(k, \xi')\mathrm{d}\xi'}{\xi' - \xi + i\epsilon} + P_{N-1}(k, \xi^{-1}) \\ &\equiv \sum_{J=0}^{\infty} \xi^{-J}\, T^{J\pm}(k), \end{aligned} \tag{2.10}$$

the expansion coefficients

$$T^{J\pm}(k) = 2 \int_{0}^{1} \xi^{J-1}\, F^{A,S}(k, \xi)\mathrm{d}\xi \tag{2.11}$$

* Or one invokes the Froissart bound on strong amplitudes.

can be continued analytically in J into the whole half-plane $\mathrm{Re}\, J \geqslant N(k)$ such that $|T^{J\pm}(k)| < e^{\gamma|J|}$ with $\gamma < \pi$. This latter property makes the continuation unique according to Carlson's theorem.

The only quite weak technical assumption needed for the proof is that $F(k, \xi)$ is bounded for $\xi \approx 1$. This assumption can be justified by extrapolating from what we have learned in the discussion of deep-inelastic e-p scattering. Here the structure functions close to $\xi = 1$ can appearently be described as a superposition of resonances only and $F_2(\xi)$ tends to zero as $(1 - \xi)^{2p-1}$ if the form factors drop as $(q^2)^{-p}$. The same picture should be expected to hold also for all other structure functions $F(k, \xi)$.

With this weak assumption, eq. (2.1) written in the form

$$T^{J\pm}(k) = 2 \int_0^1 e^{(J-1)\log\xi}\, F^{A,S}(k, \xi) \tag{2.12}$$

is seen to supply indeed the desired continuation. Since $F^{A,S}$ is positive, $T^{J\pm}(k)$ is bounded for large $|J|$ by

$$|T^{J\pm}(k)| < \frac{c}{|J|}. \tag{2.13}$$

It is obvious that eq. (2.12) is the exact analogue of the Froissart-Gribov projection in standard Regge pole analyses [5]. Just as usual, this analytic continuation can be employed to extend the validity of the expansion (2.10) from the outside region of the circle $|\xi| = 1$* to the whole plane cut from $\xi = 0$ to infinity. One first performs the Sommerfeld-Watson transform:

$$T^{\pm}(k, \xi) = \sum_{J=0}^{N(k)} \xi^{-J}\, T^{J\pm}(k)$$
$$- \frac{1}{2i} \int_{N(k)-i\infty}^{N(k)+i\infty} \mathrm{d}J\, \frac{e^{-i\pi J}\xi^{-J}}{\sin \pi J}\, T^{J\pm}(k). \tag{2.14}$$

Then one makes the standard assumption that the Compton amplitude possesses in the J-plane only poles between the lines $\mathrm{Re}\, J = -\frac{1}{2}$ and $\mathrm{Re}\, J = N(k)$, moving Regge poles as well as possibly fixed poles, and that these poles survive the scaling limit. ** Then $T^{J\pm}(k)$ will have the form***

$$T^{J\pm}(k) = \sum_i \frac{R^i(k)}{J - \alpha_i(k)} + \ldots, \tag{2.15}$$

where $R^i(k)$ denote the Regge residue of the trajectory $\alpha_i(k)$. The sum may run over infinitely many daughters. Every pole contributes a term

* This is the "Lehmann ellipse" of this expansion.

** Out of ignorance, we shall completely neglect diffraction effects.

*** Notice that we write $\alpha_i(\boldsymbol{k})$ instead of the conventional form $\alpha_i(-\boldsymbol{k}^2)$, for convenience.

$$T^{\pm}(k,\xi) \xrightarrow[\xi\to 0^+]{} -\pi \frac{e^{-i\pi\alpha_i(k)}}{\sin \pi\alpha_i(k)} R^i(k)\, \xi^{-\alpha_i(k)} \tag{2.16}$$

and the highest trajectory dominates in the limit $\xi \to 0^+$.

In complete analogy with $T^{\pm}(k, \xi)$ we may introduce *signatured bilocal form factors*

$$F^{\pm}(k, z) \equiv 2 \int_0^1 e^{i\xi z}\, F^{A,S}(k, \xi) d\xi\,. \tag{2.17}$$

Their Taylor coefficients in an expansion like (1.7)

$$F^{J\pm}(k) = 2 \int_0^1 \xi^{J-1}\, F^{A,S}(k, \xi) d\xi \tag{2.18}$$

are called *signatured form factors.* They coincide with $T^{J\pm}(k)$ for $J \geqslant N(k)$ and have therefore the same singularities in the J-plane.

Notice that the pleasant asymptotic behaviour (2.13) of $F^{J\pm}(k)$ makes the bilocal form factors (2.17) analytic in the whole z-plane due to the strong convergence factor $(J-1)!^{-1}$. As a consequence, also the original bilocal form factors

$$F^{A,S}(k, z) = \tfrac{1}{2}(F^{\pm}(k, z) \mp F^{\pm}(k, -z)) \tag{2.19}$$

are analytic in z everywhere. It is a curious fact that the Fourier transform of the imaginary part of an amplitude has Taylor coefficients which are by a factor $(J-1)!^{-1}$ smaller than the Taylor coefficients of the amplitude itself. One often finds the statement that $F(k, z)$ has a Regge limit for $|z| \to \infty$ analogous to the $\xi \to 0^+$ limit (2.16). This cannot be proved in general and counterexamples are easily given. Also for this, the reason is the occurrence of the denominator $(J-1)!$ in the expansion (1.7). While this denominator is pleasant in enforcing the convergence of the series (1.7) for all z, it makes the Sommerfeld-Watson transformation of this series convergent only inside the half-plane orthogonal to the cut * of the asymptotic term $(-iz)^{\alpha_i(k)}$. There one has for large $|z|$:

$$F^{\pm}(k, z) = -\pi \frac{1}{\sin \pi\alpha_i \Gamma(\alpha_i)} R^i(k)\, (-iz)^{\alpha_i(k)} + \ldots\, i\,;\;\; 0 < \arg z < \pi\,. \tag{2.20}$$

The limit $|z| \to \infty$ inside the other half-plane may have no relation to the Regge pole. As a consequence, the original bilocal form factors $F^{A,S}(k, z)$ will not, in general, have a Regge limit as $|z| \to \infty$.

Take as a counterexample the confluent hypergeometric function

$$M(-\alpha, -\alpha+1; x) = 1 - \frac{\alpha}{1-\alpha}\frac{x}{1!} - \frac{\alpha}{2-\alpha}\frac{x^2}{2!} - \ldots$$

* Since $(J-1)!$ decreases like $e^{-\frac{1}{2}\pi|J|}$ along the imaginary axis of J, killing half of the convergence factor $1/\sin \pi J \sim e^{-\pi|J|}$.

It can be considered a form factor $F^{\pm}(k, x)$. Careless reggeization would give the asymptotic behaviour

$$M(-\alpha, -\alpha+1; x) \xrightarrow[|x|\to\infty]{} \pi \frac{1}{\sin \pi\alpha\Gamma(\alpha)} (-x)^{\alpha} = \Gamma(-\alpha+1)(-x)^{\alpha}. \tag{2.21}$$

A short look at the tables * shows us that this is indeed only true in the half-plane $\frac{1}{2}\pi < \arg x < \frac{3}{2}\pi$. In the other half-plane one rather [6] finds

$$M(-\alpha, -\alpha+1; x) \xrightarrow[|x|\to\infty]{} -\alpha\, \mathrm{e}^{x} \frac{1}{x} \tag{2.22}$$

which fastly outgrows the Regge limit. The factor e^{x} is due to Kummer's reflection identity

$$M(a, b; x) = \mathrm{e}^{x} M(b-a, b, -x). \tag{2.23}$$

As a consequence we see that the symmetric and antisymmetric combinations of $M(a, b; x)$ (which would correspond to $F^{\mathrm{S,A}}(k, x)$) do not have a Regge limit as $|x| \to \infty$.

The important point for our future discussion is now the following: The signatured form factors $F^{J\pm}(k)$ carrying all Regge information fulfill the same algebra (1.12) as the unsignatured ones, with ± signs distributing according to the scheme $[++] = -$, $[+-] = +$, $[--] = -$. The proof is quite simple. We shall show it only for the first commutator (1.11) and drop the argument k for brevity. According to (1.11), the symmetric and antisymmetric combinations satisfy

$$\begin{aligned}
[F^{\mathrm{S}}_a(z), F^{\mathrm{S}}_b(z')] &= \tfrac{1}{2} i f_{abc}\, \{F^{\mathrm{S}}(z+z') + F^{\mathrm{S}}(z-z')\},\\
[F^{\mathrm{S}}_a(z), F^{\mathrm{A}}_b(z')] &= \tfrac{1}{2} i f_{abc}\, \{F^{\mathrm{A}}(z+z') - F^{\mathrm{A}}(z-z')\},\\
[F^{\mathrm{A}}_a(z), F^{\mathrm{A}}_b(z')] &= \tfrac{1}{2} i f_{abc}\, \{F^{\mathrm{S}}(z+z') - F^{\mathrm{S}}(z-z')\}.
\end{aligned} \tag{2.25}$$

Fourier transforming according to (1.5) we find that the structure functions commute according to

$$\begin{aligned}
[F^{\mathrm{S}}_a(\xi), F^{\mathrm{S}}_b(\xi')] &= \tfrac{1}{2} i f_{abc} (\delta(\xi-\xi') F^{\mathrm{S}}(\xi) + \delta(\xi+\xi') F^{\mathrm{S}}(\xi)),\\
[F^{\mathrm{S}}_a(\xi), F^{\mathrm{A}}_b(\xi')] &= \tfrac{1}{2} i f_{abc} (\delta(\xi-\xi') F^{\mathrm{A}}(\xi) - \delta(\xi+\xi') F^{\mathrm{A}}(\xi)),\\
[F^{\mathrm{A}}_a(\xi), F^{\mathrm{A}}_b(\xi')] &= \tfrac{1}{2} i f_{abc} (\delta(\xi-\xi') F^{\mathrm{S}}(\xi) - \delta(\xi+\xi') F^{\mathrm{S}}(\xi)).
\end{aligned} \tag{2.24}$$

If we now form the moments of eq. (2.25) according to eq. (2.18) we indeed find the algebra for the signatured form factors

$$\begin{aligned}
[F^{J+}_a(k), F^{J'+}_b(k')] &= i f_{abc}\, F^{J+J'-1,-}_c(k+k'),\\
[F^{J+}_{\mathrm{a}}(k), F^{J-}_b(k')] &= i f_{abc}\, F^{J+J'-1,+}(k+k'),\\
[F^{J-}_a(k), F^{J-}_b(k')] &= i f_{abc}\, F^{J+J'-1,-}(k+k'),
\end{aligned} \tag{2.26}$$

* See for example p. 504 of ref. [6].

with similar commutators for $[F_a(k), F_{5b}(k')]$ and $[F_{a5}(k), F_{b5}(k')]$.

The algebra (2.26) together with the Regge properties (2.15) of $F^{J\pm}(k)$ have some immediate consequences which will be discussed in the next section.

3. Consequences of the algebra of signatured form factors

There is a number of interesting sum rules which can immediately be obtained from the algebra of signatured form factors. As usual, these sum rules will be valid only for certain regions of the momentum transfer $-(k+k')^2$ which depend on the properties of the Regge trajectory that can be exchanged in the t-channel. When the angular momentum $\alpha_k(k+k')$ of the exchanged Regge pole moves too high up, the sum rules will break down and the sum over intermediate states diverges. In this situation only certain finite-energy sum rules can be written. These points will all be discussed in detail in the next section. Here we shall proceed by deriving some sum rules in a purely formal manner and not worry about their range of validity.

