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The Invisibility of Torsion in Gravity*

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Abstract

It is shown that torsion can be moved partially or totally into the curvature n a new kind of gauge transformation without changing the physical content of Einstein's theory of gravitation. This explains its invisibility in any gravitational experiment.

The question why torsion does not show up in gravity has plagued many researchers and has been discussed controversially in the literature [1]. Last year. Hammond won a prize from the Gravity Research Foundation for an essay arguing for the necessity of torsion in gravity [2]. The purpose of this note is to hold against it and to show that his arguments are invalid.

1. The second of Hammond's arguments states that torsion is necessary to absorb the gradient of the phase arising if the chiral transformation of the Dirac field $\psi(x) \to e^{i\alpha\gamma_5}\psi(x)$. However, this transformation is already local symmetry due to the presence of the vector bosons of weak interactions. These easily absorb the gradient $\alpha(x)$.

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2. The first of Hammond's arguments is that torsion is necessary to derive the conservation of total angular momentum

$$\frac{d}{d\tau} J_{\mu\nu} = 0. \tag{1}$$

However, this law follows directly from the fact that the symmetric energy-momentum tensor of Einstein's theory $G^{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$, and that the total angular momentum is given by the integral

$$J_{\mu\nu} = \int d^3x \sqrt{g} (x_\mu T_{\nu 0} - x_\nu T_{\mu 0}). \tag{2}$$

The conservation law is a direct consequence of the local conservation law (1) $D^\nu T_{\mu\nu} = 0$, which implies that $D^\lambda (x_\mu T_{\nu\lambda} - x_\nu T_{\mu\lambda}) = 0$, whose spatial integral yields (1).

It is useful to elaborate this issue a little further to better understand Hammond's mistake. We re-express Einstein's gravity in Cartan's spacetime in terms of two gauge fields, one of translations, the vierbein field $h^\alpha{}_\mu$, and one of local Lorentz transformations, the spin connection $A_{\mu\alpha}{}^\beta$ [3, 4, 5]. The metric is related to these by $g_{\mu\nu} = h^\alpha{}_m \Lambda_\alpha{}^a \Lambda_a{}^\beta h_{\beta\nu}$ where $\Lambda_a{}^\beta$ is an arbitrary local Lorentz transformation and $\Lambda_\alpha{}^a$ its inverse. There is a pure gauge field associated with $\Lambda_a{}^\beta$:

$$A_{\mu\alpha}{}^\beta = \Lambda_\alpha{}^a \partial_\mu \Lambda_a{}^\beta, \tag{3}$$

and we can form a covariant derivative of any vector v_α

$$D^L_\alpha v_\beta = \partial_\alpha v_\beta - A_{\mu\alpha}{}^\beta v_\beta, \tag{4}$$

which transforms like a tensor under local Lorentz transformations $\Lambda_a{}^\beta$. The affine connection of the Cartan spacetime is given by

$$\Gamma^C_{\mu\alpha}{}^\beta = h^\beta{}_\lambda D^L_\mu h_\alpha{}^\lambda = h^\beta{}_\lambda (\partial_\mu - A_{\mu\alpha}{}^\beta) h_\beta{}^\lambda. \tag{5}$$

By converting the indices α and β to ν and λ by contracting with $h_\alpha{}^\lambda$ or $h_\alpha{}^\lambda$, we can create from this the affine *Cartan connection* $\Gamma^C_{\mu\nu}{}^\lambda = h^\alpha{}_\nu \Gamma^C_{\mu\alpha}{}^\beta h_\beta{}^\lambda$. Its antisymmetric part is defined as the *torsion tensor* of the Cartan spacetime: $S_{\mu\nu}{}^\lambda \equiv \frac{1}{2}(\Gamma_{\mu\nu}{}^\lambda - \Gamma_{\nu\mu}{}^\lambda)$. The Christoffel connection is now obtained by forming $\Gamma_{\mu\nu}{}^\lambda = \Gamma^C_{\mu\nu}{}^\lambda - K_{\mu\nu}{}^\lambda$, where $K_{\mu\nu\lambda} \equiv h^\alpha{}_\nu h^\beta{}_\lambda K_{\mu\alpha\beta} \equiv S_{\mu\nu\lambda} - S_{\nu\lambda\mu} + S_{\lambda\mu\nu}$ is the *contortion* field. The relation between the Cartan and the Christoffel connection is $\Gamma^C_{\mu\nu}{}^\lambda = \Gamma_{\mu\nu}{}^\lambda + K_{\mu\nu}{}^\lambda$.

