THE MEMBRANE PROPERTIES OF CONDENSING STRINGS ☆

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It is pointed out that near a phase transition the string action requires an extra term to control the fluctuations and it is shown that the associated coupling constant, which is dimensionless, is asymptotically free in the ultraviolet. Attention is drawn to previous studies of the same term in biophysical membranes and microemulsions.

The physics of strings is governed by the string tension $m^2 \equiv (2\pi\alpha')^{-1}$, a quantity of the order of the squared Planck mass. Gravity is obtained in the limit of infinite tension m^2 corresponding to the zero-slope ($\alpha' \to 0$) limit of the particle trajectories. The higher excited states carry masses of the order of m^2 and will be unobservable in the foreseeable future.

In this note we would like to draw attention to the opposite limit of very small string tension, i.e. the infinite-slope ($\alpha' \to \infty$) limit. In this limit the Nambu–Goto action [1] ‡1

$$A_{\mathrm{NG}} = -m^2 \int \mathrm{d}^2\xi \sqrt{-g} = -m^2 \int \mathrm{d}^2\xi \, [-\det(\partial_\alpha x^\mu \partial_\beta x_\mu)]^{1/2} \tag{1}$$

loses control over the space–time surface swept out by the string, the surface becomes wrinkled and acquires a higher Hansdorff dimension. A higher-gradient action is necessary to prevent a catastrophy.

In euclidean space, this phenomenon is well known. Microemulsions form when surfactants remove the surface tension between oil and water [2]. In biomembranes, the surface tension is absent due to the incompressibility of the molecules within the membrane [3,4] ‡2,3. This absence leads to gigantic fluctuations which for red blood cells have been seen as early as 1890 in an ordinary light microscope ‡4. They prevent the cells from sticking to each other [10,11], in spite of their attractive van der Waals forces [12]. Similar phenomena occur in various lattice models of phase transitions. In U(1) lattice gauge theories in four dimensions, the surfaces of the strong-coupling expansion become prolific and wrinkled for $\beta > \beta_c$. The same thing happens to the domain walls in the three-dimensional Ising model.

In all these cases, the fluctuations at vanishing m^2 are controlled by what we shall call *membrane action*

$$A_{\mathrm{M}} = -\frac{\kappa}{2}\int \mathrm{d}^2\xi \sqrt{-g}\, \mathrm{D}^2 x_\mu \mathrm{D}^2 x^\mu - \frac{\bar{\kappa}}{2}\int \mathrm{d}^2\xi \sqrt{-g}\,(\mathrm{D}^2 x_\mu \mathrm{D}^2 x^\mu - \mathrm{D}_\alpha \mathrm{D}_\beta x_\mu \mathrm{D}^\alpha \mathrm{D}^\beta x^\mu)\,. \tag{2}$$

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‡1 Our notation: $x^\mu(\sigma, \tau) \equiv x^\mu(\xi^\alpha)$ parametrizes the string in D space–time dimensions with the metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1, \ldots)$. The string itself has the metric $g_{\alpha\beta} = \partial_\alpha x^\mu \partial_\beta x_\mu$. The slope of the Regge trajectory is $\alpha' = 1/2\pi m^2$.

‡2 See ref. [5] for a phenomenological calculation of surface tensions.

‡3 It is the same mechanism that has recently been proposed in ref. [6] to explain the absence of the cosmological constant in gravity.

‡4 In human erythrocytes they have been recorded as early as 1890 by Browicz [7], and interpreted first as a manifestation of life. See ref. [8]. For a recent discussion see ref. [9].

The membrane action plays a similar role with respect to the Nambu–Goto action as the Einstein–Cartan action does with respect to the Einstein action. Due to the higher gradients of x, this action brings about additional ghost states which are hard to interprete. Since the conventional string theory had to live with ghost states for a long time and finally resolved this problem in certain critical dimensions, it may well be that we shall eventually learn to deal also with the additional ghost states of the membrane action. For this reason, we shall ignore the ghost problem for the time being and study the new action $A_{\mathrm{NG}} + A_{\mathrm{M}}$ in a purely perturbative way.