(i) The first sum rule is obtained by an argument of the type of de Alfaro, Fubini, Furlan and Rosetti [7]. If we continue k^2, k'^2 to places where the form factors $F_a^{J\pm}(k)$ and $F_{5a}^{J\pm}(k)$ have particle poles, the left-hand side of (2.26) will diverge quadratically and the right-hand drops out. As a result, the particle couplings satisfy a superconvergence relation

$$[G^{i,J}(\hat{k}), G^{i',J'}(\hat{k}')] = 0. \tag{3.1}$$

Here we have denoted the residue $R^i(k)$ at the particle pole $\alpha_i(k) = J$ by $G^{i,J}(\hat{k})$ *. As we shall see more clearly in the next section this statement is equivalent to saying that in the scattering amplitudes of strongly interacting particles of any spin $J \geqslant 1$ there can be no fixed poles of spin $J' + J - 1$. Previously this result was obtained from current algebra for vector mesons only. Here it holds for most mesons of spin $J \geqslant 1$, just as long as they can be found as a pole in some form factor $F^{J\pm}(k)$, $F_5^{J\pm}(k)$.

(ii) Another interesting sum rule is obtained by continuing the algebra (2.26) analytically not in k but in J and J' for *arbitrary fixed* k, k' with restrictions to be specified in the next section. Then we obviously obtain superconvergence relations for the residues of the Regge poles

$$[R^i(k), R^j(k')] = 0. \tag{3.2}$$

(iii) As a third possibility we may continue the index J' to a Regge pole and leave J fixed. Then we pick up a single pole on the left-hand side of (2.26). The right-hand side will, in general, not have a pole except for certain discrete values of t. Thus except for those isolated situations we find that the form factors $F^{J\pm}(k)$ and $F_5^{J\pm}(k)$

* Note that the particle couplings $G^{i,J}(\hat{k})$ still depend on the direction $\hat{k}$ of the transverse momentum transfer in the infinite-momentum frame.

commute with all Regge residues $R^i(k')$. The most important exception arises for the commutator of the charges $F^{J,(-)^J}(0)$. Going to the pole $J = \alpha_i(k)$, the right-hand side will diverge if there is a daughter trajectory of $\alpha_i(k)$ spaced $J-1$ units below. If we denote its residue by $R^{i,J-1}(k)$ (with $R^{i,0}(k)$ being again the original residue $R^i(k)$) we find the sum rule

$$[F_a^{J,(-)^J}(0), R_b^i(k)] = i\, f_{abc}\, R_c^{i,J-1}(k). \tag{3.3}$$

This can be interpreted by saying that the charge $F_a^{J,(-)^J}(0)$ of the current $j^{\mu_1 \dots \mu_J}$ acts as a daughter lowering operator on the Regge residues. In the particular case $J = 1$, $F_a^{1-}(0)$ becomes the SU(3) charge. The Regge coupling is transformed into itself and (3.3) just shows that R_a^i transforms as an octet under SU(3).

4. Validity of the superconvergence relations and algebraization of finite-energy sum rules

The full meaning of the sum rules derived in the last section can be understood best by studying the scattering amplitudes of the currents $j^{\nu_1 \dots \nu_{J'}}(R')$, $j^{\mu_1 \dots \mu_J}(R)$ on any target

$$T_{ba}^{\nu_1 \dots \nu_{J'};\mu_1 \dots \mu_J}(p',Q';p,Q) \equiv i \int dR'\, e^{iQ'R'}\, \langle \beta(p')|T(j_b^{\nu_1 \dots \nu_{J'}}(R')\, j_a^{\mu_1 \dots \mu_J}(0))|\alpha(p)\rangle. \tag{4.1}$$

It is this amplitude that directly describes the process

$$\text{“}\gamma\text{”}(q_1 - Q) + \text{“}\gamma\text{”}(q_2 + Q') + \beta(p') \leftarrow \text{“}\gamma\text{”}(q_1) + \text{“}\gamma\text{”}(q_2) + \alpha(p)$$

in the deep-inelastic limit $q_1^2, q_2^2 \to \infty$, $pq_1 \to \infty$, $p'q_2 \to \infty$ and Q, Q' = finite. If the momenta q_1, q_2 are taken to infinity along the z-direction, only the good components of this equation

$$T_{ba}^{J',J}(p', Q'; p, Q) = i \int dR'\, e^{iQ'R'}\, \langle \beta(p')|T(j_b^{J'}(R')\, j_a^J(0))|\alpha(p)\rangle \tag{4.2}$$

are measurable where we have introduced the abbreviation

$$T^{J',J} \equiv T^{\overbrace{+ + \dots +}^{J'} ; \overbrace{+ \dots +}^{J}}, \qquad j^J \equiv j^{\overbrace{+ + \dots +}^{J}}. \tag{4.3}$$

The whole kinematic range of the free variables $t = (Q' - Q)^2 = (p' - p)^2$ and $\nu \equiv \frac{1}{2}(p' + p)(Q' + Q) = \frac{1}{2}(s - u)$ at spacelike Q', Q, $(Q - Q')$ can be reached by setting

$$Q^{+\prime} = Q^+ \equiv 0\,. \tag{4.4}$$

In this case the invariants become

$$\nu = \tfrac{1}{2}(m_\beta^2 - m_\alpha^2) + P^+ Q^{-\prime} + P^\perp k'$$
$$= -\tfrac{1}{2}(m_\beta^2 - m_\alpha^2) + P^+ Q^- - P^\perp k\,, \tag{4.5}$$

$$t = -(k + k')^2, \tag{4.6}$$

$$Q^2 = -k^2,\ Q'^2 = -k'^2, \tag{4.7}$$

where $P \equiv p' + p$ and k', k are the transverse momenta of Q', Q i.e.

$$Q'^\mu = (Q^{+\prime} = 0, -k', Q^{-\prime}),\ \ Q^\mu = (Q^+ = 0, k, Q^-). \tag{4.8}$$

It is obvious from the form (4.2) that at fixed k', k, p', p, the amplitude is analytic in Q^- and hence in ν. Another way of seeing this is by noticing that $T^{J',J}$ coincides, up to a constant factor $(P^+)^{J'+J}$, exactly with the first invariant amplitude $A(\nu, t)$ in a kinematic decomposition

$$T^{\nu_1 \ldots \nu_{J'};\, \mu_1 \ldots \mu_J}(p', Q'; p, Q) = P^{\nu_1} \ldots P^{\nu_{J'}} P^{\mu_1} \ldots P^{\mu_J} A(\nu, t)$$
$$+ P^{\nu_1} \ldots P^{\nu_{J'}} P^{\mu_1} \ldots Q^{\mu_J} B(\nu, t) \tag{4.9}$$
$$+ \ldots$$

But this decomposition is well known to yield invariant functions A, B, C, analytic in ν (since they are essentially t-channel helicity amplitudes).

We now use the standard method of deriving fixed-mass sum rules from light cone commutators. Performing the Bjorken limit on (4.2) we find the asymptotic behaviour *

$$T_{ba}^{J',J}\Big|_{Q^{+\prime}=Q^+=0} \xrightarrow[Q^- \to \infty]{} -\frac{1}{Q^-} \langle \beta(p')| [\textstyle\int dR^\perp\, dR^-\, e^{ik'R^\perp}\, j_b^{J'}(R), j_a^J(0)]_{R^+=0} |\alpha(p)\rangle. \tag{4.10}$$

Upon using the commutation rule of the type (1.12) in the original operator version (i.e. before saturation with single particle states) one obtains

$$T_{ba}^{J',J}\Big|_{Q^{+\prime}=Q^+=0} \xrightarrow[Q^- \to \infty]{} -\frac{i f_{bac}}{Q^-} \langle \beta(p')| j_c^{J'+J-1}(0) |\alpha(p)\rangle\,. \tag{4.11}$$

This allows that the amplitude has a fixed pole at angular momentum $J' + J - 1$ in the Q^- or ν plane. In the special case of the scattering of vector currents, $J' = J = 1$, this is a well-known result. If the imaginary part of T is sufficiently well-behaved, i.e.

$$\mathrm{Im}\, T_{ba}^{J'J}\Big|_{Q^{+\prime}=Q^+=0} \underset{Q^- \to \infty}{\leqslant} C\,(Q^-)^{-1-\epsilon},\ \epsilon > 0 \tag{4.12}$$

* We remind the reader that the so called class II graphs have to be well-behaved in order to allow the integration of the local equal-R^+ commutation rule. This is usually assumed to be true.

the limit (4.11) can be phrased in form of a dispersion relation

$$\frac{1}{\pi}\int \mathrm{d}Q^- \ \mathrm{Im}\ T^{J'J}_{ba} = i\, f_{bac}\, \langle \beta(p')|\, j_c^{J'+J-1}(0)|\alpha(p)\rangle\,. \tag{4.13}$$

Saturating the left-hand side with physical intermediate states yields *

$$[F^{J'}_b(k'), F^J_a(k)] = i\, f_{bac}\, F_c^{J'+J-1}(k'+k)\,. \tag{4.14}$$

In this way of deriving the commutation rules (1.12) of the form factors one can immediately judge their validity. The sum rule holds if the integral over the imaginary part converges, or equivalently, as long as the sum over all intermediate states in the commutator gives a finite result. The behaviour of this sum is judged on the basis of Regge theory. If the trajectory, $\alpha_k(k'+k)$ can be exchanged, the asymptotic behaviour of Im $T^{J'J}$ is expected to be

$$\mathrm{Im}\ T^{J'J}\Big|_{Q^{+\prime}=Q^+=0}\ \xrightarrow[Q^-\to\infty]{}\ g_k^{J'J}\, R^k(k'+k)\,(s/m_0^2)^{\alpha_k(k'+k)-J'-J}\,. \tag{4.15}$$

Such a Regge behaviour may be read directly off eq. (4.9) upon forming helicity amplitudes which are known ro behave like $(s/m_0^2)^{\alpha_k-J'-J}$. Hence the algebra (4.13) is valid as long as

$$\alpha_k(k'+k) < J' + J - 1\,. \tag{4.16}$$

Since $J', J \geqslant 1$ and all known trajectories lie below $\alpha_k = 1$ for $t = -(k'+k)^2 < 0$, the sum rules (4.14) hold for sure as long as k', k are in the physical region of the infinite momentum frame.

Let us now do the analytic continuation of k', k to physical spin J', J poles in $F^{J'}(k'), F^J(k)$ **. This is justified as long as one remains inside the region (4.16) where the sum rule keeps converging. Hence the superconvergence relations (3.1) for the hadronic couplings $G^{i,J}(\hat{k})$ are correct for small enough momentum transfers $t = -(k'+k)^2$ fulfilling (4.16). In particular this is true in the scattering region $t < 0$.

Let us now discuss the Regge-Regge and form factor-Regge commutation rules. In order to do so we may continue the infinite set of amplitudes $T^{J'J}$ analytically in J' and J just as we have done for the individual form factors. For this we have to introduce signatured amplitudes ***

$$T^{J'+;J+},\ T^{J'+;J-},\ T^{J'-;J+},\ T^{J'-;J-}\,.$$

These can be defined in complete analogy with $F^{J\pm}(k)$ by forming the analytic function

$$T(z', z) = \sum_{J',J=1}^{\infty} \frac{(iz')^{J'-1}}{(J-1)!}\,\frac{(iz)^{J-1}}{(J-1)!}\, T^{J'J} \tag{4.17}$$

* Upon using eq. (1.8).

** This *does* carry us outside the physical region of the infinite-momentum frame, in which the virtual masses of all three form factors are spacelike.

