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# Coordinate independence of quantum-mechanical path integrals ☆, ☆☆

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**Abstract**

We develop simple rules for performing integrals over products of distributions in coordinate space. Such products occur in perturbation expansions of path integrals in curvilinear coordinates, where the interactions contain terms of the form $\dot{q}^2 q^n$, which give rise to highly singular Feynman integrals. The new rules ensure the invariance of perturbatively defined path integrals under coordinate transformations. © 2000 Published by Elsevier Science B.V.

## 1. Introduction

In the previous Letters [1;2], we have presented a diagrammatic proof of reparametrization invariance of perturbatively defined quantum-mechanical path integrals. The proper perturbative definition of path integrals was shown to require an extension to a functional integral in $D$ spacetime, and a subsequent analytic continuation to $D = 1$. In Ref. [1] the perturbative calculations were performed in momentum space, where Feynman integrals in a continuous number of dimensions $D$ are known from the prescriptions of 't Hooft and M. Veltman [3]. In Ref. [2] we have found the same results directly from the Feynman integrals in the $1-\varepsilon$-dimensional time space with the help of the Bessel representation of Green functions. The coordinate space calculation is interesting for many applications, for instance, if one wants to obtain the effective action of a field system in curvilinear coordinates, where the kinetic term depends on the dynamic variable. Then one needs rules for performing temporal integrals over Wick contractions of local fields.

In this Letter we want to show that the reparametrization invariance of perturbatively defined quantum-mechanical path integrals can be obtained in the coordinate space with the help of a simple but quite general arguments based on the inhomogeneous field equation for the Green function, and rules of the partial integration. The prove does not require the calculation of the Feynman integrals separately and remains valid for the functional integrals in an arbitrary space-time dimension $D$.

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## 2. Problem with coordinate transformations

Recall the origin of the difficulties with coordinate transformations in path integrals. Let $x(\tau)$ be the euclidean coordinates of a quantum-mechanical point particle of unit mass in a harmonic potential $\omega^2 x^2/2$ as a function of the imaginary time $\tau = -it$. Under a coordinate transformation $x(\tau) \to q(\tau)$ defined by $x(\tau) = f(q(\tau)) = q(\tau) + \sum_{n=2}^{\infty} a_n q^n(\tau)$, the kinetic term $\dot{x}^2(\tau)/2$ goes over into

$$\dot{q}^2(\tau) f'^{\,2}(q(\tau))/2.$$

If the path integral over $q(\tau)$ is performed perturbatively, the expansion terms contains temporal integrals over Wick contractions which, after suitable partial integrations, are products of the following basic correlation functions

$$\Delta(\tau - \tau') \equiv \langle q(\tau) q(\tau') \rangle = \text{—}, \tag{1}$$

$$\partial_\tau \Delta(\tau - \tau') \equiv \langle \dot{q}(\tau) q(\tau') \rangle = \text{- - —}, \tag{2}$$

$$\partial_\tau \partial_{\tau'} \Delta(\tau - \tau') \equiv \langle \dot{q}(\tau) \dot{q}(\tau') \rangle = \text{- - - -}. \tag{3}$$

The right-hand sides define the line symbols to be used in Feynman diagrams for the interaction terms.

Explicitly, the first correlation function reads

$$' \quad 1 \quad -\omega|\tau - \tau'|$$

and an interacting part, which reads to second order in $g$:

$$\mathcal{A}_{\rm int}[q] = \tfrac{1}{2}\int d\tau \Bigg\{ -g\Bigg[2\dot{q}^2(\tau)q^2(\tau) + \frac{2\omega^2}{3}q^4(\tau)\Bigg] + g^2\Bigg[(1+2a)\dot{q}^2(\tau)q^4(\tau) + \omega^2\left(\frac{1}{9}+\frac{2a}{5}\right)q^6(\tau)\Bigg]\Bigg\}. \tag{12}$$

The exponent in (10) contains an additional effective action $\mathcal{A}_J[q]$ coming from the Jacobian of the coordinate transformation:

