Quantum Field Theory of Particle Orbits with Large Fluctuations

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We show that an ensemble of particle orbits with large fluctuations around their classical paths are described by quantum field theory in the strong-coupling limit.

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Grand-canonical ensembles of gaussian random walks can be described by quantum field theory [1–5]. Indeed, the relativistic scalar free-particle propagator of mass m in D-dimensional euclidean momentum space $\mathbf{p}$ has the form

$$G(\mathbf{p}) = \frac{1}{\mathbf{p}^2 + m^2} = \int_0^\infty ds\, e^{-sm^2} e^{-s\mathbf{p}^2}. \tag{1}$$

The Fourier transform of $e^{-s\mathbf{p}^2}$ is the distribution of gaussian random walks of length s in D euclidean dimensions

$$P(s, \mathbf{x}) = (4\pi s)^{-D/2} e^{-\mathbf{x}^2/4s}, \tag{2}$$

so that the propagator (1) is a superposition of gaussian random walks whose lenghts are distributed like e^{-sm^2}:

$$G(\mathbf{x}) = \int_0^\infty ds\, e^{-sm^2} P(s, \mathbf{x}). \tag{3}$$

If δ-function-like interaction are added to the random walks, their statistical mechanics is described by the euclidean action

$$\mathcal{A} = \int d^3x \left[\phi(\mathbf{x})(\hat{\mathbf{p}}^2 + m^2)\phi(\mathbf{x}) + \frac{g_c}{4!}\phi^4(\mathbf{x})\right]. \tag{4}$$

The evaluation of the partition function based on this action is usually done approximately by perturbation theory order by order in the coupling strength g. The results are divergent power series in g from which the physical properties must be extracted by renormalization. There exists ao-called *critical dimension*, here $D_c = 4$, where the action is scale invariant. If the physical dimension D lies slightly below D_c, say at $D = D_c - \epsilon$, the theory appears to be still scale invariant, with fields acquiring an anomalous dimension $[\phi] = 1 - \eta/2$, and the interaction becoming effectively $\phi^{\delta+1}(\mathbf{x})$, where $\delta = (D + 2 - \eta)/(D - 2 + \eta)$. This is known as the renormalization group approach to critical phenomena [6]. Alternatively, it can be formulated as to strong-coupling limit of the quantum field theory [7, 8]. This theory has exlpained critical phenomena with high accuracy. In particular it has predicted the value of the critical exponent α in the singularity of the specific heat $C \propto |T - T_c|^{-\alpha}$ to be $\alpha \approx -0.0127$ in excellent agreement with the satellite measurement $\alpha_{\rm exp} \approx -0.0129$ [9].

Thus gaussian random walks are a natural starting point for many stochastic processes. For instance, they form the basis of the most important tool in the theory of financial markets, the Black-Scholes option price theory [10] (Nobel Prize 1997), by which a portfolio of assets is intended to grow steadily via hedging. In fact, the famous *central-limit theorem* permits us to prove that many independent random movements of finite variance always pile up to display a gaussian distribution [11].

However, since the last crash and the ensuing financial crisis, it has become clear that realistic stochastic distributions in nature belong to a more general universality class, the so-called Lévy distribution. While gaussian distributions alway arise from a pile up of arbitrary finite steps, Lévy distribution emerge if these step have an infinite variance. They describe that rare events, which initiate crashes, are much more frequent than in gaussian distributions. Such tail-events also occur in the distribution of earthquakes, with catastrophic consequences [12].

These are events in the so-called power tails $\propto 1/|x|^{1+\alpha}$ of the distributions, with $\alpha \leq 2$, whose description requires a fractional Fokker-Planck equation [13]

$$[\partial_s + (\hat{\mathbf{p}}^2)^{\alpha/2}] P_\alpha(s, \mathbf{x}) = 0. \tag{5}$$

In the limit $\alpha \to 2$, the Lévy distributions reduce to gaussian distributions. In general the solution for $P_\alpha(s, \mathbf{x})$ is [14]

$$P_\alpha(s, \mathbf{x}) = \frac{1}{\pi^{D/2}|\mathbf{x}|^{D/2}} H^{2,1}_{2,3}\left(\frac{|\mathbf{x}|^\alpha}{2^\alpha s}\,\middle|\,\begin{matrix}(1,1);(1,1)\\ (1,1),(\frac{D}{2},\frac{\alpha}{2});(1,\frac{\alpha}{2})\end{matrix}\right), \tag{6}$$

where $H^{2,1}_{2,3}$ is a *Fox H-function* [15]. In the limits $\gamma = 0$ and $\alpha = 2$, this reduces to the standard quantum mechanical gaussian expression (2). For $\gamma = 0, \alpha = 1$, the result is

$$P_\alpha(s, \mathbf{x}) = \frac{s}{\pi^{(D+1)/2}|\mathbf{x}|^{D+1}} H^{1,1}_{1,1}\left(\frac{s^2}{|\mathbf{x}|^2}\,\middle|\,\begin{matrix}(1/2-D/2,1)\\ (0,1)\end{matrix}\right),$$

which is simply the Cauchy distribution function $[\Gamma(D/2 + 1/2)/\pi^{(D+1)/2}] s/(s^2 + |\mathbf{x}|^2)^{D/2+1/2}$.

From what we explained above it is clear that these nongaussian Lévy walks are contained in the theory based on the action (4) in the strong-coupling limit. Using the results of Ref. [7], and the textbook [8] the effective action of this strong-coupling limit reads

$$\mathcal{A}^{\rm eff} = \int d^3x \left[\phi(\mathbf{x})(\hat{\mathbf{p}}^2)^{1-\eta/2}\phi(\mathbf{x}) + \frac{g_c}{4!}|\phi|^{\delta+1}(\mathbf{x})\right], \quad (7)$$

where $1-\eta/2 \approx 0.985$, $\delta = (D+2-\eta)/(D-2+\eta) \approx 5(1-6\eta/5) \approx 4.83$, and $g_c \approx 1.4$.

The original field theory based on the action (4) is extremal for fields that satisfy the typical Schrödinger boundary conditions that $\phi(\mathbf{x})$ must be single valued. In the quantum theory this is the origin of all quantum numbers. In the strong-coupling limit, on the other hand, these conditions are no longer satisfied. Instead, the field is multivalued [16], and by proposing such wave funtions for strongly interacting electron systems in two dimensions has earned Laughlin [17] the Nobel prize in 1998. The strong-coupling limit may also be needed for an understanding of high-T_c conductivity, which is often associated with a non-Fermi liquid behavior of the electrons [18].

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