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SOME PROPERTIES OF THE $(\bar{\psi}\psi)^2$ MODEL IN TWO DIMENSIONS

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We show that the two-dimensional $(\bar{\psi}\psi)^2$ model is, for $N = 3$, equivalent to the supersymmetric sine-Gordon equation, and for $N = 4$, equivalent to two decoupled sine-Gordon systems. In addition, we argue that the kinks of this model are isospinors, and we construct some of the higher conservation laws that are responsible for the exact solubility of this system.

1. Introduction

Recently Zamolodchikov [1], Karowski, Weisz, Truong, and Thun [2], and Zamolodchikov and Zamolodchikov [3] have shown, in some very interesting work, how to determine what are apparently the exact S matrices of certain two-dimensional models, including the sine-Gordon equation, the nonlinear sigma model, and the $(\bar{\psi}\psi)^2$ model. It is also possible, by the same methods, to determine the S matrices of the supersymmetric generalizations of those models [4].

Of particular interest are the sigma model and the $(\bar{\psi}\psi)^2$ model, which resemble realistic four-dimensional models in a number of ways, including asymptotic freedom and dynamical mass generation.

The purpose of this paper is to describe some properties of the $(\bar{\psi}\psi)^2$ model which may aid in understanding and interpreting the S matrix of the model **. In addition, so far as this S matrix has not yet been fully determined (the S matrix is known for the elementary particles, but not yet for the kink states), some of the results might be helpful in completing the determination of it. We will pay particular attention to the behavior of the theory when the number N of components of the field is 3 or 4, because there are hints (for instance, the instantons of the sigma model) that these models resemble four-dimensional theories most when the number

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** A considerable amount is already known about this model. The large N expansion was studied in ref. [5a] and a semiclassical analysis, exhibiting what appears to be the exact spectrum was made in ref. [56].

of components of the field is small. Our main results are as follows.

In sect. 2 we apply the boson representation of fermions to the $N = 3$ and $N = 4$ theories. The $N = 3$ theory is equivalent to the supersymmetric form of the sine-Gordon equation, while the $N = 4$ theory is equivalent to a pair of decoupled sine-Gordon models, all at the critical coupling ($\beta^2 = 8\pi$ in the case of $N = 4$).

A. Zamolodchikov has suggested that the spectrum for $N = 3$ and $N = 4$ consists only of kink states. In sect. 3, therefore, using the semiclassical method of Jackiw and Rebbi [6], we attempt to determine the quantum numbers of the kink states, to see if the kink spectrum is compatible with the results of sect. 2. Surprisingly, the kinks turn out to be isospinors. The isospinor kink spectrum is perfectly compatible with the results of sect. 2.

In sect. 4 we describe a new conservation law of the model. For $N = 3$ the new charges are supersymmetry charges; for $N = 4$ they are the difference between the energy and momentum of the two decoupled sine-Gordon theories. For $N > 4$ the new charges seem to generate γ_5 transformations in isospin space for the isospinor kinks.

Finally, in sect. 5, we discuss, and attempt to justify, one of the basic assumptions of the Zamolodchikov-Zamolodchikov work – the existence of an infinite number of conserved local currents transforming with arbitrarily large weight under Lorentz transformations. Although our discussion is incomplete, we claim that such currents do exist and that they are manifestations of the conformal invariance of the underlying classical field theory.

2. Bosonization of the three- and four-component models

The model under consideration is described by the Lagrangian

$$\mathcal{L} = \int d^2x \left[\tfrac{1}{2} \sum_{i=1}^{N} (\bar{\psi}^i i \not\partial \psi^i) + g\left(\sum_{i=1}^{N} \bar{\psi}^i \psi^i\right)^2\right] , \tag{1}$$

where ψ^i_α is an N-component Majorana Fermi field. The theory has an obvious $O(N)$ invariance, corresponding to rotations of the ψ^i.

Since two Majorana fermions are equivalent to one Dirac fermion, the theory for $N = 2$ is a theory of a single interacting Dirac field; in fact, it is just the Thirring model.

Here we will use the boson representation of fermions [7] to analyze the model (1) in the cases $N = 3$ and $N = 4$. A similar analysis has been carried out by Banks et al. and by M.B. Halpern for a Lagrangian similar to the above $N = 4$ model [8] and the decoupling result that we will derive for $N = 4$ has been found independently by A.M. Polyakov (unpublished) and by A. Luther and M.B. Halpern (unpublished).

To formulate a boson representation, it is necessary to group the Majorana fermions in pairs (two Majorana fermions being equivalent to a Dirac fermion). This procedure will unfortunately not be manifestly O(3) or O(4) covariant.

The basic rules of the boson representation are as follows. Given two Majorana spinors ψ_1 and ψ_2, it is possible to introduce a canonical real scalar field ϕ such that

$$\tfrac{1}{2}(\bar\psi_1 i\partial\!\!\!/ \psi_1 + \bar\psi_2 i\partial\!\!\!/ \psi_2) = \tfrac{1}{2}(\partial_\mu\phi)^2 ,$$

$$\bar\psi_1\psi_1 + \bar\psi_2\psi_2 =: \cos\sqrt{4\pi}\,\phi : ,$$

$$\bar\psi_1\gamma^\mu\psi_2 = -\frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\phi . \tag{2}$$

We will also need to use a rather less familiar equation which can be deduced from (2). By a Fierz transformation,

$$(\bar\psi_1\psi_1 + \bar\psi_2\psi_2) = 2(\bar\psi_1\gamma^\mu\psi_2)^2 + c \text{ number} \tag{3}$$

(the c number, which would be absent classically, is present quantum mechanically because of the normal ordering or other procedure that is needed to define the composite operators carefully). From (2) and (3) it follows that

$$:\cos\sqrt{4\pi}\,\phi:^2 = \frac{-2}{\pi}(\partial_\mu\phi)^2 + c \text{ number} . \tag{4}$$

Eq. (4) has no classical counterpart and is certainly rather hard to understand intuitively. It is discussed more fully in the appendix.

