Nuclear Physics B280 [FS18] (1987) 225–254
North-Holland, Amsterdam

EXACT BETHE ANSATZ SOLUTION OF O(2N) SYMMETRIC THEORIES

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Received 11 July 1986

We construct exact eigenvectors and eigenvalues of O(2N) symmetric two-dimensional theories (vertex models and quantum field theories) by means of a generalization of the nested Bethe ansatz method. The spectrum of the physical states as well as the structure of the eigenvectors are explicitly derived. Their group theoretical properties are investigated, the free energy of the vertex model is given and its properties analysed.

1. Introduction

Since the exact solution of the eight-vertex model [1] considerable progress has been made on integrable multistate vertex models [2, 3]. By multistate we mean models where each link in the two-dimensional lattice can be in q different states with $q > 2$.

Although the transfer matrix eigenvalues were derived for a large class of models by the "unitarity method" and by the analytic Bethe ansatz the construction of exact eigenvectors has been done only for the Z_N symmetric [2, 4] and U(N) symmetric models [5, 6] and recently for the Sp(2N) symmetric model [7].

The purpose of this paper is to present the explicit construction of eigenvectors and eigenvalues for the O(2N) symmetric vertex model. This is done using a generalization of the nested Bethe ansatz method [2]. The construction of the exact eigenvectors besides their own interest is an important step in the program of the exact resolution of these models. The calculation of form factors and correlation functions will benefit from the knowledge of the eigenvectors constructed in the present paper.

We present our construction in the language of two-dimensional vertex models. So, we give the eigenvectors and eigenvalues of the O(2N) multistate vertex model

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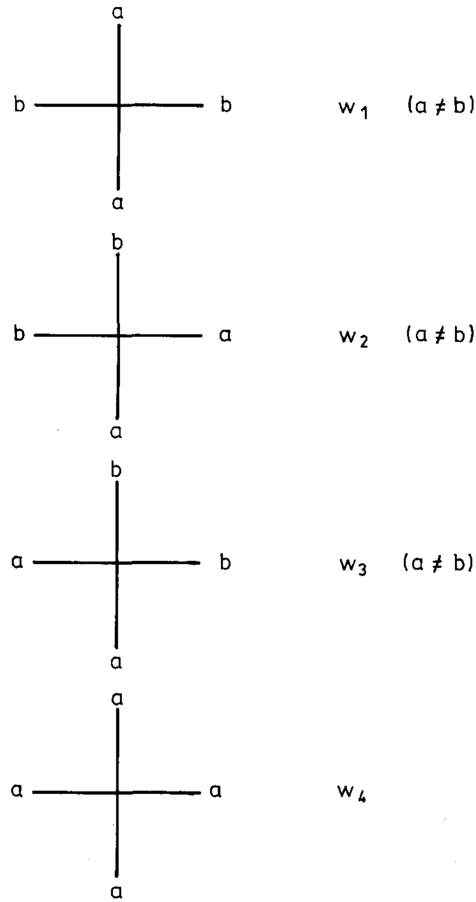

Fig. 1. The four allowed configurations at one site of a square lattice of the O(2N) vertex model.

where the statistical weights are depicted in fig. 1 and given by the matrix

$$S(\theta)=1-\frac{2}{\theta}P-\frac{2}{2N-2-\theta}K, \tag{1.1}$$

where

$$P_{ab,cd}=\delta_{ad}\delta_{bc}, \qquad K_{ab,cd}=\delta_{ab}\delta_{cd}.$$

Our results can be easily translated to fermionic quantum field theories with O(2N) symmetry. More precisely, the model defined by [8, 9]

$$\mathscr{L}=i\bar{\psi}_a \not\partial \psi_a-\left(\bar{\psi}_a\gamma_\mu\psi_c\right)_{ab}V_{cd}\left(\bar{\psi}_b\gamma^\mu\psi_d\right) \tag{1.2}$$

is directly related to the vertex model of fig. 1. Here ψ_a ($a=1,\dots,2N$) is a Dirac field and $V = g(1 + P - K)$. The eigenstates and eigenvalues of the hamiltonian of the model (1.2) can be written in terms of the eigenstates of the transfer matrix corresponding to the vertex model defined by (1.1) or fig. 1.

The construction presented here is purely algebraic and is based on the Yang-Baxter algebra of monodromy matrices [2,10]. A monodromy matrix $T_A^\Omega(\theta)$ is associated with a horizontal line of n sites of the two-dimensional lattice (eq. (2.4)). T_A^Ω acts in two spaces: A and Ω. A is a $2N$-dimensional space for the O($2N$) model and Ω a tensor product of n $2N$-dimensional spaces. The basic algebraic properties of the operators and their submatrices A_a^Ω, B_a^Ω, C_a^Ω and D_a^Ω are summarized by the Yang-Baxter algebra (for details, see sect. 2)

$$S_{AB}(\theta-\theta')T_A^\Omega(\theta)T_B^\Omega(\theta') = T_B^\Omega(\theta')T_A^\Omega(\theta)S_{AB}(\theta-\theta'), \tag{1.3}$$

$$T_A^\Omega(\theta) = \begin{pmatrix} A_a^\Omega(\theta) & B_a^\Omega(\theta) \\ C_a^\Omega(\theta) & D_a^\Omega(\theta) \end{pmatrix}. \tag{1.4}$$

The operators B_a (C_a) play the role of creation (annihilation) operators of interacting excitations (pseudoparticles). Hence the bare vacuum or reference state is defined by

$$C_a^\Omega \varphi^\Omega = 0 .$$

Eigenvectors Ψ^Ω of the transfer matrix

$$\tau^\Omega(\theta) = \mathrm{Tr}_A\left[T_A^\Omega(\theta)\right] \tag{1.5}$$

follow by filling the reference state with pseudoparticles

$$\Psi^\Omega(\{v\}) = \prod_{r=1}^{m} \sum_{i_r, j_r=1}^{N} B_{i_r j_r}^\Omega(v_r)\varphi_{\{i,j\}}^\Omega \tag{1.6}$$

(i_r, j_r are the indices of B_a^Ω as an operator on the space a). The eigenvalue problem for $\tau^\Omega(\theta)$ is solved in a recurrent way (nested Bethe ansatz method). The requirement that $\Psi^\Omega(\{v\})$ is an eigenvector of $\tau^\Omega(\theta)$ leads to another eigenvalue problem for a new family of commuting transfer matrices $\tau^{\Omega_1}(\theta)$ and $\tilde{\tau}^{\Omega_1}(\theta)$ (cf. (2.29) and (2.32)). Writing an ansatz analogous to (1.6), this eigenvalue problem leads to another eigenvalue problem for a new transfer matrix $\tau^{\Omega_2}(\theta)$ and so on. The process stops at the $(N+1)$ step since $\tau^{\Omega_N}(\theta)$ is a number. At each step parameters $\{u_r^{(k)}, 1 \leqslant r \leqslant n_k\}$ are introduced as arguments of the creation operators $B_{a_k}^{\Omega_k}$. When these numbers obey a set of algebraic equations derived in sect. 2, Ψ^Ω will be an eigenvector of $\tau^\Omega(\theta)$. It turns out that the arguments $u_i^{(N-1)}$ $(1 \leqslant i \leqslant m \leqslant n_{N-1})$ in the last step must coincide with the v_i of the first step. This would in general imply the vanishing of the eigenvector. We overcome this problem by introducing a more general set of intermediate monodromy matrices

$$T^{\Omega_k}(\theta,\{v,w\}), \qquad k=1,2,\dots,N-1$$

that reduce to the usual ones when the new parameters w_r coincide with the v_r. Then, it is possible to construct non-vanishing eigenvectors of $\tau^{\Omega}(\theta)$ by using the creation operators

$$B^{\Omega_k}\big(u_i^{(k)},\{v,v+a\varepsilon\}\big) \qquad \text{for } k=1,2,\dots,N-1$$

and letting $\varepsilon\to 0$ while the parameters a_r are defined by a set of algebraic equations (eqs. (2.75)).

The parameters $u_i^{(k)}$ ($i=1,\dots,n_k$; $1\leqslant k\leqslant N-1$) must fulfill eqs. (2.70) and (2.74). These equations have many sets of solutions describing the different eigenstates and eigenvalues of $\tau^{\Omega}(\theta)$. The maximum eigenvalue provides the free energy of the vertex model. This is computed in sects. 3 and 4. In sect. 3 the excitation spectrum is analyzed. Particular attention is paid to the group theoretical properties of the physical states formed by holes and strings. It is proved that all states constructed by the generalized Bethe ansatz (1.6) are of maximal weight. A complete catalog of excitations is found: antisymmetric tensors of rank k with $1\leqslant k\leqslant N-2$ and spinors of chirality ± 1. A natural understanding of the bound states is derived in the context of the Bethe ansatz construction of the states.

The free energy of the vertex model as well as its symmetries are investigated in sect. 4. The free energy does not reduce in general to the free energy of other models except in the particular cases $N=2$, 1 and 0, where it can be related to the free energy of the SU(2) symmetric vertex model.

2. Construction of Bethe eigenvectors

We construct in this section the eigenvectors and eigenvalues of the transfer matrix for the O(2N) symmetric vertex model. This model is defined by the following solution of the Yang-Baxter equations [11]

$$S_{AB}(\theta)=\mathbb{1}_{AB}-\frac{2}{\theta}P_{AB}-\frac{2}{\hat{\theta}}K_{AB}\,; \qquad \hat{\theta}\equiv 2N-2-\theta\,. \tag{2.1}$$

Here, and in the following, a subscript like A of an operator means that it acts on a vector space V_A. If there are more subscripts it acts on the corresponding tensor product space. Here the operator S_{AB} acts in the tensor product of real spaces

$V_A^{2N} \otimes V_B^{2N}$ where $\dim V_A^{2N} = \dim V_B^{2N} = 2N$. The operator P_{AB} exchanges both spaces and $K_{AB} = P_{AB}^{\mathrm{t}}$ is the transposed operator of P_{AB} with respect to one of the spaces.

$$P_{i_A i_B}^{j_A j_B} = \delta_{i_A}^{j_B} \delta_{i_B}^{j_A}, \qquad K_{i_A i_B}^{j_A j_B} = \delta_{i_A i_B} \delta^{j_A j_B}, \tag{2.2}$$

where $1 \leqslant i_A, i_B, j_A, j_B \leqslant 2N$. The matrix S_{AB} fulfils the crossing and reality properties

$$S_{AB}(\theta) = S_{AB}^{\mathrm{t}}(\hat{\theta}) = S_{AB}^{*}(\theta^{*}). \tag{2.3}$$

The matrix elements of S_{AB} $(\delta_{i_A i_B}^{j_A j_B})$ define the statistical weights of the vertex configurations in an $n \times n$ square lattice. So each link can be in $2N$ different states. The four allowed vertex configurations are depicted in fig. 1. The monodromy matrix of the model reads

$$T_A^{\Omega}(\theta, \{\alpha\}) = S_{An}(\theta - \alpha_n) \ldots S_{A1}(\theta - \alpha_1). \tag{2.4}$$

On the r.h.s. we have a matrix product in the space V_A^{2N} and a tensor product with respect to the spaces V_i^{2N} $(i = n, \ldots, 1)$. So, T_A^{Ω} is an operator acting both in the "auxiliary" space V_A^{2N} and in the tensor product "quantum" space.