The Cartan connection written as a matrix $\Gamma^C_{\mu\nu}{}^\lambda \equiv (\Gamma^C_\mu)_\alpha{}^\beta$ has a covariant curl

$$R^C_{\mu\nu\lambda}{}^\kappa \equiv \{\partial_\mu \Gamma^C_\nu - \partial_\nu \Gamma^C_\nu - [\Gamma^C_\mu, \Gamma^C_\nu]\}_\lambda{}^\kappa, \tag{6}$$

which is the *Cartan curvature tensor*. The same expression without superscript C is the *Riemannian curvature* tensor $R_{\mu\nu\lambda}{}^\kappa$. The two are related by

$$R_{\mu\nu\lambda}{}^\kappa = R^{\mathrm{C}}_{\mu\nu\lambda}{}^\kappa - (D_\mu K_{\nu\lambda}{}^\kappa - D_\nu K_{\mu\lambda}{}^\kappa - [K_\mu, K_\nu]_\lambda{}^\kappa). \tag{7}$$

So far, the vector field arising from the local Lorentz-transformation $\Lambda_a{}^\beta$ is a pure gauge field (3). If we form the covariant curl of the matrix $A_{\mu\alpha}{}^\beta \equiv (A_\mu)_\alpha{}^\beta$ we obtain a vanishing field strength

$$F_{\mu\nu\beta}{}^\gamma \equiv \{\partial_\mu A_\nu - \partial_\nu A_\nu - [A_\mu, A_\nu]\}_\lambda{}^\kappa. \tag{8}$$

As a consequence, also the Cartan curvature tensor vanishes, which is related to this by $R^{\mathrm{C}}_{\mu\nu\lambda}{}^\kappa \equiv h^\beta{}_\lambda h_\gamma{}^\kappa F_{\mu\nu\beta}{}^\gamma$. Then we see from Eq. (7) that the Riemannian curvature scalar is expressible entirely in terms of the tensor as

$$R = R_{\mu\nu}{}^{\nu\mu} = -4D_\mu S^\mu + S_{\mu\nu\lambda}S^{\mu\nu\lambda} + 2S_{\mu\nu\lambda}S^{\mu\lambda\nu} - 4S^\mu S_\mu, \tag{9}$$

If we form the Einstein-Hilbert action $\mathcal{A}_{EH} = -1/(2\kappa)\int dx\sqrt{g}R$, we obtain the famous teleparallel formulation of Einstein's theory [6, 7], which is thus completely equivalent to Einstein's theory, which is a field theory only of the metric $g_{\mu\nu}(x)$. This observation has led to the conclusion that torsion is just another language for expressing the laws of gravity in Einstein's theory [8, 9].

The important point of this essay is to go beyond this well-known equivalence and exhibit a new type of gauge symmetry which emphasizes even more the unobservability of torsion [10]. We realize that the decomposition (7) of the Riemannian curvature tensor is not only valid for any local Lorentz transformation ${\Lambda_a}^\beta$ where ${A_{\mu\nu\alpha}}^\beta$ is a pure gauge (3) and ${R^C_{\mu\nu\alpha}}^\beta \equiv 0$, but it is an invariant under arbitrary *multivalued Lorentz transformations* ${\Lambda_a}^\beta$, i.e., those which have noncommuting derivatives $[\partial_\mu, \partial_\nu]{\Lambda_a}^\beta \neq 0$ [11]. Then the curl (8) is no longer zero and we can shuffle *part or all* of the torsion into the Cartan curvature without changing a given Einstein tensor. The other extreme of this procedure is the choice of zero torsion, ${S_{\mu\nu}}^\lambda \equiv 0$, from which we find

$$ {A_{\mu\alpha}}^\beta = {\bar{A}_{\mu\alpha}}^\beta \equiv {h_\alpha}^\nu h^{\beta\lambda}(\Omega_{\mu\nu\lambda} - \Omega_{\nu\lambda\mu} + \Omega_{\lambda\mu\nu}), \tag{10} $$

where $\Omega_{\mu\nu\lambda}$ are the *objects of anholonomity*: $\Omega_{\mu\nu\lambda} = (1/2)[h_{\alpha\lambda}\partial_\mu {h^\alpha}_\nu - (\mu \leftrightarrow \nu)]$. The choice of a multivalued function ${\Lambda_a}^\beta$ in Eq. (3) which leads to a nonzero gauge field ${A_{\mu\alpha}}^\beta$ may be called hypergauge transformation.

The invariance of the Riemann tensor under this new type of gauge transformations is the key to understanding the invisibility of torsion in a physical experiment. Any expression containing torsion is noninvariant under hypergauge transformation since torsion can be hypergauged away into the curvature of the gravitational field.

How about the coupling of matter in such a reformulation of gravity? Since the contortion is a tensor, we can multiply it by any coupling constant q to form an infinity of q-dependent connection ${\Gamma^q_{\mu\nu}}^\lambda = {\Gamma^C_{\mu\nu}}^\lambda - q{K_{\mu\nu}}^\lambda$, and thus an infinity of covariant derivatives. For a vector field A_μ, these are

$$D^q_\mu A_\nu \equiv \partial_\mu A_\nu - \Gamma^q_{\mu\nu}{}^\lambda A_\lambda. \tag{11}$$

Experiment must tell which of these is physically correct. Thus we consider the electromagnetic field strength $F^q_{\mu\nu} \equiv D^q_\mu A_\nu - D^q_\nu A_\mu$, and find $F^q_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - 2(1-q)S_{\mu\nu}{}^\lambda A_\lambda$. This shows the well-known fact that the electromagnetic action $-\frac{1}{4}F^q_{\mu\nu}F^{q\,\mu\nu}$ has a mass term unless we choose the coupling strength $q=1$.