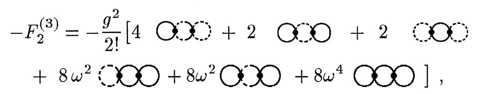
$$\mathcal{A}_J[q] = -\delta(0)\int d\tau \log \frac{\delta f(q(\tau))}{\delta q(\tau)}. \tag{13}$$

This has the power series expansion

$$\mathcal{A}_J[q] = -\delta(0)\int d\tau\Big[-gq^2(\tau) + g^2\left(a-\tfrac{1}{2}\right)q^4(\tau)\Big]. \tag{14}$$

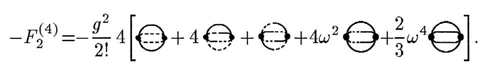
For $g=0$, the transformed partition function (10) coincides with (8). When expanding $Z$ of Eq. (10) in powers of $g$, we obtain a sum of Wick contractions with associated Feynman diagrams contributing to each order $g^n$. This sum must vanish to ensure coordinate invariance of the path integral.

By considering only connected Feynman diapossible combinations of the three line types (1)–(3). The former are



(19)

and the latter:



(20)

Since the equal-time expectation value $\langle \dot{q}(\tau)\, q(\tau) \rangle$ vanishes by Eq. (5), diagrams with a local contraction of a mixed line (2) are trivially zero, and have been omitted.

In our previous Letters [1;2], all integrals were calculated individually in $D = 1 - \varepsilon$ dimensions, taking the limit $\varepsilon \to 0$ at the end. Here we set up simple rules for finding the same results, which make the sum of all Feynman diagrams contributing to each order $g^n$ vanish.

## 5. Basic properties of dimensionally regularized distributions

The path integral (10) is extended to an associated functional integral in a $D$-dimensional coordinate space $x$, with coordinates $x_\mu \equiv (\tau, x_2, x_3, \ldots)$, by replacing $\dot{q}^2(\tau)$ in the kinetic term by $(\partial_\mu q(x))^2$, where $\partial_\mu = \partial / \partial x_\mu$. The Jacobian action term (13) is omitted in dimensional regularization because of Veltman's rule [3]:

$$\delta^{(D)}(0) = \int \frac{d^D k}{(2\pi)^D} = 0. \tag{21}$$

In our calculations, we shall encounter generalized $\delta$-functions, which are multiple derivatives of the ordinary $\delta$-function:

$$\delta^{(D)}_{\mu_1 \ldots \mu_n}(x) \equiv \partial_{\mu_1 \ldots \mu_n} \delta^{(D)}(x) = \int \bar{d}^D k (ik)_{\mu_1} \ldots (ik)_{\mu_n} e^{ikx}, \tag{22}$$

with $\partial_{\mu_1 \ldots \mu_n} \equiv \partial_{\mu_1} \ldots \partial_{\mu_n}$, and with $\bar{d}^D k \equiv d^D k / (2\pi)^D$. In dimensional regularization, alle these vanish at the origin as well:

$$\delta^{(D)}_{\mu_1 \ldots \mu_n}(0) = \int \bar{d}^D k (ik)_{\mu_1} \ldots (ik)_{\mu_n} = 0, \tag{23}$$

which is a more general way of expressing Veltman's rule. In the extended coordinate space, the correlation function (1) becomes

$$\Delta(x) = \int \frac{d^D k}{(2\pi)^D} \frac{e^{ikx}}{k^2 + \omega^2}, \tag{24}$$

At the origin, it has the value

$$\Delta(0) = \int \frac{\bar{d}^D k}{k^2 + \omega^2} = \frac{\omega^{D-2}}{(4\pi)^{D/2}} \Gamma\left(1 - \frac{D}{2}\right) \underset{D=1}{=} \frac{1}{2\omega}. \tag{25}$$

The extension of the time derivative (2),

$$\Delta_\mu(x) = \int \bar{d}^D k \frac{ik_\mu}{k^2 + \omega^2} e^{ikx} \tag{26}$$

vanishes at the origin, $\Delta_\mu(0) = 0$. This follows directly from a Taylor series expansion of $1/(k^2 + \omega^2)$ in powers of $k^2$, together with Eq. (23).