$$\Omega = V_n^{2N} \otimes \cdots \otimes V_1^{2N}. \tag{2.5}$$

The α_i, $(1 \leqslant i \leqslant n)$ are given parameters that describe possible inhomogenities of the lattice. The transfer matrix writes as usual

$$\tau^{\Omega}(\theta, \{\alpha\}) = \mathrm{tr}_A\left[T_A^{\Omega}(\theta, \{\alpha\})\right]. \tag{2.6}$$

It is directly connected with the partition function through

$$Z(\theta, \{\alpha\}) = \mathrm{Tr}_{\Omega}\left[\tau^{\Omega}(\theta, \{\alpha\})^n\right]. \tag{2.7}$$

Our main task is to solve the eigenvalue problem of the transfer matrix

$$\tau^{\Omega}(\theta, \{\alpha\}) \Psi^{\Omega} = \lambda \Psi^{\Omega}, \tag{2.8}$$

where Ψ^{Ω} is a state in the tensor product space Ω. In order to construct the eigenvectors it is convenient to introduce instead of the real basis $|A\rangle_{\mathrm{r}}$ the complex basis

$$|a\rangle = \sqrt{\tfrac{1}{2}}\left(|2a-1\rangle_{\mathrm{r}} + i|2a\rangle_{\mathrm{r}}\right) \in V_a^N,$$

$$|\bar{a}\rangle = \sqrt{\tfrac{1}{2}}\left(|2a-1\rangle_{\mathrm{r}} - i|2a\rangle_{\mathrm{r}}\right) \in \bar{V}_a^N. \tag{2.9}$$

We identify the spaces $V_A^{2N} = V_a^N \oplus \bar{V}_a^N$ where $\dim V_a^N = \dim \bar{V}_a^N = N_0$.

The S-matrix (2.1) reads in this complex basis

$$S_{AB}(\theta)=\begin{pmatrix} S_{ab} & 0 & 0 & 0 \\ 0 & Q_{ab} & U_{ab} & 0 \\ 0 & U_{ab} & Q_{ab} & 0 \\ 0 & 0 & 0 & S_{ab} \end{pmatrix}, \tag{2.10}$$

where

$$S_{ab}=\mathbb{1}_{ab}-\frac{2}{\theta}P_{ab}, \qquad U_{ab}=-\frac{2}{\theta}P_{ab}-\frac{2}{\hat{\theta}}K_{ab},$$

$$Q_{ab}=\mathbb{1}_{ab}-\frac{2}{\hat{\theta}}K_{ab}, \qquad \hat{\theta}=2N-2-\theta. \tag{2.11}$$

Here, the operators S_{ab} (Q_{ab}) act in the tensor product of complex spaces $V_a^N\otimes V_b^N$ or $\overline{V}_a^N\otimes\overline{V}_b^N$ ($V_a^N\otimes\overline{V}_b^N$ or $\overline{V}_a^N\otimes V_b^N$) and U_{ab} maps $V_a^N\otimes\overline{V}_b^N$ onto $\overline{V}_a^N\otimes V_b^N$ or $\overline{V}_a^N\otimes V_b^N$ onto $V_a^N\otimes\overline{V}_b^N$. In order to apply the matrix Bethe ansatz method of refs. [2,4,5,7,11] we need some additional properties of the matrices S_{ab}, Q_{ab}, and U_{ab}. From eqs. (2.11) we derive

$$Q_{ab}^{-1}(\theta)=\mathbb{1}_{ab}-\frac{2}{\theta+2}K_{ab}, \tag{2.12}$$

$$P_{ab}P_{cd}U_{cd}(\theta)Q_{cd}^{-1}(\theta)=\frac{1}{\theta}\operatorname*{Res}_{\theta'=0}\left[S_{ab}(\theta')Q_{cd}^{-1}(\theta')\right]. \tag{2.13}$$

The Yang-Baxter algebra for the monodromy matrix T_A^{Ω} follows from the Yang-Baxter algebra fulfilled by the S-matrix

$$S_{AB}(\theta-\theta')S_{AC}(\theta)S_{BC}(\theta')=S_{BC}(\theta')S_{AC}(\theta)S_{AB}(\theta-\theta'). \tag{2.14}$$

One finds from eqs. (2.4) and (2.11) and the properties of tensor products

$$S_{AB}(\theta-\theta')T_A^{\Omega}(\theta)T_B^{\Omega}(\theta')=T_B^{\Omega}(\theta')T_A^{\Omega}(\theta)S_{AB}(\theta-\theta'). \tag{2.15}$$

The monodromy matrix naturally decomposes in the complex basis (2.9) as

$$T_A^{\Omega}(\theta)=\begin{pmatrix} A_a^{\Omega}(\theta) & B_a^{\Omega}(\theta) \\ C_a^{\Omega}(\theta) & D_a^{\Omega}(\theta) \end{pmatrix}, \tag{2.16}$$

so

$$\tau^{\Omega}(\theta)=\operatorname{tr}_a\left[A_a^{\Omega}(\theta)\right]+\operatorname{tr}_a\left[D_a^{\Omega}(\theta)\right]. \tag{2.17}$$

The commutation relations for the submatrices A_a^Ω, B_a^Ω, C_a^Ω and D_a^Ω of T_A^Ω follow from the Yang-Baxter algebra. Inserting eqs. (2.10) and (2.16) in eq. (2.15) yields:

$$A_a^\Omega(\theta)B_b^\Omega(\theta') = S_{ba}(\theta'-\theta)B_b^\Omega(\theta')A_a^\Omega(\theta)Q_{ba}^{-1}(\theta'-\theta)$$
$$-B_a^\Omega(\theta)A_b^\Omega(\theta')U_{ba}(\theta'-\theta)Q_{ba}^{-1}(\theta'-\theta), \qquad (2.18)$$

$$D_a^\Omega(\theta)B_b^\Omega(\theta') = Q_{ab}^{-1}(\theta-\theta')B_b^\Omega(\theta')D_a^\Omega(\theta)S_{ab}(\theta-\theta')$$
$$-Q_{ab}^{-1}(\theta-\theta')U_{ab}(\theta-\theta')B_a^\Omega(\theta)D_b^\Omega(\theta'). \qquad (2.19)$$

We will not need the other commutation rules, so we do not write them down. By means of eq. (2.13) the second terms in eqs. (2.18) and (2.19) can be rewritten as

$$-\frac{1}{\theta'-\theta}\operatorname*{Res}_{\theta''=\theta'}\left\{S_{ba}(\theta''-\theta')B_b^\Omega(\theta)A_a^\Omega(\theta'')Q_{ba}^{-1}(\theta''-\theta')\right\}, \qquad (2.18a)$$

$$-\frac{1}{\theta'-\theta}\operatorname*{Res}_{\theta''=\theta'}\left\{Q_{ab}^{-1}(\theta'-\theta'')B_b^\Omega(\theta)D_a^\Omega(\theta'')S_{ab}(\theta'-\theta'')\right\}. \qquad (2.19a)$$

We have now enough machinery to start the construction of the eigenvectors of $\tau^\Omega(\theta,\{\alpha\})$. We seek for an eigenvector with an algebraic Bethe ansatz structure

$$\psi^\Omega = \prod_{r=1}^{m} B_{i_r j_r}^\Omega(v_r)\varphi_{\{i,j\}}^\Omega \qquad (2.20)$$

(summation over the indices i_l, $j_l = 1,2,\dots,N$ is assumed). We denote by a subscript Ω vectors and operators of the space Ω (c.f. (2.5)). In eq. (2.20) the parameters $v_1,\dots,v_m$ are arbitrary for the moment and we require the vectors $\varphi_{\{i,j\}}^\Omega$ for each set of the indices $i_1,\dots,i_m$; $j_1,\dots,j_m$ to verify

$$C_{kl}^\Omega(\theta)\varphi_{\{i,j\}}^\Omega = 0 \qquad (2.21)$$

for all $k,l = 1,\dots,N$. It can be checked from eqs. (2.4), (2.10) and (2.16) that eq. (2.21) holds if all "bar" components of $\varphi_{\{i,j\}}^\Omega$ vanish. That means the vectors $\varphi_{\{i,j\}}^\Omega$ lie in a subspace $\Omega^{(0)}$ of Ω called "reference space" where

$$\Omega^{(0)} = \mathrm{V}_n^N \otimes \cdots \otimes \mathrm{V}_1^N, \qquad (2.22)$$

while

$$\Omega = (\mathrm{V}^N \otimes \bar{\mathrm{V}}^N)_n \otimes \cdots \otimes (\mathrm{V}^N \otimes \bar{\mathrm{V}}^N)_1.$$

Now we start to solve the eigenvalue problem eq. (2.8) and, following the general strategy of the algebraic Bethe ansatz [2,4,12] we apply the transfer matrix $\tau^{\Omega}(\theta)$ in terms of the decomposition (2.17) to the state given by eq. (2.20). We begin by acting with $A_a^{\Omega}(\theta)$ on Ψ^{Ω} and we push it to the right through the $B_{i_r j_r}^{\Omega}(v_r)$ using the commutations rule (2.18). Then we shall do the same for $D_a^{\Omega}(\theta)$.

Finally taking trace over V_a we will relate in this way the action of $\tau^{\Omega}(\theta)$ on Ψ^{Ω} to the action of two new transfer matrices τ^{Ω_1} and $\tilde{\tau}^{\Omega_1}$ (corresponding to A^{Ω} and D^{Ω}) on a new state Ψ^{Ω_1}.

Two types of terms arise when A_a^{Ω} passes through $B_{i_r j_r}^{\Omega}(v_r)$ (see eq. (2.18)). The first type comes from the first term in eq. (2.18) and $A^{\Omega}(\theta)$ and $B_{i_r j_r}^{\Omega}(v_r)$ keep their original arguments unchanged. They are "wanted" terms, and they will finally give a vector proportional to Ψ^{Ω}.