*** We have dropped the other labels $b, a; p', Q', p, Q$ in $T^{J'J}$.

and its symmetrized and antisymmetrized combinations

$$T^{\eta',\eta}(z', z) \equiv \tfrac{1}{4}\{(T(z', z) + \eta' T(-z', z)) + \eta(T(z', -z) + \eta' T(-z', -z))\}; \qquad \eta', \eta = \pm 1.$$

These functions are Fourier transformed according to

$$T^{\eta',\eta}(\xi', \xi) = \frac{1}{(2\pi)^2} \int dz'\, dz\, e^{-i(\xi' z' + \xi z)}\, T^{\eta',\eta}(z', z). \tag{4.18}$$

The signatured amplitude $T^{J'\eta',J\eta}$ are now obtained as the positive frequency moments of $T^{\eta',\eta}(\xi', \xi)$:

$$T^{J'\eta',J\eta} \equiv 4 \int_0^\infty \int_0^\infty d\xi'\, d\xi\, \xi'^{J'-1}\, \xi^{J-1}\, T^{-\eta',-\eta}(\xi', \xi). \tag{4.19}$$

Clearly the amplitudes $T^{J',J}$ coincide with $T^{J'\pm,J\pm}$ or $T^{J\pm,J\mp}$ depending on wbther J', J are $\left\{\begin{smallmatrix}\text{even, even}\\ \text{odd, odd}\end{smallmatrix}\right\}$ or $\left\{\begin{smallmatrix}\text{even, odd}\\ \text{odd, even}\end{smallmatrix}\right\}$, respectively.

The integral over the imaginary parts of the signatured amplitudes is now given by the left-hand side of the commutators (2.26). The right-hand side determines the fixed poles.

If we continue the amplitudes $T^{J'\pm,J\pm}$ and $T^{J'\pm,J\mp}$ in J', J to Regge poles, the left-hand side of the dispersion relation (4.13) becomes the commutator of the corresponding residue functions and we obtain the left-hand side of sum rule (3.2). The right-hand side, however, has to be treated with caution. While the fixed pole contribution obviously drops out since it cannot produce the quadratic divergence as J' and J both hit a Regge pole, the Regge asymptotic behaviour (4.15)

$$(s/m_0^2)^{\alpha_k(k'+k)-J'-J}$$

continues analytically to

$$(s/m_0^2)^{\alpha_k(k'+k)-\alpha_j(k')-\alpha_i(k)}. \tag{4.20}$$

As a consequence we may conclude that Reggeons fulfill the superconvergence sum rule (3.2) as long as

$$\alpha_k(k'+k) < \alpha_j(k') + \alpha_i(k) - 1. \tag{4.21}$$

Certainly, the whole argument holds only under the presently popular assumption that the couplings $g_k^{J'J}$ have poles when J', J hit Regge trajectories and that it is really the emerging triple-Regge couplings that are responsible for the high-energy behaviour of the Regge-particle scattering amplitude. Explicitly we may assume that close to the Regge poles

$$\operatorname{Im} T^{J'\eta',J\eta}\Big|_{Q^{+\prime}=Q^+=0}\ \xrightarrow[Q^-\to\infty]{}\ \frac{g_k^{ji}(k',k)}{(J'-\alpha_j(k'))(J-\alpha_i(k))}\left(\frac{s}{m_0^2}\right)^{\alpha_k(k'+k)-J'-J} R^k(k'+k). \tag{4.22}$$

There is now an important difference between the condition (4.16) and (4.21). While in the case of scattering on the currents both J' and J were $\geqslant 1$ and (4.15) was converging to zero like (4.12) for any $t = -(k' + k)^2 < 0$, their continued values $\alpha_j(k')$, $\alpha_i(k)$ may be small and even become negative. Thus Im $T^{J'\eta',J\eta}$ will diverge worse and worse if one considers the commutator between lower and lower trajectories.

Even at forward scattering of zero-mass Reggeons the right-hand side will, in general, grow too fast for the sum rules to converge. For example, if α_j and α_i are ρ trajectories the leading trajectory exchanged can be ρ again and $\alpha_k - \alpha_i - \alpha_j \approx -\frac{1}{2}$.

In this situation we can still obtain interesting results by taking recourse to an algebraic version of finite-energy sum rules. Eq. (4.22) can be understood as carrying two types of information on the commutator of two Regge couplings

$$\sum_{m_\gamma^2=0}^{N} \{R^j(k')_{\beta\gamma}\, R^i(k)_{\gamma\alpha} - R^i(k)_{\beta\gamma}\, R^j(k')_{\gamma\alpha}\}. \tag{4.23}$$

(i) The intermediate sum over m_γ^2 diverges in a well-defined way as a function of the cutoff mass N.

(ii) There will be terms diverging with different powers in N, everyone of them giving a certain Regge residue $R_k(k' + k)$. Explicitly, we obtain [8] the following type of sum rule*

$$\sum_{m_\gamma^2=0}^{N} [R^j(k'), R^i(k)] = \sum_k g_k^{ji}(k', k) \left(\frac{N}{m_0^2}\right)^{\alpha_k(k'+k)-\alpha_j(k')-\alpha_i(k)+1} R^k(k'+k). \tag{4.24}$$

Notice that under the former condition (4.21) the cutoff may again be removed to infinity and the superconvergence relation is recovered. Otherwise, however, truncation is *necessary* to make the commutators exist. If the set of commutators among these Regge couplings closes, we obviously obtain for every fixed N a Lie algebra, whose structure functions are proportional to the triple Regge couplings multiplied by a power of the cutoff parameter N. As an example, the commutator

$$[v_a, v_b] = i\epsilon_{abc}\, v_c\, N^{\frac{1}{2}} \tag{4.25}$$

is solved by matrixes

$$(v_a)_{mn} = \frac{1}{\pi}\, [mn]^{-\frac{1}{4}}\, \frac{\sigma_a}{2} \tag{4.26}$$

in the sense

$$\sum_{l=1}^{N} (v_a)_{kl}\, (v_b)_{lm} - (v_b)_{kl}\, (v_a)_{lm} \underset{N\,\text{large}}{\approx} i\epsilon_{abc}\, v_c\, N^{\frac{1}{2}}. \tag{4.27}$$

* We have dropped a factor $(\alpha_k - \alpha_j - \alpha_i + 1)^{-1}$ on the right-hand side in order to simplify the later formulas. It would have been convenient to define $g_k^{ji}(k', k)$ with such a factor to begin with since there can be in general no pole at $\alpha_k = \alpha_j + \alpha_i - 1$ (see the Veneziano amplitude as an illustration of how the pole is avoided). We have used the conventional definitions.

Finally let us keep one J integer and continue J' to a Regge pole $\alpha_j(k')$. Then the left-hand side of the sum rule (4.13) becomes

$$[F^J(k), R^j(k')]. \tag{4.28}$$

On the right-hand side consider first the Regge contribution. It is expected to have the form

$$\sum_k g_k^{J,j}(k,k')\left(\frac{N}{m_0^2}\right)^{\alpha_k(k+k')-J-\alpha_j(k')+1} R^k(k+k'). \tag{4.29}$$

Thus if

$$\alpha_k(k+k') < J + \alpha_j(k') - 1. \tag{4.30}$$

This contribution vanishes and the sum rule obtained previously by continuing directly the algebra of form factors holds.

$$[F^J(k), R^j(k')] = 0; \qquad \alpha_k < J + \alpha_i - 1. \tag{4.31}$$

Notice that under the condition (4.30) the right-hand side of (2.26), $F^{J+J'-1}(k+k')$, cannot contribute a Regge residue as J' hits the pole $\alpha_j(k')$.

Let us now increase t such that $\alpha_k(k+k')$ passes the line

$$\alpha_k(k+k') = J + \alpha_j(k') - 1 \tag{4.32}$$

to

$$\alpha_k(k+k') > J + \alpha_j(k') - 1. \tag{4.33}$$

Then the Regge term will diverge and a finite-energy sum rule can be written:

$$\sum_{m_\gamma^2=0}^{N} [F^J(k), R^j(k')] = \sum_k g_k^{J,j}(k,k')\,(N/m_0^2)^{\alpha_k(k+k')-J-\alpha_j(k')+1} R^k(k+k'). \tag{4.34}$$

Again, there is no contribution from $F^{J+J'-1}(k+k')$. Observe that the sum rule (4.34) can be continued analytically in t into the region (4.30). As α_k hits the critical line (4.32) the trajectory R^k contributes without any N dependence. Exactly the same type of term is obtained by formally continuing the commutator (2.26) to the line (4.32). Thus we understand why in sect. 3 the right-hand side of form factor Regge commutators would pick up another Regge term only at certain discrete values of $t = -(k'+k)^2$ for fixed k'^2. At this point the exchanged Reggeon passes through the value where usually a fixed pole would be sitting. If we thus accept the formal result which the commutators (2.26) try to tell us at the singular point where Regge theory fails to justify it, we find the following interesting normalization condition for the current-Regge-Regge coupling

$$g_k^{J,i}(k,k') = 1 \qquad \text{for} \qquad \alpha_k(k+k') - \alpha_j(k') = J-1. \tag{4.34}$$

This relation can be tested for $J = 1$ in a high-energy reaction like

$$\pi p \to \pi p\, e^+e^-$$

when the e^+e^- pair is coming out fast with respect to the proton but slow with respect to the pion.

At arbitrary J but for $k = 0$ we reobtain the "daughter lowering" relation written down before [eq. (3.3)].

5. The algebra of vector, axial vector and Regge charges

In this section we want to illustrate our algebraic methods by applying them to the commutators between vector and axial vector from factors $F^{1-}(k)$, $F_5^{1-}(k)$ and the couplings $R^\rho(k)$, $R^{A_1}(k)$, $R^\pi(k)$ and $R^f(k)$ of ρ, A_1, π and f trajectories. We shall restrict ourselves to SU(2) multiplets and consider only the charges by setting $k = 0$ everywhere. For the sake of brevity, the argument k will then be dropped in all equations. The charges F_a^{1-}, F_{5a}^{1-} are commonly denoted by T_a and X_a. According to eq. (2.26) they fulfill the well-known algebra of SU(2)×SU(2):

$$\begin{aligned}
[T_a, T_b] &= i\epsilon_{abc}\, T_c, \\
[T_a, X_b] &= i\epsilon_{abc}\, X_c, \\
[X_a, X_b] &= i\epsilon_{abc}\, T_c.
\end{aligned} \tag{5.2}$$