The second derivative of $\Delta(x)$ has the Fourier representation

$$\Delta_{\mu\nu}(x) = -\int \bar{d}^D k \frac{k_\mu k_\nu}{k^2 + \omega^2} e^{ikx}. \tag{27}$$

Contracting the indices yields

$$\Delta_{\mu\mu}(x) = -\int \bar{d}^D k \frac{k^2}{k^2 + \omega^2} e^{ikx} = -\delta^{(D)}(x) + \omega^2 \Delta(x), \tag{28}$$

which follows from the definition of the correlation function by the inhomogeneous field equation

$$\left(-\partial_\mu^2+\omega^2\right)q(x)=\delta^{(D)}(x). \tag{29}$$

From (28) we have the relation between integrals

$$\int d^D x\, \Delta_{\mu\mu}(x)=-1+\omega^2\int d^D x\, \Delta(x), \tag{30}$$

Inserting Veltman's rule (21) into (28), we obtain

$$\Delta_{\mu\mu}(0)=\omega^2\,\Delta(0)\underset{D=1}{=}\frac{\omega}{2}. \tag{31}$$

This ensures the vanishing of the first-order contribution (16) to the free energy

$$-F_1=-g\left[-\Delta_{\mu\mu}(0)+\omega^2\Delta(0)\right]\Delta(0)=0. \tag{32}$$

The same Eq. (28) allows us to calculate immediately the second-order contribution (17) from the local diagrams

$$\begin{aligned}-F_2^{(1)}&=-3g^2\left[\left(\tfrac{1}{2}+a\right)\Delta_{\mu\mu}(0)\right.\\&\quad\left.-5\left(\frac{1}{18}+\frac{a}{5}\right)\omega^2\Delta(0)\right]\Delta^2(0)\\&=-\frac{2}{3}\,\omega^2\Delta^3(0)\underset{D\to1}{=}-\frac{1}{12\,\omega}.\end{aligned} \tag{33}$$

The other contributions to the free energy in the expansion (15) require rules for calculating products of two and four distributions, which we are now going to develop.

## 6. Integrals over products of two distributions

The simplest integrals of this type are

$$\begin{aligned}\int d^D x\, \Delta^2(x)&=\int đ^D p\, đ^D k\,\frac{\delta^{(D)}(k+p)}{(p^2+\omega^2)(k^2+\omega^2)}\\&=\int\frac{đ^D k}{(k^2+\omega^2)^2}\\&=\frac{\omega^{D-4}}{(4\pi)^{D/2}}\Gamma\left(2-\frac{D}{2}\right)\\&=\frac{(2-D)}{2\,\omega^2}\,\Delta(0),\end{aligned} \tag{34}$$

and

$$\begin{aligned}&\int d^D x\, \Delta_\mu^2(x)\\&\quad=-\int d^D x\, \Delta(x)\left[-\delta^{(D)}(x)+\omega^2\Delta(x)\right]\\&\quad=\Delta(0)-\omega^2\int d^D x\, \Delta^2(x)=\frac{D}{2}\,\Delta(0).\end{aligned} \tag{35}$$

To obtain the second result we have performed a partial integration and used (28).

In contrast to the integrals (34) and (35), the integral

$$\begin{aligned}\int d^D x\, \Delta_{\mu\nu}^2(x)&=\int đ^D p\, đ^D k\,\frac{(kp)^2\,\delta^{(D)}(k+p)}{(k^2+\omega^2)(p^2+\omega^2)}\\&=\int đ^D k\,\frac{(k^2)^2}{(k^2+\omega^2)^2}\\&=\int d^D x\, \Delta_{\mu\mu}^2(x)\end{aligned} \tag{36}$$

diverges formally in $D=1$ dimension. In dimensional regularization, however, we may decompose $(k^2)^2=(k^2+\omega^2)^2-2\,\omega^2(k^2+\omega^2)+\omega^4$, and use (23) to evaluate further

$$\begin{aligned}\int d^D x\, \Delta_{\mu\mu}^2(x)&=\int đ^D k\frac{(k^2)^2}{(k^2+\omega^2)^2}\\&=-2\,\omega^2\int\frac{đ^D k}{(k^2+\omega^2)}\\&\quad+\omega^4\int\frac{đ^D k}{(k^2+\omega^2)^2}\\&=-2\,\omega^2\Delta(0)+\omega^4\int d^D x\,\Delta^2(x).\end{aligned} \tag{37}$$