The terms arising from the second term in eq. (2.18) are called "unwanted" since they contain $B^{\Omega}(\theta)$ and so they can never give a vector proportional to Ψ^{Ω}. One gets from eq. (2.20), after repeated use of eq. (2.18),

$$A_a^{\Omega}(\theta)\Psi^{\Omega} = \left[S_{1'a}(v_1-\theta)\dots S_{m'a}(v_m-\theta)\right]_{\{i,k\}}\prod_{r=1}^{m} B_{k_r l_r}^{\Omega}(v_r)A_a^{\Omega}(\theta)$$
$$\times\left[Q_{m''a}^{-1}(v_m-\theta)\dots Q_{1''a}^{-1}(v_1-\theta)\right]_{\{l,j\}}\varphi_{\{i,j\}}^{\Omega} + \text{unwanted terms}. \quad (2.23)$$

The indices $\{i,k\}$ and $\{j,l\}$ refer here to the auxiliary spaces $V_{r'}^N$ and $V_{r''}^N$ $(1 \leqslant r', r'' \leqslant m)$, respectively. Taking the trace w.r.t. V_a, eq. (2.23) yields:

$$\operatorname{tr}_a\left[A_a^{\Omega}(\theta)\right]\Psi^{\Omega} = \prod_{r=1}^{m} B_{k_r l_r}^{\Omega}(v_r)\operatorname{tr}_a\left\{\left[S_{1'a}^{t}(v_1-\theta)\dots S_{m'a}^{t}(v_m-\theta)\right]_{\{k,i\}}\right.$$
$$\left.\times A_a^{\Omega}(\theta)\left[Q_{m''a}^{-1}(v_m-\theta)\dots Q_{1''a}^{-1}(v_1-\theta)\right]_{\{l,j\}}\right\}\varphi_{\{i,j\}}^{\Omega}$$
$$+\text{unwanted terms}. \quad (2.24)$$

The superscript t means transposition w.r.t. to $V_{r'}^N$ $(1 \leqslant r' \leqslant m)$. Since here $A_a^{\Omega}(\theta)$ acts on the vectors φ^{Ω} in the reference state space, i.e. obeying the constraint (2.21) it can be replaced by

$$A_a^{\Omega}(\theta) \to S_{an}(\theta-\alpha_n)\dots S_{a1}(\theta-\alpha_1). \quad (2.25)$$

It is convenient to enlarge the space $\Omega^{(0)}$ such that the "auxiliary" indices $\{i, j\}$ also become "quantum" ones

$$\Omega_1 = \left(V_{1'}^N\otimes\cdots\otimes V_{m'}^N\right)\otimes\left(V_n^N\otimes\cdots\otimes V_1^N\right)\otimes\left(V_{m''}^N\otimes\cdots\otimes V_{1''}^N\right). \quad (2.26)$$

The set of vectors $\varphi_{\{i,j\}}^{\Omega} \in \Omega^{(0)}$ is reinterpreted as one-vector $\Psi^{\Omega_1} \in \Omega_1$ where

$$\left[\Psi^{\Omega_1}\right]_{\{i,j\}} = \varphi_{\{i,j\}}^{\Omega} \quad (2.27)$$

are the components of Ψ^{Ω_1} with respect to the subspace defined by the first and third group of factors in eq. (2.26). In order to rewrite eq. (2.24) in the space Ω_1 we define a new monodromy operator $T^{\Omega_1}(\theta)$ acting in Ω_1 and V_a and the corresponding transfer matrix by

$$T_a^{\Omega_1}(\theta,\{\alpha,v,w\}) = \left[S^{t}_{1'a}(v_1-\theta)\dots S^{t}_{m'a}(v_m-\theta)\right]\left[S_{an}(\theta-\alpha_n)\dots S_{a1}(\theta-\alpha_1)\right]$$
$$\times\left[Q^{-1}_{m''a}(w_m-\theta)\dots Q^{-1}_{1''a}(w_1-\theta)\right], \qquad (2.28)$$

$$\tau^{\Omega_1}(\theta,\{\alpha,v,w\}) = \mathrm{tr}_a\left[T_a^{\Omega_1}(\theta,\{\alpha,v,w\})\right]. \qquad (2.29)$$

For later convenience we have introduced the parameters $w_1,\dots,w_m$; finally they have to be identified with the $v_1,\dots,v_m$. So we can rewrite eq. (2.24) as

$$\mathrm{tr}_a\left[A_a^{\Omega}(\theta,\{\alpha\})\right]\Psi^{\Omega} = \prod_{r=1}^{m} B^{\Omega}_{i_rj_r}(v_r)\left[\tau^{\Omega_1}(\theta,\{\alpha,v,v\})\Psi^{\Omega_1}\right]_{\{i,j\}}$$
$$+\,\text{unwanted terms}, \qquad (2.30)$$

where the notation (2.27) has been used.

The unwanted terms can be worked out in an analogous way. One finds from eqs. (2.18a) and (2.25) for the unwanted term containing $A_a^{\Omega}(v_1)$ analogously to eq. (2.30)

$$\left\{tr_a\left[A_a^{\Omega}(\theta)\right]\Psi^{\Omega}\right\}^{(1)}_{\mathrm{unw}}$$
$$= -B^{\Omega}_{i_1j_1}(\theta)\prod_{r=2}^{m}B^{\Omega}_{i_rj_r}(v_r)\frac{1}{\theta-v_1}\operatorname*{Res}_{\theta'=v_1}\left[\tau^{\Omega_1}(\theta',\{\alpha,v,v\})\Psi^{\Omega_1}\right]_{\{i,j\}}. \qquad (2.31)$$

Analogous expressions follow for the remaining $m-1$ unwanted terms. They can explicitly be derived as in ref. [2]. We interrupt the analysis of the first term in the decomposition (2.27), the A-term. Using eqs. (2.19) and (2.19a) instead of eqs. (2.18) and (2.18a) completely analogous results will be obtained for the second, the D-term. In eqs. (2.30) and (2.31) A has to be replaced by D and τ by another transfer matrix $\tilde{\tau}$ defined by:

$$\tilde{\tau}^{\Omega_1}(\theta,\{\alpha,v,w\}) = \mathrm{tr}_a\left[\tilde{T}_a^{\Omega_1}(\theta,\{\alpha,v,w\})\right], \qquad (2.32)$$

$$\tilde{T}_a^{\Omega_1}(\theta,\{\alpha,v,w\}) = \left[Q^{-1t}_{a1'}(\theta-v_1)\dots Q^{-1t}_{am'}(\theta-v_m)\right]\left[Q_{an}(\theta-\alpha_n)\dots Q_{a1}(\theta-\alpha_1)\right]$$
$$\times\left[S_{am''}(\theta-w_m)\dots S_{a1''}(\theta-w_1)\right]. \qquad (2.33)$$

The Yang-Baxter algebra for the monodromy matrices T^{Ω_1} and $\tilde{T}^{\Omega_1}$ can be obtained from the Yang-Baxter algebra of the matrices S and Q which follow from eqs. (2.10) and (2.15)

$$S_{ab}(\theta-\theta')T_a^{\Omega_1}(\theta)T_b^{\Omega_1}(\theta') = T_b^{\Omega_1}(\theta')T_a^{\Omega_1}(\theta)S_{ab}(\theta-\theta'), \tag{2.34}$$

$$Q_{ab}(\theta-\theta')\tilde{T}_a^{\Omega_1}(\theta)T_b^{\Omega_1}(\theta') = T_b^{\Omega_1}(\theta')\tilde{T}_a^{\Omega_1}(\theta)Q_{ab}(\theta-\theta'), \tag{2.35}$$

where eq. (2.34) also holds for $\tilde{T}^{\Omega_1}$. From these equations we conclude that the transfer matrices commute.

$$\left[\tau^{\Omega_1}(\theta), \tau^{\Omega_1}(\theta')\right] = \left[\tilde{\tau}^{\Omega_1}(\theta), \tilde{\tau}^{\Omega_1}(\theta')\right] = \left[\tau^{\Omega_1}(\theta), \tilde{\tau}^{\Omega_1}(\theta')\right] = 0. \tag{2.36}$$

Hence τ^{Ω_1} and $\tilde{\tau}^{\Omega_1}$ can be diagonalized simultaneously. Eqs. (2.30) and (2.31) tell us that the eigenvalue problem

$$\tau^{\Omega}(\theta,\{\alpha\})\Psi^{\Omega}(\{a,v\}) = \lambda\Psi^{\Omega}(\{\alpha,v\})$$

has been transformed to two eigenvalue problems

$$\tau^{\Omega_1}(\theta,\{\alpha,v,v\})\Psi^{\Omega_1} = \lambda_1\Psi^{\Omega_1},$$

$$\tilde{\tau}^{\Omega_1}(\theta,\{\alpha,v,v\})\Psi^{\Omega_1} = \tilde{\lambda}_1\Psi^{\Omega_1}, \tag{2.37}$$

to be solved simultaneously with the following constraints that eliminate the unwanted terms

$$\operatorname*{Res}_{\theta=v_j}(\lambda_1+\tilde{\lambda}_1) = 0; \qquad j=1,\dots,n. \tag{2.38}$$

For all solutions $\{v\}$ of these equations the unwanted terms cancel and the corresponding eigenvalue and eigenvector of τ^{Ω} are

$$\lambda = \lambda_1 + \tilde{\lambda}_1, \tag{2.39}$$

$$\Psi^{\Omega} = \sum_{r=1}^{m} B_{i_r j_r}^{\Omega}(v_r)[\Psi^{\Omega_1}]_{\{i,j\}}. \tag{2.40}$$

As we will see later the eqs. (2.37)–(2.40) will not always give the correct result. The reason is that for certain solutions of the Bethe ansatz equation the vector Ψ^{Ω} (given by eq. (2.40)) vanishes if the parameters v_r and w_r are taken to be equal (as in eq. (2.37)). We overcome this problem by taking the limit $w_r \to v_r$ in a specified way (see below). Let us now consider the generalized eigenvalue problem

$$\tau^{\Omega_1}(\theta,\{\alpha,v,w\})\Psi^{\Omega_1} = \lambda_1\Psi^{\Omega_1}. \tag{2.41}$$

From the explicit expressions (2.11) for S_{ab} we see that eq. (2.34) defines an SU(N) symmetric Yang-Baxter algebra for T^{Ω_1}. This implies that the diagonalization of τ^{Ω_1} should be possible by means of the nested Bethe ansatz known for the SU(N) case [4–6]. But since T^{Ω_1} also contains K operators, there are new features and therefore we will explain the procedure in more detail. We split from the N-dimensional space V_a^N a one-dimensional one

$$\mathrm{V}_a^N = \mathrm{V} \oplus \mathrm{V}_{a_1}^{N-1}. \tag{2.42}$$

The corresponding decomposition of the monodromy matrix $T_a^{\Omega_1}(\theta, \{\alpha, v, w\})$ reads

$$T_a^{\Omega_1} = \begin{pmatrix} A^{\Omega_1} & B_{a_1}^{\Omega_1} \\ C^{\Omega_1} & D_{a_1}^{\Omega_1} \end{pmatrix}, \tag{2.43}$$

where $A^{\Omega_1}, B^{\Omega_1}, C^{\Omega_1}, D^{\Omega_1}$ are with respect to the "auxiliary" space $1\times 1, 1\times (N-1), (N-1)\times 1, (N-1)\times(N-1)$ matrices, respectively. Their commutation rules follow from the Yang-Baxter algebra (2.34)

$$A^{\Omega_1}(\theta) B_{a_1}^{\Omega_1}(\theta') = \frac{\theta'-\theta-2}{\theta'-\theta} B_{a_1}^{\Omega_1}(\theta') A^{\Omega_1}(\theta) + \frac{2}{\theta'-\theta} B_{a_1}^{\Omega_1}(\theta) A^{\Omega_1}(\theta'), \tag{2.44}$$

$$D_{a_1}^{\Omega_1}(\theta) B_{b_1}^{\Omega_1}(\theta') = B_{b_1}^{\Omega_1}(\theta') D_{a_1}^{\Omega_1}(\theta) S_{a_1 b_1}(\theta-\theta') + \frac{2}{\theta'-\theta} B_{a_1}^{\Omega_1}(\theta) D_{b_1}^{\Omega_1}(\theta'), \tag{2.45}$$

where $S_{b_1 a_1}$ is the restriction of S_{ba} (given by eq. (2.11)) to the space $\mathrm{V}_{b_1}^{N-1} \otimes \mathrm{V}_{a_1}^{N-1}$ of dimension $(N-1)^2$. The algebraic Bethe ansatz for the eigenvectors of τ^{Ω_1} writes

$$\psi^{\Omega_1} = \prod_{r=1}^{n_1} B_{i_r}^{\Omega_1}\big(u_r^{(1)}\big) \varphi_{\{i\}}^{\Omega_1} \qquad \big(0 \leqslant i_1, \dots, i_{n_1} \leqslant N-1\big), \tag{2.46}$$

where the (level 1) reference states fulfil

$$C^{\Omega_1} \varphi_{\{i\}}^{\Omega_1} = 0. \tag{2.47}$$

From the definition of T^{Ω_1}, (eqs. (2.28) and (2.11)) it follows that the subspace of Ω_1 defined by this condition is

$$\Omega_1^{(0)} = \big(\mathrm{V}_{1'}^{N-1} \otimes \cdots \otimes \mathrm{V}_{m'}^{N-1}\big) \otimes \big(\mathrm{V}_n \otimes \cdots \otimes \mathrm{V}_1\big) \otimes \big(\mathrm{V}_{m''}^{N-1} \otimes \cdots \otimes \mathrm{V}_{1''}^{N-1}\big). \tag{2.48}$$