(Note that we have not specified precisely the procedure for defining our normal ordered products. In principle the second equation in (2) contains a free constant depending on the definition of the normal ordered product; we have used our liberty to set this constant equal to one. For a fuller discussion of this and other mathematical aspects of the bosonization procedure, see ref. [7].)

We are now ready to describe the boson representation of the $N = 3$ and $N = 4$ models. For the $N = 3$ model, the Lagrangian can be written

$$\mathcal{L} = \int d^2x[\tfrac{1}{2}(\bar\psi_1 i\partial\!\!\!/ \psi_1 + \bar\psi_2 i\partial\!\!\!/ \psi_2 + \bar\psi_3 i\partial\!\!\!/ \psi_3) + g(\bar\psi_1\psi_1 + \bar\psi_2\psi_2)^2$$

$$+ 2g(\bar\psi_1\psi_1 + \bar\psi_2\psi_2)\,\bar\psi_3\psi_3] . \tag{5}$$

Note that we have dropped a term $(\bar\psi_3\psi_3)^2$. This vanishes by Fermi statistics, since ψ_3 is a Majorana fermion.

We may now follow (2) to replace ψ_1 and ψ_2 by a boson field ϕ. The resulting form of the Lagrangian is

$$\mathcal{L} = \int d^2x[\tfrac{1}{2}(\partial_\mu\phi)^2 + \tfrac{1}{2}\,\bar\psi_3 i\partial\!\!\!/ \psi_3 + g : \cos\sqrt{4\pi}\,\phi:^2 + 2g : \cos\sqrt{4\pi}\,\phi : \bar\psi_3\psi_3] . \tag{6}$$

Let us compare (6) with the supersymmetric form of the sine-Gordon equation,

$$\mathcal{L} = \int d^2\chi [\tfrac{1}{2}(\partial_\mu \phi)^2 + \tfrac{1}{2}\, \bar{\psi} i \partial\!\!\!/\, \psi + 2A^2 \cos^2 B\phi + AB(\cos B\phi)\, \bar{\psi}\psi] \,, \tag{7}$$

which possesses a conserved supersymmetry current

$$\partial_\mu [((\gamma^\lambda \partial_\lambda \phi) - 2iA \sin B\phi]\, \gamma^\mu \psi] = 0\,. \tag{8}$$

(The cosine squared interaction in (7) can be written as the more familiar cosine interaction, by using the identity $\cos^2 B\phi = \frac{1}{2}\cos 2B\phi + \frac{1}{2}$.)

Comparing (6) and (7), we see that, naively, (6) is supersymmetric if and only if $g = \frac{1}{2}\pi$. However, in a renormalizable theory such as this one, the coupling constant changes under a scale transformation, and we do not expect $g = \frac{1}{2}\pi$ to be special. In fact, when $g \neq \frac{1}{2}\pi$, the Lagrangian (6) naively fails to be supersymmetric only because the coefficient of the $:\cos 4\pi\phi:^2$ term is incorrect. But in view of (4), the $:\cos 4\pi\phi:^2$ term is, in quantum mechanics, equivalent to a renormalization of the kinetic energy (which in turn can be absorbed in a rescaling of the field ϕ), so this coefficient can be adjusted at will. Thus, the theory (6) is actually supersymmetric, and is equivalent to (7), with appropriate values of the parameters. It turns out that, after using (4) and rescaling ϕ, (6) can be rewritten as

$$\mathcal{L} = \int d^2\chi [\tfrac{1}{2}(\partial_\mu \sigma)^2 + \tfrac{1}{2}\, \bar{\psi}_3 i \partial\!\!\!/\, \psi_3 + 2A^2 :\cos\gamma\sigma:^2 + A\gamma :\cos\gamma\sigma:\, \bar{\psi}_3\psi_3] \,, \tag{9}$$

where in terms of

$$\lambda = \frac{2g^2(1 - 4g/\pi)}{\pi - 8g^2/\pi} \,,$$

we find

$$A = \sqrt{\tfrac{1}{2}\lambda}\,, \qquad \gamma = \sqrt{4\pi}\,\frac{gA}{\sqrt{\pi}} \,,$$

and where

$$\sigma = \frac{\sqrt{\pi}}{gA}\,\phi\,.$$

So the $N = 3$ model is, unexpectedly, equivalent to the supersymmetric form of the sine-Gordon equation.

