Critical exponents from seven-loop strong-coupling ϕ^4 theory in three dimensions

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Using strong-coupling quantum field theory, we calculate highly accurate critical exponents ν,η following from new seven-loop expansions in three dimensions. Our theoretical value for the critical exponent α of the specific heat near the λ point of superfluid helium is $\alpha=-0.01294\pm0.00060$, in excellent agreement with the space shuttle experimental value $\alpha=-0.01285\pm0.00038$. [S0556-2821(99)01312-0]

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The accurate calculation of critical exponents from field theory presents a theoretical challenge, since the relevant information is available only from divergent power series expansions. The results are also of practical relevance, since they predict the outcome of many possible future experiments on many second-order phase transitions. In recent work [1] we have developed a novel method for extracting these exponents from such expansions via a strong-coupling theory of scalar fields with a ϕ^4 interaction. The fields are assumed to have n components with an action which is O(n) symmetric. As an application, we have used available six-loop perturbation expansions of the renormalization constants in three dimensions [2–4] to calculate the critical exponents for all O(n) universality classes with high precision. Strong-coupling theory works also in $4-\epsilon$ dimensions [6], and is capable of interpolating between the expansions in $4-\epsilon$ with those in $2+\epsilon$ dimensions of the nonlinear σ model [7].

The purpose of this note is to improve significantly the accuracy of our earlier results in three dimensions [1] by making use of new seven-loop expansion coefficients for the critical exponents ν and η [8] and, most importantly, by applying a more powerful extrapolation method to infinite order than before. The latter makes our results as accurate as those obtained by Guida and Zinn-Justin [9] via a more sophisticated resummation technique based on analytic mapping and Borel transformations, which in addition takes into account information on the large-order growth of the expansion coefficients. We reach this accuracy without using that information which, as we shall demonstrate at the end in Sec. V, has practically no influence on the results, except for lowering ω slightly (by less than $\sim0.2\%$). The reason for the little importance of the large-order information in our approach is that the critical exponents are obtained from evaluations of expansions at infinite bare couplings. The information on the large-order behavior, on the other hand, specifies the discontinuity at the tip of the left-hand cut which starts at the origin of the complex-coupling constant plane [10]. This is too far from the infinite-coupling limit to be of relevance. In our resummation scheme for expansion in powers of the bare coupling constant, an important role is played by the critical exponent of approach to scaling ω, whose precise calculation by the same scheme is crucial for obtaining high accuracies in all other critical exponents. It is determined by the condition that the renormalized coupling strength g goes against a constant g^* in the strong-coupling limit. The knowledge of ω is more yielding than the large-order information in previous resummation schemes in which the critical exponents are determined as a function of the renormalized coupling constant g near g^* which is of order unity, thus lying a finite distance away from the left-hand cut in the complex g plane. Although these determinations are sensitive to the discontinuity at the top of the cut, it must be realized that the influence of the cut is very small due to the smallness of the fugacity of the leading instanton, which carries a Boltzmann factor $e^{-\text{const}/g}$.

We briefly recall the available expansions [4] of the renormalized coupling $\bar{g}\equiv g/m$ in terms of the bare coupling $\bar{g}_0\equiv g_0/m$ for all O(n),

$$\begin{aligned}\bar{g}/\bar{g}_0={}&1-\bar{g}_0(8+n)+\bar{g}_0^2(2108/27+514n/27+n^2)+\bar{g}_0^3(-878.7937193-312.63444671n-32.54841303n^2-n^3)\\&+\bar{g}_0^4(11068.06183+5100.403285n+786.3665699n^2+48.21386744n^3+n^4)\\&+\bar{g}_0^5(-153102.85023-85611.91996n-17317.702545n^2-1585.1141894n^3-65.82036203n^4-n^5)\\&+\bar{g}_0^6(2297647.148+1495703.313n+371103.0896n^2+44914.04818n^3+2797.291579n^4+85.21310501n^5+n^6),\end{aligned} \tag{1}$$

and of the critical exponents [5],

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TABLE I. Fluctuation determinants and integrals over extremal field solution.

D	D_L	D_T	I_1	I_4	I_6	H_3
3	10.544 ± 0.004	$1.4571 \pm .0001$	31.691522	75.589005	659.868352	13.563312
2	135.3 ± 0.1	1.465 ± 0.001	15.10965	23.40179	71.08023	9.99118

$$\begin{aligned}\omega(\overline{g_0}) = & -1 + 2\overline{g_0}(8+n) - \overline{g_0}^2(1912/9 + 452n/9 + 2n^2) + \overline{g_0}^3(3398.857964 + 1140.946693n + 95.9142896n^2 + 2n^3) \\ & + \overline{g_0}^4(-60977.50127 - 26020.14956n - 3352.610678n^2 - 151.1725764n^3 - 2n^4) \\ & + \overline{g_0}^5(1189133.101 + 607809.998n + 104619.0281n^2 + 7450.143951n^3 + 214.8857494n^4 + 2n^5) \\ & + \overline{g_0}^6(-24790569.76 - 14625241.87n - 3119527.967n^2 - 304229.0255n^3 - 14062.53135n^4 - 286.3003674n^5 - 2n^6),\end{aligned} \tag{2}$$

$$\begin{aligned}\eta(\overline{g}) = & \overline{g_0}^2(16/27 + 8n/27) + \overline{g_0}^3(-9.086537459 - 5.679085912n - 0.5679085912n^2) \\ & + \overline{g_0}^4(127.4916153 + 94.77320534n + 17.1347755n^2 + 0.8105383221n^3) \\ & + \overline{g_0}^5(-1843.49199 - 1576.46676n - 395.2678358n^2 - 36.00660242n^3 - 1.026437849n^4) \\ & + \overline{g_0}^6(28108.60398 + 26995.87962n + 8461.481806n^2 + 1116.246863n^3 + 62.8879068n^4 + 1.218861532n^5),\end{aligned} \tag{3}$$

$$\begin{aligned}\eta_m(\overline{g}) = & \overline{g_0}(2+n) + \overline{g_0}^2(-523/27 - 316n/27 - n^2) + \overline{g_0}^3(229.3744544 + 162.8474234n + 26.08009809n^2 + n^3) \\ & + \overline{g_0}^4(-3090.996037 - 2520.848751n - 572.3282893n^2 - 44.32646141n^3 - n^4) \\ & + \overline{g_0}^5(45970.71839 + 42170.32707n + 12152.70675n^2 + 1408.064008n^3 + 65.97630108n^4 + n^5) \\ & + \overline{g_0}^6(-740843.1985 - 751333.064n - 258945.0037n^2 - 39575.57037n^3 - 2842.8966n^4 - 90.7145582n^5 - n^6),\end{aligned} \tag{4}$$

where $\eta_m \equiv 2 - \nu^{-1}$. To save space we have omitted a factor $1/(n+8)^n$ accompanying each power $\overline{g_0^n}$ on the right-hand sides. The additional seventh-order coefficients have been calculated for $n=0,1,2,3$ and are [8] [these without a factor $1/(n+8)^7$ on the right-hand side]

$$\eta^{(7)} = \begin{Bmatrix} -0.2164239372 \\ -0.2395467913 \\ -0.2414247646 \\ -0.2333645418 \end{Bmatrix} \overline{g_0^7},$$

$$\nu^{-1(7)} = \begin{Bmatrix} -6.0998295658 \\ -7.0482198342 \\ -7.3780809849 \\ -7.3808485089 \end{Bmatrix} \overline{g_0^7} \quad \text{for} \quad \begin{Bmatrix} n=0 \\ n=1 \\ n=2 \\ n=3 \end{Bmatrix}. \tag{5}$$

It is instructive to see how close the new coefficients are to their large-order limiting values derived from instanton calculations, according to which the expansion coefficients with respect to the renormalized coupling $\overline{g}$ should grow for large order k as follows [11]:

$$\omega^{(k)} = \gamma_\omega(-a)^k k! k \Gamma(k+b_\omega)\left(1 + \frac{c_\beta^{(1)}}{k} + \frac{c_\beta^{(2)}}{k^2} + \dots\right), \tag{6}$$

$$\eta^{(k)} = \gamma_\eta(-a)^k k! k \Gamma(k+b_\eta)\left(1 + \frac{c_\eta^{(1)}}{k} + \frac{c_\eta^{(2)}}{k^2} + \dots\right), \tag{7}$$

$$\overline{\eta}^{(k)} = \gamma_{\overline{\eta}}(-a)^k k! k \Gamma(k+b_{\overline{\eta}})\left(1 + \frac{c_{\overline{\eta}}^{(1)}}{k} + \frac{c_{\overline{\eta}}^{(2)}}{k^2} + \dots\right), \tag{8}$$

TABLE II. Growth parameter of $D=3$ perturbation expansions of $\beta(\overline{g})$, $\eta(\overline{g})$, and $\overline{\eta} = \eta + \nu^{-1} - 2$.