We again follow the general strategy of the algebraic Bethe ansatz. We apply A^{Ω_1} and $\mathrm{Tr}_{a_1}[D_{a_1}^{\Omega_1}]$ to ψ^{Ω_1} and push them through all the B^{Ω_1} using the commutation rules ((2.44) and (2.45)). From the first terms of eqs. (2.44) and (2.45) we get the "wanted" contributions

$$A^{\Omega_1}(\theta)\psi^{\Omega_1} = \prod_{i=1}^{n_1} \frac{u_i^{(1)} - \theta - 2}{u_i^{(1)} - \theta} \prod_{r=1}^{n_1} B_{i_r}^{\Omega_1}\left(u_r^{(1)}\right) A^{\Omega_1}(\theta) \varphi_{\{i\}}^{\Omega_1} + \text{unwanted terms}, \tag{2.49}$$

$$\operatorname{Tr}_{a_1}\left[D_{a_1}^{\Omega_1}(\theta)\right]\psi^{\Omega_1} = \prod_{r=1}^{n_1} B_{i_r}^{\Omega_1}\left(u_r^{(1)}\right)\left[\tau^{\Omega_2}(\theta)\psi^{\Omega_2}\right]_{\{i\}} + \text{unwanted terms}. \tag{2.50}$$

In the second equation we have introduced the new transfer matrix τ^{Ω_2} as the a_1 trace of the new monodromy matrix

$$\begin{aligned} T_{a_1}^{\Omega_2}\left(\theta, \{v, w, u^{(1)}\}\right) = &\left(S_{1'a_1}^{t}(v_1 - \theta) \dots S_{m'a_1}^{t}(v_m - \theta)\right) \\ &\times \left(Q_{m''a_1}^{-1}(w_m - \theta) \dots Q_{1''a_1}^{-1}(w_1 - \theta)\right) \\ &\times \left(S_{a_1 n_1}\left(\theta - u_{n_1}^{(1)}\right) \dots S_{a_1 1}\left(\theta - u_1^{(1)}\right)\right) \end{aligned} \tag{2.51}$$

acting in the $(N-1)$-dimensional "auxiliary" space $V_{a_1}^{N-1}$ and the tensor product space

$$\begin{aligned} \Omega_2 &= \Omega_1^{(0)} \otimes \left(V_{n_1}^{N-1} \otimes \cdots \otimes V_1^{N-1}\right) \\ &= \left(V_{1'}^{N-1} \otimes \cdots \otimes V_{m'}^{N-1}\right) \otimes \left(V_{m''}^{N-1} \otimes \cdots \otimes V_{1''}^{N-1}\right) \otimes \left(V_{n_1}^{N-1} \otimes \cdots \otimes V_1^{N-1}\right). \end{aligned} \tag{2.52}$$

This means that the auxiliary indices i_r introduced by the B-operators in eq. (2.50) have been reinterpreted as additional quantum indices for the next Bethe ansatz step. The set of reference vectors $\varphi_{\{i\}}^{\Omega_1} \in \Omega_1^{(0)}$ has been reinterpreted as one-vector $\psi^{\Omega_2} \in \Omega_2$

$$\left[\psi^{\Omega_2}\right]_{\{i\}} = \varphi_{\{i\}}^{\Omega_1}. \tag{2.53}$$

By $[\cdot]_{\{i\}}$ we denote the components of a vector in Ω_2 with respect to the third group of spaces in eq. (2.52). Note that the reference states φ^{Ω_1} are eigenstates of the operator A^{Ω_1}. If the unwanted terms of eqs. (2.45) and (2.50) cancel we therefore obtain for the eigenvalue of τ^{Ω_1}

$$\lambda_1\left(\theta, \{\alpha, v, w, u^{(1)}\}\right) = \prod_{i=1}^{n} \frac{\theta - \alpha_i - 2}{\theta - \alpha_i} \prod_{i=1}^{n_1} \frac{\theta - u_i^{(1)} + 2}{\theta - u_i^{(1)}} + \lambda_2, \tag{2.54}$$

where λ_2 is defined by the eigenvalue problem of the next step

$$\tau^{\Omega_2}\psi^{\Omega_2} = \lambda_2 \psi^{\Omega_2}. \tag{2.55}$$

The unwanted terms of eqs. (2.49) and (2.50) can be written analogously to eq. (2.31) in terms of residues of the wanted ones. Those which contain the operator $B_{i_1}^{\Omega_1}(\theta)$ are

$$\left[A^{\Omega_1}(\theta)\psi^{\Omega_1}\right]^{(1)}_{\text{unwanted}} = -B_{i_1}^{\Omega_1}(\theta)\prod_{r=2}^{n_1} B_{i_r}\left(u_r^{(1)}\right)$$
$$\times \frac{1}{\theta - u_1^{(1)}} \operatorname*{Res}_{\theta' = u_1^{(1)}} \prod_{i=1}^{n_1} \frac{u_i^{(1)} - \theta' - 2}{u_i^{(1)} - \theta'} A^{\Omega_1}(\theta')\varphi^{\Omega_1}_{\{i\}},$$

$$\left[\mathrm{Tr}_{a_1} D_{a_1}^{\Omega_1}(\theta)\psi^{\Omega_1}\right]^{(1)}_{\text{unwanted}} = -B_{i_1}^{\Omega_1}(\theta)\prod_{r=2}^{n_1} B_{i_r}\left(u_r^{(1)}\right)$$
$$\times \frac{1}{\theta - u_1^{(1)}} \operatorname*{Res}_{\theta' = u_1^{(1)}} \left[\tau^{\Omega_2}(\theta')\psi^{\Omega_2}\right]_{\{i\}}. \tag{2.56}$$

Analogous expressions follow for the remaining $n_1 - 1$ unwanted terms. Therefore the eigenvalue conditions for the parameters $u_i^{(1)}$ read

$$\operatorname*{Res}_{\theta = u_j^{(1)}} \lambda_1\left(\theta, \{\alpha, v, w, u^{(1)}\}\right) = 0; \qquad j = 1, \dots, n_1. \tag{2.57}$$

(These analyticity requirements are of course necessary since in eq. (2.41) the operator τ^{Ω_1} does not depend on the $u_j^{(1)}$ and the state ψ^{Ω_1} is independent of θ.)

In order to continue the nested Bethe ansatz procedure we note that the eigenvalue problems step 1 (c.f. eqs. (2.41), (2.28), (2.29)) and step 2 (c.f. eqs. (2.55), (2.51)) are completely analogous, $(V^N, \Omega_1, \lambda_1, \{\alpha\}, n, \{u^{(1)}\}, n_1)$ correspond to $(V^{N-1}, \Omega_2, \lambda_2, \{u^{(1)}\}, n_1, \{u^{(2)}\}, n_2)$ where the new parameters $u_1^{(2)}, \dots, u_{n_2}^{(2)}$ are introduced analogously to the $u_r^{(1)}$ in eq. (2.46). Therefore the solution of step 2 is obtained from that of step 1 eq. (2.54) and the conditions (2.57), replacing in addition λ_2 by λ_3. (The different order of two groups of factors in eq. (2.28) compared to eq. (2.51) will not affect this statement.) Hence the recursion relation (2.54) solves the diagonalization problem of τ^{Ω_1} completely. The dimension of the V-space decreases in each step by one which means that τ^{Ω_N} reduces to a number

$$\tau^{\Omega_N} = \lambda_N = \prod_{i=1}^{n_{N-1}} \frac{\theta - u_i^{(N-1)} - 2}{\theta - u_i^{(N-1)}} \prod_{i=1}^{m} \frac{\theta - v_i + 2}{\theta - v_i} \frac{\theta - w_i}{\theta - w_i - 2}. \tag{2.58}$$

The final result for the eigenvalue of $\tau^{\Omega_1}(\theta,\{\alpha,v,w\})$ reads

$$\lambda_1\big(\theta,\{\alpha,v,w,u^{(1)},\dots,u^{(N-1)}\}\big)=\sum_{k=0}^{N-1}F_k(\theta),\tag{2.59}$$

where

$$F_k(\theta)=\prod_{i=1}^{n_k}\frac{\theta-u_i^{(k)}-2}{\theta-u_i^{(k)}}\prod_{i=1}^{n_{k+1}}\frac{\theta-u_i^{(k+1)}+2}{\theta-u_i^{(k+1)}},\qquad k<N-1,$$

$$F_{N-1}(\theta)=\lambda_N,\qquad \{u^{(0)}\}=\{\alpha\},\qquad n_0=n.\tag{2.60}$$

The parameters $u_j^{(k)}$ are the roots of the "Bethe ansatz equations" which follow from the cancellation conditions of the unwanted terms

$$\operatorname*{Res}_{\theta=u_j^{(k)}}\lambda_1=0;\qquad j=1,\dots,n_k,\ k=1,\dots,N-1$$

or

$$\left.\frac{F_{k-1}}{F_k}\right|_{\theta=u_j^{(k)}}=\prod_{i=1}^{n_k}\frac{u_j^{(k)}-u_i^{(k)}+2}{u_j^{(k)}-u_i^{(k)}-2}\prod_{l=k\pm1}\prod_{i=1}^{n_l}\frac{u_j^{(k)}-u_i^{(l)}-k+l-1}{u_j^{(k)}-u_i^{(l)}-k+l+1}=-1.\tag{2.61}$$

For $k=N-1$, the factor corresponding to $l=k+1$ has to be replaced by

$$\prod_{i=1}^{m}\frac{u_j^{(k)}-v_i}{u_j^{(k)}-w_i}\,\frac{u_j^{(k)}-w_i-2}{u_j^{(k)}-v_i+2}.\tag{2.61a}$$