For the $N = 4$ model, in addition to the field ϕ defined by (2), we introduce a second boson field ϕ' that satisfies the analogous equations

$$\tfrac{1}{2}(\bar{\psi}_3 i\partial\!\!\!/\,\psi_3 + \bar{\psi}_4 i\partial\!\!\!/\,\psi_4) = \tfrac{1}{2}(\partial_\mu\phi')^2 \,,$$

$$\bar\psi_3\psi_3 + \bar\psi_4\psi_4 = :\cos\sqrt{4\pi}\,\phi':\,, \tag{10}$$

$$\bar\psi_3\gamma^\mu\psi_4 = -\frac{1}{\sqrt{\pi}}\,\epsilon^{\mu\nu}\partial_\nu\phi\,.$$

In terms of ϕ and ϕ', the $N = 4$ Lagrangian takes the form

$$\mathcal{L} = \int d^2\chi\left[\frac{1}{2}\left(1+\frac{2g}{\pi}\right)\left((\partial_\mu\phi)^2 + (\partial_\mu\phi')^2\right) + 2g\,:\cos\sqrt{4\pi}\,\phi:\,:\cos\sqrt{4\pi}\,\phi':\right]\,, \tag{11}$$

where use has been made of (4) in treating the $(\bar\psi_1\psi_1 + \bar\psi_2\psi_2)^2$ and $(\bar\psi_3\psi_3 + \bar\psi_4\psi_4)^2$ terms. Letting

$$\phi_\pm = \sqrt{\tfrac{1}{2} + g/\pi}\,(\phi \pm \phi')\,, \tag{12}$$

the Lagrangian becomes

$$\mathcal{L} = \int d^2\chi[\tfrac{1}{2}(\partial_\mu\phi_+)^2 + \tfrac{1}{2}(\partial_\mu\phi_-)^2$$

$$+ 2g\,:\cos\sqrt{2\pi/(1+2g/\pi)}\,(\phi_+ + \phi_-):\,:\cos\sqrt{2\pi/(1+2g/\pi)}\,(\phi_+ - \phi_-):]\,. \tag{13}$$

Using the formula $\cos(A+B)\cos(A-B) = \frac{1}{2}(\cos 2A + \cos 2B)$, this is

$$\mathcal{L} = \int d^2\chi[\tfrac{1}{2}(\partial_\mu\phi_+)^2 + \tfrac{1}{2}(\partial_\mu\phi)^2 + g\,:\cos\sqrt{8\pi/(1+2g/\pi)}\,\phi_+:$$

$$+ g\,:\cos\sqrt{8\pi/(1+2g/\pi)}\,\phi_-:]\,. \tag{14}$$

Thus, the $N = 4$ model is a theory of two decoupled sine-Gordon equations.

This rather surprising decoupling makes somewhat more sense in the light of the following group theoretical facts. The symmetry algebra O(4) of this theory can be expressed as a direct product, $O(4) \sim O(3) \times O(3)$, where the generators of O(4) are arbitrary real antisymmetric 4×4 matrices and the generators of $O(3)_L$ and $O(3)_R$ are, respectively, the self-dual and anti-self-dual 4×4 matrices.

Let us consider the definitions of the fields ϕ_+ and ϕ_-,

$$\epsilon^{\mu\nu}\partial_\nu\phi_+ = -\sqrt{\tfrac{1}{2}\pi + g}\,(\bar\psi_1\gamma^\mu\psi_2 + \bar\psi_3\gamma^\mu\psi_4)\,,$$

$$\epsilon^{\mu\nu}\partial_\nu\phi_- = -\sqrt{\tfrac{1}{2}\pi + g}\,(\bar\psi_1\gamma^\mu\psi_2 - \bar\psi_3\gamma^\mu\psi_4)\,. \tag{15}$$

These currents generate, respectively, the transformations represented by

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

and by

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

on the four-component isovector field ψ^i_α. The first of these matrices is self-dual and so is a generator of $O(3)_R$, while the second is anti-self-dual and is a generator of $O(3)_L$. The currents in (14) thus transform as (1, 0) and 0, 1), respectively, under $O(3)_L \times O(3)_R$, and ϕ_+ and ϕ_- transform in the same way.

Thus, this theory decouples into fields, and excitations, that are singlets under $O(3)_L$, and fields, and excitations, that are singlets under $O(3)_R$.

It is also possible to see this decoupling in perturbation theory. If J^i_μ are the $O(3)_R$ currents and $K_{\nu i}$ are the $O(3)_L$ currents, then it is possible to establish the factorization

$$\langle 0|T(\sum_{i=1}^{N} J_{\mu i}(x_i)\sum_{j=1}^{M} K_{\nu j}(y_j))|0\rangle = \langle 0|T(\sum_{i=1}^{N} J_{\mu i}(x_i))|0\rangle \, \langle 0|T(\sum_{j=1}^{M} K_{\nu j}(y_j))|0\rangle \tag{16}$$

for the current matrix elements. That this factorization is satisfied order by order in perturbation theory can be established by using the Ward identities for the currents J^i_μ and K^i_μ, plus those for the axial currents $\epsilon_{\mu\nu}J^{\nu i}$ and $\epsilon_{\mu\nu}K^{\nu i}$, keeping track of all possible anomalies. We will omit the details here. (For a prototype for this argument, see ref. [9].) The factorization (16) implies that the excitations created from the vacuum by the J^i_μ propagate independently of those created by the K^j_ν.

3. The kink spectrum of the model

The results of the previous section have striking consequences for the particle spectrum of the $N = 3$ and $N = 4$ models. For $N = 3$, the states must form supersymmetric multiplets of bosons and fermions. For $N = 4$, all one particle states must transform as $(p, 0)$ or as $(0, q)$ under $O(3)_L \times O(3)_R$, since the $O(3)_L$ and $O(3)_R$ degrees of freedom are decoupled.

The "elementary fermions" of the model, whose exact S matrix has been determined [3], do not satisfy these conditions. For $N = 4$ the elementary fermions would transform under isospin as $(\frac{1}{2}, \frac{1}{2})$, which is inconsistent with the decoupling theorem. For $N = 3$ the problem is that, according to the S matrix of ref. [3], there are no bosons degenerate with the elementary fermions, to form a supersymmetry multiplet. Thus, the elementary fermions must be absent for $N = 3$ and $N = 4$.