	$n=0$	$n=1$	$n=2$	$n=3$
a	0.1662460	0.14777422	0.1329968	0.12090618
b_ω	4	9/2	9	11/2
$b_{\overline{\eta}}$	3	7/2	4	9/2
b_η	2	5/2	3	7/2
$10^2 \times \gamma$	8.5489(16)	3.9962(6)	1.6302(3)	0.59609(10)
$10^3 \times \gamma_{\overline{\eta}}$	10.107	6.2991	3.0836	1.2813
$10^3 \times \gamma_\eta$	2.8836	1.7972	0.8798	0.3656

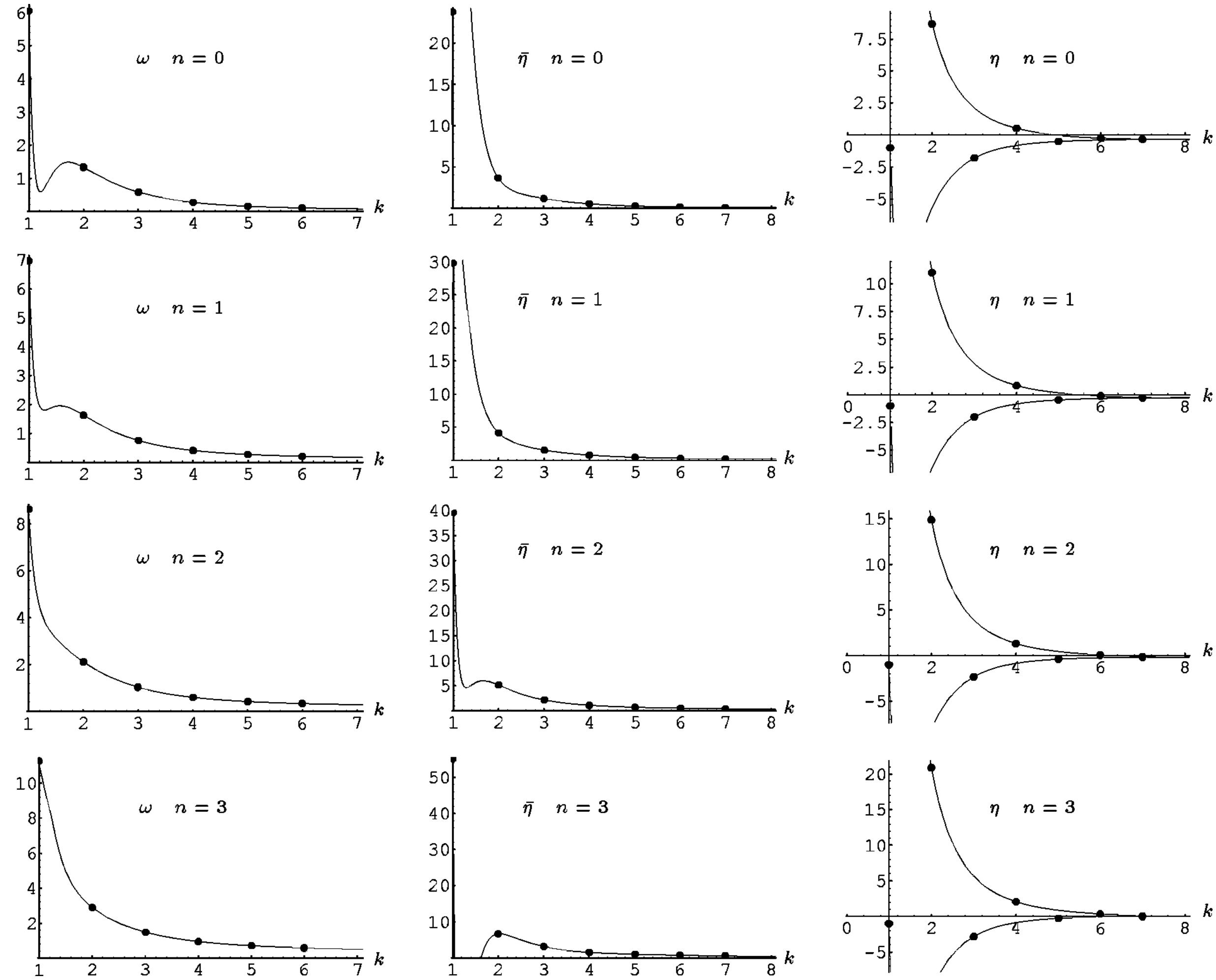

FIG. 1. Precocity of large-order behavior of coefficients of the expansions of the critical exponents ω, $\overline{\eta} \equiv \nu^{-1} + \eta - 2$, and η in powers of the renormalized coupling constant. The dots show the relative deviations exact/asymptotic-1. The curves are plots of the asymptotic expressions in Eqs. (6)–(8) listed in Table III. The curve for ω is the smoothest, promising the best extrapolation to the next orders, with consequences to be discussed in Sec. V.

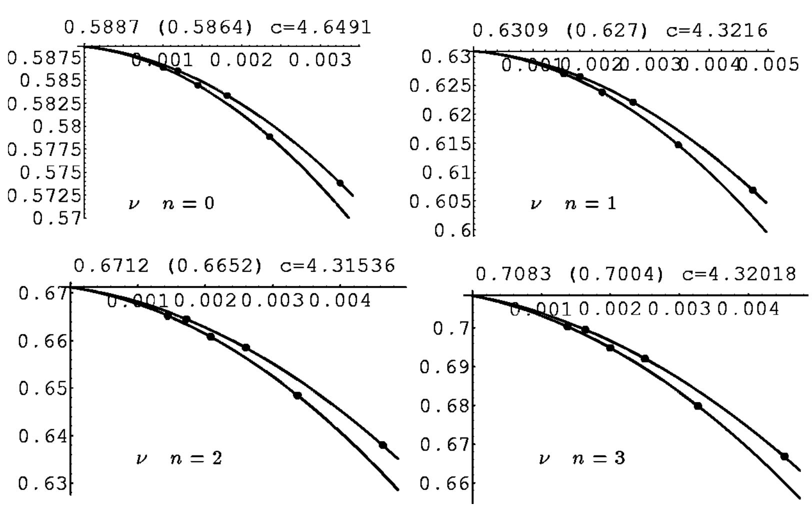

FIG. 2. Strong-coupling values for the critical exponent ν^{-1} obtained from expansion (4) via formula (14), for increasing orders $N=2,3,\ldots,7$ of the approximation. The exponents are plotted against the variable $x_N = e^{-cN^{1-\omega}}$ and should for large n lie on a straight line. Here at finite N, even and odd approximants may be connected by slightly curved parabolas whose common intersection determines the critical exponents for $N=\infty$. More details on the determination of the constant c are given in the text. The numbers on top give the extrapolated critical exponents and, in parentheses, the highest approximants, to illustrate the extrapolation distance.

TABLE III. Coefficients of the large-order expansions (6)–(8), to fit the known expansion coefficients of ω, η, $\overline{\eta}$. The coefficients $\eta^{(k)}$ possess two shorter expansions for even and odd k.

	n	$c^{(1)}$	$c^{(2)}$	$c^{(3)}$	$c^{(4)}$	$c^{(5)}$	$c^{(6)}$
ω	0	0.2630147511231	3.440818282282	−31.7673335904347	209.9430468590877	−387.982076950413	212.156573953838
	1	1.6353509905175	−8.762940856111	32.5298724631003	49.5693979855620	−198.550118637547	130.539359765928
	2	4.1903240993241	−32.521882201016	159.2316083453243	−271.5678237829086	185.521462986276	−36.220362093550
	3	8.0659054235535	−69.138003762384	356.1987017927173	−773.4084307341978	787.410568298674	−297.863117806916
$\overline{\eta}$	0	15.4745287323349	−263.105249597920	1695.85217994178	−4797.25478881458	6198.21126891018	−2825.37877442787
	1	10.9470420638543	−169.697930580512	1074.82692242305	−2886.57808941584	3577.48655305529	−1577.19837665961
	2	1.2481454871524	60.932456514040	−409.59535356475	1526.62040773429	−2300.49464074955	1163.42553732492
	3	−25.8032867124555	508.523659337565	−3253.93912011988	9876.17157690861	−13307.48621904672	6257.59065449436
$\eta_{\rm o}$	0	−6.3634296712273	54.796985733992	−209.212694395258	159.7791383324933		
	1	−5.8608156341154	58.173292227872	−237.158174423958	183.8456978302008		
	2	−5.1086981057007	64.465105150609	−285.116154230741	224.7597471858332		
	3	−4.2039863427233	76.269147128915	−364.452995945739	291.3878351595474		
$\eta_{\rm e}$	0	−5.6929922203758	15.551243915764	61.12469347544379			
	1	−5.3245881267711	14.110708087849	81.2312043328075			
	2	−4.5203425601138	9.799960635959	117.4131477198922			
	3	−3.1970976073075	1.705210978430	176.4615812743069			

where $\overline{\eta}\equiv\eta+\nu^{-1}-2$. The growth parameter a is proportional to the inverse Euclidean action of the classical *instanton* solution $\varphi_c(\mathbf{x})$ to the field equations

$$a=(D-1)\frac{16\pi}{I_4}\frac{1}{N+8}=0.14777423\frac{9}{N+8}. \tag{9}$$

The quantity I_4 denotes the integral $I_4=\int d^Dx[\varphi_c(\mathbf{x})]^4$. Its numerical values in two and three dimensions D are listed in Table I. The growth parameters $b_\omega, b\_\eta, b_{\overline{\eta}}$ are directly related to the number $D+n$ of zero-modes in the fluctuation determinant around the instanton (associated with D translations, $n-1$ rotations, and one dilation). Their values are

$$b_\omega=b_\beta+1=\tfrac{1}{2}(D+5+n),$$

$$b_\eta=\tfrac{1}{2}(D+1+n),\quad b_{\overline{\eta}}=\tfrac{1}{2}(D+3+n). \tag{10}$$

The prefactors $\gamma_\beta,\gamma_\eta,\gamma_{\overline{\eta}}$ in Eqs. (6)–(8) require the calculation of the full fluctuation determinants. This yields

$$\gamma_\beta\equiv\frac{(n+8)2^{(n+D-5)/2}3^{-3(D-2)/2}}{\pi^{3+D/2}\Gamma[2+(1/2)n]}\left(\frac{I_1^2}{I_4}\right)^2$$
$$\times\left(\frac{I_6}{I_4}-1\right)^{D/2}D_L^{-1/2}D_T^{-(n-1)/2}e^{-1/a}. \tag{11}$$