Together with (34), we obtain the finite integrals

$$\begin{aligned}\int d^D x\, \Delta_{\mu\nu}^2(x)&=\int d^D x\, \Delta_{\mu\mu}^2(x)\\&=-2\,\omega^2\Delta(0)+\omega^4\int d^D x\, \Delta^2(x)\\&=-\left(1+D/2\right)\omega^2\Delta(0).\end{aligned} \tag{38}$$

An alternative way of deriving the equality (36) is to use partial integrations and the identity

$$\partial_\mu \Delta_{\mu\nu}(x) = \partial_\nu \Delta_{\mu\mu}(x), \tag{39}$$

which follows directly from the Fourier representation (26).

Finally, from Eqs. (34), (35), and (38), we observe the useful identity

$$\int d^D x \left[ \Delta^2_{\mu\nu}(x) + 2\omega^2 \Delta^2_\mu(x) + \omega^4 \Delta^2(x) \right] = 0, \tag{40}$$

which together with the inhomogeneous field Eq. (28) reduces the calculation of the second-order contribution of all three-bubble diagrams (19) to zero:

$$-F_2^{(3)} = -g^2 \Delta^2(0) \int d^D x \left[ \Delta^2_{\mu\nu}(x) + 2\omega^2 \Delta^2_\mu(x) + \omega^4 \Delta^2(x) \right] = 0. \tag{41}$$

## 7. Integrals products of four distributions

More delicate integrals arise from the watermelon diagrams in (20) which contain products of four distributions, a nontrivial tensorial structure, and overlapping divergences [1;2]. Consider the first three diagrams:

 $= \int d^D x\, \Delta^2(x) \Delta^2_{\mu\nu}(x).$ (42)

4  $= 4 \int d^D x \Delta(x) \Delta_\mu(x) \Delta_\nu(x) \Delta_{\mu\nu}(x),$ (43)

 $= \int d^D x \Delta_\mu(x) \Delta_\mu(x) \Delta_\nu(x) \Delta_\nu(x),$ (44)

To exhibit the subtleties with the tensorial structure, we introduce the integral

$$I_D = \int d^D x\, \Delta^2(x) \left[ \Delta^2_{\mu\nu}(x) - \Delta^2_{\mu\mu}(x) \right]. \tag{45}$$

In $D=1$ dimension, the bracket vanishes formally, but the limit $D \to 1$ of the integral is nevertheless finite. We now decompose the Feynman diagram (42), into the sum

$$\int d^D x\, \Delta^2(x) \Delta^2_{\mu\nu}(x) = \int d^D x\, \Delta^2(x) \Delta^2_{\mu\mu}(x) + I_D. \tag{46}$$

To obtain an analogous decompositions for the other two diagrams (43) and (44) we derive a few useful relations using the inhomogeneous field Eq. (28), partial integrations, and Veltman's rule (23). First there is the relation

$$-\int d^D x\, \Delta_{\mu\mu}(x) \Delta^3(x) = \Delta^3(0) - \omega^2 \int d^D x\, \Delta^4(x). \tag{47}$$

By a partial integration, the left-hand side becomes

$$\int d^D x\, \Delta_{\mu\mu}(x) \Delta^3(x) = -3 \int d^D x\, \Delta^2_\mu(x) \Delta^2(x), \tag{48}$$

leading to

$$\int d^D x\, \Delta^2_\mu(x) \Delta^2(x) = \tfrac{1}{3} \Delta^3(0) - \tfrac{1}{3} \omega^2 \int d^D x\, \Delta^4(x). \tag{49}$$