A. Zamolodchikov has suggested (private communication) a very attractive resolution of this problem. In addition to the elementary fermions and their bound states, this model contains also kink states, associated with the spontaneous breaking of the discrete chiral symmetry. The kinks interpolate between the two possible vacuum states at spatial infinity, and a semiclassical formula for their mass is known from work of Callan, Coleman, Gross and Zee [10].

In the sine-Gordon equation, it is known that the elementary particle disappears from the spectrum when the coupling constant becomes strong enough that the elementary particle mass reaches the kink-antikink threshold. For $\beta^2 > 4\pi$ the spectrum of the sine-Gordon equation consists only of kinks and antikinks.

Zamolodchikov has suggested that a similar phenomenon occurs in the $(\bar{\psi}\psi)^2$ model. When N is small enough, the elementary particle becomes unstable against decay into a pair of kinks, and disappears from the spectrum. The semiclassical formula for the kink mass indicates that this occurs at $N = 4$ *. Then the spectrum for $N = 3$ and $N = 4$ would consist only of kinks.

This provides a welcome explanation for the absence of the elementary particle states for $N = 3$ and $N = 4$. However, we must still ask whether the kink quantum numbers are consistent with the decoupling results for $N = 4$ and with supersymmetry for $N = 3$. In fact, what are the quantum numbers of the kinks?

Following Jackiw and Rebbi [6], we will attempt to determine the isospin quantum numbers of the kinks from a semiclassical point of view. Let us attempt semiclassical quantization in a kink state, that is, in a topologically nontrivial configuration of the composite field $\sigma = \bar{\psi}\psi$. The N Majorana fermion species ψ_i interact with σ via a coupling $\sigma\Sigma_{i=1}^N \bar{\psi}_i\psi_i$. As shown by Jackiw and Rebbi, a fermion interacting with a topological kink by such a coupling will possess a single, normalizable zero energy mode f_0, in addition to the non-zero energy solutions f_n (f_0 and f_n are spinors; we will suppress spinor indices). The non-zero energy solutions of definite frequency are complex; the zero energy solution is non-degenerate and real. The semiclassical expansion for the field ψ^i is

$$\psi^i = \sum (f_n a_n{}^i + f_n{}^* a_n{}^{i*}) + f_0 b^i \, . \tag{17}$$

Here $a_n{}^i$ and $a_n{}^{i*}$ are creation and annihilation operators for particles of type i in the states corresponding to f_n. The operators b^i are hermitean, since the ψ^i are, and they transform according to the vector representation of O(N), since the ψ^i do. Moreover, the canonical commutation relations for ψ^i imply that the b^i satisfy the Clifford algebra relations

$$\{b^i, b^j\} = 2\delta_{ij} \, . \tag{18}$$

* Properly speaking, the semiclassical analysis is only valid for large N. However, Dashen, Hasslacher and Neveu speculated [5] that the semiclassical results become exact with the substitution $N \to N - 2$, and this seems to be confirmed by the analysis in ref. [3]. It is the improved semiclassical formula, with N replaced by $N - 2$, that indicates that $N = 4$ is the critical value.

In short, the b_i have the transformation properties and the anticommutation relations appropriate for the gamma matrices of O(N).

The kink states, on which the b^i act, therefore transform in the spinor representation of O(N). The kinks are isospinors.

Before proceeding, a few mathematical remarks about the algebra (18) may be in order.

In any representation of (18), it is possible to define operators $M_{ij} = b_i b_j - b_j b_i$ which are generators of O(N). It is also possible to define an operator $\gamma_5 = ib^1 b^2 ... b^N$ which commutes with the group generators M_{ij}. Moreover, $\gamma_5{}^2 = 1$, so the possible eigenvalues of γ_5 are ± 1.

If N is odd, then γ_5 commutes with all the Clifford algebra generators b^i, and in an irreducible representation of the Clifford algebra γ_5 is a constant. There are for odd N two inequivalent irreducible representations of the Clifford algebra, one with $\gamma_5 = -1$ and one with $\gamma_5 = +1$. These representations each have dimension $2^{(N-1)/2}$.

If N is even, γ_5 does not commute with the Clifford generators, and any representation of the Clifford algebra must contain both "left-handed" isospinors with $\gamma_5 = -1$, and "right-handed" ones with $\gamma_5 = +1$. In this case there is a single irreducible representation of the Clifford algebra; it has dimension $2^{N/2}$. However, since γ_5 commutes with the group generators, these $2^{N/2}$ states transform reducibly under the O(N) group; the left-handed and right-handed states transform independently.

Returning now to physics, we would like to decide whether the kink states form an irreducible representation of the Clifford algebra. In this regard there is an additional, discrete symmetry of the theory that should be taken into account – the operation $\psi \to -\psi$. Corresponding to this symmetry, there is an operator P with $P\psi P = -\psi$ and $P^2 = 1$. Of course, P commutes with the O(N) group.

It follows from the formula (17) or from the Jackiw-Rebbi analysis that the fermion field operators have nonzero matrix elements among the kink states $|\alpha\rangle$. In fact

$$\langle\beta|\psi^i|\alpha\rangle = f\langle\beta|b^i|\alpha\rangle\,, \tag{19}$$

where the b^i, as we have said, generate a Clifford algebra. Since ψ^i is odd under P, the b^i must also be odd under P.