The constants I_1, I_2, I_6 are are generalizations of the above integral I_4: $I_p=\int d^Dx[\varphi_c(\mathbf{x})]^p$, and D_L and D_T are found from the longitudinal and transverse parts of the fluctuation determinants. Their numerical values are given in Table I. The constant γ_β is the prefactor of growth in the expansion coefficients of the β function [the integral over $\omega(\overline{g})$]:

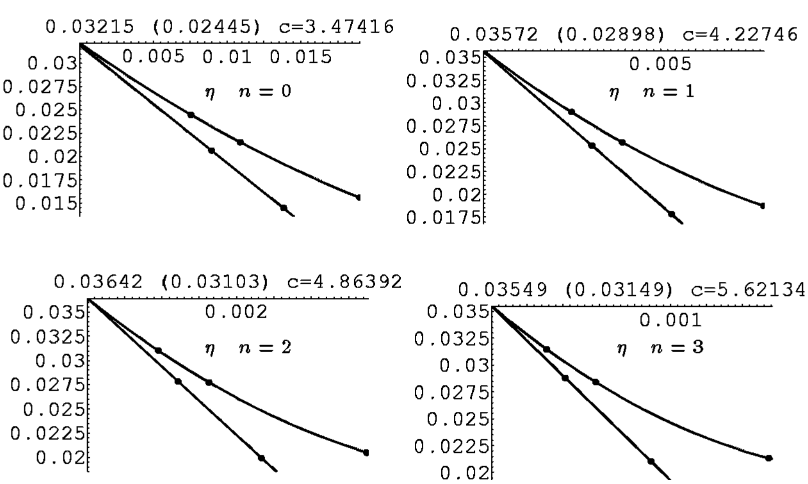

FIG. 3. Strong-coupling values for the critical exponent η obtained from the expansion (3) via formula (14) for increasing orders $N=3,\ldots,7$ of the approximation. The exponents are plotted against $x_N=e^{-cN^{1-\omega}}$. Even approximants are connected by straight line and odd approximats by slightly curved parabolas, whose common intersection determines the critical exponents expected for $N=\infty$.

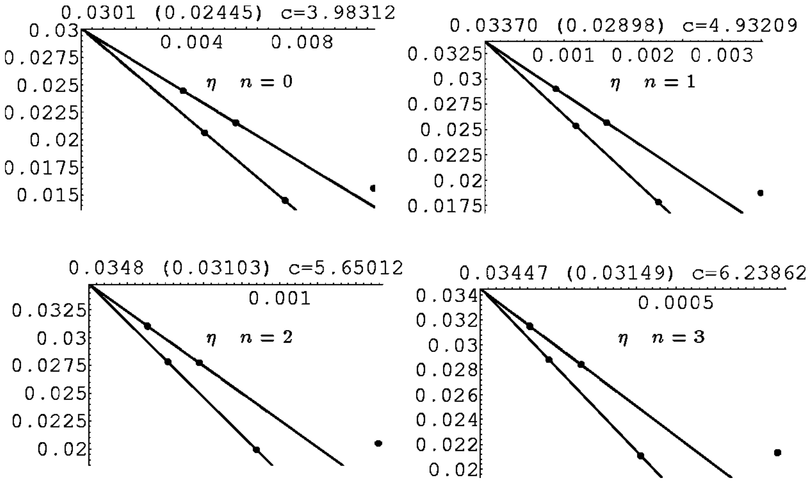

FIG. 4. Plot analogous to Fig. 3, but the extrapolation is found from the intersection of the straight lines connecting the last two even and odd approximants. The resulting critical exponents differ only little from those obtained in Fig. 3, the differences given an estimate for the systematic error of our results.

$\beta^{(k)} \approx \gamma_\beta(-a)^k k!\Gamma(k+b_\beta)$. The prefactors in γ_ω, γ_η, and $\gamma_{\bar\eta}$ in Eqs. (6)–(8) are related to γ_β by

$$\gamma_\omega = -a\gamma_\beta, \quad \gamma_\eta = \gamma_{\bar\eta}\frac{2H_3}{I_1 D(4-D)},$$

$$\gamma_{\bar\eta} = \gamma_\beta \frac{n+2}{n+8}(D-1)4\pi\frac{I_2}{I_1^2}, \tag{12}$$

where $I_2 = (1-D/4)I_4$ and H_3 are listed in Table I. The numerical values of all growth parameters for are listed in Table II. In Fig. 1 we show a comparison between the exact coefficients and their asymptotic forms (8).

The critical exponents are derived from the divergent expansions (1)–(5) by going to the limit $\bar g_0 \to \infty$. In a theory with scaling behavior, the renormalized coupling constant $\bar g$ tends to a limiting value $\bar g^*$ as follows:

$$\bar g(\bar g_0) = \bar g^* - \frac{\text{const}}{\bar g_0^{\omega/\epsilon}} + \cdots, \tag{13}$$

where g^* is commonly referred to as the infrared-stable fixed point, and ω is called the critical exponent of the approach to scaling. The same exponent governs the approach to scaling of every function $G(\bar g)$ which behaves like $G(\bar g) = G(\bar g^*) + G'(\bar g^*) \times \text{const}/\bar g_0^{\omega} + \cdots$.

How do we recover the $\bar g_0 \to \infty$ limits of a function $f(\bar g_0)$ if we know the first N terms of its asymptotic expansion $f_N(\bar g_0) = \Sigma_{n=0}^N a_n \bar g_0^n$? Extending systematically the behavior (13) we shall assume that $f(\bar g_0)$ approaches its constant limiting value f^* in the form of an inverse power series [12] $f_M(\bar g_0) = \Sigma_{m=0}^M b_m (\bar g_0^{-\omega})^m$. This strong-coupling expansion has usually a finite convergence radius g_s (see [1,10,13]). The Nth approximation to the value f^* is obtained from the formula

$$f_N^* = \text{opt}_{\hat g_0}\left[\sum_{j=0}^{N} a_j \hat g_0^j \sum_{k=0}^{N-j} \binom{-j/\omega}{k} (-1)^k\right], \tag{14}$$

where the expression in brackets has to be optimized in the variational parameter $\hat g_0$. The optimum is the smoothest among all real extrema. If there are no such extrema, which happens for the even approximants, the turning points serve the same purpose.

From the theory [1], we expect the exact values to be approached exponentially fast with the order N of the available expansions, with the error decreasing like $e^{-cN^{1-\omega}}$. In

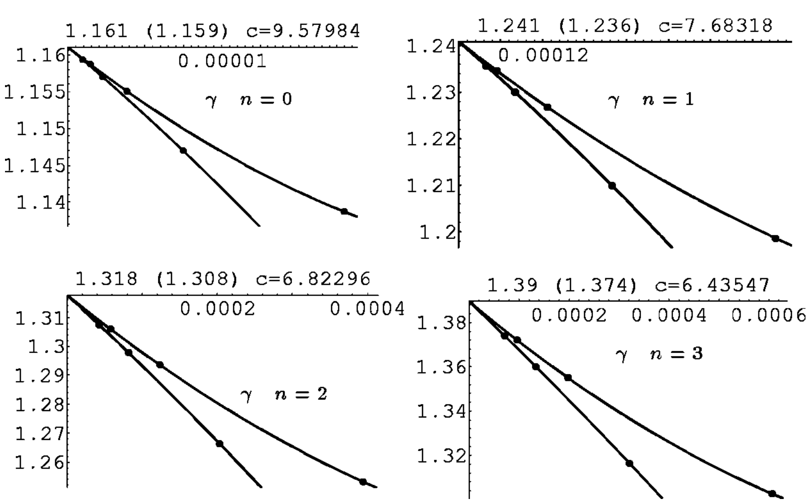

FIG. 5. Strong-coupling values for the critical exponent $\gamma = \nu(2-\eta) = (2-\eta)/(2-\eta_m)$ obtained from a combination of the expansions (3) and (4) via formula (14) for increasing orders $N = 2,3,\ldots,7$ of the approximation. The exponents are plotted against the variable $x_N = e^{-cN^{1-\omega}}$ and should lie on a straight line in the limit of large N. Even and odd approximants are connected by slightly curved parabolas whose common intersection with the vertical axis determines the critical exponents expected for $N = \infty$. The determination of the constant c is described in the text.

TABLE IV. Our seven-loop critical exponents (superscript s), compared with results obtained by other techniques. The superscripts f and g refer to other seven-loop expansions in $D=3$ dimensions (f ∈ [9], g ∈ [8]), the other superscripts a–e refer to six-loop results of a Padé-Borel resummation (a ∈ [4], b ∈ [3], c ∈ [15]), and to five-loop expansions in $\epsilon=4-D$ (d∈ [16], e∈ [17]). For each of our results we give the highest approximation before the extrapolated one in parentheses. Only the first three rows and the ω values with superscript a in the entries for $n=0,1,2,3$ are new with respect to the table in Ref. [1].

