Nonfactorizing Saturation of Current Algebra

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A general method is devised by which nonfactorizing solutions (i.e., those in which the currents are not proportional to the charges) of the charge-current commutation rules between arbitrary states and the current-current commutation rules between states of equal momenta can be constructed, given any set of factorizing solutions of the current-current rules. A theorem is proved according to which such factorizing solutions can be obtained from a large class of infinite-component wave equations.

The considerable success of current algebras in deriving sum rules has led to an intensive search for simple algebraic models of currents satisfying the SU(2) x SU(2) or SU(3) x SU(3) current-current commutation relations, which imply the following sum rules for form factors of the weak and electromagnetic currents in the infinite-momentum frame (i.e., $p_z \to \infty$):

$$\sum_n \sum_{\nu} \left[ g_{\alpha \beta}^{\nu}(q_1) g_{\nu \gamma}^{\nu'}(q_1) - g_{\alpha \beta}^{\nu'}(q_1) g_{\nu \gamma}^{\nu}(q_1) \right]$$

$$= i f_{\alpha \beta \gamma} g_{\gamma}^{\nu}(q_1),$$

(1)

and two similar equations, one for the commutator of an axial-vector ($A$) current and a vector ($V$) current giving an axial-vector current, and another one in which the commutator of two axial-vector currents becomes a vector current in the usual way.\(^1\)

The $g_{\alpha \beta}(q_1)$ are the infinite-momentum limits of the current matrix elements between physical single-particle states:\(^2\)

$$g_{\alpha \beta}(q_1) = \lim_{k \to \infty} \left( \frac{M_x M_y}{E_x E_y} \right)^{1/2}$$

$$\times \langle \kappa a + p_\gamma; n | j^\nu(p); \kappa a + p_\gamma; m \rangle_{\text{phys}},$$

(2)

where $a$ is a vector pointing along the direction in which the momentum becomes infinite, while $q_1$ is the momentum transfer orthogonal to the momentum $a$.

The problem we pose is to find a set of functions $g_{\alpha \beta}(q_1)$ which satisfy Eq. (1), but such that those $g_{\alpha \beta}(q_1)$ which vanish at $q_1 = 0$ do not vanish identically. The latter condition means that the solution will be "nonfactorizing." For an important subset of the relations in Eq. (1), we find a solution to this problem. The solution is based upon the use of infinite-component wave equations and is exact in those physically important limiting cases in which the spacelike solutions of such equations "decouple" from the timelike ones.

The procedure for constructing such infinite-component wave equations goes as follows: One introduces a field $\psi_\alpha(p)$ transforming under Lorentz transformations $\Lambda = (\Lambda_\mu)$ as

$$\psi_\alpha(p) \to \psi'_\alpha(p') = D_{\alpha' \alpha}(\Lambda) \psi_\alpha(\Lambda^{-1} p),$$

(3)

where $D_{\alpha' \alpha}(\Lambda)$ is some reducible representation of the Lorentz group $O(3,1)$, selects some vector operator $J_{\alpha' \mu}(p)$, and a scalar operator $\kappa_{\alpha \mu}$ in the representation space (3) which satisfy

$$D_{\alpha' \alpha}(\Lambda) J_{\rho \mu}(p) D_{\alpha' \alpha}(\Lambda^{-1}) = J_{\alpha' \mu}(\Lambda p) \Lambda_{\rho \mu},$$

$$D_{\alpha' \alpha}(\Lambda) \kappa_{\rho \mu} D_{\alpha' \alpha}(\Lambda^{-1}) = \kappa_{\alpha' \mu},$$

(4)

then postulates the field equation

$$[J_{\alpha' \mu}(p) p_\mu - \kappa_{\alpha' \mu}] \psi_\alpha(p) = 0,$$

(5)

and determines the conserved current operator $j^\mu$ by finding a Lagrangian for (5) and performing a gauge transformation.\(^4\) Our starting point will be a factorizing solution. Therefore, the $SU(3)$ octet of vector currents $j^\mu$ is introduced as the product of $j^\mu$ and the

\(^{1}\) For references on this subject see, for example, F. Coester and G. Roepstorff, Phys. Rev. 155, 1583 (1967).

\(^{2}\) The sum in (1) means summation over all internal quantum numbers and spins.