First of all, let us observe that the membrane action (2) carries only dimensionless coupling constants implying that the theory is *renormalizable* by power counting. In fact, the κ term plays a similar role for strings at small m^2 as the $g\varphi^4$ term does for the φ^4 theory at small mass. Contrary to that theory, however, the new coupling in the extended string action turns out to be asymptotically free in the ultraviolet, a fact, which will be a major result of this note. At present, we are unable to answer the question of an infrared stable fixed point. Just as in quantum chromodynamics, such a point can probably be located only by Monte Carlo techniques. Its properties would be crucial for the understanding of microemulsions. In the membrane action (2), the symbols D_α are the covariant derivatives $\partial_\alpha - \Gamma_\alpha$ and Γ_α is the connection. The second term depends only on the boundary and is locally irrelevant. It measures the genus of the surface. Another way to rewrite this action is by introducing the $D-2$ orthogonal normal vectors of the surface N_n^μ, $n = 3, ..., D$, and observing that the covariant derivatives of the tangent vectors $\mathrm{D}_\alpha x^\mu$ can be spanned by the normal vectors as follows [‡5] [13]

$$\mathrm{D}_\alpha \mathrm{D}_\beta x^\mu = C^n_{\alpha\beta} N^\mu_n \, . \tag{3}$$

The $D-2$ 2×2 matrices $C^n_{\alpha\beta} = N^n_\mu \mathrm{D}_\beta \mathrm{D}_\beta x^\mu = -\mathrm{D}_\alpha N^n_\mu \mathrm{D}_\beta x^\mu$ are known as *extrinsic curvature* and satisfy the Gauss–Codazzi relations [13]

$$R_{\alpha\beta} = C^n_{\alpha\beta} C^{\gamma n}_\gamma - C^{\gamma n}_\alpha C^n_{\gamma\beta} \, , \qquad \mathrm{D}_\alpha C^n_{\beta\gamma} = \mathrm{D}_\beta C^n_{\alpha\gamma} \, , \tag{4}$$

where $R_{\alpha\beta}$ is the intrinsic Ricci tensor of the surface whose trace $R = R^\alpha_\alpha$ is the scalar curvature. Introducing also the quantities $C^n \equiv C^{\gamma n}_\gamma$ ($\cong D-2$ times the mean curvatures) we arrive at the form

$$A_{\mathrm{M}} = -\frac{\kappa}{2} \int \mathrm{d}^2\xi \sqrt{-g}\, C^n C^n - \frac{\bar{\kappa}}{2} \int \mathrm{d}^2\xi \sqrt{-g}\, R \, , \tag{5}$$

where the pure boundary character of the second term is the content of the Gauss–Bonnet theorem ($\equiv -4\pi\kappa(1-h)$ with h = number of handles). What has been studied in membranes [14,15] is the effect of quadratic *normal* fluctuations

$$\delta x^\mu = \nu_n N^\mu_n \, , \tag{6}$$

upon this action. These change the metric by

$$\begin{aligned} \Delta g_{\alpha\beta} &\equiv \delta g_{\alpha\beta} + \delta^2 g_{\alpha\beta} = [\mathrm{D}_\alpha \delta x_\mu \mathrm{D}_\beta x^\mu + (\alpha\beta)] + \mathrm{D}_\alpha \delta x_\mu \mathrm{D}_\beta \delta x^\mu \\ &= -2\nu_n C^n_{\alpha\beta} + (\mathrm{D}_\alpha \nu_n \mathrm{D}_\beta \nu_n + C^n_{\alpha\gamma} C^{\gamma n}_\beta \nu_n \nu_m) \, , \end{aligned} \tag{7}$$

and the area element by

$$\Delta\sqrt{-g} = -\nu_n C^n + \tfrac{1}{2}(\mathrm{D}\nu_n)^2 + \tfrac{1}{2}(C^n C^m - C^{\gamma n}_\alpha C^{\alpha m}_\gamma)\nu_n \nu_m + \dots \, , \tag{8}$$

such that the Nambu–Goto action has the quadratic fluctuations

$$\delta^2 A_{\mathrm{NG}} = -\frac{m^2}{2} \int \mathrm{d}^2\xi \sqrt{-g}\,[(\mathrm{D}\nu_n)^2 + R^{nm}\nu_n \nu_m] \, , \tag{9}$$

‡5 Since $N^\mu_n N_{m\mu} = \delta_{nm}$ the n, m part of the metric is δ_{nm}.

where R^{nm} is the $(D-2)\times(D-2)$ matrix $C^nC^m - C^{\gamma n}_{\alpha}C^{\alpha m}_{\gamma}$ whose trace is the scalar curvature R. The connection has the variation

$$\delta\Gamma^{\gamma\alpha}_{\gamma} = D^2\delta x_\mu D^\alpha x^\mu + D^2 x_\mu D^\alpha S x^\mu + \ldots = C^n D^\alpha \nu_n - D^\alpha C^n \nu_n - 2C^{\alpha\beta n}D_\beta \nu_n + O(\nu^2)\,. \tag{10}$$