n	g_c	$\gamma(\gamma_{6,7})$	$\eta(\eta_{6,7})$	$\nu(\nu_{6,7})$	α	β	$\omega\ (\omega_6)$
0		1.161(1.159) [s]	0.0311±0.001 [s]	0.5883(0.5864) [s]			0.810(0.773)
	1.413±0.006 [f]	1.160±0.002 [f]	0.0284±0.0025 [f]	0.5882±0.0011 [f]	0.235±0.003 [f]	0.3025±0.0008 [f]	0.812±0.016 [f]
	1.39 [g]	1.1569±0.0004 [g]	0.0297±0.0009 [g]	0.5872±0.0004 [g]			
	1.402 [a]	1.160 [a]	0.034 [a]	0.589 [a]	0.231 [a]	0.305 [a]	
	1.421±0.004 [b]	1.161±0.003 [b]	0.026±0.026 [b]	0.588±0.001 [b]	0.236±0.004 [b]	0.302±0.004 [b]	0.794±0.06 [b]
	1.421±0.008 [c]	1.1615±0.002 [c]	0.027±0.004 [c]	0.5880±0.0015 [c]		0.3020±0.0015 [c]	0.80±0.04 [c]
		1.160±0.004 [e]	0.031±0.003 [e]	0.5885±0.0025 [e]		0.3025±0.0025 [e]	0.82±0.04 [e]
1		1.241(1.236) [s]	0.0347±0.001 [s]	0.6305(0.6270) [s]			0.805(0.772)
	1.411±0.004 [f]	1.240±0.001 [f]	0.0335±0.0025 [f]	0.6304±0.0013 [f]	0.109±0.004 [f]	0.3258±0.0014 [f]	0.799±0.011 [f]
	1.40 [g]	1.2378±0.0006 [g]	0.0355±0.0009 [g]	0.6301±0.0005 [g]			
	1.419 [a]	1.239 [a]	0.038 [a]	0.631 [a]	0.107 [a]	0.327 [a]	0.781 [a]
	1.416±0.0015 [b]	1.241±0.004 [b]	0.031±0.011 [b]	0.630±0.002 [b]	0.110±0.008 [b]	0.324±0.06 [b]	0.788±0.003 [b]
	1.416±0.004 [c]	1.2410±0.0020 [c]	0.031±0.004 [c]	0.6300±0.0015 [c]		0.3250±0.0015 [c]	0.79±0.03 [c]
			0.035±0.002 [d]	0.628±0.001 [d]			0.80±0.02 [d]
		1.1239±0.004 [e]	0.037±0.003 [e]	0.6305±0.0025 [e]		0.3265±0.0025 [e]	0.81±0.04 [e]
2		1.318(1.306) [s]	0.0356±0.001 [s]	0.6710(0.6652) [s]			0.800(0.772)
	1.403±0.003 [f]	1.317±0.002 [f]	0.0354±0.0025 [f]	0.6703±0.0013 [f]	−0.011±0.004 [f]	0.3470±0.0014 [f]	0.789±0.011 [f]
	1.40 [g]	1.3178±0.001 [g]	0.0377±0.0006 [g]	0.6715±0.0007 [g]			
	1.408 [a]	1.315 [a]	0.039 [a]	0.670 [a]	−0.010 [a]	0.348 [a]	0.780 [a]
	1.406±0.005 [b]	1.316±0.009 [b]	0.032±0.015 [b]	0.669±0.003 [b]	−0.007±0.009 [b]	0.346±0.009 [b]	0.78±0.01 [b]
	1.406±0.004 [c]	1.3160±0.0025 [c]	0.033±0.004 [c]	0.6690±0.0020 [c]		0.3455±0.002 [c]	0.78±0.025 [c]
			0.037±0.002 [d]	0.665±0.001 [d]			0.79±0.02 [d]
		1.315±0.007 [e]	0.040±0.003 [e]	0.671±0.005 [e]		0.3485±0.0035 [e]	0.80±0.04 [e]
3		1.390(1.374) [s]	0.0350±0.0005 [s]	0.7075(0.7004) [s]			0.797(0.776)
	1.391±0.004 [f]	1.390±0.005 [f]	0.0355±0.0025 [f]	0.7073±0.0030 [f]	−0.122±0.009 [f]	0.3662±0.0025 [f]	0.782±0.0013 [f]
	1.39 [g]	1.3926±0.001 [g]	0.0374±0.0004 [g]	0.7096±0.0008 [g]			
	1.392 [a]	1.386 [a]	0.038 [a]	0.706 [a]	−0.117 [a]	0.366 [a]	0.780 [a]
	1.392±0.009 [b]	1.390±0.01 [b]	0.031±0.022 [b]	0.705±0.005 [b]	−0.115±0.015 [b]	0.362 [b]	0.78±0.02 [b]
	1.391±0.004 [c]	1.386±0.004 [c]	0.033±0.004 [c]	0.705±0.003 [c]		0.3645±0.0025 [c]	0.78±0.02 [c]
			0.037±0.002 [d]	0.79±0.02 [d]			0.79±0.02 [d]
		1.390±0.010 [e]	0.040±0.003 [e]	0.710±0.007 [e]		0.368±0.004 [e]	0.79±0.04 [e]
4		1.451(1.433)	0.031(0.0289)	0.737(0.732)			0.795(0.780)
	1.375 [a]	1.449 [a]	0.036 [a]	0.738 [a]	−0.213 [a]	0.382 [a]	0.783 [a]
5		1.511(1.487)	0.0295(0.0283)	0.767(0.760)			0.795(0.785)
	1.357 [a]	1.506 [a]	0.034 [a]	0.766 [a]	−0.297 [a]	0.396 [a]	0.788 [a]
6		1.558(1.535)	0.0276(0.0273)	0.790(0.785)			0.797(0.792)
	1.339 [a]	1.556 [a]	0.031 [a]	0.790 [a]	−0.370 [a]	0.407 [a]	0.793 [a]
7		1.599(1.577)	0.0262(0.0260)	0.810(0.807)			0.802(0.800)
	1.321 [a]	1.599 [a]	0.029 [a]	0.811 [a]	−0.434 [a]	0.417 [a]	0.800 [a]
8		1.638(1.612)	0.0247(0.0246)	0.829(0.825)			0.810(0.808) 0.848
	1.305 [a]	1.637 [a]	0.027 [a]	0.830 [a]	−0.489 [a]	0.426 [a]	0.808 [a]

TABLE IV. (*Continued*).

n	g_c	$\gamma(\gamma_6)$	$\eta(\eta_6)$	$\nu(\nu_6)$	α	β	ω (ω_6)
9		1.680(1.643)	0.0233(0.0233)	0.850(0.841)			0.817(0.815) 0.854
	1.289 [a]	1.669 [a]	0.025 [a]	0.845 [a]	−0.536 [a]	0.433 [a]	0.815 [a]
10		1.713(1.670)	0.0216(0.0220)	0.866(0.854)			0.824(0.822)0.860
	1.275 [a]	1.697 [a]	0.024 [a]	0.859 [a]	−0.576 [a]	0.440 [a]	0.822 [a]
12		1.763(1.716)	0.0190(0.0198)	0.890(0.877)			0.838(0.835)
	1.249 [a]	1.743 [a]	0.021 [a]	0.881 [a]	−0.643 [a]	0.450 [a]	0.836 [a]
14		1.795(1.750)	0.0169(0.0178)	0.905(0.894)			0.851(0.849)
	1.227 [a]	1.779 [a]	0.019 [a]	0.898 [a]	−0.693 [a]	0.457 [a]	0.849 [a]
16		1.822(1.779)	0.0152(0.0161)	0.918(0.907)			0.862(0.860)
	1.208 [a]	1.807 [a]	0.017 [a]	0.911 [a]	−0.732 [a]	0.463 [a]	0.861 [a]
18		1.845(1.803)	0.0148(0.0137)	0.929(0.918)			0.873(0.869)
	1.191 [a]	1.829 [a]	0.015 [a]	0.921 [a]	−0.764 [a]	0.468 [a]	0.871 [a]
20		1.864(1.822)	0.0125(0.0135)	0.938(0.927)			0.883(0.878)
	1.177 [a]	1.847 [a]	0.014 [a]	0.930 [a]	−0.789 [a]	0.471 [a]	0.880 [a]
24		1.890(1.850)	0.0106(0.0116)	0.950(0.939)			0.900(0.894)
	1.154 [a]	1.874 [a]	0.012 [a]	0.942 [a]	−0.827 [a]	0.477 [a]	0.896 [a]
28		1.909(1.871)	0.009232(0.01010)	0.959(0.949)			0.913(0.906)
	1.136 [a]	1.893 [a]	0.010 [a]	0.951 [a]	−0.854 [a]	0.481 [a]	0.909 [a]
32		1.920(1.887)	0.00814(0.00895)	0.964(0.955)			0.924(0.915)
	1.122 [a]	1.908 [a]	0.009 [a]	0.958 [a]	−0.875 [a]	0.483 [a]	0.919 [a]

[a]Six-loop results of a Pade-Borel resummation, ∈ [4].
[b]Six-loop results of a Pade-Borel resummation, ∈ [3].
[c]Six-loop results of a Pade-Borel resummation, ∈ [15].
[d]Five-loop expansions in $\epsilon=4-D$, ∈ [16].
[e]Five-loop expansions in $\epsilon=4-D$, ∈ [17].
[f]Seven-loop expansions in $D=3$ dimensions, ∈ [9].
[g]Seven-loop expansions in $D=3$ dimensions, ∈ [8].
[h]Our seven-loop critical exponents.

order to extrapolate our results to $N=\infty$, we plot the data against the variables $x_N=e^{-cN^{1-\omega}}$ (see also Addendum to Ref. [1]). This is done separately for even and odd approximants, since the former stem from extrema, the latter from turning points. The unknown constants c c are determined by fitting to each set of points a slightly curved parabola and making them intersect the vertical axis at the same point, which yields the extrapolated critical exponent listed on top of each figure (together with the seventh-order value in parentheses, and the optimal parameter c).