Then commutators of the charges with the reggeons and of the reggeons among each other satisfy the sum rules (4.34) and (4.24). They form a closed algebra if we neglect the influence of Regge daughters and assume the leading trajectory to be dominant on the right-hand side of every sum rule. With the intercepts $\alpha_\rho = \alpha_f = \frac{1}{2}$; $\alpha_{A_1} = \alpha_\pi = 0$, this algebra reads [8–13]

$$[X_a, R^f] = (N/m_0^2)^{-\frac{1}{2}}\, g_\pi^{Xf}\, iR_a^\pi, \tag{5.3a}$$

$$[X_a, R_b^\pi] = (N/m_0^2)^{\frac{1}{2}}\, g_f^{X\pi}\, i\delta_{ab}\, R^f, \tag{5.3b}$$

$$[X_a, R_b^\rho] = (N/m_0^2)^{-\frac{1}{2}}\, g_A^{X\rho}\, i\epsilon_{abc}\, R_c^{A_1}, \tag{5.3c}$$

$$[X_a, R_b^{A_1}] = (N/m_0^2)^{\frac{1}{2}}\, g_\rho^{XA}\, i\epsilon_{abc}\, R_c^\rho, \tag{5.3d}$$

and [12]

$$[R_a^\rho, R_b^\rho] = (N/m_0^2)^{\frac{1}{2}}\, g_\rho^{\rho\rho}\, i\epsilon_{abc}\, R_c^\rho, \tag{5.4a}$$

$$[R_a^\rho, R_b^{A_1}] = (N/m_0^2)^{\frac{1}{2}}\, g_A^{\rho A}\, i\epsilon_{abc}\, R_c^{A_1}, \tag{5.4b}$$

$$[R_a^{A_1}, R_b^{A_1}] = (N/m_0^2)^{\frac{3}{2}}\, g_\rho^{AA}\, i\epsilon_{abc}\, R_c^\rho, \tag{5.4c}$$

$$[R_a^\rho, R_b^\pi] = (N/m_0^2)^{\frac{1}{2}}\, g_\pi^{\rho\pi}\, i\epsilon_{abc}\, R_c^\pi, \tag{5.4d}$$

$$[R_a^\pi, R_b^\pi] = (N/m_0^2)^{\frac{3}{2}}\, g_\rho^{\pi\pi}\, i\epsilon_{abc}\, R_c^\rho, \tag{5.4e}$$

$$[R^\rho_a, R^f] = 0, \tag{5.4f}$$
$$[R^{A_1}_a, R^f] = (N/m_0^2)^{\frac{1}{2}}\, g_\pi^{Af}\, i R^\pi_a, \tag{5.4g}$$
$$[R^{A_1}_a, R^\pi_b] = -(N/m_0^2)^{\frac{3}{2}}\, g_f^{A\pi}\, i\delta_{ab} R^f \tag{5.4h}$$
$$[R^\pi_a, R^f] = -(N/m_0^2)^{\frac{1}{2}}\, g_A^{\pi f}\, i R^{A_1}_a, \tag{5.4i}$$
$$[R^f, R^f] = 0. \tag{5.4j}$$

The commutators of the isospin charge T_a with the Regge couplings according to (3.3) has been left out since it shows merely that R^ρ_a, $R^{A_1}_a$, R^π_a are isovectors while R^f is an isosinglet operator.

At first there appear 4 unknown charge-Regge-Regge and 8 unknown triple-Regge couplings in these commutation rules. However, the sum rules are consistent with each other only if they satisfy the Jacobi identity. This cuts the number of independent couplings down to six. In the commutators (5.4), one finds the relations:

$$g_\rho^{\rho\rho} = g_A^{\rho A} = g_\pi^{\rho\pi}, \tag{5.5}$$
$$g_f^{A\pi}\, g_A^{\pi f} = g_\rho^{\pi\pi}\, g_\rho^{\rho\rho}, \tag{5.6}$$
$$g_f^{A\pi}\, g_\pi^{Af} = g_\rho^{AA}\, g_\rho^{\rho\rho}. \tag{5.7}$$

Only $g_\rho^{\rho\rho}$, $g_\rho^{\pi\pi}$, g_ρ^{AA} and g_π^{Af} remain free parameters. This freedom amounts to the normalization of the four Regge trajectories being undetermined by algebraic considerations.

The couplings occurring in eq. (5.3), on the other hand, have to satisfy:

$$g_A^{X\rho}\, g_\rho^{XA} = 1, \tag{5.8}$$
$$g_\pi^{Xf},\, g_f^{X\pi} = -1. \tag{5.9}$$

Inspection of the commutation rules shows that we are dealing with the group SU(2)×SU(2)×O(5). This can be seen most explicitly by forming the auxiliary operators *

$$\begin{aligned}
&\widetilde{T}_a \equiv T_a - \rho_a, \qquad \widetilde{X}_a \equiv X_a - A_a,\\
&\rho_a \equiv R^\rho_a\, [(N/m_0^2)^{\frac{1}{2}}\, g_\rho^{\rho\rho}]^{-1},\\
&A_a \equiv R^{A_1}_a\, [(N/m_0^2)^2\, g_\rho^{AA}\, g_\rho^{\rho\rho}]^{-\frac{1}{2}},\\
&\pi_a \equiv R^\pi_a\, [(N/m_0^2)^2\, g_\rho^{\pi\pi}\, g_\rho^{\rho\rho}]^{-\frac{1}{2}},\\
&f \equiv R^f_a\, [(N/m_0^2)(g_\pi^{Af})^2 g_\rho^{\pi\pi}/g_\rho^{AA}]^{-\frac{1}{2}}.
\end{aligned} \tag{5.10}$$

Then $\widetilde{T}_a$, $\widetilde{X}_a$ completely decouple from the remaining 10 generators of O(5) while still obeying the same commutation rules of SU(2)×SU(2) as T_a and X_a did before.

* Remember the commutation rules of O(5): $[L_{AB}, L_{AC}] = iL_{BC}$, $A, B, C = 1, ..., 5$.

The physical chiral SU(2)×SU(2) is recovered by adding the two commuting SU(2)×SU(2) groups formed by $\tilde{T}_a$, $\tilde{X}_a$ and by ρ_a, A_a of O(5).

Nature provides us with prominent particle states only for isospins $T \leqslant \frac{3}{2}$. All other states are called exotic. In a first approximation of solving the algebra we shall reject such exotic states.

The set of all allowed unitary irreducible representations of our algebra can then easily be listed. As a manner of speaking, we shall call representations of integer isospins boson, those of half-integer fermion representations (as if we were dealing with mesons and baryons of strangeness zero only). For the SU(2)×SU(2) subgroup, the only non-exotic representations are for bosons

$$(0,0);\quad (\tfrac{1}{2},\tfrac{1}{2});\quad (1,0);\quad (0,1) \tag{5.11}$$

and for fermions *

$$(\tfrac{1}{2},0);\quad (1,\tfrac{1}{2});\quad (\tfrac{3}{2},0)\ +\text{conjugate}. \tag{5.12}$$

The group O(5) has the two fundamental representations [14], a four dimensional and a five dimensional one. Their chiral contents with respect to the SU(2)×SU(2) formed by ρ_a and A_a are

$$\begin{aligned} \underline{4} &= (\tfrac{1}{2},0) + (0,\tfrac{1}{2}),\\ \underline{5} &= (0,0) + (\tfrac{1}{2},\tfrac{1}{2}). \end{aligned} \tag{5.13}$$

Notice that the group O(5) is only a little smaller than SU(4). There the fundamental representations are

$$\begin{aligned} \underline{4} &= (\tfrac{1}{2},0) + (0,\tfrac{1}{2}),\\ \underline{6} &= (0,0) + (0,0) + (\tfrac{1}{2},\tfrac{1}{2}), \end{aligned}$$

i.e. there is only one more singlet in the six-dimensional representation. Arbitrary representations can be obtained by applying the generators of the group to the state of maximal weight in the direct product $\underbrace{4\times4\times4\ldots\times4}_{p}\times\underbrace{5\times5\times\ldots\times5}_{q}$. The resulting representations may be denoted by (p, q) and have dimension

$$d_{[p,q]} = \tfrac{1}{6}\,(p+1)(q+1)(p+q+2)(p+2q+3). \tag{5.14}$$

The only non-exotic representations of O(5) can thus be found to be for bosons

$$\begin{aligned} [0,0] &= \underline{1} = (0,0),\\ [0,1] &= \underline{5} = (0,0) + (\tfrac{1}{2},\tfrac{1}{2}),\\ [2,0] &= \underline{10} = (\tfrac{1}{2},\tfrac{1}{2}) + (1,0) + (0,1), \end{aligned}$$

* By conjugation we mean $(n, m) \to (m, n)$.

and for fermions

$$[1,0]=\underline{4}=(\tfrac{1}{2},0)+(0,\tfrac{1}{2}),$$
$$[1,1]=\underline{16}=(\tfrac{1}{2},0)+(1,\tfrac{1}{2})+(\tfrac{1}{2},1)+(0,\tfrac{1}{2}),$$
$$[3,0]=\underline{20}=(1,\tfrac{1}{2})+(\tfrac{3}{2},0)+(0,\tfrac{3}{2})+(\tfrac{1}{2},1). \tag{5.15}$$

The representations of SU(2)XSU(2) and O(5) can now be coupled, again with only non-exotic combinations being allowed. This leads to the following representations:

A) Boson representations

1a) $(0,0)\times\underline{1} = \underline{1} = (0,0)$

b) $(0,0)\times\underline{5} = \underline{5} = (0,0)+(\tfrac{1}{2},\tfrac{1}{2})$

c) $(0,0)\times\underline{10} = \underline{10} = (\tfrac{1}{2},\tfrac{1}{2})+(1,0)+(0,1)$

2) $(\tfrac{1}{2},0)\times\underline{4} = \underline{8} = (0,0)+(\tfrac{1}{2},\tfrac{1}{2})+(1,0)$

3) $(\tfrac{1}{2},\tfrac{1}{2})\times\underline{1} = \underline{4} = (\tfrac{1}{2},\tfrac{1}{2})$

4) $(1,0)\times\underline{1} = \underline{3} = (1,0)$

B) Fermion representations

1a) $(0,0)\times\underline{4} = \underline{4} = (\tfrac{1}{2},0)+(0,\tfrac{1}{2})$

b) $(0,0)\times\underline{16} = \underline{16} = (\tfrac{1}{2},0)+(1,\tfrac{1}{2})+(\tfrac{1}{2},1)+(0,\tfrac{1}{2})$

c) $(0,0)\times\underline{20} = \underline{20} = (1,\tfrac{1}{2})+(\tfrac{3}{2},0)+\text{conj.}$

2a) $(\tfrac{1}{2},0)\times\underline{1} = \underline{2} = (\tfrac{1}{2},0)$

b) $(\tfrac{1}{2},0)\times\underline{5} = \underline{10} = (\tfrac{1}{2},0)+(0,\tfrac{1}{2})+(1,\tfrac{1}{2})$

c) $(\tfrac{1}{2},0)\times\underline{10} = \underline{20}' = \{(\tfrac{1}{2},0)+(1,\tfrac{1}{2})+\text{conj}\}+(\tfrac{3}{2},0)$

3) $(\tfrac{1}{2},\tfrac{1}{2})\times\underline{4} = \underline{16}' = (\tfrac{1}{2},0)+(1,\tfrac{1}{2})+\text{conj}$

4) $(1,0)\times\underline{4} = \underline{12} = (\tfrac{1}{2},0)+(1,\tfrac{1}{2})+(\tfrac{3}{2},0)$

5) $(1,\tfrac{1}{2})\times\underline{1} = \underline{6} = (1,\tfrac{1}{2})$

6) $(\tfrac{3}{2},0)\times\underline{1} = 4' = (\tfrac{3}{2},0)$ (5.16)

The ordering has been peformed according to increasing representations of the SU(2)XSU(2) generated by $\widetilde{T}_a$, $\widetilde{X}_a$. In the representations 1a)b)c) these matrices vanish for bosons and fermions. In these cases the chiral SU(2)XSU(2) generated by T_a; X_a coincides with the Regge SU(2)XSU(2) generated by ρ_a and A_a. Hence the

ρ trajectory conserves isospin and its coupling strength is the same for all states of the representations. This state of affairs may be called "ρ universality". In the alternative representations in which O(5) is represented by $\underline{1}$, all Regge couplings vanish.

Clearly, the physical representations will consist of some unitary mixture of these irreducible representation. Observe that due to $k = 0$ all operators leave helicity invariant. Thus a different representation has to be constructed for the states of every helicity separately. They can only be connected upon including $k \neq 0$ operators in the algebra.