Calculating further

$$D^2\delta x_\mu D^2\delta x^\mu = (D^2\nu_n)^2 - 2C^{\beta n}_{\alpha}C^{\alpha m}_{\beta}\nu_n D^2\nu_m + 4C^{\alpha n}_{\gamma}C^{\gamma\beta m}D_\alpha\nu_n D_\beta\nu_m + C^{\alpha\beta r}C^{\gamma\delta r}C^n_{\alpha\beta}C^m_{\gamma\delta}\nu_n\nu_m$$
$$+ (DC^n)(DC^m)\nu_n\nu_m - 4C^{\alpha\beta n}(D_\beta C^m)\nu_n D_\alpha\nu_m\,, \tag{11}$$

we find the second-order change in the membrane part of the action

$$\delta^2 A_{\mathrm{M}} = -\frac{\kappa}{2}\int d^2\xi\sqrt{-g}[(D^2\nu_n)^2 - 2C^{\beta n}_{\alpha}C^{\alpha m}_{\beta}\nu_n D^2\nu_m + 4C^{\alpha n}_{\gamma}C^{\gamma\beta m}D_\alpha\nu_n D_\beta\nu_m - C^nC^m D\nu_n D\nu_m$$
$$+ 4C^{\gamma n}_{\gamma}C^{\alpha\beta m}D_\alpha\nu_n D_\beta\nu_m - 4C^{\alpha n}_{\gamma}C^{\gamma\beta m}D_\alpha\nu_n D_\beta\nu_m + 4C^{\beta n}_{\alpha}C^{\alpha m}_{\beta}\nu_n D^2\nu_m + 4C^{\gamma n}_{\gamma}C^{\alpha\beta m}\nu_n D_\alpha D_\beta\nu_m$$
$$+ \tfrac{1}{2}C^{m\,2}(D\nu_n)^2 - 2C^nC^{\alpha\beta m}D_\alpha\nu_n D_\beta\nu_m - 2C^nC^m\nu_n D^2\nu_m + \ldots]\,. \tag{12}$$

The omitted terms contain fewer than two derivatives in ν_n and are irrelevant in the ultraviolet.

Short-wavelength fluctuations are dominated by $(D^2\nu_n)^2$. They have the correlations

$$\langle D_\alpha\nu_n D_\beta\nu_m\rangle = (1/\kappa)\,\delta_{\alpha\beta}\delta_{nm}L + \text{finite terms}\,, \tag{13}$$

where $L \equiv \int d^2k/(2\pi)^2k^2 = (1/2\pi)\log(k_{\max}/k_{\min})$. Thus, integrating out the ν_n fields leads to the additional one-loop action

$$A^{\text{one-loop}} = -(D-2)\,\mathrm{tr}\log(D^2) - \int d^2\xi\sqrt{-g}(\delta m^2 + \tfrac{1}{2}\delta\kappa C^{n2} + \tfrac{1}{2}\delta\bar{\kappa}R)\,, \tag{14}$$

where

$$\delta m^2 = m^2[(D-2)/2\kappa]L\,,\quad \delta\kappa = -\tfrac{1}{2}DL\,,\quad \delta\bar{\kappa} = 2L\,, \tag{15a,b,c}$$

and $\mathrm{tr}\log(D^2)$ is the usual conformal anomaly [16]. In the conformal gauge where $g_{\alpha\beta} = e^\varphi\delta_{\alpha\beta}$ it takes the local form [17]

$$-\int d^2\xi\,[\sqrt{-g}\,Q + (1/48\pi)(\partial_\alpha\varphi)^2] - \frac{L}{6}\int d^2\xi\sqrt{-g}R\,, \tag{16}$$

with $Q = \int d^2k/(2\pi)^2$ giving a further, quadratically divergent, contribution to δm^2 (which vanishes in dimensional regularization). The last term of which is missed in (14) because of the method used there (see ref. [15] for a simple derivation) renormalizes and changes eq. (15c) into

$$\delta\bar{\kappa} = [2-(D-2)/3]L\,. \tag{15c$'$}$$

The negative sign in $\delta\kappa$ shows that the fluctuations *soften* the membrane implying *asymptotic freedom* of the coupling constant κ, just as announced in the beginning of this note.

Concerning the infrared behavior, the only information comes from Monte Carlo simulations. The critical behavior of the $D=4$ U(1) Villain lattice gauge theory and the $D=3$ Ising model teaches us that in these models there exists a stable fixed point also in the infrared at which the system becomes critical.

Notice that in the usual cosine form, the U(1) lattice gauge theory seems to be close to tricritical ‡6 implying

‡6 This point is elaborated in ref. [18], see also ref. [19].

that, there, even the curvature terms κ are close to zero such that also the gradients in $C^n_{\alpha\beta}$ (of sixth order in $\boldsymbol{k}$) will eventually have to be considered.

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