Following this procedure, we find from the expansion (4) for ν^{-1} the approximants ν_N^{-1} via formula (14). Extrapolating separately even and odd approximants ν_N, we determine the limiting value ν, as shown in Fig. 2. The ω values used for this extrapolation are those of Ref. [6], listed in the last column of Table IV:

$$\omega_6=\begin{Bmatrix}0.810\\0.805\\0.797\\0.790\end{Bmatrix}\quad\text{for}\quad\begin{Bmatrix}n=0\\n=1\\n=2\\n=3\end{Bmatrix}. \qquad (15)$$

They lead to the ν values $\nu_7=\{0.5883,\ 0.6305,\ 0.6710,\ 0.7075\}$, the entries in this vector referring to $n=0,1,2,3$. Since these results depend on the critical exponents ω, it is useful to study the dependence of the extrapolation on ω, with the result

$$\nu_7=\begin{Bmatrix}0.5883+0.0417\times(\omega-0.810)\\0.6305+0.0400\times(\omega-0.805)\\0.6710+0.0553\times(\omega-0.800)\\0.7075+0.1891\times(\omega-0.797)\end{Bmatrix}\quad\text{for}\quad\begin{Bmatrix}N=0\\N=1\\N=2\\N=3\end{Bmatrix}. \qquad (16)$$

For the critical exponent η, we cannot use the same extrapolation procedure, since the expansion (3) starts out with $\bar{g}_0^2$, so that there exists only an odd number of approximants η_N. We therefore use two alternative extrapolation procedures. In the first we connect the even approximants η_2 and η_4 by a straight line and the odd ones η_3,η_5,η_7 by a

TABLE V. Coefficients of extended perturbation expansions obtained from the large-order expansions (6)–(8) for ω, η_m, η up to g^{12}.

	k	$n=0$	$n=1$	$n=2$	$n=3$
$\omega^{(k)}$	0	-1	-1	-1	-1
	1	2	2	2	2
	2	$-95/72$	$-308/243$	$-272/225$	$-1252/1089$
	3	1.559690758	1.404278391	1.259667768	1.131786725
	4	-2.236580484	-1.882634142	-1.589642400	-1.351666500
	5	3.803133000	2.973285060	2.346615000	1.875335400
	6	-7.244496000	-5.247823000	-3.867143000	-2.904027000
	7	15.0706772	10.0938530	6.9384728	4.8954471
	8	-33.8354460	-20.9045761	-13.3833570	-8.8630280
	9	81.4263429	46.2983010	27.5543342	17.1018561
	10	-209.0371337	-109.1428445	-60.2679848	-34.9985085
	11	570.2558985	272.8574773	139.5403648	75.6925030
	12	-1647.63898	-721.159283	-340.986931	-172.506443
	13	5027.12671	2009.473994	877.142753	413.269514
	14	-16154.2792	-5888.53514	-2369.63316	-1038.433113
	15	54539.7867	18105.83253	6708.76515	2731.28823
	16	-193034.402	-58292.0930	-19865.5739	-7505.78230
	17	714771.195	196130.5369	61414.0151	21513.8526
	18	2.7637289×10^{6}	-688418.829	-197883.530	-64215.5872
	19	1.1139530×10^{7}	2.5166119×10^{6}	663509.086	199303.824
	20	-4.6728706×10^{7}	-9.5668866×10^{6}	-2.3117713×10^{6}	-642301.398
	21	2.0370346×10^{8}	3.7765630×10^{7}	8.3581769×10^{6}	2.1465643×10^{6}
	22	-9.2152712×10^{8}	-1.5460268×10^{8}	-3.1317941×10^{7}	-7.4301887×10^{6}
	23	4.3206669×10^{9}	6.5552885×10^{8}	1.2147059×10^{8}	2.6607749×10^{7}
	24	-2.0970132×10^{10}	-2.8755155×10^{9}	-4.8714560×10^{8}	-9.8469119×10^{7}
	25	1.0523676×10^{11}	$1.3035111\times10^{1}0$	2.0178960×10^{9}	3.7621336×10^{8}
$\overline{\eta}^{(k)}$	1	$-1/4$	$-1/3$	$-2/5$	$-5/11$
	2	1/16	2/27	2/25	10/121
	3	-0.0357672729	-0.0443102531	-0.0495134446	-0.0525519564
	4	0.0343748465	0.0395195688	0.0407881055	0.0399640005
	5	-0.0408958349	-0.0444003474	-0.0437619509	-0.0413219917
	6	0.0597050472	0.0603634414	0.0555575703	0.0490929344
	7	-0.09928487	-0.09324948	-0.08041336	-0.06708630
	8	0.18143353	0.15857090	0.12955711	0.10413882
	9	-0.35946458	-0.29269274	-0.22839265	-0.17925852
	10	0.76759881	0.58218392	0.43525523	0.33488318
	11	-1.75999735	-1.24181846	-0.88911482	-0.66904757
	12	4.31887516	2.82935836	1.93487570	1.41644564
	13	-11.3068155	-6.86145603	-4.46485563	-3.15991301
	14	31.4831400	17.65348358	10.8846651	7.40110473
	15	-92.9568675	-48.04185493	-27.9476939	-18.1528875
	16	290.205144	137.9015950	75.3808299	46.5326521
	17	-955.369710	-416.4425396	-213.088140	-124.454143
	18	3308.08653	1319.8954890	630.008039	346.784997
	19	-12019.6749	-4380.9238169	-1944.51060	-1005.36571
	20	45726.095	15196.764595	6254.75115	3028.67211
	21	-181763.39	-54989.750148	-20934.4636	-9469.48945
	22	753530.79	207207.59430	72800.2529	30694.0685
	23	-3.2522981×10^{6}	-811759.779	-262684.705	-103030.713
	24	1.4590604×10^{7}	3.3014377×10^{6}	982242.312	357779.77
	25	-6.7936016×10^{7}	-1.3919848×10^{7}	-3.8016399×10^{6}	-1.2840285×10^{6}

TABLE V. (*Continued*).

	k	$n=0$	$n=1$	$n=2$	$n=3$
$\eta^{(k)}$	1	0	0	0	0
	2	1/108	8/729	8/675	40/3267
	3	0.0007713749	0.0009142223	0.0009873600	0.0010200000
	4	0.0015898706	0.0017962229	0.0018368107	0.0017919257
	5	−0.0006606149	−0.0006536980	−0.0005863264	−0.0005040977
	6	0.0014103421	0.0013878101	0.0012513930	0.0010883237
	7	−0.001901867	−0.0016976941	−0.001395129	−0.001111499
	8	0.003178395	0.0026439888	0.002043629	0.001544149
	9	−0.006456700	−0.0049783320	−0.003585593	−0.002532983
	10	0.012015200	0.0084255120	0.005570210	0.003647578
	11	−0.029656348	−0.0194143738	−0.012066168	−0.007451622
	12	0.064239639	0.0378738590	0.021403479	0.012148673
	13	−0.180415293	−0.0992734993	−0.0527914785	−0.0282931664
	14	0.4519047994	0.22304200134	0.1074844332	0.0528085190
	15	−1.4092869972	−0.6472476781	−0.2928360472	−0.135567321
	16	4.0214900375	1.65386975	0.6774143887	0.287414739
	17	−13.758814405	−5.24609037	−2.01071514	−0.801301742
	18	44.090284529	15.0426293	5.21919799	1.907241838
	19	−164.205876	−51.7544723	−16.7458885	−5.728643910
	20	583.728411	164.571258	48.2146655	15.13540671
	21	−2352.30706	−610.647520	−166.308263	−48.72074256
	22	9182.66367	2131.95908	525.890013	141.4691142
	23	−39836.8326	−8491.50902	−1941.60261	−486.0716246
	24	169338.243	32279.4193	6686.67654	1538.009823
	25	−787352.117	−137442.343	−26325.2747	−5621.263980

slightly curved parabola, and vary c until there is an intersection at $x=0$. This yields the critical exponents η shown in Fig. 3.

Allowing for the inaccurate knowledge of ω, the results may be stated as

$$\eta_7=\begin{Bmatrix} 0.03215+0.1327\times(\omega-0.810)\\ 0.03572+0.0864\times(\omega-0.805)\\ 0.03642+0.0655\times(\omega-0.800)\\ 0.03549+0.0320\times(\omega-0.797)\end{Bmatrix}\quad \text{for}\quad \begin{Bmatrix} n=0\\ n=1\\ n=2\\ n=3\end{Bmatrix}. \tag{17}$$

Alternatively, we connect the last odd approximants η_5 and η_7 also by a straight line and choose c to make the lines intersect at $x=0$. This yields the exponents

$$\eta_7=\begin{Bmatrix} 0.03010+0.08760\times(\omega-0.810)\\ 0.03370+0.03816\times(\omega-0.805)\\ 0.03480+0.01560\times(\omega-0.800)\\ 0.03447+0.00588\times(\omega-0.797)\end{Bmatrix}\quad \text{for}\quad \begin{Bmatrix} n=0\\ n=1\\ n=2\\ n=3\end{Bmatrix}, \tag{18}$$

as shown in Fig. 4, the ω dependences being somewhat weaker than in Eq. (17).

Combining the two results and using the difference to estimate the systematic error of the extrapolation procedure, we obtain for η the values

$$\eta_7=\begin{Bmatrix} 0.0311\pm 0.001\\ 0.0347\pm 0.001\\ 0.0356\pm 0.001\\ 0.0350\pm 0.001\end{Bmatrix}\quad \text{for}\quad \begin{Bmatrix} n=0\\ n=1\\ n=2\\ n=3\end{Bmatrix}, \tag{19}$$

whose ω dependence is the average of that in Eqs. (17) and (18).

For our extrapolation procedure, the power series for the critical exponent $\gamma=\nu(2-\eta)$ are actually better suited than those for η, since they possess three even and three odd approximants, just as ν^{-1}. Advantages of this expansion have been observed before [14].