\(^{3}\) We use the normalization of states:

$$\langle p' | p \rangle = (2\pi)^3 (2\pi M)^{3/2} (p' - p).$$


\(^{5}\) For fixed spatial momentum $p$, we define the eigenstates of $\not{L}_3$, $\not{L}_4$ with eigenvalues s\(\gamma\) and \(\not{L}_4\) as particles. Possible additional quantum numbers are omitted. Note that $s = \frac{1}{2}$ corresponding to the little spin group 0(2,1) if Eq. (5) has spacelike solutions at that $p$, which is the case for almost all wave equations discussed until now.
eight generators of $SU(3)$, $\lambda_5$, i.e.,
\[ j^a = \lambda_5 j^a. \tag{6} \]
The content of the model is to define the functions $g_{a_1}(\mathbf{q}_1)$ in terms of the “spinor” quantities related to the wave equation as
\[ g_{a_1}(\mathbf{q}_1) = \lim_{\varepsilon \to 0} \left( \frac{M a M M}{E a E M} \right)^{1/2} \mathcal{X} \left( \mathbf{a} + \mathbf{p}_a; m \right) \left( \mathbf{a} + \mathbf{p}_m; m \right)_{\text{spinor}}. \tag{2'} \]
This equation is the analog of Eq. (2), except that the states are now the timelike solutions of the wave equation which are considered in one-to-one correspondence with physical states; the operator $j^a$ in Eq. (2') is defined in Eq. (6). Then the quantities $g_{a_1}(\mathbf{q}_1)$ can be shown to satisfy the sum rules (1) for an important class of (algebraic as well as nonalgebraic) current operators.\(^2\)

First note that the commutation rule (1) implies that the matrix elements of $j^0$ itself,
\[ g(\mathbf{q}_1) = \lim_{\varepsilon \to 0} \left( \frac{M a M M}{E a E M} \right)^{1/2} \langle \mathbf{p}' \mathbf{s}' \mathbf{s}^1 | j^0 | \mathbf{p} \mathbf{s} \mathbf{s}^2 \rangle, \tag{7} \]
have to satisfy the product rule
\[ \sum_a g_{a}(\mathbf{q}_1)g_{a}(\mathbf{q}_1') = g_{a}(\mathbf{q}_1' + \mathbf{q}_1). \tag{8} \]
In order to discuss this relation, let us rewrite the limit (7) in the more explicit form (indices being understood)
\[ g(\mathbf{q}_1) = \lim_{\varepsilon \to 0} \left( \frac{M a M M}{E a E M} \right)^{1/2} \langle 0 \mathbf{s}' \mathbf{s}, | e^{-i \mathbf{r} \cdot \mathbf{M}} j^0 e^{i \mathbf{r} \cdot \mathbf{M}} | 0 \mathbf{s} \mathbf{s} \rangle, \]
\[ = \left( \frac{M}{M} \right)^{1/2} \langle 0 \mathbf{s}' \mathbf{s}, | (j^0 + j^0) e^{i \mathbf{r} \cdot \mathbf{M}} | 0 \mathbf{s} \mathbf{s} \rangle, \tag{9} \]
where $\mathbf{r}$, $\mathbf{r}'$ are the rapidities $[-= \tanh^{-1}(v/c)]$ of the initial and final particle, and $e^{i \mathbf{r} \cdot \mathbf{M}}$ is defined in the

\(^{4}\) Note that in such a scheme there is a natural way to introduce nonconserved strangeness changing currents simply by letting $g_{a,b}$ in Eq. (5) depend on the strangeness. This dependence also causes an $SU(3)$ mass splitting which can be fitted to the observed one (compare Ref. 18).

\(^{5}\) It will turn out [Eqs. (17)–(19)] that the states in the solution space of the wave equation have in general different completeness properties from those of the physical states. This, however, causes no difficulties, since we only have to show that the functions defined in (2') in terms of quantities related to a wave equation satisfy the sum rules (1), and correspond to a relativistically covariant physical current. There is certainly no additional requirement that the physical metric and the metric in the solution space of the wave equation must be the same.