We now come to the diagonalization problem of the transfer matrix $\tilde{\tau}^{\Omega_1}$. We do not need to perform long calculations since the eigenvalues $\tilde{\lambda}_1$ of $\tilde{\tau}^{\Omega_1}$ follow from those of τ^{Ω_1} using the crossing and reality properties of the matrix S_{AB}, eq. (2.3). For the monodromy matrix defined by eq. (2.4) we obtain from eq. (2.3)

$$T_A^{\Omega}(\theta,\{\alpha\})=T_A^{\Omega\mathrm{t}}(\hat{\theta},\{-\alpha\})=T_A^{\Omega*}(\theta^*,\{\alpha^*\}),\tag{2.62}$$

where $\hat{\theta}=2N-2-\theta$ and t means transposition with respect to the space Ω. These relations imply for the eigenvalues of the transfer matrix τ^{Ω}

$$\lambda\big(\theta,\{\alpha,v,u^{(k)}\}\big)=\lambda^*\big(\hat{\theta}^*,\{-\alpha^*,v,u^{(k)}\}\big).\tag{2.63}$$

Moreover we find from the explicit expressions (2.11) and (2.12)

$$Q_{ab}(\theta)=S_{ab}^{\mathrm{t}}(\hat{\theta}),$$

$$Q_{ab}^{-1}(\theta)=S_{ab}^{\mathrm{t}}(\theta+2).\tag{2.64}$$

So the monodromy matrix $\tilde{T}^{\Omega_1}$ can be expressed in terms of T^{Ω_1}

$$\tilde{T}^{\Omega_1}(\theta,\{\alpha,v,w\}) = T^{\Omega_1 t}(\hat{\theta},\{-\alpha,\tilde{w},\tilde{v}\}), \tag{2.65}$$

where $\tilde{v}_i = 2N - v_i$ and $\tilde{w}_i = 2N - 4 - w_i$. Therefore the eigenvalues $\tilde{\lambda}_1$ of $\tilde{\tau}^{\Omega_1}$ obtained by those of τ^{Ω_1} are

$$\begin{aligned}\tilde{\lambda}_1(\theta,\{\alpha,v,w,\tilde{u}^{(k)}\}) &= \lambda_1^*(\hat{\theta}^*,\{-\alpha^*,\tilde{v}^*,\tilde{w}^*,\tilde{u}^{(k)}\})\\ &= \lambda_1(\hat{\theta},\{-\alpha,\hat{w},\hat{v},\tilde{u}^{(k)}\})\end{aligned} \tag{2.66}$$

for sets of parameters $\tilde{u}_i^{(k)}$ fulfilling Bethe ansatz equations. The last equality in eq. (2.66) follows from the explicit expression for λ_1 given by eqs. (2.58)–(2.60). The eigenvalue of the transfer matrix τ^{Ω}

$$\lambda(\theta,\{\alpha,v,u^{(k)}\}) = \lambda_1(\theta,\{\alpha,v,v,u^{(k)}\}) + \tilde{\lambda}_1(\theta,\{\alpha,v,v,\tilde{u}^{(k)}\}) \tag{2.67}$$

fulfils the crossing relation (2.63) provided that the following relations hold

$$v_i^* = \hat{v}_i = 2N-2-v_i, \qquad \tilde{u}_i^{(k)} = u_i^{(k)}; \qquad i=1,\dots,n_k,\ k=1,\dots,N-1. \tag{2.68}$$

On the r.h.s. of eq. (2.67) the limit $w_i \to v_i$ has been assumed. As we will discuss now this has to be done in a very specific way.

If we proceeded naively we would set $w_i = v_i$ and try to calculate the roots v_i from the Bethe ansatz equations (2.38). But eq. (2.58) shows that the poles at $\theta = w_i$ are cancelled by zeros at $\theta = v_i$ and we would not get any equations for the parameters v_i. On the other hand, there are poles at $\theta = v_i + 2$ ($\theta = v_i - 2$) in λ_1 ($\tilde{\lambda}_1$) which are not related to any unwanted terms. These singularities must of course be absent in the eigenvalue $\lambda = \lambda_1 + \tilde{\lambda}_1$ since in eq. (2.8) the operator $\tau^{\Omega}(\theta,\{\alpha\})$ does not depend on the v_i and the state $\psi^{\Omega}(\{\alpha,v\})$ is independent of θ. The solution of this puzzle is that the state ψ^{Ω} constructed in this naive way would be (in general) identical to zero. We overcome these problems by writing the state ψ^{Ω} as a limit

$$\begin{aligned}\Psi^{\Omega}(\{\alpha,v,u^{(k)}\}) = \lim_{\varepsilon\to 0}\prod_{i=1}^{m} B^{\Omega}(v_i)&\\ \times\prod_{k=1}^{N-1}\left[\prod_{i=1}^{n_k} B^{\Omega_k}(u_i^{(k)},\{v,v+a\varepsilon\})\right]\varphi^{\Omega_{N-1}}&.\end{aligned} \tag{2.69}$$

We have introduced the parameters $a_1,\dots,a_m$ which have to be determined together with the roots v_i, $u_i^{(k)}$ from the analyticity condition of λ as a function of θ. Details are given in appendix A. The result can be formulated as follows.

The poles at $\theta = v_i \pm 2$ are cancelled by zeros at $\theta = u_i^{(N-1)} \pm 2$. This means each v_i coincides with a $u_i^{(N-1)}$. Introducing a new notation we write (after reordering the $u_i^{(N-1)}$)

$$\begin{aligned} u_i^{(N-1)} &= v_i = iq_i^{(+)} + N - 1, && i = 1,\dots, n_+ = m, \\ u_{i+m}^{(N-1)} &= iq_i^{(-)} + N - 1, && i = 1,\dots, n_- = n_{N-1} - m. \end{aligned} \tag{2.70}$$

Then after the substitutions

$$\begin{aligned} \alpha_i &= iq_i^{(0)}, && i = 1,\dots, n_0 = n, \\ u_i^{(k)} &= iq_i^{(k)} + k, && i = 1,\dots, n_i,\ k = 1,\dots, N-2, \end{aligned} \tag{2.71}$$

the eigenvalue of the transfer matrix $\tau^{\Omega}(\theta, \{iq^{(0)}\})$ reads

$$\lambda\left(\theta, \{q^{(k)}\}\right) = \sum_{k=0}^{2N-1} G_k(\theta), \tag{2.72}$$

where the functions G_k follow from the F_k defined by eqs. (2.60) and (2.58) and eqs. (2.70)–(2.71)

$$\begin{aligned} G_k(\theta) &= \prod_{i=1}^{n_k} \frac{\theta - iq_i^{(k)} - k - 2}{\theta - iq_i^{(k)} - k} \prod_{i=1}^{n_{k+1}} \frac{\theta - iq_i^{(k+1)} - k + 1}{\theta - iq_i^{(k+1)} - k - 1}, \qquad k = 0,\dots, N-3, \\ G_{N-2}(\theta) &= \prod_{i=1}^{n_{N-2}} \frac{\theta - iq_i^{(N-2)} - N}{\theta - iq_i^{(N-2)} - N + 2} \prod_{i=1}^{n_-} \frac{\theta - iq_i^{(-)} - N + 3}{\theta - iq_i^{(-)} - N + 1} \prod_{i=1}^{n_+} \frac{\theta - iq_i^{(+)} - N + 3}{\theta - iq_i^{(+)} - N + 1}, \\ G_{N-1}(\theta) &= \prod_{i=1}^{n_-} \frac{\theta - iq_i^{(-)} - N - 1}{\theta - iq_i^{(-)} - N + 1} \prod_{i=1}^{n_+} \frac{\theta - iq_i^{(+)} - N + 3}{\theta - iq_i^{(+)} - N + 1}, \\ G_k(\theta) &= G^*_{2N-1-k}(2N - 2 - \theta). \end{aligned} \tag{2.73}$$

The last equation follows from eqs. (2.66)–(2.67). The Bethe ansatz equations which determine the parameters $q_j^{(k)}$, $j = 1,\dots, n_k$, $k = 1,\dots, N-2, -, +$ are

$$\prod_{i=1}^{n_k} \frac{i\left(q_j^{(k)} - q_i^{(k)}\right) + 2}{i\left(q_j^{(k)} - q_i^{(k)}\right) - 2} \prod_{l \in \mathrm{L}_k} \prod_{i=1}^{n_l} \frac{i\left(q_j^{(k)} - q_i^{(l)}\right) - 1}{i\left(q_j^{(k)} - q_i^{(l)}\right) + 1} = -1, \tag{2.74}$$

where $\mathrm{L}_k = \{k-1, k+1\}, \{N-3, -, +\}, \{N-2\}$ for $1 \leqslant k \leqslant N-3$, $k = N-2$, $k = \mp$, respectively. The parameters a_j needed for the limit in eq. (2.69) are given by

$$a_j = 1 - \prod_{i=1}^{n_-} \frac{i\left(q_j^{(+)} - q_i^{(-)}\right) + 2}{i\left(q_j^{(+)} - q_i^{(-)}\right) - 2} \prod_{i=1}^{n_+} \frac{i\left(q_j^{(+)} - q_i^{(+)}\right) - 2}{i\left(q_j^{(+)} - q_i^{(+)}\right) + 2}. \tag{2.75}$$

They vanish for the ground state and all antisymmetric tensor excitations of rank $\leqslant N-2$. They do not vanish only for excitations which are non-symmetric with respect to + and − chirality spinors.

The eigenvalues of the O($2N$) transfer matrix given by eqs. (2.72)–(2.73) and the Bethe ansatz equations (2.74) coincide with the corresponding expressions obtained by the "analytic Bethe ansatz" in ref. [7]. Our derivation provides a rigorous proof of these equations.

3. Solution of the eigenvalue equations

In this section we present solutions of the O($2N$) Bethe ansatz equations. We discuss the ground state of the O($2N$) transfer matrix, the one-particle excitations and their quantum numbers. The well known [2,4,5,10] procedure to solve Bethe ansatz equations was applied in refs. [8,9] to the O($2N$) case. Here, we report some results and then analyse the spectrum.

Taking the logarithm of eqs. (2.74)

$$2\pi J_j^{(k)}+\sum_{i=1}^{n_k}\phi\left(\left(q_j^{(k)}-q_i^{(k)}\right)/2\right)-\sum_{l\in \mathrm{L}_k}\sum_{i=1}^{n_l}\phi\left(q_j^{(k)}-q_i^{(l)}\right)=0,$$

$$k=1,\dots,N-2,-,+ \qquad (3.1)$$

(where $\phi(q)=2\operatorname{arctg}(q)+\pi$) the roots $q^{(k)}$ are given by sets of numbers $J^{(k)}$ (integers or half odd integers). The density of real roots (for $n\to\infty$) defined by

$$\frac{\mathrm{d}J^{(k)}}{\mathrm{d}q}=\rho^{(k)}(q)+\sum_h\delta(q-h) \qquad (3.2)$$

(where h are "holes") fulfil the system of integral equations

$$\rho(q)+\int \mathrm{d}q'\,K(q-q')\rho(q')=\sigma_0(q)+\sigma_{\mathrm{h}}(q)+\sigma_{\mathrm{c}}(q), \qquad (3.3)$$

which is derived from eq. (3.1) by taking the derivative $\mathrm{d}/\mathrm{d}q$ and substituting $\sum_{i=1}^{n_k}\to\int \mathrm{d}q\,\rho^{(k)}(q)$. The inhomogenities are due to the interaction with the external $\alpha_i=iq_i$

$$\sigma_0^{(l)}(q)=\frac{1}{2\pi}\delta_{l1}\sum_{i=1}^{n}\phi'(q-q_i), \qquad (3.4)$$

the holes σ_{h} and complex roots σ_{c} (see below). The system (3.3) can be solved in Fourier space

$$\tilde{\rho}(x)=\int \mathrm{d}q\,\mathrm{e}^{iqx}\rho(q), \qquad (3.5)$$

where the integral kernel is given by

$$\tilde{K}_{kj}(x) = \delta_{kj} e^{-2|x|} - \sum_{l \in L_k} \delta_{lj} e^{-|x|}. \tag{3.6}$$