The associated plots are shown in Fig. 5. The extrapolated exponents are, including the ω dependence,

$$\gamma_7=\begin{Bmatrix} 1.161-0.049\times(\omega-0.810)\\ 1.241-0.063\times(\omega-0.805)\\ 1.318-0.044\times(\omega-0.800)\\ 1.390-0.120\times(\omega-0.797)\end{Bmatrix}\quad \text{for}\quad \begin{Bmatrix} n=0\\ n=1\\ n=2\\ n=3\end{Bmatrix}. \tag{20}$$

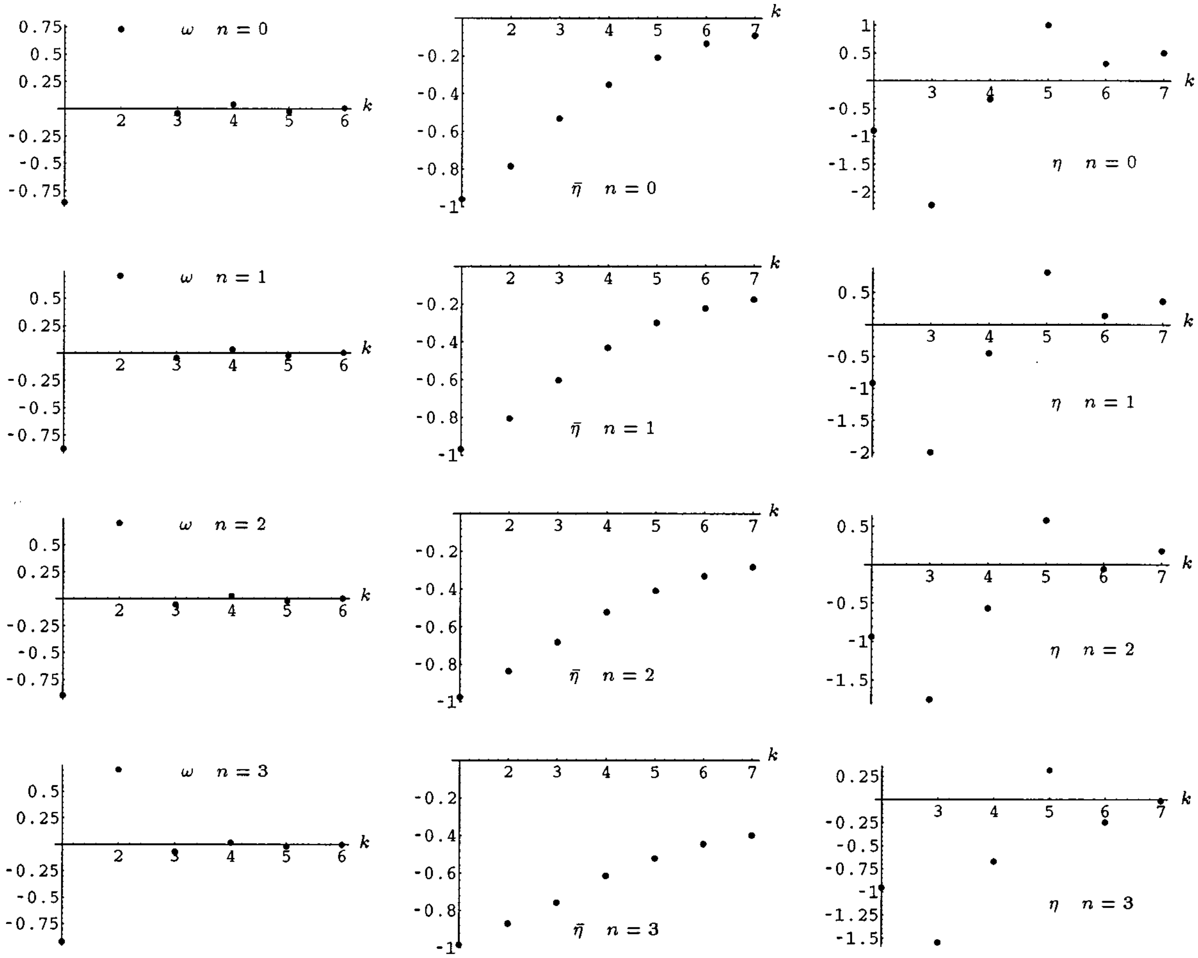

FIG. 6. Relative errors in predicting the kth expansion coefficient by fitting the strong-coupling expansions (6)–(8) for ω, $\bar{\eta}\equiv\nu^{-1}+\eta-2$, and η to the first $k-1$ expansion coefficients.

Unfortunately, the exponent $\gamma=\nu(2-\eta)$ is not very insensitive to η, since this is small compared to 2, so that the extrapolation results (17) are more reliable than those obtained from γ via the scaling relation $\eta=2-\gamma/\nu$. By combining Eqs. (16) and (17), we find from $\gamma=\nu(2-\eta)$:

$$\gamma_7=\begin{Bmatrix}1.1589\\1.2403\\1.3187\\1.3932\end{Bmatrix}\quad\text{for}\quad\begin{Bmatrix}n=0\\n=1\\n=2\\n=3\end{Bmatrix},\tag{21}$$

the difference with respect to Eq. (20) showing the typical small errors of our approximation, which are of the same order as those of the exponents obtained in Ref. [9]. As mentioned in the beginning, the knowledge of the large-order behavior does not help to improve significantly the accuracy of the approximation. In our theory, the most important exploited information is the knowledge of the exponentially fast convergence which leads to a linear behavior of the resummation results of order N in a plot against $x_N=e^{-cN^{1-\omega}}$. This knowledge, which allows us to extrapolate our approximations for $N=2,3,4,5,6,7$ quite well to infinite order N, seems to be more powerful than the knowledge of the large-order behavior exploited by other authors (quoted in Table IV).

The complete updated list of exponents is shown in Table IV, which also contains values for the other critical exponents $\alpha=2-D\nu$ and $\beta=\nu(D-2+\eta)/2$.

Let us now show that the large-order information is indeed rather irrelevant to the critical exponents within strong-coupling theory. For this purpose we choose the coefficients $c^{(i)}$ in the asymptotic formulas (6)–(8) to fit exactly the six known expansion coefficients of $\omega(\bar{g})$ and the seven of $\bar{\eta}(\bar{g})$ and $\eta(\bar{g})$. The coefficients are listed in Table III, and the associated fits are shown in Fig. 1. Since even and odd coefficients $\eta^{(k)}$ lie on two separate smooth curves, we fit the two sets separately. These fits permit us to extend the presently available coefficients and predict the results of future higher-loop calculations, listed in Table V up to order 25. The errors in these predictions are expected to be smallest for $\omega^{(k)}$, as illustrated in Fig. 6.

At this place we observe an interesting phenomenon: According to Table V, the expansion coefficients $\omega^{(k)}$ of $\omega(\bar{g})$ have alternating signs and grow rapidly, reaching precociously their asymptotic form (6), as we have seen in Fig. 1.

TABLE VI. Coefficients of $\bar{g}_0(\bar{g})$ obtained from extended perturbation expansions obtained from the large-order expansions (6)–(8) for $\omega(\bar{g})$ up to g^{25}.

	k	$n=0$	$n=1$	$n=2$	$n=3$
$\bar{g}_0^{(k)}$	1	1	1	1	1
	2	+1	+1	+1	+1
	3	+337/432	+575/729	+539/675	+2641/3267
	4	+0.61685694588	+0.62411053351	+0.63484885720	+0.64721832545
	5	+0.44266705709	+0.45557995443	+0.47149516705	+0.48876206059
	6	+0.35597494073	+0.35927512536	+0.36876801981	+0.38195333853
	7	+0.21840619207	+0.23668638696	+0.25507866294	+0.27372501773
	8	+0.23516444398	+0.22010271935	+0.21833333377	+0.22423492600
	9	+0.02522653990	+0.07797541233	+0.11146939079	+0.13619054953
	10	+0.32466738893	+0.21722566733	+0.17071122132	+0.15281461436
	11	−0.46084539160	−0.17781419227	−0.04796874299	+0.01851106465
	12	+1.36111296151	+0.62177013621	+0.32371445346	+0.19688967179
	13	−3.42004319798	−1.33935153089	−0.55625249070	−0.23297770291
	14	+9.68597708110	+3.55457753745	+1.44715002648	+0.65263956302
	15	−28.5286709455	−9.51594412468	−3.51833733708	−1.41477489238
	16	+88.9376821020	+27.1477264424	+9.31404148366	+3.53850316476
	17	−291.235785543	−81.0609653416	−25.6008150903	−9.00262320492
	18	+1000.66241399	+253.799830529	+73.8458792207	+24.1544067361
	19	−3599.15484483	−830.784519325	−222.359395181	−67.4743858406
	20	+13526.5566605	+2838.71379781	+698.348588943	+196.518945901
	21	−53025.6841577	−10107.5962344	−2283.46544025	−595.358754523
	22	+216470.154554	+37445.8720302	+7762.54138666	+1873.85881729
	23	−918905.735057	−144134.115732	−27396.7807350	−6119.04352841
	24	+4050397.96349	+575646.134976	+100259.282083	+20705.5670994
	25	−18514433.0840	−2382463.70507	−379975.758849	−72517.4413857

Now, from $\omega(\bar{g})$ we can derive the so called β function $\beta(\bar{g})\equiv\int d\bar{g}\,\omega(\bar{g})$, and from this the expansion for the bare coupling constant $\bar{g}_0(\bar{g})=-\int d\bar{g}/\beta(\bar{g})$, with coefficients $\bar{g}_0^{(k)}$ listed in Table VI. From the standard instanton analysis [10], we know that the function $\bar{g}_0(\bar{g})$ has the same left-hand cut in the complex $\bar{g}$-plane as the functions $\omega(\bar{g}), \bar{\eta}(\bar{g}), \eta(\bar{g})$, with the same discontinuity proportional to $e^{-\text{const}/g}$ at the tip of the cut. Hence, the coefficients $\bar{g}_0^{(k)}$ must have asymptotically a similar alternating signs and a factorial growth. Surprisingly, this expectation is not borne out by the explicit seven-loop coefficients $\bar{g}_0^{(k)}$ following from Eq. (6) in Table VI. If we, however, look at the higher-order coefficients derived from the extrapolated $\omega^{(k)}$ sequence which are also listed in that table, we see that sign change and factorial growth do eventually set in at the rather high order 11. Before this order, the coefficients $\bar{g}_0^{(k)}$ look like those of a convergent series. Thus, if we would make a plot analogous to those in Fig. 1 for $\bar{g}_0^{(k)}$, we would observe huge deviations from the asymptotic form up to an order much larger than 10. In contrast, the inverse series $\bar{g}(\bar{g}_0)$ has expansion coefficients $\bar{g}_k$ which do approach rapidly their asymptotic form, as seen in Table VII. This is the reason why our resummation of the critical exponents $\omega, \bar{\eta}, \eta$ as power series in $\bar{g}_0$ yields good results already at the available rather low order seven.