So, in the space of kink states, there must exist an operator P which anticommutes with all the Clifford generators. Can such an operator exist if the kinks form an irreducible representation of the Clifford algebra? For even N such an operator certainly exists – we have already seen that for even N, there is an operator γ_5 that anticommutes with the Clifford generators. But for odd N, an operator P that anticommutes with the Clifford generators must anticommute with γ_5; in an irreducible representation, such an operator does not exist because γ_5 is a c-number. Therefore, for odd N, because of the existence of P, the kink states do not form an irreducible representation of the Clifford algebra, but rather a reducible representation, with the $\gamma_5 = +1$ and $\gamma_5 = -1$ representations each occurring at least once.

So the minimal possibility, which we believe is actually realized, is that for even N the kink states form an irreducible representation, of dimension $2^{N/2}$, of the Clifford algebra, while for odd N, they form a reducible representation, of dimension $2(2^{(N-1)/2})$.

In fact, the operator P simply distinguishes bosons from fermions. We are saying that for odd N, there are two isospinors of kinks, an even P isospinor of boson kinks and an odd P isospinor of fermion kinks. For even N, there is a single isospinor of kink states. Of these, the left-handed and right-handed spinors have opposite behavior under P and therefore have opposite statistics. We have not determined whether the right-handed spinors are bosons and the left-handed ones fermions or vice-versa, and this point may well be ambiguous, because of the difficulty in some cases of distinguishing bosons from fermions in one space dimension.

Are these results about the kink states compatible with the conclusions of sect. 2 about the $N = 3$ and $N = 4$ models?

For $N = 3$ – odd N – we claim that the spectrum consists of an isodoublet of boson kinks and an isodoublet of fermion kinks. This is certainly compatible with a possible supersymmetry.

For $N = 4$, the spinor representation transforms as $(\frac{1}{2}, 0) + (0, \frac{1}{2})$ under $O(3)_L \times O(3)_R$. So isospinor kinks are certainly compatible with the decoupling of the $O(3)_L$ and $O(3)_R$ variables.

Having determined the kink quantum numbers, it is natural to try to use the method of ref. [3] to determine the kink-particle and kink-kink S matrices. This has been done by Shankar and Witten [11] who argue that the semiclassical bound states, with the right masses and isospin quantum numbers, appear as poles in the kink-kink S matrix.

4. A new conservation law

In this section, we will describe a new conservation law of the $(\bar{\psi}\psi)^2$ model, which will shed light on the results of the preceeding two sections.

It will be convenient to use light cone coordinates. We define $x_+ = x_0 + x_1$, $x_- = x_0 - x_1$, and let ψ_+ and ψ_- be the chiral components of the fermi fields. Under a Lorentz transformation

$$\begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}$$

these variables transform as $x_+ \to e^{\theta} x_+$, $x_- \to e^{-\theta} x_-$, $\psi_+ \to e^{\theta/2}\psi_+$, $\psi_- \to e^{-\theta/2}\psi_-$. An object that transforms as $f \to e^{p\theta} f$ we will consider to have Lorentz weight p, so ψ_+, for example, has weight $\frac{1}{2}$. In these coordinates, the Lagrangian is

$$\mathcal{L} = \int dx_+ dx_- [\psi_+^a i\partial_- \psi_+^a + \psi_-^a i\partial_+ \psi_-^a + g\psi_+^a \psi_-^a \psi_+^b \psi_-^b] \ . \tag{20}$$

We can now easily describe the following conservation law *. Let $\epsilon_{i_1 \ldots i_N}$ be the completely antisymmetric tensor. Let

$$J^+ = \epsilon_{i_1 \ldots i_N} \psi_{+i_1} \cdots \psi_{+i_N} ,$$

$$J^- = \epsilon_{i_1 \ldots i_N} \psi_{-i_1} \cdots \psi_{-i_N} . \tag{20}$$

J^+ is the product of all N positive chirality Fermi fields, and J^- is the product of all N negative chirality fields. Remarkably, at the classical level, J^+ and J^- satisfy free field equations,

$$\partial_- J^+ = 0 , \qquad \partial_+ J^- = 0 . \tag{21}$$

To check these statements, one simply uses the equations of motion, together with the fact that an arbitrary product of $N + 1$ positive chirality fermi fields (or $N + 1$ negative chirality Fermi fields) vanishes by fermi statistics.

If (21) is not modified by anomalies, it implies that the theory contains massless particles, because J^+ and J^-, being free, massless fields, will create free, massless particles when acting on the vacuum.

To what extent may (21) contain an anomaly? $\partial_- J^+$, if not zero, is a local operator of dimension $\frac{1}{2}N + 1$ and Lorentz weight $\frac{1}{2}N - 1$, and moreover it is a pseudoscalar, rather than scalar, in isospin space, because of the factor $\epsilon_{\mu_1 \ldots \mu_N}$. The only isotopic pseudoscalar with this dimension and weight is

$$z^+ = \epsilon_{\mu_1 \ldots \mu_N} \psi_{\mu_1 -} \psi_{\mu_2 -} (\partial_+ \psi_{\mu_3 +}) \, \psi_{\mu_4 +} \cdots \psi_{\mu_{N+}} .$$

But z^+ is in turn a total divergence, $z^+ = \partial_+ R^-$, where

$$R^- = \frac{1}{N-2} \epsilon_{\mu_1 \ldots \mu_N} \psi_{\mu_1 -} \psi_{\mu_2 -} \psi_{\mu_3 +} \cdots \psi_{\mu_{N+}} .$$

Likewise $\partial_+ J^-$, if not zero, must be proportional to $\partial_- R^+$, where

$$R^+ = \frac{1}{N-2} \epsilon_{\mu_1 \ldots \mu_N} \psi_{\mu_1 +} \psi_{\mu_2 +} \psi_{\mu_3 -} \cdots \psi_{\mu_-} .$$

Consequently, the only possible anomaly in (21) is that (21) might be modified to read

$$\partial_- J^+ + f(g) \, \partial_+ R^- = 0 , \qquad \partial_+ J^- + f(g) \, \partial_- R^+ = 0 , \tag{22}$$

where $f(g)$ is some function of the coupling constant. In perturbation theory, an anomaly such as (22) does occur.