The solution

$$\tilde{\rho}(x) = \tilde{R}(x)\left[\tilde{\sigma}_0(x) + \tilde{\sigma}_h(x) + \tilde{\sigma}_c(x)\right] \tag{3.7}$$

is expressed in terms of the (symmetric) resolvent

$$\tilde{R}_{kl}(x) = e^{|x|} \frac{\operatorname{ch}(N-1-k)x \operatorname{sh} l_x}{\operatorname{ch}(N-1)x \operatorname{sh} x}, \qquad 1 \leqslant l \leqslant k \leqslant N-2,$$

$$\tilde{R}_{k\pm}(x) = \tfrac{1}{2} e^{|x|} \frac{\operatorname{sh} kx}{\operatorname{ch}(N-1)x \operatorname{sh} x}, \qquad 1 \leqslant k \leqslant N-2,$$

$$\tilde{R}_{-+}(x) = \tfrac{1}{2} e^{|x|} \frac{\operatorname{sh}(N-2)x}{\operatorname{ch}(N-1)x \operatorname{sh} 2x},$$

$$\tilde{R}_{\pm\pm}(x) = \tfrac{1}{2} e^{|x|} \frac{\operatorname{sh} Nx}{\operatorname{ch}(N-1)x \operatorname{sh} 2x}. \tag{3.8}$$

The (antiferromagnetic) ground state is obtained for absence of holes and complex roots $\sigma_h = \sigma_c = 0$

$$\tilde{\rho}^{(l)}(x) = \tilde{R}_{l1}(x) e^{-|x|} \sum_{i=1}^{n} e^{i q_i x}. \tag{3.9}$$

An excitation corresponding to a k-level hole is obtained from eq. (3.7) with

$$\tilde{\sigma}_h^{(l)}(x) = -\delta_{kl} e^{ihx}. \tag{3.10}$$

A k-level two-string $q^{(k)} = c \pm i$ with real center c is given by

$$\tilde{\sigma}_c^{(l)}(x) = -\tilde{R}_{lk}^{-1} e^{-|x|} e^{icx}. \tag{3.11}$$

This is a simple case of complex roots. (Similar expressions hold for longer strings, quartets etc. They will not be used in this paper.) The solution of Bethe ansatz equations are additive with respect to holes and complex roots in the sense that $\sigma_{h \cdot h'} = \sigma_h + \sigma_{h'}$ and $\sigma_{c \cdot c'} = \sigma_c + \sigma_{c'}$ The locations of holes are free parameters, but the centers of strings fulfil "higher level" [14] Bethe ansatz equations expressing interactions with each other and the holes. These equations are derived from eq. (2.74) eliminating the ground state contributions. For holes $h_1^{(k)}, \ldots, h_{\mu_k}^{(k)}$ and two-string centers $c_1^{(k)},\dots,c_{\nu_k}^{(k)}$ they read

$$\prod_{i=1}^{\nu_k}\frac{i\left(c_j^{(k)}-c_i^{(k)}\right)+2}{i\left(c_j^{(k)}-c_i^{(k)}\right)-2}\prod_{l\in \mathrm{L}_k}\prod_{i=1}^{\nu_l}\frac{i\left(c_j^{(k)}-c_i^{(l)}\right)-1}{i\left(c_j^{(k)}-c_i^{(l)}\right)+1}\prod_{i=1}^{\mu_k}\frac{i\left(c_j^{(k)}-h_i^{(k)}\right)-1}{i\left(c_j^{(k)}-h_i^{(k)}\right)+1}=-1. \tag{3.12}$$

The same type of equation holds for other complex roots as "quartets" and "wide pairs" [14].

There are two further constraints for Bethe states:

(i) The states have maximal weights [15].

(ii) The numbers of roots n_k are integers.

The latter will be considered at the end of this section. Appendix B contains an analysis of group theoretic properties of Bethe states. The results are the following. The "weights" of the states defined by eqs. (2.69)–(2.75) are given by

$$(w_1,w_2,\dots,w_{N-1},w_N)=(n-n_1,n_1-n_2,\dots,n_{N-2}-n_--n_+,n_--n_+). \tag{3.13}$$

They have to fulfil the "maximality" condition

$$w_1\geqslant w_2\geqslant\cdots\geqslant w_{N-1}\geqslant|w_N|. \tag{3.14}$$

In order to find allowed states, i.e. those which fulfil the constraints (i) and (ii) we determine the weights of the ground state, holes and two-strings.

The numbers of roots n_l ($l=1,\dots,N-2,-,+$) are calculated by the formula

$$n_l=\int \mathrm{d}q\,\rho^{(l)}(q)=\tilde\rho^{(l)}(0). \tag{3.15}$$

From eqs. (3.8)–(3.9) we obtain for the ground state

$$n_l^{(0)}=n;\qquad l=1,\dots,N-2;\qquad n_\pm^{(0)}=\tfrac12 n. \tag{3.16}$$

These numbers change for excitations by $\Delta n_l=n_l-n_l^{(0)}$. For a k-hole ($1\leqslant k\leqslant N-2$) we find

$$\begin{aligned}\Delta n_l&=-\min(l,k),\qquad l\leqslant N-2,\\ \Delta n_\pm&=-\tfrac12 k;\end{aligned} \tag{3.17}$$

for σ-holes ($\sigma=\pm$)

$$\begin{aligned}\Delta n_l&=-\tfrac12 l,\qquad && l\leqslant N-2,\\ \Delta n_\tau&=-\tfrac14(N-1+\sigma\tau)\qquad && \tau=\pm;\end{aligned} \tag{3.18}$$

and for a k two-string

$$\Delta n_l = \quad \delta_{kl} . \tag{3.19}$$

By means of eq. (3.16) we obtain the weights for the ground state

$$(0,\ldots,0), \tag{3.20}$$

a k-hole

$$\begin{aligned} &(1,\ldots, w_k = 1, 0,\ldots,0), && 1 \leqslant k \leqslant N-2, \\ &\left(\tfrac{1}{2},\ldots,\tfrac{1}{2},-\tfrac{1}{2}\right), && k = -, \\ &\left(\tfrac{1}{2},\ldots,\tfrac{1}{2},\tfrac{1}{2}\right), && k = +, \end{aligned} \tag{3.21}$$

and a k two-string

$$\begin{aligned} &(0,\ldots, w_k = -1, 1, 0, \ldots 0), && k = 1,\ldots, N-2, -, \\ &(0,\ldots,0,-1,-1), && k = + . \end{aligned} \tag{3.22}$$

It follows that the ground state is an O($2N$) scalar and a k-hole an antisymmetric tensor of rank k for $k \leqslant N-2$ and a spinor of O($2N$) chirality ± 1 for $k = \pm$.

As opposed to these states the two strings have not maximal weights in the sense of eq. (3.14). So they can exist only in combination with holes. By the principle of maximal weights we get then one particle spectrum as follows. Let $h^{(k)}$ and $c^{(k)}$ denote k-holes and k two-strings, respectively. Then

$$\begin{aligned} (k, r) = {} & h^{(h)} c^{(r+1)} (2c^{(r+2)}) \ldots ((k-r) c^{(k)}) \ldots ((k-r) c^{(N-2)}) \\ & \times \tfrac{1}{2}(k-r) c^{(-)} \tfrac{1}{2}(k-r) c^{(+)}, \\ & r = k-2, k-4, \ldots \geqslant 0, \qquad k = 1, 2, \ldots, N-2, \end{aligned} \tag{3.23}$$

characterize a k-hole decorated with two-strings. It is a one-particle antisymmetric tensor state of rank r. The string centers have to fulfil eq. (3.12). As a simple example we find for the state (2,0)

$$c^{(l)} = h^{(1)} \pm \sqrt{(N-1-l)(l-1)}\ ; \qquad l = 2, \ldots, N-2, \qquad c^{(1)} = c^{(-)} = c^{(+)} = h^{(1)} .$$

The tensor states (k, r) give exactly that spectrum obtained by semi-classical methods [16] for the O($2N$) Gross-Neveu (GN) model. In addition we have the $\pm$ spinor states which correspond to the GN kinks [7, 18].

Finally we remark that the bound state picture discussed in [18] for the GN model can also be understood in terms of the Bethe ansatz equations. As an example we consider the antisymmetric two-vector particle scattering state $h_1^{(1)}h_2^{(1)}c^{(1)}$ where the string center is given by $c = \frac{1}{2}(h_1 + h_2)$. For the complex values of the hole positions $h_{1,2} = h \pm i$ this state can be identified with the one-particle state $h^{(2)}$. This follows directly from the Bethe ansatz equations (2.74) if we write the holes $h_{1,2}^{(1)}$ in eql (2.74) as the inverse factors $(i(q_j^{(1)} - h_{1,2}^{(1)}) - 2)/(i(q_j^{(1)} - h_{1,2}^{(1)}) + 2)$ and $(i(q_j^{(k)} - h_{1,2}^{(1)}) + 1)/(i(q_j^{(k)} - h_{1,2}^{(1)}) - 1)$ and take the limit $h_{1,2}^{(1)} \to h \pm (1-\varepsilon)i$. Then the holes $h_{1,2}^{(1)}$ cancel the string $c^{(1)}$ and in $\rho^{(2)}(q)$ appears a hole at $q = h$.

Let us consider the selection rules which follow from the condition that the numbers of roots n_l (cf. eq. (3.15)) have to be integers. From eq. (3.16) we conclude that for the ground state the number of sites n has to be even. For a k-hole, eq. (3.17) yields the constraint $(n-k)$ even. From eq. (3.18), it follows that spinor excitations can appear only by pairs. In general one can have an arbitrary number of even k-holes and even spinor pairs but n + # odd k-holes + # odd spinor pairs has to be even. An even spinor pair means for even N a $(+,+)$ or $(-,-)$ spinor pair and for odd N a$(+,-)$ spinor pair and vice versa for an odd spinor pair.

It must be stressed that the classification of states and their quantum numbers given here hold for three theories associated with the R-matrix (2.1): the vertex model (fig. 1), the QFT (eq. (1.2)) and the magnetic chain discussed in ref. [9].

4. Vertex model and free energy

We investigate in this section the thermodynamical properties of the O($2N$) symmetric vertex model. The three allowed vertex configurations are depicted in fig. 1 as well as their respective weights. These weights are not positive definite so one cannot give a direct statistical interpretation to them. However, it is possible to relate them to a set of positive definite weights using the symmetries of the model. Let us write as usual the partition functions as

$$Z(w_1, w_2, w_3, w_4) = \sum_{\text{config}} \prod_{\text{sites}} w_i, \tag{4.1}$$

where the product is over all sites of an $n \times n$ square lattice with even n. Under these conditions

$$Z(w_1, w_2, w_3, w_4) = Z(-w_1, -w_2, -w_3, -w_4), \tag{4.2}$$

since the total number of sites is even.