Given the extrapolated list of expansion coefficients in Table V, we may wonder how much these change the seven-loop results. In Fig. 7 we show the results. The known six-loops coefficients of $\omega(\bar{g}_0)$ and $\eta(\bar{g}_0)$ were extended by one extrapolated coefficient, since this produces an even number of approximants which can be most easily extrapolated to infinite order. For $\bar{\eta}(\bar{g}_0)$ we use two more coefficients for the same reason. The extrapolations are shown in Fig. 7. The resulting ω_8 values are lowered somewhat with respect to ω_6 from Eq. (15) to

$$\omega_8=\begin{Bmatrix}0.7935\\0.7916\\0.7900\\0.7880\end{Bmatrix}\quad\text{for}\quad\begin{Bmatrix}n=0\\n=1\\n=2\\n=3\end{Bmatrix}. \tag{22}$$

The new η values are

$$\eta_8=\begin{Bmatrix}0.02829-0.01675\times(\omega-0.7935)\\0.03319-0.01523\times(\omega-0.7916)\\0.03503-0.02428\times(\omega-0.7900)\\0.03537-0.01490\times(\omega-0.7880)\end{Bmatrix}\quad\text{for}\quad\begin{Bmatrix}n=0\\n=1\\n=2\\n=3\end{Bmatrix}, \tag{23}$$

TABLE VII. Coefficients of $\overline{g}(\overline{g}_0)$ obtained from extended perturbation expansions obtained from the large-order expansions (6)–(8) for $\omega(\overline{g})$ up to g^{25}.

	k	$n=0$	$n=1$	$n=2$	$n=3$
$\overline{g}^{(k)}$	1	1	1	1	1
	2	−1	−1	−1	−1
	3	+527/432	+883/729	+811/675	+3893/3267
	4	−1.7163939829	−1.680351960126292	−1.642256264617284	−1.60528382897736
	5	+2.7021635328	+2.591685040643859	+2.481604560563785	+2.378891143794822
	6	−4.6723281932	−4.363908063002809	−4.073635397816119	−3.813515390028028
	7	+8.7648283753	+7.926093595753771	+7.180326093595318	+6.539645290718699
	8	−17.684135663	−15.39841276963578	−13.47981441366666	−11.90293506879397
	9	+38.129348202	+31.80063328573243	+26.79259688548747	+22.86325133485651
	10	−87.419391225	−69.48420478282783	−56.1279351033013	−46.14596304145893
	11	+212.28789113	+160.0400066477353	+123.4985362910675	+97.5437851896555
	12	−544.33806227	−387.4479410496121	−284.6297746951519	−215.3826650602743
	13	+1470.2445538	+983.719405302971	+685.668309006505	+495.7770927688912
	14	−4175.1804881	−2614.933427024693	−1723.672999416843	−1187.794187410145
	15	+12447.739474	+7268.064649337187	+4516.120357408118	+2958.336103932099
	16	−38915.141370	−21101.49568383381	−12320.85534817637	−7652.516371929849
	17	+127440.33105	+63943.24392789235	+34975.98186824855	+20545.02631707489
	18	−436738.21140	−202094.1329180427	−103252.1798678474	−57215.98372843337
	19	+1564637.2472	+665710.523944826	+316810.7604431689	+165210.8008728902
	20	−5853354.4104	−2283830.09806744	−1009811.938755735	−494409.476944406
	21	+22839087.694	+8153184.95412866	+3341698.327836095	+1532757.736028176
	22	−92830002.172	−30260412.9590709	−11473421.52345331	−4920271.757368278
	23	+392524311.64	+116646023.338810	+40840739.00033049	+16345368.25243382
	24	−1724406456.3	−466498175.446816	−150595276.6851763	−56159032.51385756
	25	+7860313710.5	+1933471826.94197	+574727529.9905997	+199417525.2243582

lying reasonably close to the previous seven-loop results (17), (18) for the smaller ω values (22). The first set yields $\eta_8=\{0.0300,0.0356,0.0360,0.0354\}$, the second $\eta_8=\{0.03150.0342,0.0349,0.0345\}$.

For $\overline{\eta}$ we find the results

$$\overline{\eta}_9=\begin{Bmatrix} -0.2711+0.0400\times(\omega-0.810) \\ -0.3803+0.0974\times(\omega-0.805) \\ -0.4735+0.1240\times(\omega-0.800) \\ -0.5506+0.4761\times(\omega-0.797) \end{Bmatrix} \quad \text{for} \quad \begin{Bmatrix} N=0 \\ N=1 \\ N=2 \\ N=3 \end{Bmatrix}. \tag{24}$$

It is interesting to observe how the resummed values $\omega_N,\overline{\eta}_N,\eta_N$ obtained from the extrapolated expansion coefficients in Table V continue to higher orders in N This is shown in Fig. 8. The dots converge against some specific values which, however, are different from the extrapolation results in Fig. 7 based on the theoretical convergence behavior error $\approx e^{-cN^{1-\omega}}$. We shall argue below that these results are worse than the properly extrapolated values.

All the above numbers agree reasonably well with each other and with other estimates in the literature listed in Table IV. The only comparison with experiment which is sensitive enough to judge the accuracy of the results and the reliability of the resummation procedure is provided by the measurement of ν for $n=2$, where the critical exponent $\alpha=2-3\nu$ has been extracted from the singularity $C\propto|1-T/T_c|^{-\alpha}$ in the specific heat at the λ point of superfluid helium with high accuracy [18]:

$$\alpha=-0.01285\pm0.00038. \tag{25}$$

Since ν is of the order 2/3, this measurement is extremely sensitive to ν. It is therefore useful to do the resummations and extrapolations for $N=2$ directly for the approximate α values $\alpha_N=2-3\nu_N$, once for the six-loop ω value ω=0.8, and once for a neighboring value ω=0.790, to see the ω dependence. The results are shown in Fig. 9. The extrapolated values for our ω=0.8 in Table IV yield

$$\alpha=-0.01294\pm0.00060, \tag{26}$$

in very good agreement with experiment.

The extrapolated expansion coefficients for orders larger than 11 do not carry significant information on the critical

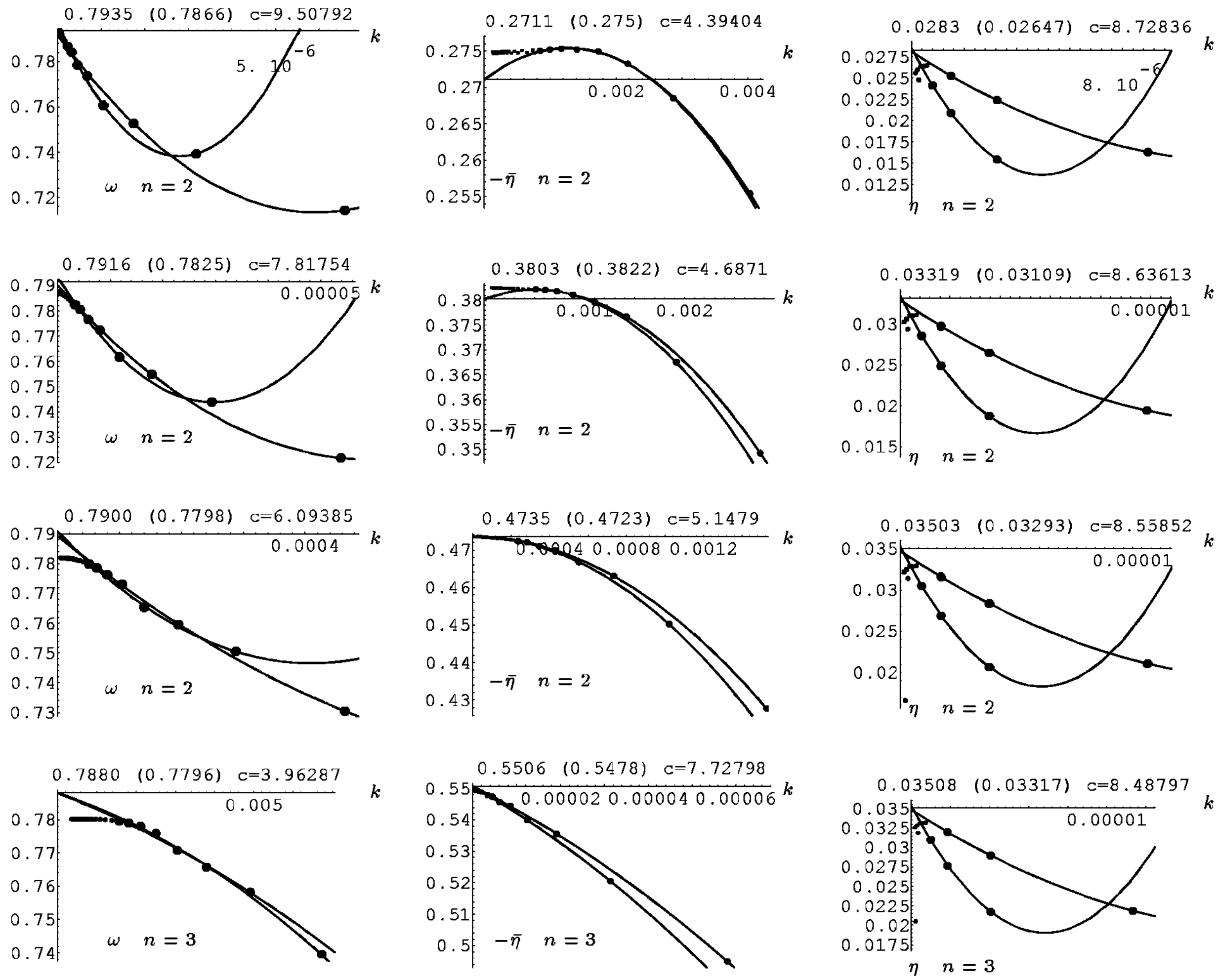

FIG. 7. Extrapolation of resummed $\omega, \bar{\eta}, \eta$ values if one (ω, η) or two $(\bar{\eta})$ more expansion coefficients of Table V are taken into account. The fat dots show the resummed values used for extrapolation, the small dots indicate higher resummed values not used for the extrapolation. The numbers on top specify the extrapolated values and the values of the last approximation, corresponding to the leftmost fat dot.