As a simple example we take the boson representation $\underline{10}$ and identify the chiral states as the particles π, A_1, ρ and σ according to the old mixing scheme of Gilman and Harari, and Weinberg [16]:

$$\begin{aligned}
\pi &= \cos\psi \,\frac{(10)-(01)}{\sqrt{2}} + \sin\psi\,(\tfrac{1}{2},\tfrac{1}{2})_1\,,\\
A_1 &= -\sin\psi\,\frac{(10)-(01)}{\sqrt{2}} - \cos\psi\,(\tfrac{1}{2},\tfrac{1}{2}),\\
\rho &= \frac{(10)+(01)}{\sqrt{2}}\,,\quad \sigma = (\tfrac{1}{2},\tfrac{1}{2})_0\,.
\end{aligned} \tag{5.17}$$

For charges and Regge couplings we then find the reduced matrices *

$$T = \begin{array}{c c} & \begin{array}{cccc} \pi & A_1 & \rho & \sigma \end{array} \\ \begin{array}{c} \pi \\ A_1 \\ \rho \\ \sigma \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} = \rho, \tag{5.18}$$

$$X = \begin{array}{c c} & \begin{array}{cccc} \pi & A_1 & \rho & \sigma \end{array} \\ \begin{array}{c} \pi \\ A_1 \\ \rho \\ \sigma \end{array} & \begin{pmatrix} 0 & 0 & \cos\psi & \sin\psi \\ 0 & 0 & -\sin\psi & \cos\psi \\ \cos\psi & -\sin\psi & 0 & 0 \\ \sin\psi & \cos\psi & 0 & 0 \end{pmatrix} \end{array} = A, \tag{5.19}$$

$$\pi = i\begin{pmatrix} 0 & 0 & -\sin\psi & \cos\psi \\ 0 & 0 & -\cos\psi & -\sin\psi \\ \sin\psi & \cos\psi & 0 & 0 \\ -\cos\psi & \sin\psi & 0 & 0 \end{pmatrix}, \tag{5.20}$$

$$f = \begin{pmatrix} \sin 2\psi & \cos 2\psi & 0 & 0 \\ \cos 2\psi & -\sin 2\psi & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.21}$$

* Defined in ref. [9].

Eq. (5.19) shows the property of ρ universality of this saturation scheme.

Let us recall that the same Regge couplings have been found in quite a different fashion before in refs. [8, 9]. There it was shown that in any chiral saturation scheme the SU(2)XSU(2) breaking part in the mass matrix

$$m_4^2\,\delta_{ab} \equiv [X_b\,, [X_a, m^2]] \tag{5.22}$$

converges against the coupling of the f trajectory as the size of the scheme N goes to infinity*:

$$m_4^2 \propto (N/m_0^2)^{\frac{1}{2}}\, R^{\mathrm{f}}. \tag{5.23}$$

Algebraically, the commutator (5.22) states that m_4^2 is a fourth component of a $(\frac{1}{2}, \frac{1}{2})$ representation. This is exactly what we obtain by combining our new commutators (5.3 a and b). But there is additional information in (5.3a) itself: According to this commutator the chiral partner of f is given by the residue of the π trajectory. The chiral partner of the matrix m_4^2, on the other hand, is the isovector matrix m_a^2:

$$(m_a^2)_{\beta\alpha} = -i\,[X_a, m_4^2]_{\beta\alpha} = -i[X_a, m^2]_{\beta\alpha} = i(m_\beta^2 - m_\alpha^2)\,X_{a_{\beta\alpha}}. \tag{5.24}$$

Hence (5.3a) is telling us that m_a^2 should represent coupling of the Reggeized pion at zero mass

$$m_a^2 \propto R_a^{\pi}. \tag{5.25}$$

This result is not really astonishing at all. If $\pi = (1/f_\pi \mu^2)\partial A$ is used as an interpolating pion field, m_a^2 is well known to give the coupling of the elementary pion continued to zero mass:

$$\begin{aligned}\langle\beta|(\Box+\mu^2)\pi|\alpha\rangle\Big|_{q^2=0} &= \frac{1}{f_\pi}\langle\beta|\partial A|\alpha\rangle = \frac{i}{f_\pi}(m_\beta^2 - m_\alpha^2)X_{a_{\beta\alpha}}\\ &= \frac{1}{f_\pi}m_a^2\,.\end{aligned} \tag{5.26}$$

Since both extrapolations, Regge and PCAC, are expected to be quite smooth and since they agree on the near-by pion mass shell, they should indeed lead to about the same result.

The next two commutators can be compared with the old approach in a completely analogous fashion. There it was shown that the commutator $[m_a^2, m_b^2]$ converges to the ρ residue [8, 9]

$$[m_a^2, m_b^2] \propto (N/m_0^2)^{\frac{1}{2}}\, R^{\rho}. \tag{5.27}$$

Now with m_a^2 being a $(\frac{1}{2}, \frac{1}{2})$ vector, the commutator is necessarily a member of a (1,0)+(0,1) representation. This is also what eqs. (5.3c and d) are telling us. In addition (5.3c) itself states that the chiral partner of ρ must converge to the A_1 residue.

* Remember it is the good high-energy behaviour of the $I_t = 2$ amplitude that forces any $I_t = 2$ part to be absent in (5.22).

Commuting X_a with (5.27) we find the new result:

$$-i\,[m_4^2, m_a^2] \propto \frac{N}{m_0^2} R^{A_1}. \tag{5.28}$$

Summarizing we can say: Given any chiral saturation scheme of SU(2)×SU(2) (eq. 5.2) we can construct approximations to ρ, A_1, π and f Regge couplings purely by additional commutation with the mass matrix

$$m_{\beta\alpha}^2 = m_\alpha^2\, \delta_{\beta\alpha}, \tag{5.29}$$

First one makes sure the mass matrix contains at most a $(\frac{1}{2},\frac{1}{2})$ chiral breaking part m_4^2 by enforcing the double commutator to be free of isospin 2. Then one uses this matrix m_4^2 together with its chiral partner m_a^2 to calculate the Regge residue of ρ, A_1, π and f trajectories according to

$$\begin{aligned}
&\rho_a \propto -i\,[m_b^2, m_c^2]\;(N/m_0^2)^{-\frac{3}{2}},\\
&A_{1_a} \propto -i\,[m_4^2, m_a^2]\;(N/m_0^2)^{-1},\\
&\pi_a \propto m_a^2\,,\\
&f \propto m_4^2\;(N/m_0^2)^{-\frac{1}{2}}.
\end{aligned} \tag{5.30}$$

One may now wonder under what conditions do the matrices (5.30) satisfy also the new O(5) algebra of Regge residues. A short glance at eq. (5.4) shows us that two of the commutators, eqs. (5.4e and g) are fulfilled by definition (5.30) while eq. (5.4j) is true trivially. It can now easily be demonstrated that the necessary and sufficient condition for all others to be fulfilled is given by the commutator:

$$-i\,[[m_a^2, m_b^2], m_c^2] \propto (N/m_0^2)^2\,(\delta_{ac}\, m_b^2 - \delta_{bc}\, m_a^2). \tag{5.31}$$

That this is necessary is seen from eq. (5.4d). It is certainly sufficient for eq. (5.4a) such that ρ generate rotations. For eq. (5.4f) we have to make sure that m_4^2 is invariant under ρ. For this take (5.30) in the form

$$[[m_1^2, m_2^2], m_3^2] = 0$$

and commute with X_3. This leads to

$$[[m_1^2, m_2^2], m_4^2] = 0 \tag{5.32}$$

proving eq. (5.4f). With m_4^2 being invariant and (5.31), A_1 is a vector under ρ and eq. (5.4b) is seen to hold. For the commutators (5.4g) and (5.4h) we take (5.31) in the forms

$$\begin{aligned}
&[[m_1^2, m_2^2] m_1^2] \propto -(N/m_0^2)^2\, m_2^2,\\
&[[m_1^2, m_2^2] m_3^2] \propto 0,
\end{aligned}$$

and apply X_2 in both cases obtaining

$$-i\,[[m_4^2, m_1^2], m_1^2] \propto -i\,(N/m_0^2)^2\, m_4^2, \tag{5.33}$$

$$-i\,[[m_4^2, m_1^2], m_3^2] \propto 0, \tag{5.34}$$

which demonstrates eq. (5.4h). Commuting these last equations with X_1, and X_3, respectively verifies eq (5.4g). Finally, eq. (5.4c) can be checked by using the Jacobi identity and the other commutators. Thus the Regge couplings (5.29) of any saturation scheme will indeed form the group O(5) if and only if m_a^2 is a vector operator with respect to $[m_a^2, m_b^2]$.

A *practical* way of constructing chiral saturation schemes satisfying the whole algebra proceeds in the following steps:

(i) One chooses certain representations of the whole supergroup SU(2)XSU(2)XO(5).

(ii) One introduces representation mixing of the chiral states.

(iii) One uses the selection rules to enforce m_4^2 to be a $(\frac{1}{2}, \frac{1}{2})$ representation. Between the unmixed chiral states, certain matrix elements will have to vanish. This gives rise to mass formulas. The allowed matrix elements of m_4^2 are free parameters at this stage. Given m_4^2, the Regge couplings (5.30) can be calculated.

(iv) The group O(5) is now enforced by making the matrix elements of m_4^2 between the unmixed states coincide with the generator f of O(5). This amounts to fixing the relative sizes of many of the formerly independent matrix elements of m_4^2. It gives rise to additional mass formulas.

Notice that in the above example for mesons, the group O(5) is fulfilled automatically by the matrices (5.30). The reason is first of all that we are dealing with a representation space of O(5) and second that in this representation there is *only one* non-zero reduced matrix element of m_4^2.

It will be interesting to discuss a saturation scheme in which the new mass formulas can be tested [13].

Let us finally point out that in the former work two of the proportionality factors occurring in eqs. (5.29) have been determined. The results are

$$\rho_a = -i\,C_\rho\,[m_b^2, m_c^2]\,(N/m_0^2)^{-\frac{3}{2}}, \qquad C_\rho = \pi\,\frac{3}{4f_\pi^2 m_0^2}\,R^\rho_{\pi\pi}, \tag{5.35}$$

$$f = C_f\, m_4^2 (N/m_0^2)^{-\frac{1}{2}}, \qquad C_f = \pi\,\frac{1}{4f_\pi^2}\,R^f_{\pi\pi}, \tag{5.36}$$

where $R^\rho_{\pi\pi}$ and $R^f_{\pi\pi}$ are the couplings of ρ and f trajectories to two pions. In addition we know from PCAC:

$$\pi = \frac{1}{f_\pi}\, m_a^2. \tag{5.37}$$

With these three constants being known, our new algebra (5.4) allows us to calculate directly two of the independent triple-Regge couplings, $g_\rho^{\rho\rho}$ and $g_\rho^{\pi\pi}$ and two relations involving $g_\pi^{\mathrm{A}f}$, $g_f^{\mathrm{A}\pi}$ and the unknown normalization C_{A_1}. If also C_{A_1} was known, all triple-Regge couplings would be determined [using eqs. (5.5)–(5.7)]. Of the two independent charge-Regge-Regge couplings of eq. (5.3), on the other hand, one is determined in terms of C_f. The other depends again on the unknown C_{A_1}.

6. Conclusion and outlook

We have gained some combined algebraic view of current and Regge aspects of hadronic interactions. There are many open questions that will have to be answered in the future.