Eq. (22) is no longer a free field equation, but it is still a conservation law. What are its consequences?

* A similar point has been made by Callan and Gross in connection with two-dimensional gauge theories (unpublished).

For $N = 2$, the conservation law (22) is just the axial vector current of the massless Thirring model (and in this case only, $f(g) = 0$).

For $N = 3$, the new current is cubic in the Fermi fields and so is an anticommuting spinor current. It also has dimension $\frac{3}{2}$ and Lorentz weight $\frac{3}{2}$; these are the approriate values for a supersymmetry current satisfying the usual algebra. Actually, if one rewrites (22) in the boson language, one obtains the supersymmetry current of eq. (8). (Some care is required; (8) itself is modified by an anomaly to include a term $(\partial_\mu\phi)\ \psi_3$.) It is also possible to verify directly in the fermion representation that the charges associated with (22) satisfy, for $N = 3$, the usual supersymmetry algebra – their anticommutators are the energy-momentum operators. (This also requires some care; in evaluating the anticommutators, it is useful to define the composite operators by a point splitting method.)

For $N = 4$ the new currents have weight two and dimension two, as is appropriate for an energy-momentum tensor. In the boson language, these currents turn out to be simply the difference between the energy-momentum tensors of the decoupled fields ϕ_+ and ϕ_-.

How are the new conservation laws to be interpreted for $N > 4$? Let Q^+ and Q^- be the conserved charges and $|f^a\rangle$ the elementary fermion states (which do exist for $N > 4$!). Now, consider the states $Q^+|f^a\rangle$ and $Q^-|f^a\rangle$. If these states are not zero, they must be degenerate in mass with the $|f^a\rangle$.

For odd N, the $Q^\pm$ are anticommuting charges, and the $Q^\pm|f^a\rangle$, if not zero, would be boson states degenerate with the $|f^a\rangle$. For even or odd N, the states $Q^\pm|f^a\rangle$ would transform as pseudovectors under isospin (as opposed to the $|f^a\rangle$, which transform as vectors), because $Q^\pm$ are isotopic pseudoscalars.

The S matrix proposed in ref. [3] for the elementary particles shows no hint of bosons degenerate with the elementary fermions for odd N, or of isospin pseudovectors degenerate with them for even or odd N. Therefore, we are led to suspect that $Q^\pm$ may annihilate the elementary particle states.

If $Q^\pm$ annihilate the elementary fermions, they must also annihilate all the bound states of the elementary fermions. Therefore $Q^\pm$ can have a non-zero action only on the remaining states, namely the kink states.

Given that $Q^\pm$ are isotopic pseudoscalars, what action can they have on the kink states? The only matrix in the space of kink states that transforms like a pseudoscalar under isospin is the matrix γ_5 defined in sect. 3. Taking account of the Lorentz transformation properties of $Q^\pm$, we are therefore led to conjecture that

$$Q^\pm|\alpha, p\rangle = p_\pm^{(N-2)/2}(\gamma_5)_{\alpha\beta}|\beta, p\rangle\,, \tag{23}$$

where $|\alpha, p\rangle$ is a kink of type α and momentum p.

Although formally the same, (23) has a somewhat different import depending on whether the $Q^\pm$ are commuting or anticommuting charges (even or odd N). For odd N, (23) implies a sort of supersymmetry in the scattering of Bose and Fermi kinks.

For even N, (23) implies that the number of left-handed (or right-handed) kinks is conserved. More than that, it implies that in a collision of a left-handed and a right-handed kink, there is no backward scattering. For in a collision of a left-handed kink of momentum p and a right-handed one of momentum q to give a left-handed kink of momentum r and a right-handed one of momentum s, conservation of $Q^{\pm}$ gives

$$(p_{\pm})^{(n-2)/2} - (q_{\pm})^{(N-2)/2} = (r_{\pm})^{(N-2)/2} - (s_{\pm})^{(N-2)/2} \,. \tag{24}$$

This, plus momentum conservation, implies $p = r$, $q = s$.

5. Conservation laws of arbitrary weight

Finally, we wish to ask this question about the $(\bar{\psi}\psi)^2$ model: why does the construction of the S matrix in ref. [3] work? This construction involves, as a basic assumption, the existence of local conserved currents transforming under the Lorentz group with arbitrarily high weight *. Such currents are known in the sine-Gordon equation. They must exist also in the $(\bar{\psi}\psi)^2$ and non-linear sigma models, or the construction of the exact S matrix would fail. But what are they?

(Of course, in the previous section we described one new conservation law, but the calculation of ref. [3] requires an infinite number.)

Since this section was originally written, a considerable literature has developed on this subject. We will here review the relationship of the literature to the analysis presented below.