Invoking once more the inhomogeneous field Eq. (28) and Veltman's rule (21), we obtain the integrals

$$\int d^D x\, \Delta^2_{\mu\mu}(x) \Delta^2(x) = -2\omega^2 \Delta^3(0) + \omega^4 \int d^D x\, \Delta^4(x), \tag{50}$$

and

$$\int d^D x\, \Delta_{\mu\mu}(x) \Delta^2_\mu(x) \Delta(x) = \omega^2 \int d^D x\, \Delta^2_\mu(x) \Delta^2(x). \tag{51}$$

Due to Eq. (49), the integral (51) takes the form

$$\int d^D x\, \Delta_{\mu\mu}(x)\, \Delta_\mu^2(x)\, \Delta(x)$$

$$= \tfrac{1}{3}\omega^2\Delta^3(0) - \tfrac{1}{3}\omega^4 \int d^D x\, \Delta^4(x). \qquad (52)$$

Partial integration, together with Eqs. (50) and (52), leads to

$$\int d^D x\, \partial_\mu \Delta_{\lambda\lambda}(x)\, \Delta_\mu(x)\, \Delta^2(x)$$

$$= -\int d^D x\, \Delta_{\lambda\lambda}^2(x)\, \Delta^2(x)$$

$$-2\int d^D x\, \Delta_{\lambda\lambda}(x)\, \Delta_\mu^2(x)\, \Delta(x)$$

$$= \tfrac{4}{3}\omega^2\Delta^3(0) - \tfrac{1}{3}\omega^4 \int d^D x\, \Delta^4(x), \qquad (53)$$

A further partial integration, and use of Eqs. (39), (51), and (53), produces the decompositions of the second and third Feynman diagrams (43) and (44):

$$4\int d^D x\, \Delta(x)\, \Delta_\mu(x)\, \Delta_\nu(x)\, \Delta_{\mu\nu}(x)$$

$$= -2\, I_D + 4\omega^2 \int d^D x\, \Delta^2(x)\, \Delta_\mu^2(x), \qquad (54)$$

and

$$\int d^D x\, \Delta_\mu^2(x)\, \Delta_\nu^2(x)$$

$$= I_D - 3\omega^2 \int d^D x\, \Delta^2(x)\, \Delta_\mu^2(x). \qquad (55)$$

We now make the important observation that the subtle integral $I_D$ of Eq. (45) appears in Eqs. (46), (54) and (55) in such a way that it drops out from the sum of the watermelon diagrams in (20):

 $+\,4$  $+$ 

$$= \int d^D x\, \Delta^2(x)\Delta_{\mu\mu}^2(x) + \omega^2 \int d^D x\, \Delta^2(x)\Delta_\mu^2(x). \qquad (56)$$

Using (49) and (50), the right-hand side becomes a sum of completely regular expressions. Moreover, adding to this sum the last two watermelon-like diagrams in Eq. (20):

$4\omega^2$  $= 4\omega^2 \int d^D x\, \Delta^2(x)\Delta_\mu^2(x), \qquad (57)$

and

$\frac{2}{3}\omega^4$  $= \frac{2}{3}\omega^4 \int d^D x\, \Delta^4(x), \qquad (58)$

we obtain for the contribution of all watermelon-like diagrams (20) the simple expression

$$-F_2^{(4)} = -2g^2 \int d^D x\, \Delta^2(x) \Big[\, \Delta_{\mu\mu}^2(x)$$

$$+5\omega^2\, \Delta_\mu^2(x) + \tfrac{2}{3}\omega^4\, \Delta^2(x)\Big]$$

$$= \tfrac{2}{3}\,\omega^2\Delta^3(0) \underset{D\to 1}{=} \frac{1}{12\,\omega}. \qquad (59)$$

This cancels the finite contribution (33), thus making also the second-order free energy in (15) vanish, and confirming the invariance of the perturbatively defined path integral under coordinate transformations up to this order.

## 8. Summary

In this Letter we have set up simple rules for calculating integrals over products of distributions in configuration space which produce the same results as dimensional regularization in momentum space. For a path integral of a quantum-mechanical point particle in a harmonic potential, we have shown that these rules lead to a reparametrization-invariant perturbation expansions of path integral.

Let us end with the remark that in the time-sliced definition of path integrals, reparametrization invariance has been established as long time ago in the textbook [4].

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