On the current side it appears as if the infinitely many local tensor currents $j^{\mu_1 \dots \mu_J}(x)$ will provide an important tool for the study of many strong interaction phenomena. It is exciting to speculate that there may well be a current-current type of effective Hamiltonian, analogous to that the weak interactions, that allows for a rather complete description of scattering processes:

$$S \propto \sum_{J=1}^{\infty} a_J \langle p_4 | j^{\mu_1 \dots \mu_J} | p_2 \rangle \langle p_3 | j_{\mu_1 \dots \mu_J} | p_1 \rangle \,. \tag{6.1}$$

There are now certainly enough spins exchanged to build up Regge poles and cuts (from cuts in the form factors). The phenomenon of duality will make sure that the s and u channel pole structure will be contained properly in a pure t-channel exchange picture like (6.1). As a first step in this direction one may try and construct a theory of currents of the Sugawara type [17] containing only the energy momentum tensor $\theta^{\mu\nu}$ and the infinitely many tensor currents $j^{\mu_1 \dots \mu_J}$. The canonical commutators would all be replaced by light-cone commutators among the currents. There is also the possibility that the traceless part of $\theta^{\mu\nu}$ coincides with $j^{\mu\nu}$ itself, which is the content of the no-gluon hypothesis.

Also we would like to learn how diffraction scattering enters into the form factors $F^{J\pm}(k)$. We have left this point completely out of our discussion since we don't know what singularity it would cause in the J-plane.

On the Regge side we would certainly like to understand more the mathematical features of a system of diverging matrix commutators with a cutoff N. Are solutions determined uniquely? Maybe in the limiting case $N \to \infty$? What are the errors to be expected? The study of infinitely rising saturation schemes will probably provide some hints to this question. We have neglected Regge daughters in our algebra in order to make the group as small as possible. Is this a grave omission? What is the role of the pomeron here? Since we are saturating the sum rules with resonances we should not expect to see any pomeron in the t-channel. But what about the external Regge legs?

Appendix A: Angular conditions for the form factors $F^J(k)$

The connection of the form factors $F^J(k)$ with the standard spin-J helicity couplings is derived in the following fashion: One starts out with the infinite-momentum definition of $F^J_{\beta\alpha}(k)$ [3, 18]:

$$F^J_{\beta\alpha}(k) = \lim_{x\to\infty} \frac{1}{(p^{0\prime}+p^0)^J} \langle \beta\, xe_3 + p^{\perp\prime} | j^{\overbrace{00\ldots0}^{J}}(0) | \alpha\, xe_3 + p^{\perp} \rangle, \tag{A.1}$$

where $k = p^{\perp\prime} - p^{\perp}$ is the transverse momentum transfer. Then one transforms the external states to their rest frames

$$F^J_{\beta\alpha}(k) = \lim_{x\to\infty} \frac{1}{(p^{0\prime}+p^0)^J} \langle \beta 0 | B^{-1}(p')\, j^{\overbrace{00\ldots0}^{J}} B(p) | \alpha 0 \rangle \tag{A.2}$$

with $B(p) \equiv e^{-i\xi M}$. Using now the formula

$$\lim_{x\to\infty} \frac{1}{(p^{0\prime}+p^0)^J} B^{-1}(p')\, j^{\overbrace{00\ldots0}^{J}} B(p') = \frac{1}{(2m_\beta)^J} j^{\overbrace{++\ldots+}^{J}} \tag{A.3}$$

and the fact that

$$\lim_{x\to\infty} B^{-1}(p')B(p) = K(m_\beta, m_\alpha, k) \tag{A.4}$$

with a finite transformation $K(m_\beta, m_\alpha, k)$, we can write

$$F^J_{\beta\alpha}(k) = \frac{1}{(2m_\beta)^J} \langle \beta 0 | j^{\overbrace{++\ldots+}^{J}} K(m_\beta, m_\alpha, k) | \alpha 0 \rangle\,. \tag{A.5}$$

For the next step it is useful to assume k to point in the x-direction (i.e. $k = (k, 0)$, arbitrary transverse momenta $k = (k^1, k^2)$ can be reached by a mere phase transformation $e^{i\,\mathrm{arctg}(k^2/k^1)}$ (see appendix B). Then $K(m_\beta, m_\alpha, k)$ can be decomposed in the following fashion [3, 18]

$$K(m_\beta, m_\alpha, k) = e^{i\omega_\beta J_2}\, e^{-i\xi M_3}\, e^{i\omega_\alpha J_2} \tag{A.6}$$

with the angles

$$\sin\omega_{\alpha,\beta} = \frac{1}{2m_\beta m_\alpha \mathrm{sh}\xi} 2m_{\alpha,\beta} k, \qquad \cos\omega_{\alpha,\beta} = \frac{1}{2m_\beta\, m_\alpha \mathrm{sh}\xi} [\pm k^2 + m_\beta^2 - m_\alpha^2],$$

$$\mathrm{sh}\xi = \frac{1}{2m_\beta m_\alpha} [(k^2 + (m_\beta + m_\alpha)^2)(k^2 + (m_\beta - m_\alpha)^2)]^{\frac{1}{2}}. \tag{A.7}$$

Notice that ξ is the relative rapidity of particles α and β. The rotation on the left-hand side can now be passed through the operator $j^{\underbrace{++\ldots+}_{J}}$ with the result

$$e^{-i\omega_\beta J_2}\, j^{++\ldots+}\, e^{i\omega_\alpha J_2} = j^{++\ldots+\prime} \tag{A.8}$$

with

$$j^{+\prime} = j^0 + \cos\omega_\beta\, j^3 + \sin\omega_\beta\, j^1 \tag{A.9}$$

and every index + in $j^{++\cdots +}$ treated in the same way. Initial and final rotations can now easily be performed. For this we introduce the third componenets h_β, h_α of the spins of particles β and α explicitly and $F^J(k)$ takes the form

$$F^J_{\beta h_\beta,\alpha h_\alpha}(\boldsymbol{k}) = \frac{1}{(2m_\beta)^J} d^{s_\beta}_{h_\beta h'_\beta}(-\omega_\beta)\, \langle \beta h'_\beta \mathbf{0}|\, j^{++\dots+'}\, \mathrm{e}^{-i\xi M_3} |\alpha h_\alpha \mathbf{0}\rangle \times d^{s_\alpha}_{h_\alpha' h_\alpha}(\omega_\alpha). \tag{A.10}$$

As a final step we merely have to decompose $j^{++\dots+'}$ into contents of pure spin orientation M in the z-direction. For simplicity we assume current conservation. In the frame in which the matrix element in (A.10) is written, particle β is at rest and particle α runs with rapidity ξ in the z-direction. Hence

$$q^\mu = (m_\beta - m_\alpha \mathrm{ch}\xi, 0, 0, -m_\alpha\, \mathrm{sh}\,\xi). \tag{A.11}$$

Then current conservation tells us

$$j^3 = -\frac{m_\beta - m_\alpha\, \mathrm{ch}\,\xi}{m_\alpha \mathrm{sh}\,\xi} j^0 = -\cos\omega_\beta\, j^0 \tag{A.12}$$

such that (A.9) can be rewritten as

$$j^{+\prime} = \sin^2\omega_\beta\, j^0 + \sin\omega_\beta\, j^1. \tag{A.13}$$

For j^1 the decomposition in states $j^{(\pm)} \equiv \mp(1/\sqrt{2})\,(j^1 \pm i j^2)$ of pure spin orientation is trivial

$$j^1 = -\frac{1}{\sqrt{2}}(j^{(+)} - j^{(-)}). \tag{A.14}$$

For the state of spin orientation zero we have to specify the phase. At timelike momentum transfer we would have

$$j^{(0)} = \epsilon_\mu(0,q)\, j^\mu \tag{A.15}$$

with

$$\epsilon^\mu(0,q) = -(q^3, 0, 0, q^0)/\sqrt{q^2}. \tag{A.16}$$

Continuing $q^2 = -k^2$ negative values *below* the square root cut gives

$$\epsilon^\mu(0,q) = -i(q^3, 0, 0, q^0)/k \tag{A.17}$$

such that

$$j^{(0)} = -i\frac{k}{q^3} j^0 = i\frac{k}{m_\alpha\, \mathrm{sh}\,\xi} j^0 = i\sin\omega_\beta\, j^0. \tag{A.18}$$

Hence (A.13) can be rewritten simply as

$$j^{+\prime} = -i \sin \omega_\beta \left(j^{(0)} - \frac{i}{\sqrt{2}} j^{(+)} + \frac{i}{\sqrt{2}} j^{(-)} \right). \tag{A.19}$$

Similarly $j^{++\dots+\prime}$ is given by

$$\begin{aligned} j^{++\dots+\prime} &= (-i \sin \omega_\beta)^J \sum_{n=0}^{J} \binom{J}{n} i^{J-n} j^{(\overbrace{0,\dots,0}^{n},\ \overbrace{1,\dots,1}^{J-n})} \\ &= (-i \sin \omega_\beta)^J \sum_{n=0}^{J} \binom{J}{n} \frac{i^{J-n}(-)^k}{\sqrt{2^{J-n}}} \binom{J-n}{k} j^{(\overbrace{0\dots0}^{n},\overbrace{+\dots+}^{k},\overbrace{-\dots-}^{J-k-n})} \end{aligned} \tag{A.20}$$

Let us reorder the sum according to states of pure spin orientation [19]

$$\begin{aligned} j^{JM} &= \sum_{\substack{h_1,\dots,h_J \\ \Sigma h_i = M}} \left[2^{J-\Sigma|h_i|} \frac{(J+M)!\,(J-M)!}{(2J)!} \right]^{\frac{1}{2}} j^{(h_1\dots h_J)} \\ &= \sqrt{\frac{(J+M)!\,(J-M)!}{(2J)!}} \sum_{n=0,\dots,J} \sqrt{2^n}\, j^{(\overbrace{0,\dots,0}^{n},\ \overbrace{+\dots+}^{k},\ \overbrace{-\dots-}^{J-k-n})} \end{aligned} \tag{A.22}$$

Then (A.20) becomes

$$j^{++\dots+\prime} = (-i \sin \omega_\beta)^J \sum_{M=-J}^{J} (-\mathrm{i})^M \sqrt{\frac{(2J)!}{2^J (J+M)!\,(J-M)!}}\, j^{JM}, \tag{A.23}$$

where we have used

$$(-i)^M = (-i)^{k-(J-k-n)} = (-)^k\, i^{J-n}. \tag{A.24}$$

Finally we introduce the standard form factors of the current

$$G^{J,M}_{\beta h_\beta, \alpha h_\alpha}(q^2) = \langle \beta h_\beta \mathbf{0} | j^{JM} | \alpha h_\alpha\, p e_z \rangle\, \delta_{M, h_\beta - h_\alpha} \tag{A.25}$$

and obtain the result referred to in sect. 1:

$$F^J_{\beta h_\beta, \alpha h_\alpha} = \left(-i \frac{\sin \omega_\beta}{2 m_\beta} \right)^J \sum_{M=-J}^{J} (-i)^M \sqrt{\frac{(2J)!}{2^J (J+M)!\,(J-M)!}}$$

$$d^{s_\beta}_{h_\beta h'_\beta}(-\omega_\beta)\, G^{JM}_{\beta h_{\beta'}, \alpha h_{\alpha'}}\, d^{s_\alpha}_{h_{\alpha'} h_\alpha}(\omega_\alpha). \tag{A.26}$$

Notice that the factor in front is symmetric in β and α:

$$\frac{\sin \omega_\beta}{2 m_\beta} = \frac{q}{2 m_\beta m_\alpha \operatorname{sh} \xi} \tag{A.27}$$

as it should be.

The analytic continuation of (A.26) in J leads to the angular condition for the Regge couplings $R^i(k)$.