Pohlmeyer [12] and Neveu and Papanicolaou [13] have constructed conserved local currents in the classical sigma and $(\bar{\psi}\psi)^2$ models, respectively. These currents, however, are fractional functions of the fields and, on the face of things, one would not expect them to be present in quantum theory. Luscher and Pohlmeyer [14], however, were able to construct nonlocal conservation laws by using the construction of ref. [12], and Luscher showed, in an elegant treatment [15], that these non-local conservation laws survive in quantum theory and explain the factorization of the S matrix and the absence of particle production. Polyakov, using Pohlmeyer's work as a starting point [16], constructed a conserved local current for the quantum nonlinear sigma model. (Polyakov actually constructed the same conserved current that we construct below from a completely different point of view.) Finally, Araf'eva, Kulish, Nissimov and Pacheva [17], whose point of view is similar to the one that we follow below but whose treatment is much more mathematical, have derived a general result of which the result that follows, derived by a much simpler method, is a special case.

Our point of view is that for the existence of conserved local currents in the $(\bar{\psi}\psi)^2$ and nonlinear sigma models, deep properties of the classical theories are not essential. The main property that is essential is the conformal invariance.

* This is pointed out in refs. [2,3], and reviewed in detail in ref. [4].

In terms of the notation introduced at the beginning of the previous section, the components of the energy-momentum tensor in the $(\bar{\psi}\psi)^2$ model are (classically) $T_{++} = \psi_+{}^a \partial_+ \psi_+{}^a$, $T_{--} = \psi_-{}^a \partial_- \psi_-{}^a$, $T_{+-} = 0$. T_{+-} is the trace of the energy-momentum tensor (since in these coordinates the metric tensor is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$), and its vanishing is the sign of conformal invariance.

Because $T_{+-} = 0$, the conservation of energy-momentum reduces to

$$\partial_- T_{++} = 0 , \qquad \partial_+ T_{--} = 0 . \tag{25}$$

Thus, the energy-momentum tensor is a free field.

Eq. (25) states that the energy and momentum densities propagate as if there were no interaction. Consequently, the scattering, classically, can consist only of isospin exchange.

Thus, any conformally invariant theory in one space dimension will automatically possess the basic property assumed in refs. [2,3] in deriving S matrices: the scattering consists only of isospin exchange. This statement is equally true in classical or quantum physics.

We have seemingly made no reference to the existence of conserved currents of high weight, but in reality this is implicit in (25). For (25) implies the existence of an infinite number of conserved currents of arbitrarily high weight, such as

$$\partial_-(T_{++}{}^2) = 0 , \qquad \partial_-(T_{++}{}^3) = 0 , \qquad \partial_-((\partial_+ T_{++})^2) = 0 . \tag{26}$$

However, as we know, the theories under discussion here are not conformally invariant in quantum mechanics; T_{+-}, because of anomalies, is not zero; and (25) is not satisfied. (25) is modified by the anomalies to read

$$\partial_- T_{++} + \partial_+ T_{-+} = 0 , \qquad \partial_+ T_{--} + \partial_- T_{+-} = 0 . \tag{27}$$

From (27) we cannot infer higher conservation laws such as (26), and (27), of course, does not restrict the scattering to isospin exchange.

In fact, in most theories in which anomalies modify (25) into (27), our statements above are all invalid in the quantum theory, and the Zamolodchikov-Zamolodchikov assumptions would also not be valid.

What is special about the sigma and $(\bar{\psi}\psi)^2$ models, is, apparently, that although (25), from which (26) was deduced, is ruined by anomalies, (26) still survives. More exactly, the currents in (26) are modified by anomalies and cease to be free fields, but the conservation laws survive.

We do not have a general proof of this, but will show explicitly that it is true for the first conservation law stated in (26).

For the $(\bar{\psi}\psi)^2$ model, this conservation law is, naively, $\partial_-((\psi_+^a \partial_+ \psi_+^a)(\psi_+^b \partial_+ \psi_+^b)) = 0$. We must show that all possible anomalies in this equation are themselves total divergences.

Any possible anomaly must be an operator of dimension five and Lorentz weight three. Such an operator must be linear in ∂_- and zeroth order in ψ_-, or zeroth order in ∂_- and quadratic in ψ_-. Operators of the first type can always be rewritten as operators of the second type by using the equations of motion, so we will restrict our attention to operators quadratic in ψ_- and zeroth order in ∂_-.

So we must consider the operators of dimension five, quadratic in $\psi_-^{\cdot}$, zeroth order in ∂_-, and prove that each is a total divergence. Of course, we may drop those operators that vanish by Fermi statistics or by the equations of motion. Of the remaining operators, it turns out that most can be written as total divergences even without using the equations of motion. In fact, the only one that cannot be so written without use of the equations of motion is $(\psi_+^a\psi_-^a)^2(\psi_-^b\partial_-\psi_-^b)$. But it is easy to see, by use of the equations of motion, that this operator is a linear combination of $\partial_-(\psi_+^a\partial_+^3\psi_+^a)$, $\partial_+((\psi_-^a\psi_+^a)\,\partial_+^2(\psi_-^b\psi_+^b))$, $\partial_+((\partial_+(\psi_-^a\psi_+^a))^2)$ and $\partial_+((\psi_-^a\psi_+^a)^2\,\psi_+^b\partial_+\psi_+^b)$. This completes the argument that the conservation law associated with $\partial_-(T_{++}{}^2) = 0$ survives in the quantum mechanical $(\bar{\psi}\psi)^2$ model, although the current is modified and ceases to be a free field.