$SL(2,C)$ representation of the Lorentz group with $M = -\frac{1}{i} \sigma$ by
\[ e^{ia \cdot M} = \frac{1}{\sqrt{(M'M)}} \begin{pmatrix} 0 & q \sqrt{M'} \\ q \sqrt{M'} & 0 \end{pmatrix}, \quad q = \frac{1}{1+iq_2}. \tag{10} \]
But $e^{i \mathbf{r} \cdot \mathbf{M}}$ can be factorized\(^8\) as
\[ e^{i \mathbf{r} \cdot \mathbf{M}} = e^{-i \mathbf{r} \cdot \mathbf{M}} e^{i \mathbf{r} \cdot \mathbf{M}} \tag{11} \]
with
\[ e^{i \mathbf{r} \cdot \mathbf{M}} = \begin{pmatrix} \sqrt{(M_0/M)} & 0 \\ 0 & \sqrt{(M/M_0)} \end{pmatrix} \tag{12} \]
and
\[ Q = \begin{pmatrix} 1 \\ -q_1 / M_0 \\ | M_0 > 0, \text{ arbitrary} \end{pmatrix}, \tag{13} \]
and if one commutes $e^{-i \mathbf{r} \cdot \mathbf{M}}$ through $(j^0 + j^0)$ to the left, one obtains
\[ g(\mathbf{q}_1) = \frac{\sqrt{(M'M)}}{M_0} \langle 0 \mathbf{s}' \mathbf{s}, | e^{-i \mathbf{r} \cdot \mathbf{M}} (j^0 + j^0) e^{i \mathbf{r} \cdot \mathbf{M}} | 0 \mathbf{s} \mathbf{s} \rangle. \]

At this point it is convenient to introduce a new set of states:
\[ | \mathbf{s} \mathbf{s} \rangle = Q \sqrt{(M/M_0)} e^{i \mathbf{r} \cdot \mathbf{M}} | 0 \mathbf{s} \mathbf{s} \rangle, \quad | \mathbf{s} \mathbf{s} \rangle = | 0 \mathbf{s} \mathbf{s} \rangle. \tag{14} \]
In terms of these, $g(\mathbf{q}_1)$ becomes\(^9\)
\[ g(\mathbf{q}_1) = \{ \langle s' \mathbf{s}' | (j^0 + j^0) | \mathbf{s} \mathbf{s} \rangle \} = \{ -q_1 s' s' \} | (j^0 + j^0) | \mathbf{s} \mathbf{s} \rangle. \tag{15} \]
Let us now consider the consequences of current conservation. Since $j^0 q_a = 0$, we find for states at equal momenta
\[ \langle p_0 - p_0 | \langle \mathbf{s}' \mathbf{s}' | j_0 | \mathbf{s} \mathbf{s} \rangle | p_0 - p_0 \rangle = 0, \tag{16} \]
such that suitably normalized states with different mass and spin are orthogonal:

\[ \langle \mathbf{s}' \mathbf{s}' | j_0 | \mathbf{s} \mathbf{s} \rangle = \delta_{s'} \delta_{s'} \delta_{M' \cdot M} (p_0/M). \tag{17} \]

This leads to the infinite-momentum limit to the orthogonality relation
\[ \{ 0, s' s' | j^0 + j^0 | 0, s \mathbf{s} \} = \delta_{s'} \delta_{s'} \delta_{M' \cdot M}. \tag{18} \]
Because of this property, the completeness of the solutions $| \mathbf{s} \mathbf{s} \rangle$ of the wave equation at any fixed momentum $\mathbf{p}$ can be expressed in the infinite-momentum...\(^{8}\) See H. Bebi and H. Leutwyler, Ref. 4.
\(^{9}\) Note that $Q$ and $(j^0 + j_0)$ commute.
limit as \(10\)

\[
\sum_{s\bar{s}} (j_0 + j_3) \{0,ss_s\} \{0,ss_s\} - \sum_{s\bar{s}} \{0,ss_s\} \{0,ss_s\} (j_0 + j_3) = 1. \tag{19}
\]

Note that this relation holds up to now only in the space spanned by the states \(|0,ss_s\rangle\). The crucial point of the proof is that for a large class of currents \(j^a\), this relation can be extended to hold in the much larger Hilbert space \(|qss_s\rangle\). To see this, let us assume \(q\) to point in \(\hat{x}\) direction and apply the expression (19) to the state \(|qss_s\rangle\). We observe that the most general current following from the wave equation (5) may consist of so-called algebraic vectors \(I^a\) which have no explicit momentum dependence and nonalgebraic ones \(j^a(\hat{p}; \hat{q})\) which explicitly use the momenta \(\hat{p}\) and \(\hat{q}\) of the external states to couple to a vector. Since the momentum of the states \(|qss_s\rangle\) is

\[
\hat{p} = \left(\frac{(M_0^2 + M^2 + q^2)}{2M_0}, 0, 0, \frac{(M_0^2 - M^2 - q^2)}{2M_0}\right), \tag{20}
\]

the matrix elements of \(j^a(\hat{p}; \hat{q})\) between \(|0ss_s\rangle\) and \(|qss_s\rangle\) can be written in more detail as

\[
\{0ss_s\} \langle j^a(\hat{p}; \hat{q}) \frac{(M_0^2 + M^2)}{2M_0}, 0, 0, \frac{(M_0^2 - M^2)}{2M_0};
\]

\[
\frac{(M_0^2 + M^2 + q^2)}{2M_0}, 0, 0, \frac{(M_0^2 - M^2 - q^2)}{2M_0}\}
\]