The set of equations (A.26) is in general singular. The left-hand side contains $(2s_\beta + 1)(2s_\alpha + 1)$ components, the right-hand side only the kinematically independent form factors. From angular momentum conservation we have

$$G^{J,M}_{\beta h_\beta, \alpha h_\alpha} = 0, \qquad M \neq h_\beta - h_\alpha \tag{A.28}$$

and hence G vanishes for $|h_\beta - h_\alpha| > J$. A direct count shows that the number of independent G's is

$$\begin{aligned} &(2J + 1)(2s + 1) \quad \text{for } J \leqslant \Delta s\,, \\ &(2J + 1)(2s + 1) - (J - \Delta s)(J + 1 - \Delta s) \quad \text{for } \Delta s < J < s_\beta + s_\alpha\,, \\ &(2s_\beta + 1)(2s_\alpha + 1) \quad \text{for } J \geqslant s_\beta + s_\alpha\,, \end{aligned} \tag{A.29}$$

where

$$s \equiv \min(s_\beta, s_\alpha), \qquad \Delta s \equiv |s_\beta - s_\alpha|. \tag{A.30}$$

Thus, except in the third case, there will be many constraints of the form factors $F^J_{\beta\alpha}(k)$. Certainly, parity cuts the number of independent components by a factor 2. If n_β, n_α, and n are the normalities of the external particles and the tensor $j^{\mu_1 \ldots \mu_J}$ *

$$(n = \text{parity} \times \{\begin{smallmatrix}(-)^J & \text{bosons} \\ (-)^{J-\frac{1}{2}} & \text{fermions}\end{smallmatrix}\})$$

then **

$$\begin{aligned} &F_{\beta, h_\beta; \alpha, h_\alpha}(k_1, k_2) = n n_\beta n_\alpha (-)^{\Delta h}\, F_{h_\beta; \alpha, -h_\alpha}(k_1, -k_2) \\ &G^{JM}_{\beta, h_\beta; \alpha, h_\alpha}(t) = n\, n_\beta n_\alpha\; G^{J,-M}_{\beta, -h_\beta; \alpha, -h'_\alpha}(t) \end{aligned} \tag{A.31}$$

As an example consider the well-known case of a vector current between spin-$\frac{1}{2}$ particles of parity +. Taking care of parity, there are two independent form factors $F^1_{\uparrow\uparrow}(k)$ and $F^1_{\downarrow\uparrow}(k)$ and also two independent $G^{10}_{\uparrow\uparrow}(t)$ and $G^{1-}_{\downarrow\uparrow}$. Thus there are no constraints among $F^1_{\uparrow\uparrow}(k)$ and $F^2_{\downarrow\uparrow}(k)$. The connection of G^{1M} with the standard Sachs electric and magnetic form factors is given by

$$\begin{aligned} G^{10}_{\uparrow\uparrow}(t) &= G^{10}_{\downarrow\downarrow}(t) \\ &= i\frac{k}{m \operatorname{sh} \xi} \langle \uparrow 0 | j^0 | \uparrow p e_z \rangle = i \frac{k}{2m \operatorname{sh} \frac{1}{2}\xi} G_E(t) = i G_E(t)\,, \\ G^{1-1}_{\downarrow\uparrow}(t) &= G^{11}_{\uparrow\downarrow}(t) \\ &= i \langle \downarrow 0 | j^{(-)} | \uparrow p e_z \rangle = \sqrt{2} \operatorname{sh} \tfrac{1}{2}\xi\; G_M(t) = \sqrt{2} \frac{k}{2m}\, G_M(t). \end{aligned} \tag{A.32}$$

* $n = 1$ is a tensor with normal spin parities $J^P = 0^+, 1^-, 2^+, \ldots$, $n = -1$ is a pseudotensor with $J^P = 0^-, 1^+, 2^-, \ldots$.

** Notice that the angular condition (A.26) conserves these parity relations.

With the rotation matrices

$$d^{\frac{1}{2}}(\omega) = \begin{pmatrix} \cos\frac{1}{2}\omega & -\sin\frac{1}{2}\omega \\ \sin\frac{1}{2}\omega & \cos\frac{1}{2}\omega \end{pmatrix} \tag{A.33}$$

we find

$$F^1_{\uparrow\uparrow}(k) = -i\frac{\sin\omega_\beta}{2m}(\sin\omega_\beta\, G^{10}_{\uparrow\uparrow} - i\cos\omega_\beta\, G^{1-}_{\downarrow\uparrow}/\sqrt{2})\,, \tag{A.34}$$

$$F^{1'}_{\downarrow\uparrow}(k) = -i\frac{\sin\omega_\beta}{2m}(-\sin\omega_\beta\cos\omega_\beta\, G^{10}_{\uparrow\uparrow} - i\sin\omega_\beta\, G^{1-}_{\downarrow\uparrow}/\sqrt{2}).$$

Inserting the angles

$$\sin\omega_\beta = \frac{k}{m\,\mathrm{sh}\,\xi}\,, \qquad \mathrm{sh}\,\xi = \frac{k}{m}\sqrt{1+\frac{k^2}{4m^2}} \tag{A.35}$$

or

$$\sin\omega_\beta = \left[1+\frac{k^2}{4m^2}\right]^{-\frac{1}{2}}, \qquad \cos\omega_\beta = -\frac{k}{2m}\left[1+\frac{k^2}{4m^2}\right]^{-\frac{1}{2}}, \tag{A.36}$$

we find

$$F^1_{\uparrow\uparrow}(k) = \frac{1}{2m}\frac{1}{1+k^2/4m^2}\left(G_\mathrm{E} + \frac{k^2}{4m^2}G_\mathrm{M}\right) = \frac{1}{2m}F_1(t), \tag{A.37}$$

$$F^1_{\downarrow\uparrow}(k) = \frac{1}{2m}\frac{k}{2m}\frac{1}{1+k^2/4m^2}(G_\mathrm{E} - G_\mathrm{M}) = -\frac{k}{(2m)^2}F_2(t).$$

Thus the form factors $F^1_{\beta\alpha}(k)$ are essentially equal to the standard Dirac form factors $F_1(t)$, $F_2(t)$. It is well known that these do not satisfy any constraints.

We would like to remind the reader that the procedure of transforming the tensor $j^{00\ldots0}$ of (A.2) into the combination $j^{++\ldots+'}$ of (A.23) could certainly be replaced by a Wigner rotation. It turns out that such a calculation is somewhat more tedious than the tensor method used before. For completeness we present this calculation here. The little group is generated by L_3, M_1, M_2 in the standard frame in which

$$q_s^\mu = (0,0,0,k). \tag{A.38}$$

Consider the spacelike current $j^{J,M}(q)$. In the standard frame, the polarization states of a vector with respect to the 3rd direction are given by

$$j^{(\pm)}(q_s) = \mp\frac{1}{\sqrt{2}}j^{1\pm i2},$$

$$j^{(0)}(q_s) = \epsilon_\mu(0,q_s)\,j^\mu, \tag{A.39}$$

$$\epsilon^\mu(q_s,0) = -i(1,0,0,0). \tag{A.40}$$

The phase $-i$ arises if one wants $j^{(\pm)}$, $j^{(0)}$ to have positive phases under the raising operator $M^+ = M^1 + iM^2$:

$$[M^+, (-ij^0)] = \sqrt{2}\, j^{(+)},$$

$$[M^+, j^{(-)}] = \sqrt{2}\,(-ij^0). \tag{A.41}$$

Since a spin-1 representation of O(2,1) is non-unitary, M^- has all negative phases. One may now couple vectors of this type to any higher spin J by using standard Clebsch-Gordan coefficients. The resulting spin-J object transforms under an O(2,1) Wigner "rotation" as

$$\mathrm{e}^{-iM_1\xi}\, j^{JM}(q_s)\, \mathrm{e}^{iM_1\xi} = j^{JM'}\, d^J_{M'M}(i\xi), \tag{A.42}$$

where $d^J_{M'M}(\beta)$ denotes the usual spin-J representation of the rotation group. The imaginary angle $\beta = i\xi$ has to be used since M^1, M^2 transforms $j^{(\pm)}$, $j^{(0)}$ in exactly the same way as iL^2, $-iL^1$ would do with a usual spherical three-vector. With this phase convention we can now proceed in the following fashion:

The infinite momentum value of q is (leaving out the y-component)

$$q^\mu_\infty = \left(\frac{m_\beta^2 - m_\alpha^2 + k^2}{2x}, k, 0\right). \tag{A.43}$$

It may be obtained from the standard momentum by a Lorentz transformation

$$L(q_\infty) = \mathrm{e}^{-i\frac{1}{2}\pi J^2}\, \mathrm{e}^{-i\epsilon M^3}\, q_s$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & \dfrac{m_\beta^2 - m_\alpha^2 + k^2}{2xk} \\ 0 & 1 & 0 \\ \dfrac{m_\beta^2 - m_\alpha^2 + k^2}{2xk} & 0 & 1 \end{pmatrix} q_s. \tag{A.44}$$

Under $L(q_\infty)$, the polarization vector (A.40) becomes

$$\epsilon^\mu(0, q_\infty) = -i\left(1, \frac{m_\beta^2 - m_\alpha^2 + k^2}{2xk}, 0\right). \tag{A.45}$$

Hence the spherical component $j^{10}(q_\infty)$ is

$$j^{10}(q_\infty) = -i\, j^0 + i\, \frac{m_\beta^2 - m_\alpha^2 + k^2}{2xk}\, j^1. \tag{A.46}$$

In the infinite-momentum frame, j^1 has elements of order $O(x^{-1})$, so it can be dropped in (A.46). As a consequence, the component $j^{00\ldots0}$ is identical with

$$j^{00\ldots0} = i^J\, j^{(00\ldots0)}(q_\infty). \tag{A.47}$$

Inverting eq. (A.22) and keeping trace of the tracelessness of $j^{(h_1 \dots h_J)}$ in its spherical indices

$$j^{(00h_3 \dots h_J)} - j^{(+-h_3 \dots h_J)} - j^{(-+h_3 \dots h_J)} = 0 \tag{A.48}$$

we have

$$j^{00 \dots 0} = i^J \sqrt{\frac{2^J (J!)^2}{(2J)!}}\; j^{J,0}(q_\infty)\,. \tag{A.49}$$

The Wigner "rotation" is now computed by decomposing

$$B^{-1}(p')\, L(q_\infty) = L(q^*)\, W, \tag{A.50}$$

where q^* is the momentum of the current in the frame in which particle β is at rest while particle α is running with the momentum

$$B^{-1}(p')p_\alpha = p^*_\alpha = \left(\frac{m_\beta^2 - m_\alpha^2 - k^2}{2m_\beta}\,, 1, -\frac{m_\beta^2 - m_\alpha^2 - k^2}{2m_\beta}\right). \tag{A.51}$$

As should be clear from the decomposition (A.6) of K, the rotation may be taken out of $L(q^*)$

$$L(q^*) = \mathrm{e}^{i\omega_\beta J_2}\, L(q_\mathrm{f})\,, \tag{A.52}$$

such that $L(q_\mathrm{f})$ becomes a pure Lorentz transformation in the $-z$ direction bringing the standard momentum to the final form

$$q_\mathrm{f} = (m_\beta - m_\alpha \operatorname{ch} \xi, 0, 0, -m_\alpha \operatorname{sh} \xi). \tag{A.53}$$