A little algebra shows us that $\Gamma^q_{\mu\nu}{}^\lambda$ for $q=1$ is simply the good-old Fock-Ivanenko spin connection used in Einstein's theory. Multiplication by vierbein fields produces the corresponding $A_{\mu\alpha}{}^\beta$-field, which is simply the $S\equiv 0$-gauge field (10):

$$h_\alpha{}^\nu h^\beta{}_\lambda \Gamma^1_{\mu\nu}{}^\lambda \equiv \bar{A}_{\mu\alpha}{}^\beta. \tag{12}$$

If we want to avoid that the photon acquires a mass by virtual processes in which it goes over into a ρ-meson and further into barions, we must couple also *all* other fields in the same way, i.e., via the covariant derivative $\bar{D}_\mu\psi \equiv (\partial_\mu - \frac{i}{2}\bar{A}_{\mu\alpha}{}^\beta\Sigma^\alpha{}_\beta)\psi$ where $\Sigma^\beta{}_\alpha$ are the generators of Lorentz transformations with the commutation rules $[\Sigma_{\alpha\beta},\Sigma_{\alpha\gamma}] = -i\eta_{\alpha\alpha}\Sigma_{\beta\gamma}$ (no sum over α). For Dirac fields $\Sigma_{\alpha\beta} = \frac{i}{3}[\gamma_\alpha,\gamma_\beta]$, whereas for vector fields v_β: $(\Sigma_{\alpha\beta})_{\alpha'\beta'} = i(\delta_{\alpha\alpha'}\delta_{\beta\beta'} - \delta_{\alpha\beta'}\delta_{\beta\alpha'})$, and $\bar{D}_\mu v_\beta \equiv \partial_\mu v_\beta - \bar{A}_{\mu\alpha}{}^\beta v_\beta$, corresponding to (11) for $q=1$.

In Cartan spacetime, the Einstein tensor is related to its Cartan version by the identity:

$$G_{\mu\nu} = G^{\mathrm{C}}_{\mu\nu} - \tfrac{1}{2}D^{*\lambda}\left(S_{\mu\nu,\lambda} - S_{\nu\lambda,\mu} + S_{\lambda\mu,\nu}\right), \tag{13}$$

where where the covarinant derivative is defined by $D^*_\lambda v_\mu \equiv \partial_\lambda v_\mu - \Gamma^C_{\lambda\mu}{}^\kappa v_\kappa + 2S_{\lambda\kappa}{}^\kappa v_\mu$. This is precisely the Belinfante relation [12] between canonical energy-momentum tensor and spin density. Thus the total angular momentum (2) becomes the sum of an orbital part $L_{\mu\nu} = \int d^3x\sqrt{g}(x_\mu G^C_{\nu 0} - x_\nu G^C_{\mu 0})$ and spin $S_{\mu\nu} = \int d^3x\sqrt{g}S_{\mu\nu,0}$, where $S_{\mu\nu,0}$ is the spin current density introduced by Palatini: $S_{\mu\kappa,}{}^{\tau} \equiv 2(S_{\mu\kappa}{}^{\tau} + \delta_\mu{}^\tau S_\kappa - \delta_\kappa{}^\tau S_\mu)$. From the hypergauge invariance of the Riemann curvature we may now conclude there is no sense in trying to distinguish between orbital and spin angular momentum of the gravitational field. They can be hypergauged into each other at any fixed total angular momentum, which is a hypergauge-invariant quantity. This invariance is physically necessary to give the orbital angular momentum of a planet and the sum of all spins of the elementary particles the same frame-dragging effect [13] measured in Gravity Probe B [14]. For as long as we do not know what are the true elementary particles of matter (only string theorists do, but they have even more severe problems with nature), the distinction between orbital rotation and intrinsic spin is fundamentally impossible, as explained in Ref. [10].

What we have done can be understood very easily by considering an analogous field-theoretical model of a real field ρ with an Euclidean Lagrangian $\mathcal{L} = (\partial_\mu\rho)^2 - \rho^2 + \rho^4$, and a partition function $Z = \int \mathcal{D}\psi\mathcal{D}\psi^*\, e^{-\int dx\mathcal{L}}$. We may trivially introduce an extra gauge structure by re-expressing the Lagrangian in terms of a complex field $\psi = \rho e^{i\theta}$ and a gauge field A_μ as $\bar{\mathcal{L}} = |(\partial_\mu - iA_\mu)\psi|^2 - |\psi|^2 + |\psi|^4$. Now we form the partition function $\bar{Z} = \int \mathcal{D}\psi\mathcal{D}\psi^*\mathcal{D}A_\mu\Phi\, e^{-\int dx\bar{\mathcal{L}}}$, where Φ is an arbitrary gauge-fixing functional, multiplied by the associated Faddeev-Popov determinant. The new $\bar{Z}$ is completely equivalent to the original Z. There is no way of observing A_μ. The partition function Z plays the role of Einstein's theory, whereas $\bar{Z}$ is its reformulation in terms of a gauge fields, which does not change the physical content of the theory.

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