It must be noticed that the lines of each "color" A $(1 \leqslant A \leqslant 2N)$ are continuous on the lattice for all allowed configurations of vertices w_1, w_2, w_3, w_4. Assuming periodic boundary conditions, as in sect. 2, implies that the number of intersections of lines of different color is even. So, the number of vertices w_1 is even for any allowed configuration. We then have

$$Z(w_1, w_2, w_3, w_4) = Z(-w_1, w_2, w_3, w_4), \tag{4.3}$$

and combining eqs. (4.2) and (4.3)

$$Z(w_1, w_2, w_3, w_4) = Z(w_1, -w_2, -w_3, -w_4).$$

So we can take as vertex weights for the model

$$\tilde{w}_1 = w_1 = 1, \qquad \tilde{w}_3 = -w_3 = \frac{2}{2N-\theta-2},$$

$$\tilde{w}_2 = -w_2 = \frac{2}{\theta}, \qquad \tilde{w}_4 = -w_4 = \frac{2(N-1)(\theta-2)-\theta^2}{\theta(\theta+2-2N)}, \tag{4.4}$$

keeping the same partition function.

$$Z(\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \tilde{w}_4) = Z(w_1, w_2, w_3, w_4).$$

The weights $\tilde{w}_i$ are positive definite in the domain

$$0 < \theta < 2(N-1) \qquad \text{for } N \leqslant 5 \tag{4.5}$$

and

$$0 < \theta < N-1-\sqrt{(N-1)(N-5)} \quad \text{or} \quad N-1+\sqrt{(N-1)(N-5)} < \theta < 2(N-1)$$

$$\text{for } N > 5.$$

This will be our physical domain. The free energy is defined as usual as

$$f(\theta, N) = -\lim_{n\to\infty} \frac{1}{n^2} \log Z(\theta, N). \tag{4.6}$$

Now, the use of eqs. (2.4), (2.6) and (2.72) yields for the homogeneous model ($\alpha_i \equiv 0$)

$$Z(\theta, N) = \mathrm{Tr}_{\Omega}\left[\left(\tau^{\Omega}(\theta,0)\right)^n\right] \tag{4.7}$$

and

$$f(\theta, N) = -\lim_{n\to\infty} \frac{1}{n} \log\left[\sum_{k=0}^{2N-1} G_k(\theta)\right], \tag{4.8}$$

where the $G_k(\theta)$ correspond to the maximum eigenvalue $\lambda_0(\theta, \{q^{(k)}\})$ or $\tau^{\Omega}(\theta,0)$.

So, we find in the limit $n \to \infty$ that one must pick in (4.8) the maximum $G_k(\theta)$ among the set $G_0(\theta), \dots, G_{2N-1}(\theta)$ associated to the maximum eigenvalue λ_0. λ_0 corresponds to the antiferromagnetic ground state. It can be computed (for $n = \infty$) with the help of the density of roots (3.9).

For $0 < \theta < 1$, explicit calculations show that the term $G_0(\theta)$ dominates in eq. (4.8). One finally finds from eqs. (2.73)

$$f(\theta, N) = -\log\frac{\theta - 2}{\theta} + i\int_{-\infty}^{\infty} \mathrm{d}\lambda\, \rho^{(1)}(\lambda)\phi(\lambda + i\theta). \tag{4.9}$$

This integral can be easily evaluated with the help of the Fourier representation (3.9) and

$$\phi(z) = \pi + 2\int_0^{\infty} \frac{\mathrm{d}p}{p} \sin(pz)\mathrm{e}^{-p}, \qquad |\mathrm{Im}\, z| < 1, \tag{4.10}$$

one finds for $0 < \theta < 1$

$$f(\theta, N) = \log \sigma_2(\theta, N), \tag{4.11}$$

$$\sigma_2(\theta, N) = \frac{\Gamma(\varphi + \sigma)\Gamma(1 - \varphi)\Gamma\left(\varphi + \frac{1}{2}\right)\Gamma\left(\sigma + \frac{1}{2} - \varphi\right)}{\Gamma(\sigma + 1 - \varphi)\Gamma(\varphi)\Gamma\left(\frac{1}{2} - \varphi\right)\Gamma\left(\sigma + \frac{1}{2} + \varphi\right)}, \tag{4.12}$$

where

$$\varphi = \frac{\theta}{4(N-1)}, \qquad \sigma = \frac{1}{2(N-1)}. \tag{4.13}$$

It must be noticed that $\sigma_2(\theta, N)$ exactly coincide with the S-matrix in the transmission channel for the O($2N$) invariant non-linear sigma model if one identifies φ with the rapidity divided by $2\pi i$. So, if we normalize our weights by $\sigma_2(\theta, N) = \tilde{w}_1$ the partition function per site becomes unity. This proves that the unitarity method holds for the present vertex model.

In the physical region (4.5) we have from eq. (4.13) for $N > 1$

$$0 < \varphi < \tfrac{1}{2}, \qquad 0 < \sigma < \infty. \tag{4.14}$$

So, the free energy (4.12) has no singularities in the physical domain. This is related to the fact that the eigenstates of the transfer matrix do not depend on θ. No phase transitions are expected when θ varies. Phase transitions appear in integrable statistical models when parameters other than the spectral one change. Since the present model is gapless it is probably the critical regime of a more general one. The O($2N$) asymmetric model of the last reference in [3] could be a candidate.

The vertex weights (4.4) possess the following symmetry linked to crossing

$$\begin{aligned} \tilde{w}_i(\theta) &= \tilde{w}_i(\hat{\theta}), \qquad i = 1,4, \\ \tilde{w}_2(\theta) &= \tilde{w}_3(\hat{\theta}), \end{aligned} \tag{4.15}$$

where $\hat{\theta} = 2N - 2 - \theta$ (see eq. (2.1)). The exchange $\tilde{w}_2 \leftrightarrow \tilde{w}_3$ amounts to a 90° rotation of the lattice. This rotation leaves the free energy unchanged since eqs. (4.11)–(4.12) yield

$$f(\theta, N) = f(\hat{\theta}, N). \tag{4.16}$$

It must be noticed that the physical domain (4.5) is crossing invariant.

We would like to conclude this section by considering two special cases: $N = 2$, $N = 1$ and $N = 0$. Let us first recall the isotropic SU(2) symmetric model defined by

$$S_{ab}(\varphi) = \mathbb{1}_{ab} - \frac{1}{2\varphi} P_{ab}. \tag{4.17}$$

The free energy of this model reads

$$f_{\mathrm{SU(2)}}(\varphi) = \log \frac{\Gamma(1-\varphi)\Gamma(\varphi - \frac{1}{2})}{\Gamma(\varphi)\Gamma(\frac{1}{2} - \varphi)}. \tag{4.18}$$

This is the isotropic limit of the six-vertex model. Now, setting $N = 2$ in eqs. (4.11)–(4.12) yields

$$f(\theta, 2) = \log \frac{(2\varphi + 1)^2}{2\varphi(1 - 2\varphi)} + 2 f_{\mathrm{SU(2)}}(\varphi). \tag{4.19}$$

So, the free energy of the O(4) symmetric model relates to twice the free energy of the SU(2) symmetric one as it can be expected from O(4) = SU(2) × SU(2). Although the limits $N = 1$ and $N = 0$ are singular in the sense that the construction of sects. 2 and 3 only holds for $N \geqslant 2$ it is interesting to analyse them. It follows from eqs. (4.11)–(4.12) that

$$f(\theta, N) \overset{N \to 1}{=} \log 2(N-1) + \log(\varphi + \tfrac{1}{2}) + f_{\mathrm{SU(2)}}(\varphi) + \mathrm{O}(N-1), \tag{4.20}$$

$$f(\theta, N) \overset{N \to 0}{=} \log \frac{1 - 2\varphi}{2\varphi} + 2 f_{\mathrm{SU(2)}}(\varphi) + \mathrm{O}(N). \tag{4.21}$$

In the $N \to 1$ limit we get a singular term since the weights (4.4) blow up when $N \to 1$ at fixed φ. The rest is the free energy of the SU(2) symmetric model. The result for $N = 0$ resembles that for $N = 2$. This can be traced back to the following symmetry property.

$$\sigma_2(\varphi, -\sigma) = \frac{\operatorname{tg} \pi(\varphi + \sigma)}{\operatorname{tg} \pi(\varphi - \sigma)} \sigma_2(\varphi, \sigma).$$

So, $N=2$ relates to $N=0$

$$\sigma_2\left(\varphi, -\tfrac{1}{2}\right) = \sigma_2\left(\varphi, \tfrac{1}{2}\right).$$

The free energy in the infinite volume limit follows from the eigenvalue of $\tau(\theta)$ with largest module (see eqs. (4.6)–(4.7)). In the SU(N) and Z_N symmetric models [2,4] N eigenvalues of $\tau(\theta)^{1/n}$ have the same modules for infinite volume ($n=\infty$) whereas their arguments differ in multiples of $2\pi/N$. One gets these states from each other by shifting all integers $J_j^{(k)}$ in the Bethe ansatz equation by one [4]. Let us investigate this question for the O($2N$) symmetric model (2.1). It follows that a shift on $J_j^{(k)}$ does not affect the eigenvalue of $\tau(\theta)$ for $1 \leqslant k \leqslant N-2$. Only when the $J_j^{(+)}$ or the $J_j^{(-)}$ are shifted by one does the eigenvalue $\lambda(\theta)$ change sign.

Appendix A

EIGENSTATES OF THE O($2N$) TRANSFER MATRIX

In this appendix we analyse the eigenvalue equation for τ^{Ω} (eq. (2.8)) where the state ψ^{Ω} is defined as an ε-limit given by eq. (2.69). Eq. (2.30) and the corresponding one for D^{Ω} and $\tilde{\tau}^{\Omega_1}$ yield

$$\tau^{\Omega}(\theta)\psi^{\Omega} = \lim_{\varepsilon\to 0}\prod_{i=1}^{m} B^{\Omega}(v_i)\left\{\left(\tau^{\Omega_1}(\theta,\{v,v\}) + \tilde{\tau}^{\Omega_1}(\theta,\{v,v\})\right)\psi^{\Omega_1}(\{v, v+a\varepsilon\})\right\}$$
$$+\,\text{unwanted terms}. \tag{A.1}$$