This Lorentz transformation is obviously given by

$$L(q_\mathrm{f}) = \mathrm{e}^{-i\pi J_2}\, \mathrm{e}^{-i\tilde{\xi} M_3} \tag{A.54}$$

with

$$\operatorname{ch} \tilde{\xi} = \frac{1}{2m_\beta k}\,[(k^2 + (m_\beta - m_\alpha)^2)\,(k^2 + (m_\beta + m_\alpha)^2)]^{\frac{1}{2}},$$

$$\operatorname{sh} \tilde{\xi} = \frac{m_\beta^2 - m_\alpha^2 - k^2}{2m_\beta k}\,. \tag{A.55}$$

Inserting (A.54) into (A.50), we finally obtain the Wigner "rotation"

$$W = \mathrm{e}^{-i\beta M_1}, \quad \operatorname{sh} \beta \sim \operatorname{ch} \beta \sim 2kx\; [(k^2 + (m_\beta - m_\alpha)^2)\,(k^2 + (m_\beta + m_\alpha)^2)]^{-\frac{1}{2}}. \tag{A.56}$$

We have left out lower powers in the (infinite) momentum x. The matrix elements of W are then

$$\mathcal{D}^J_{M'M}(W) = d^J_{M'M}(i\beta)$$
$$\sim \frac{(2J)!}{[(J+M')!\,(J-M')!\,(J+M)!\,(J-M)!]^{\frac{1}{2}}} \cdot \frac{i^{M'-M}}{2^J} (\mathrm{ch}\,\beta)^J, \qquad M' \geqslant M \quad \text{(A.57)}$$

with a factor $(-)^{M'-M}$ for $M' \leqslant M$. Hence, the component $j^{J,0}$ is to be transformed by (A.42) with

$$d^J_{M0}(i\beta) \sim \frac{i^M (2J)!}{[(J+M)!\,(J-M)!]^{\frac{1}{2}}} \frac{1}{J!} \frac{1}{2^J} \left(\frac{x}{m_\beta}\right)^J (\sin\omega_\beta)^J. \quad \text{(A.58)}$$

Together with the normalization factor (A.49) this renders

$$F^J_{\beta k_\beta;\alpha k_\alpha}(k) = \left(\frac{i \sin\omega_\beta}{2m_\beta}\right)^J \sum_{M=-J}^{J} \frac{i^M}{\sqrt{2^J}} \sqrt{\frac{(2J)!}{(J+M)!\,(J-M)!}} \quad \text{(A.59)}$$

$$d^{s_\beta}_{h_\beta h'_\beta}(-\omega_\beta)\,\langle \beta h'_\beta \mathbf{0} |\, j^{JM}(q_\mathrm{f}) \,|\, \alpha h'_\alpha\, p e_z\rangle\, d^{s_\alpha}_{h'_\alpha h_\alpha}(\omega_\alpha).$$

This is not yet the former result (A.26), since the polarization vectors of $j^{JM}(q_\mathrm{f})$ are

$$\epsilon^\mu(0, q_\mathrm{f}) = -i(\mathrm{ch}\,\tilde{\xi}, 0, 0, \mathrm{sh}\,\tilde{\xi}),$$
$$\epsilon^\mu(\pm, q_\mathrm{f}) = \pm\frac{1}{\sqrt{2}}(0, -1, \pm i, 0). \quad \text{(A.60)}$$

Hence there is a factor $(-)^{J-M}$ with respect to the former definition of j^{JM}. Then (A.59) agrees with (A.26).

Let us finally remark that the angular condition may also be phrased in a differential form. If we introduce the operator $\hat{J}_3$ as

$$(\hat{J}_3 \circ G^{JM})_{\beta h_\beta;\alpha h_\alpha} \equiv [J_3, G^{JM}_{\beta\alpha}]_{h_\beta h_\alpha}$$
$$= \Delta h\, G^{JM}_{\beta h_\beta;\alpha h_\alpha} \quad \text{(A.61)}$$

we can write the constraint (A.28) as

$$(\hat{J}_3 - M) \circ G^{JM}_{\beta\alpha} = 0 \quad \text{(A.62)}$$

If we invert eq. (A.26), we find a combination of G^J's which therefore satisfies

$$\hat{J}_3 \circ (\hat{J}_3^2 - 1^2) \circ \ldots \circ (\hat{J}_3^2 - J^2) \circ \mathrm{e}^{-i\omega_\beta J_2}\, F^3(k)\, \mathrm{e}^{i\omega_\alpha J_2} = 0 \quad \text{(A.63)}$$

Due to (A.29), this constraint has non-trivial content only for $J < s_\beta + s_\alpha$. Pulling the rotations outside gives

$$\hat{J}'_3 \circ (\hat{J}'^2_3 - 1^2) \circ \ldots \circ (\hat{J}'^2_3 - J^2) \circ F^J(k) = 0, \quad J < s_\beta + s_\alpha, \quad \text{(A.64)}$$

where $\hat{J}'_3$ denotes the commutator

$$\hat{J}'_3 \circ F^J \equiv J^\beta_3 F^J - F^J J^\alpha_3. \quad \text{(A.65)}$$

with

$$J_3^{\alpha,\beta} = \mathrm{e}^{i\omega_{\alpha,\beta}J_2} J_3\, \mathrm{e}^{-i\omega_{\alpha,\beta}J_2} = \cos\omega_{\alpha,\beta} J_3 - \sin\omega_{\alpha,\beta} J_1\,. \quad \text{(A.66)}$$

For $J = 1$ the old differential angular condition results [18,20]. In that case the trigonometric functions $\cos\omega_{\alpha,\beta}$ and $\sin\omega_{\alpha,\beta}$ have also been rewritten in terms of commutators with momentum and mass operator. The resulting formula is little illuminating since it is extremely complicated. So we do not write it down explicitly.

The reader not familiar with the history of the subject may wonder why so much emphasis is put on these angular conditions. The point is that it is these conditions that will make the construction of models of bilocal form factors extremely difficult. It is no problem at all to satisfy the algebra (1.11). Every ansatz

$$F(k, z) = \mathrm{e}^{i(kX+zZ)} \tag{A.67}$$

will do if the matrices X^1, X^2, Z commute with each other. But in order to satisfy the angular conditions, the matrices X, Z will have to fulfill complicated multiple commutator and anticommutator relations.

Physically, the angular conditions ensure that a solution (A.67) of the algebra can really be considered as an infinite-momentum form factor of a bilocal relativistic current $j^\mu(x, y)$. In the past, great effort has been invested to find general solutions for $z = 0$ [18, 20]. However, no solution was found that had not previously been obtained directly from of fully relativisitic currents as provided by infinite component wave equations [20, 21].

Thus attempts at constructing models of structure functions which do not take the angular conditions into account at all [24] should be viewed with great suspicion. Probably also here, the only consistent feasible models will be those derived from infinite-component fields [22, 23].

Appendix B: Analytic properties of $F^J_{\beta\alpha}(k)$

Analyticity of F in k_1, k_2 follows immediately by observing that $K(m_\beta, m_\alpha, k)$ in eq. (A.4) can be written in the form

$$K(m_\beta, m_\alpha, k) = \mathrm{e}^{i\log(m_0/m_\beta)M_3}\, \mathrm{e}^{i(k^1E_1+k^2E_2)/m_0}\, \mathrm{e}^{-i\log(m_0/m_\alpha)M_3} \tag{B.1}$$

with some arbitrary mass parameter m_0. Here E_1, E_2 are the usual Euclidean generators

$$E_1 = M_1 + L_2, \quad E_2 = M_2 - L_1 \tag{B.2}$$

leaving j^+ invariant (since $[M_1, j^0] = ij^1$ and $[L_2, j^3] = ij^1$).

The decomposition (B.1) is easily proved by writing $B(p) = \mathrm{e}^{-i\zeta M}$ in the 2X2 representation $\exp(-\zeta\sigma/2)$. In this representation the limit (A.4) gives

$$K(m_\beta, m_\alpha, k) = \frac{1}{\sqrt{m_\beta m_\alpha}}\begin{pmatrix} m_\alpha & k^* \\ 0 & m_\beta \end{pmatrix}, \quad k^* \equiv k' - ik^2, \tag{B.3}$$

which decomposes according to

$$K(m_\beta, m_\alpha, k) = \begin{pmatrix} \sqrt{m_0/m_\beta} & \\ & \sqrt{m_\beta/m_0} \end{pmatrix} \begin{pmatrix} 1 & k^*/m_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{m_\alpha/m_0} & 0 \\ 0 & \sqrt{m_0/m_\alpha} \end{pmatrix} . \quad \text{(B.4)}$$

Rewriting this in terms of generators leads indeed to (B.1).

With (B.1), the k^1, k^2 dependence of $F^J_{\beta\alpha}(k)$ is given by the power series:

$$F^J_{\beta\alpha}(k^1, k^2) = \frac{1}{(2m_0)^J} \Sigma \langle \beta \mathbf{0} | \, \mathrm{e}^{i\log(m_0/m_\beta)M_3} \, j^{++\ldots+} \, E_1^m \, E_2^n \, \mathrm{e}^{-i\log(m_0/m_\alpha)M_3} | \alpha \mathbf{0} \rangle$$

$$\frac{(k^1)^m \, (k^2)^n}{m_0^{\;m+n}} . \quad \text{(B.5)}$$

Thus $F^J_{\beta\alpha}$ is analytic in k_1, k_2 with at most dynamical singularities.

The physical interesting variable is, however, $t = -k^2$. Also here the analytic properties are very simple. If $F^J_{\beta h_\beta, \alpha h_\alpha}$ contains a spin flip

$$\Delta h = h_\beta - h_\alpha$$

we first rotate $\boldsymbol{k}$ into x direction, obtaining

$$F^J_{\beta h_\beta ; \alpha h_\alpha}(k^1, k^2) = \mathrm{e}^{i\varphi\Delta h} \, F^J_{\beta h_\beta, \alpha h_\alpha}(k, 0), \qquad \varphi = \text{arc tg} \frac{k^2}{k^1} . \quad \text{(B.6)}$$

Now we observe that the expansion for $F^J_{\beta h_\beta, \alpha h_\alpha}(k, 0)$ starts with $k^{|\Delta h|}$ and is followed by higher powers spaced by even integers. Hence we can write

$$F^J_{\beta h_\beta ; \alpha h_\alpha}(k, 0) = k^{|\Delta h|} \, f^J_{\beta h_\beta ; \alpha h_\alpha}(-k^2), \quad \text{(B.7)}$$

where $f^j_{\beta h_\beta ; \alpha h_\alpha}(t)$ is analytic in $t = -k^2$.

Thus apart from a threshold factor $k^{|\Delta h|}$, F^J is analytic in t. The reason for these pleasant analytic properties of $F^J_{\beta\alpha}(k)$ is certainly, that in the infinite-momentum frame the helicity of a particle is always well defined.

As an illustration for this theorem we refer to eq. (A.37). It is well known that $F_1(t)$ and $F_2(t)$ are analytic in t.

It is worth pointing out that the angular condition together with the nice analyticity of $F^J_{\beta\alpha}(k)$ can be used to derive all kinematic singularities and constraints of the helicity form factors $G^{JM}_{\beta\alpha}(t)$ (and therefore also of the multipole couplings): For $J \geqslant s_\beta + s_\alpha$, the angular conditions are inverted and singularities and constraints can directly be read off the system of equations. All singularities come from the Wigner rotation (as an example see again (A.37): G_E and G_M have no singularities and at $k^2 = -4m^2$ there is one constraint $G_\mathrm{E} = G_\mathrm{M}$). For $J < s_\beta + s_\alpha$, one has to first find all the constraints among $F^J_{\beta\alpha}(k)$ resulting from the selection rule (A.28). Then one may solve for $G^{JM}_{\beta\alpha}(t)$ in terms of the independent $F^J_{\beta\alpha}(k)$. The labor involved using this method should be about the same as in standard derivations [25].

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