A similar result can easily be seen to hold for the conservation law $\partial_-(T_{++}{}^2) = 0$ of the nonlinear sigma model. It too survives in the quantum theory.

We have not studied the other conservation laws of the type indicated in (26), but suspect that they also survive quantum mechanically.

Perhaps it is worthwhile to stress that the fact that these conservation laws are satisfied despite anomalies is a special property of the $(\bar{\psi}\psi)^2$ and nonlinear sigma models. It is true because the O(N) symmetry, under which the fields transform as an irreducible multiplet, restricts severely the possible anomalies. In a theory with naive conformal invariance and less internal symmetry, the conservation law $\partial_-(T_{++}{}^2) = 0$ would, in general, be completely ruined by anomalies.

6. Conclusions

We have found a number of unexpected properties of the $(\bar{\psi}\psi)^2$ model: supersymmetry for $N = 3$, decoupling into two sectors for $N = 4$, and isospinor kinks with γ_5 invariance for any value of N. Understanding these properties may help in understanding the S matrix.

Finally, in the last section, we have tried to understand why the Zamolodchikov-Zamolodchikov calculation works. Although the analysis is incomplete, this seems to be a consequence of the conformal invariance of the classical theory, not all of whose consequences are ruined by anomalies.

Appendix

Here we will give a fuller derivation of eq. (4), for which one argument has already been given in the text.

For the composite operator $\cos\beta\phi$, normal ordering is equivalent to a multiplicative renormalization [7]. In fact,

$$:\cos\beta\phi: = (\Lambda/\mu)^{\beta^2/4\pi} \cos\beta\phi \,, \tag{28}$$

where Λ is an ultraviolet cutoff, μ a scale parameter, and $\cos\beta\phi$ is defined without normal ordering. The same is true for the chiral components $\mathrm{e}^{\pm i\beta\phi}$:

$$:\mathrm{e}^{\pm i\beta\phi}: = (\Lambda/\mu)^{\beta^2/4\pi}\, \mathrm{e}^{\pm i\beta\phi} \,. \tag{29}$$

To define the product $:\cos\beta\phi:^2$, we may use a point splitting method and consider instead $\lim_{x\to y} :\cos\beta\phi(x)\colon :\cos\beta\phi(y):$, which, expanding in chiral components, is

$$\begin{aligned}\tfrac{1}{4}\lim_{x\to y}\, (&:\mathrm{e}^{i\beta\phi(x)}\colon :\mathrm{e}^{i\beta\phi(y)}: + :\mathrm{e}^{-i\beta\phi(x)}\colon :\mathrm{e}^{-i\beta\phi(y)}: \\ &+ :\mathrm{e}^{i\beta\phi(x)}\colon :\mathrm{e}^{-i\beta\phi(y)}: + :\mathrm{e}^{-i\beta\phi(x)}: + :\mathrm{e}^{i\beta\phi(y)}:) \,.\end{aligned} \tag{30}$$

We claim that the first term vanishes as $\Lambda \to \infty$:

$$\lim_{x\to y} :\mathrm{e}^{i\beta\phi(x)}\colon :\mathrm{e}^{i\beta\phi(y)}: = 0 \,. \tag{31}$$

In fact

$$\begin{aligned} :\mathrm{e}^{i\beta\phi(x)}\colon :\mathrm{e}^{i\beta\phi(y)}: &= \left(\frac{\Lambda}{\mu}\right)^{\beta^2/2\pi} (\mathrm{e}^{i\beta\phi(x)}\,\mathrm{e}^{i\beta\phi(y)}) \\ &= \left(\frac{\Lambda}{\mu}\right)^{\beta^2/2\pi} (\mathrm{e}^{2i\beta\phi(y)} + \mathrm{O}(x-y)) \\ &= \left(\frac{\Lambda}{\mu}\right)^{-\beta^2/2\pi} :\mathrm{e}^{2i\beta\phi(y)}: + \mathrm{O}(x-y) \,. \end{aligned} \tag{32}$$

As $x \to y$ and $\Lambda \to \infty$, (32) vanishes.

(From the point of view of the analogy between the sine-Gordon equation and the Coulomb gas, (31) has a simple interpretation. $:\mathrm{e}^{i\beta\phi}:$ can be regarded as a charged particle, and (31) vanishes because the amplitude to have two particles of the same charge at the same point is zero.)

Likewise, the second term in (30) vanishes as $\Lambda \to \infty$.

As for the third term,

$$:\mathrm{e}^{i\beta\phi(x)}\colon :\mathrm{e}^{-i\beta\phi(y)}: = \left(\frac{\Lambda}{\mu}\right)^{\beta^2/2\pi} (\mathrm{e}^{i\beta\phi(x)}\,\mathrm{e}^{-i\beta\phi(y)}) \,. \tag{33}$$

Naively, as x approaches y, $\mathrm{e}^{i\beta\phi(x)}\,\mathrm{e}^{-i\beta\phi(y)}$ simply approaches one. If this were the full story, the operator: $\cos\beta\phi:^2$ would be a c number, since the first two terms in (30) vanish and the last two would be c numbers. Actually, it is necessary to proceed more carefully, making an operator product expansion for the product $e^{i\beta\phi(x)}\, e^{-i\beta\phi(y)}$. In this expansion, the most singular term is a c number, but there are also less singular terms proportional to gradients of ϕ. Carrying out this expansion, keeping only the terms that do not vanish as $x \to y$ and $\Lambda \to \infty$, and averaging over the separation between x and y in a Lorentz-invariant way, we arrive at (4).

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