Using the explicit expressions for T^{Ω_1} (eq. (2.28)) and $\tilde{T}^{\Omega_1}$ (eq. (2.33)) we can replace the terms in the curled bracket by

$$\left\{\tau^{\Omega_1}(\theta,\{v,v+a\varepsilon\})\psi^{\Omega_1}(\{v,v+a\varepsilon\})\right.$$
$$\left.+\,\tilde{\tau}^{\Omega_1}(\theta,\{v+b\varepsilon,v\})\psi^{\Omega_1}(\{v+b\varepsilon,v\}) + O(\varepsilon)\right\} \tag{A.2}$$

if the parameters b_i fulfil $1-b_i = 1/(1-a_i)$, $i=1,\dots,m$. We introduce

$$\lambda_\varepsilon\left(\theta,\{v,u^{(k)}\}\right) = \lambda_1\left(\theta,\{v,v+a\varepsilon,u^{(k)}\}\right) + \tilde{\lambda}_1\left(\theta,\{v+b\varepsilon,v,u^{(k)}\}\right), \tag{A.3}$$

which approaches the eigenvalue λ of τ^{Ω} for $\varepsilon\to 0$. From the explicit expressions for λ_1 and $\tilde{\lambda}_1$ given by eqs. (2.58)–(2.60), (2.63) it can be seen that in the limit $\varepsilon\to 0$ the poles of $\lambda = \lambda_1 + \tilde{\lambda}_1$ at $\theta = v_i \pm 2$, ($i=1,\dots,m$) can only disappear if they are cancelled by zeroes at $\theta = u_i^{(N-1)} + 2$ (-2) in λ_1 ($\tilde{\lambda}_1$). Therefore each v_i, $i=1,\dots,m$ must coincide with a $u_j^{(N-1)}$ (for $\varepsilon = 0$). Introducing a new notation (after reordering the $u_i^{(N-1)}$) we write for $\varepsilon > 0$

$$u_i^{(N-1)} = u_i^{(+)} = v_i + \varepsilon, \qquad i = 1, \dots, n_+ = m,$$
$$u_{i+m}^{(N-1)} = u_i^{(-)}, \qquad i = 1, \dots, n_- = n_{N-1} - m. \tag{A.4}$$

The pole cancellations in λ_ε at $\theta = v_j$ and $\theta = u_j^{(+)}$, $j = 1, \dots, m$ imply (up to corrections of order ε)

$$\prod_{i=1}^{n_{N-1}} \frac{v_j - u_i^{(N-1)} - 2}{v_j - u_i^{(N-1)}} \left(\prod_{i=1}^{m} \frac{v_j - v_i + 2}{v_j - v_i - 2} \right)^2 = -\frac{b_j}{a_j},$$

$$\prod_{i=1}^{n_{N-2}} \frac{v_j - u_i^{(N-2)} - 2}{v_j - u_i^{(N-2)}} \prod_{i=1}^{n_{N-1}} \frac{v_j - u_i^{(N-1)} + 2}{v_j - u_i^{(N-1)} - 2} \prod_{i=1}^{m} \frac{v_j - v_i - 2}{v_j - v_i + 2} = -(1 - a_j). \tag{A.5}$$

These equations determine the parameters a_j (using $b_j = 1 - 1/(1 - a_j)$) and give the Bethe ansatz equation for the $u_j^{(+)}$

$$a_j = 1 - \prod_{i=1}^{n_-} \frac{u_j^{(+)} - u_i^{(-)} + 2}{u_j^{(+)} - u_i^{(-)} - 2} \prod_{i=1}^{n_+} \frac{u_j^{(+)} - u_i^{(+)} - 2}{u_j^{(+)} - u_i^{(+)} + 2}, \qquad j = 1, \dots, n_+,$$

$$\prod_{i=1}^{n_+} \frac{u_j^{(+)} - u_i^{(+)} + 2}{u_j^{(+)} - u_i^{(+)} - 2} \prod_{i=1}^{n_{N-2}} \frac{u_j^{(+)} - u_i^{(N-2)} - 2}{u_j^{(+)} - u_i^{(N-2)}} = -1, \qquad j = 1, \dots, n_+. \tag{A.6}$$

The parameters a_j which determine the ε-limit in the definition of the state ψ^Ω (c.f. (2.69)) vanish for the ground state and all antisymmetric tensor (rank $\leqslant N-2$) excitations. They are non-vanishing only for excitations which are not symmetric w.r.t. + and − chirality spinors. The Bethe ansatz equations for the parameters $u_i^{(-)}$ and $u_i^{(k)}$, $k = 1, \dots, N-2$ are directly obtained from the cancellation condition of the corresponding poles in λ_1.

Appendix B

GROUP THEORETIC PROPERTIES OF BETHE STATES

In this appendix we analyse some group theoretic properties of Bethe states. By asymptotic expansion ($\theta \to \infty$) of the matrix S_{AB} and the monodromy matrix T_A^Ω (c.f. eqs. (2.1) and (2.4)) we define the matrices M_{AB} and M_A^Ω

$$S_{AB}(\theta) = 1_{AB} - \frac{2}{\theta} M_{AB} + \mathrm{O}(\theta^{-2}),$$

$$T_A^\Omega(\theta) = 1_A^\Omega - \frac{2}{\theta} M_A^\Omega + \mathrm{O}(\theta^{-2}). \tag{B.1}$$

The Yang-Baxter algebra (2.15) yields for $\theta' \to \infty$

$$\left[T_A^{\Omega}(\theta), M_B^{\Omega} + M_{AB}\right] = 0, \tag{B.2}$$

and if additionally $\theta \to \infty$

$$\left[M_A^{\Omega}, M_B^{\Omega} + M_{AB}\right] = 0 \tag{B.3}$$

(where M_A^{Ω} and M_{AB} act as unit operators in the spaces B and Ω, respectively). This commutator represents the structure relations of a O($2N$) Lie algebra. This becomes obvious if we note that

$$M_{AB} = (P - K)_{AB}. \tag{B.4}$$

Then (omitting the subscript Ω) eq. (B.3) assumes the more familiar form

$$\left[M_{ik}, M_{jl}\right] = -M_{il}\delta_{kj} + M_{ij}\delta_{kl} + \delta_{il}M_{jk} - \delta_{ij}M_{lk}.$$

Therefore $\exp[\frac{1}{2}i\,\mathrm{tr}_A(\alpha_A M_A)]$, $(\alpha_{ik} = -\alpha_{ki})$ represents an O($2N$) rotation and eq. (B.2) expresses the O($2N$) invariance of T_A^{Ω}.

In the complex basis defined by eq. (2.9) $(\mathrm{V}_A^{2N} = \mathrm{V}_a^N \oplus \overline{\mathrm{V}}_a^N)$ the matrix M_A $(= M_{AB}$ or $M_A^{\Omega})$ has the form analogous to eqs. (2.10) or (2.16)

$$M_A = \begin{pmatrix} W_a & M_a^{(-)} \\ M_a^{(+)} & -W_a^{\mathrm{t}} \end{pmatrix}. \tag{B.5}$$

The diagonal elements of W_a the W_{kk} generate the Cartan subalgebra. Their eigenvalues are the "weights"

$$W_{kk}\psi = w_k\psi. \tag{B.6}$$

For the fundamental vector representation M_{AB} eqs. (2.10), (2.11) yield

$$W_{aB} = \begin{pmatrix} P_{ab} & 0 \\ 0 & -K_{ab} \end{pmatrix}, \tag{B.7}$$

which means that the complex basis vectors $|b\rangle$ and $|\bar{b}\rangle$ (defined by eq. (2.9)) have the weights ± 1, or more precisely

$$\begin{aligned} W_{kkB}|b\rangle &= \delta_{kb}|b\rangle, \\ W_{kkB}|\bar{b}\rangle &= -\delta_{kb}|\bar{b}\rangle. \end{aligned} \tag{B.8}$$

For a basis vector $|\overset{(-)}{b}_1, \ldots, \overset{(-)}{b}_n\rangle$ of Ω we have

$$w_k = \sum_{i=1}^{n} (-)\delta_{kb_i}. \tag{B.9}$$

Now we calculate the weights of the Bethe states constructed in sect. 2. Taking the $O(\theta^{-1})$ terms of eq. (2.30) (notice that the unwanted terms are $O(\theta^{-2})$) we obtain with eqs. (2.41) and (B.5), (B.6)

$$\lambda_1 = 1 - \frac{2}{\theta} \sum_{k=1}^{N} w_k + O(\theta^{-2}). \tag{B.10}$$

From the explicit expressions (2.48)–(2.60) we find for the sum of all weights

$$\sum_{k=1}^{N} w_k = n - 2m. \tag{B.11}$$

Correspondingly, we obtain for the Bethe ansatz step k ($k = 1, \dots, N-1$) from eqs. (2.49), (2.54)

$$w_k = n_{k-1} - n_k, \qquad 1 \leqslant k \leqslant N-1. \tag{B.12}$$

The interpretation of the relations (B.11), (B.12) is that in eq. (2.69) each operator B^{Ω} reduces two weights say w_k and w_l by one and B^{Ω_k} reduces w_k and lifts a w_l ($l > k$) by one. These properties are obvious from the construction of the B's by the decomposition of the spaces Ω and Ω_k. Of course they can also be derived formally from the general commutation rules (B.2). Combining eqs. (B.11) and (B.12) we obtain for a Bethe state given by eqs. (2.69)–(2.75) the weights

$$(w_1, w_2, \dots, w_{N-1}, w_N) = (n - n_1, n_1 - n_2, \dots, n_{N-2} - n_- - n_+, n_- - n_+). \tag{B.13}$$

Next we show that the cancellations of the unwanted terms in eqs. (2.38) and (2.57) are equivalent to the maximality of the weights [15]. Using the commutation rules (B.2) for $B_b^{\Omega}(v)$ and $M_a^{(+)\Omega}$, the unwanted terms of eqs. (2.18) and (2.19) become

$$\left[\tau^{\Omega}(\theta) B_b^{\Omega}(v)\right]_{\text{unwanted}} = \frac{2}{\theta - v} \operatorname{tr}_a B_a^{\Omega}(\theta)\left[M_a^{(+)\Omega}, B_b^{\Omega}(v)\right]. \tag{B.14}$$

Therefore we obtain from eqs. (2.31), (2.38), (2.39)

$$\operatorname*{Res}_{\theta = v_j} \lambda(\theta) = 0\,; j = 1, \dots, m \Leftrightarrow M_a^{(+)\Omega} \psi^{\Omega} = 0. \tag{B.15}$$

From $M^{(-)} = M^{(+)+}$ and the commutation rule

$$\left[M_{lk}^{(+)}, M_{kl}^{(-)}\right] = W_{kk} + W_{ll}, \qquad k \neq l \tag{B.16}$$

follows the positivity condition

$$0 \leqslant 2M_{lk}^{(-)}M_{kl}^{(+)} + W_{kk} + W_{ll}, \qquad k \neq l,$$

which yields for the weights of a Bethe state

$$0 \leqslant w_k + w_l, \qquad 1 \leqslant k \neq l \leqslant N. \tag{B.17}$$

This condition means that at most one weight can be negative. The analogous equations for τ^{Ω_k} and $W_{kl}^{\Omega_k}$ are

$$\left[\tau^{\Omega_k}(\theta) B_l^{\Omega_k}(u)\right]_{\text{unwanted}} = \frac{2}{\theta - u} B_{l'}^{\Omega_k}(\theta)\left[W_{l'k}^{\Omega_k}, B_l^{\Omega_k}(u)\right], \qquad l, l' > k, \, ,$$

$$0 \leqslant 2W_{kl}W_{lk} + W_{kk} - W_{ll}, \qquad k < l,$$

$$0 \leqslant w_k - w_l, \qquad k < l. \tag{B.18}$$

Together with the relation (B.17) we conclude for the weights of O($2N$) Bethe states the "maximality" condition

$$w_1 \geqslant w_2 \geqslant \cdots \geqslant w_{N-1} \geqslant |w_N|. \tag{B.19}$$

At the end of this investigation we remark that for the pole cancellation discussed before eq. (2.70), $n_- = n_{N-1} - m$ has necessarily to be positive. This follows from eq. (B.9), which yields $\Sigma_{k=1}^{N}|w_k| \leqslant n$, and eqs. (B.11) and (B.19).

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