exponent ν. The fact that the extrapolated expansion coefficients should lie rather close to the true ones as expected from the decreasing errors in the plots in Fig. 6 does not imply the usefulness of the new coefficients in Table V for obtaining better critical exponents. The errors are only relatively small with respect to the huge expansion coefficients. The resummation procedure removes the factorial growth and becomes extremely sensitive to very small deviations from thes huge coefficients. This is the numerical consequence of the fact discussed earlier that the information residing in the exponentially small imaginary part of all critical exponents near the tip of the left-hand cut in the complex $\bar{g}_0$-plane has practically no effect upon the strong-coupling results at infinite $\bar{g}_0$.

Note also that the critical exponents which one would obtain from a resummation of the extrapolated expansion coefficients of high order in Table V and their naive extrapolation performed in Fig. 8 yield slightly worse results for α in superfluid helium. Indeed, inserting $\bar{\eta}=-0.47366$ and $\eta=0.0331$ into the scaling relation $\alpha=2-3/(2+\bar{\eta}-\eta)$ we obtain $\alpha=-0.0091$, which differs by $\sim$25% from the experimental α.

The extrapolation of the approximations $\nu_1, \dots, \nu_9$ can be done similarly, as illustrated in Fig. 9.

Combining these with (23), we find from $\nu=1/(2+\bar{\eta}-\eta)$ the new values for ν:

$$\nu_8=\begin{Bmatrix} 0.5880-0.0196\times(\omega-0.7935) \\ 0.6303-0.0447\times(\omega-0.7916) \\ 0.6705-0.0267\times(\omega-0.7900) \\ 0.7072-0.3123\times(\omega-0.7880) \end{Bmatrix} \quad \text{for} \quad \begin{Bmatrix} n=0 \\ n=1 \\ n=2 \\ n=3 \end{Bmatrix}, \tag{27}$$

quite close to the seven-loop results (16). We may also sum directly the series for $\nu^{-1}=2+\bar{\eta}-\eta$ and extrapolate the resulting values for $\nu_1, \dots, \nu_9$ yielding

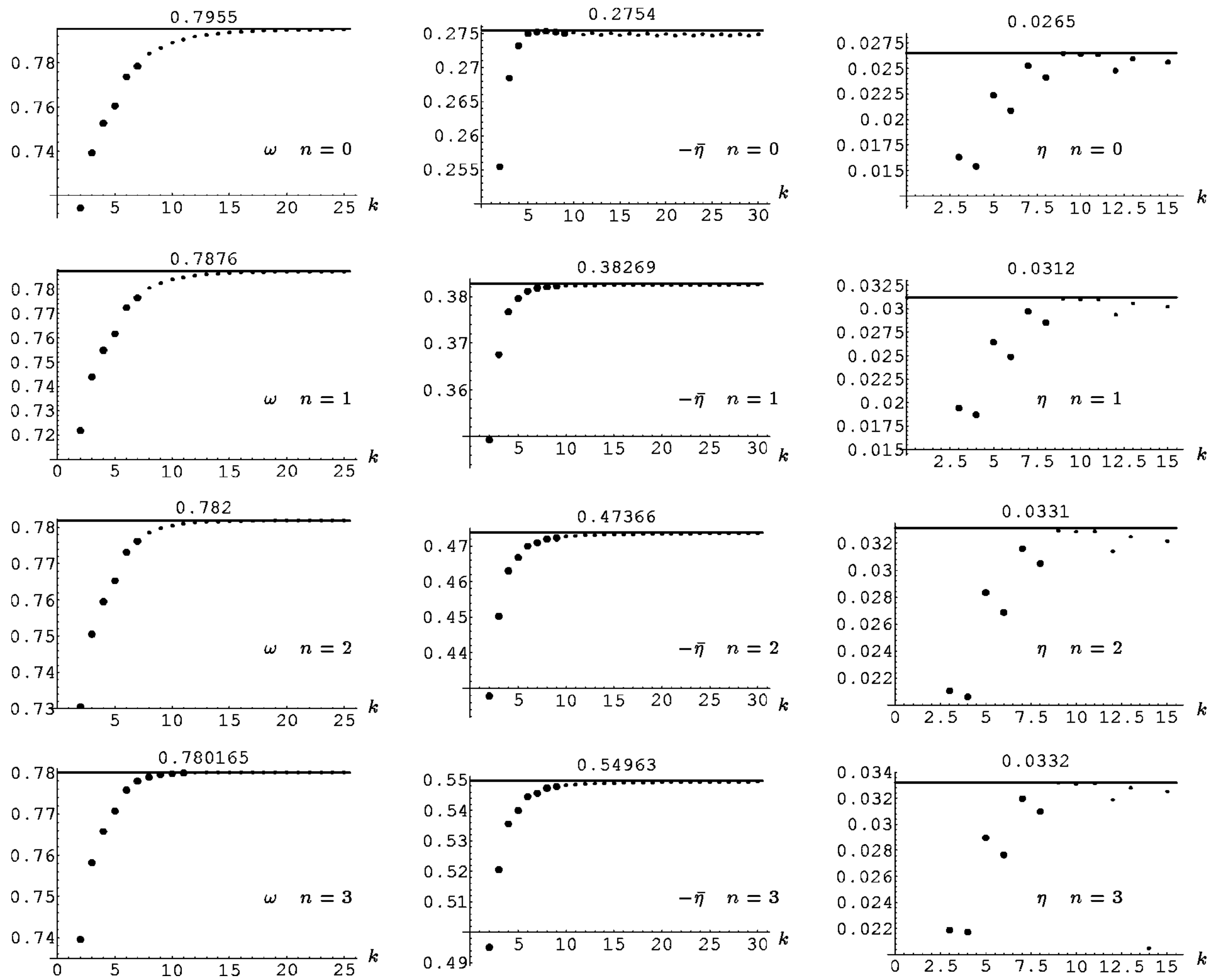

FIG. 8. Direct plots of the resummed $\omega, \overline{\eta}, \eta$ values for all resummed values from all extrapolated expansion coefficients of Table V. The line is fitted to the maximum of all dots at the place specified by the number on top. Fat and small dots distinguish the resummed exponents used in the previous extrapolations from the unused ones.

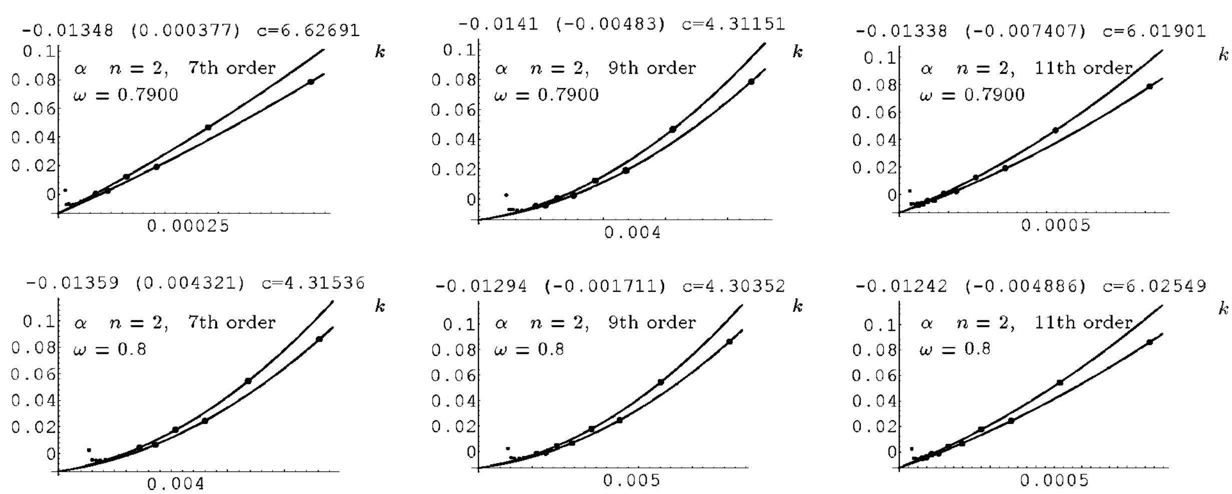

FIG. 9. Extrapolation of resummed α values if two more expansion coefficients are taken from list in Table V. The large dots show the resummed values used for extrapolation; the small dots indicate higher resummed values not used for the extrapolation.

$$\nu_9 = \begin{Bmatrix} 0.5889 - 0.0417 \times (\omega - 0.7935) \\ 0.6311 - 0.0400 \times (\omega - 0.7916) \\ 0.6714 - 0.0553 \times (\omega - 0.7900) \\ 0.7079 - 0.1891 \times (\omega - 0.7880) \end{Bmatrix} \quad \text{for} \quad \begin{Bmatrix} n=0 \\ n=1 \\ n=2 \\ n=3 \end{Bmatrix}, \tag{28}$$

even closer to the seven-loop results (16).

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