\(|qss_s\rangle\), \(\langle \hat{s}\langle \hat{p}\rangle\frac{(M_0^2 + M^2)}{2M_0}, 0, 0, \frac{(M_0^2 - M^2)}{2M_0};
\]

\[
\frac{(M_0^2 + M^2 + q^2)}{2M_0}, 0, 0, \frac{(M_0^2 - M^2 - q^2)}{2M_0}\}
\]

where \(M^2\) and \(M\) are the masses of the particles with spin \(s'\) and \(s\) respectively.

Let us now assume that the current contains besides an algebraic part \(I^a\) only so-called convective currents of the form \(S \otimes (\hat{p} + \hat{q}) \otimes \hat{p}^a\) or \(S \otimes (\hat{p} - \hat{q}) \otimes \hat{p}^a\), where \(S\) is an algebraic scalar operator. In this case, the matrix elements (21) are in fact independent of \(M', M, q\).

Furthermore, in this paper we shall restrict our attention to saturating only the vector-vector commutation rule (1).

Let \(\hat{p}^a\) be a conserved current which satisfies the product sum rule (18) with a complete set of solutions of Eq. (5). Let furthermore \(T_{\mu}^a\) be an arbitrary algebraic antisymmetric tensor operator in the Hilbert space defined by the solution of the corresponding wave equations. Such a tensor operator always exists. An example is the set of Lorentz group generators \(L_{\mu}^a\) in (3).

Then consider the current defined by

\[
k_{\mu}^a = \lambda^a j_\mu + \delta_\mu^a T_{\nu}^a q_\nu, \tag{23}\]

where \(q^a = p'^a - p^a\) is the momentum transfer. Since \(T_{\mu}^a\) is antisymmetric, this current is always conserved. In the infinite-momentum limit defined by (2), this current becomes

\[
g_{\mu}(q_{\mu}) = \lambda^a \{ s' s' \} (j^a + \hat{p}^a) Q_{ss_s} = \delta_\mu^a \{ s' s' \} (T_{\mu}^a + T_{\nu}^a) q_\nu Q_{ss_s}. \tag{24}\]

From (20) and a similar expression for \(p'^a\), we deduce that

\[
g^a + q^a = 0. \tag{25}\]

Every conserved current combined of algebraic and convective terms saturates the factorized current commutation rules at infinite momentum.\(^10\)

This type of approach to the saturation problem has as yet produced only solutions with unphysical features:

(i) Either the mass spectrum contains an infinitely degenerate mass, or the wave equation has spacelike solutions.

(ii) The currents are proportional to the charges (such that the neutron has vanishing electric and magnetic form factors).

In fact, it has been proved that (i) holds as a theorem for the vector \(J^a(\hat{p})\) is purely algebraic and does not depend on \(\hat{p}\).\(^12\) But there are also examples with unphysical solutions when \(J^a\) has a convective part.\(^13\)

It is the purpose of this paper to give a solution to the second problem for two important subalgebras of (1):

(a) \(q' = 0\). In this case Eq. (1) forms the so-called charge-current commutation rules.

(b) \(q' = -q\), \(s' = s\), which is the subalgebra of (1) taken between states of equal momenta and spin orientation.\(^14\)

\(10\) If the space of solutions contains spacelike momenta, then all steps still hold true if one replaces the particle mass \(m\) by \(\mu = \sqrt{(-\hat{p}^2) - 4M^2}\) and interprets the state \(|0ss_s\rangle\) as state with momentum \(p = (0,0,0,\mu)\).

\(12\) Any current derivable from a conventional second-order Lagrangian through a gauge transformation is at most linear in the external momenta \(\hat{p}\) and \(\hat{q}\) and hence of this form. For this reason, our theorem covers all cases discussed in the literature so far.

\(13\) In a forthcoming paper, the effects of \(SU(3)\) breaking will be included; compare Ref. 18.
Therefore

\[ (T^{\alpha} + T^{\beta}) q_r = (T^{\alpha} + T^{\beta}) q_r, \quad (r = 1, 2) \]

so that

\[ g_j(q_1) = \lambda_i s^{s} s' (j_0 + j_3) Q | s s s s \rangle + \delta_i s^{s} s' (T^{\alpha} + T^{\beta}) Q | s s s s \rangle q_r. \]  \hspace{1cm} (26)

In fact, noting that \( T^{\alpha} + T^{\beta}, Q, \) and \( Q' \) all commute with each other, we see that the commutator of \( g_j(q_1) \) and \( g_j(q_1') \) becomes

\[
[g_j(q_1'), g_j(q_1)] = [\lambda_i, \lambda_j] (s^{s} s' (j_0 + j_3) Q | s s s s \rangle + [\delta_i, \delta_j] q_r s^{s} s' | (T^{\alpha} + T^{\beta}) Q | s s s s \rangle \]  

\[ + q_r q_r \delta_j \sum_{s'' s'''} s^{s} s' (T^{\alpha} + T^{\beta}) Q | s'' s''' s s \rangle \]  

\[ - \delta_j \delta_i \sum_{s'' s'''} s^{s} s' | (T^{\alpha} + T^{\beta}) Q | s'' s''' s s \rangle. \]  \hspace{1cm} (27)

This equation shows that we can indeed fulfill two subalgebras of (1):

(a) \( q' \) or \( q = 0 \), which is called the charge-current algebra. In this case \( \lambda_i, \delta_i \) have to satisfy the commutation rules

\[
[\lambda_i, \lambda_j] = i f_{ijk} \lambda_k, \quad [\lambda_i, \delta_j] = i f_{ijk} \delta_k. \]  \hspace{1cm} (28)

Note that \([\delta_i, \delta_j]\) is completely arbitrary.\(^{10}\)

(b) \( q' = -q \) and \( s' = s_0 \), and equal \( SU(3) \) quantum numbers in initial and final state. Here we have to impose, in addition to (28), the condition\(^{11}\)

\[
[\delta_i, \delta_j] = 0. \]  \hspace{1cm} (29)

If \( \lambda_i \) are the generators of \( SU(3) \), both algebras (28) and (29) allow for a nonvanishing \( d/f \) ratio in \( \delta_i \).

Thus, neutral particles can have nonvanishing form factors. Notice that the generators \( \lambda_i \) have matrix elements only within the same \( SU(3) \) multiplet. Different \( SU(3) \) multiplets are connected by the \( \delta_i \) currents. In case (a), the matrix elements of \( \delta_i \) can be fixed by postulating a commutation rule for \([\delta_i, \delta_j]\) and specifying a representation in agreement with the observed spectrum of internal quantum numbers. Let us summarize our result in Theorem 2.

Given any conserved current \( \lambda_i j^a \) satisfying the factorized current commutation rules and then using an arbitrary antisymmetric algebraic tensor operator \( T^{a} \), one can construct a new current

\[ j^a = \lambda_i j^a + \delta_i T^{a} q_r, \]  \hspace{1cm} (30)

whose infinite momentum limit satisfies two important subalgebras of the current-current commutation rules without factorizing. The matrices \( \delta_i \) have to be a vector operator with respect to \( \lambda_i \) and are otherwise arbitrary for the charge-current subalgebra while they have to commute for the \( q' = -q \) subalgebra.

Currents of such a type have led to good agreement with experimental results in dynamical group calculations.\(^{18}\) The possibility of using them to describe transitions between different \( SU(3) \) multiplets will be exploited in a forthcoming paper.\(^{19}\)

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\(^{10}\) In particular, we may choose \( \delta_i \) to close back in the form \([\delta_i, \delta_j] = \delta_{ij} \lambda_i \), in which case \( \pi^i \) and \( \delta^i \) form the so-called non-invariance groups of T. G. Kuriyan and E. C. G. Sudarshan, Phys. Letters 21, 106 (1966). The compact form of this commutation rule is clearly preferable since only a finite number of isospins has been observed until now.

\(^{11}\) These commutation rules are the same as those used by T. Cook, C. J. Goebel, and B. Sakita, Phys. Rev. Letters 15, 35 (1965) in their strong-coupling theory. They have the disadvantage of containing only multiplets with infinitely many isospins.
