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Asymptotic factorization of n-particle SU(N) form factors

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Abstract: We investigate the high energy behavior of the SU(N) chiral Gross-Neveu model in $1+1$ dimensions. The model is integrable and matrix elements of several local operators (form factors) are known exactly. The form factors show rapidity space clustering, which means factorization, if a group of rapidities is shifted to infinity. We analyze this phenomenon for the SU(N) model. For several operators the factorization formulas are presented explicitly.

Keywords: Bethe Ansatz, Field Theories in Lower Dimensions, Integrable Field Theories, Lattice Integrable Models

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Contents

1 Introduction

The Bjorken scattering or inelastic lepton-hadron scattering at high energies has been a very important and crucial stage in the development of modern QCD [1–3]. This well known experimental investigation in high energy physics is very actual and has now a modern continuation, being part of lepton-hadron experimental research [4, 5]. The essential point in these studies is the behavior of the structure functions of the hadrons [3]. They describe the parton (quark) structure of the hadrons and the nature of the interaction between the quarks inside of the hadrons. The amplitude of the lepton-hadron interaction consists of two parts, where the lepton part is well known. The hadron part, whose invariant decomposition provides the hadron form factors or structure functions [3], is not known. In QCD the calculation of the structure function for all values of the Bjorken variable x is still an open problem.

On the other side, the existence of exact integrable models in 1+1 dimensional asymptotically free theories may be relevant, providing valuable insights into this discussion. Remarkably, due to integrability, it is possible to obtain exact form factors of local operators [6–9].[1] In the remarkable papers [19–21] Balog and Weisz define analogs of the structure functions in two-dimensional integrable quantum field theories. In particular, they consider form factors of the current operator (related to the structure-function) of the O(3) sigma model, which are accurately computed over the whole x range; in addition, the structure functions and some moments are compared with renormalized perturbation theory. They also calculate structure functions in the O(N) sigma model using 1/N expansion and make some conjectures on possible universal formulae in 4 dimensional QCD for small x. Interestingly, in [21] the authors employ the so called cluster behavior of the form factors to calculate the same structure functions. Here we mention that in all of the previously cited papers the authors use only 2,3 and 4 particle form factors in O(3) or in O(N) sigma models.

In this article we will start an investigation of the above mentioned problems in an opposite order: we will analyze the cluster behavior of the SU(N) chiral Gross-Neveu model,[2] which is an asymptotically free theory. For this, we do not only use the 2,3 and 4 particle form factors, but also the general n -particle form factors. We should point out that the first investigation of the cluster behavior of the exact form factors was performed by Smirnov [7] in the case of the sine-Gordon, the SU(2) Thirring model and the O(3) sigma model. He also applied these results to the current algebra [7]. For the sinh-Gordon model the cluster property of form factors was investigated in [22]. Here we will consider the high energy behavior of the exact form factors in 1+1 dimensional asymptotically free quantum field theories [7, 23], with connection to the factorization property and the Bjorken scattering.

The paper is organized as follows: in section 2 we recall some known formulae, which will be used in the following. In particular we present the SU(N) S-matrix and construct the form factors which are n-particle matrix elements of local operators. In section 3 we

[1] Other approaches to form factors in integrable quantum field theories can be found in [10–18].
[2] For $N = 2$ also called SU(2) Thirring model.

investigate the “rapidity space clustering” of form factors, which describes the behavior of form factors, if a group of rapitities is shifted to infinity. Several examples of operators are considered, as the Noether current, the energy-momentum tensor, the fundamental field of the SU(N) chiral Gross-Neveu model, etc. In section 4 we present the proofs. Some more technical details are delegated to the appendices.

2 Generalities

2.1 SU(N) S-matrix

The two particle S-matrix is $S(\theta) = \mathbf{1}\, b(\theta) + \mathbf{P} c(\theta)$ or in terms of matrix elements [9, 24–26]

$$S_{\alpha\beta}^{\delta\gamma}(\theta) = \delta_\alpha^\gamma \delta_\beta^\delta\, b(\theta) + \delta_\alpha^\delta \delta_\beta^\gamma\, c(\theta) \tag{2.1}$$

where $\alpha, \beta, \gamma, \delta = 1, \ldots, N$ denote fundamental particles. We introduce also

$$\tilde{S}_{\alpha\beta}^{\delta\gamma}(\theta) = S_{\alpha\beta}^{\delta\gamma}(\theta)/a(\theta) = \delta_\alpha^\gamma \delta_\beta^\delta \tilde{b}(\theta) + \delta_\alpha^\delta \delta_\beta^\gamma \tilde{c}(\theta) \tag{2.2}$$

where

$$\begin{aligned} a(\theta) = b(\theta) + c(\theta) &= \frac{\Gamma\left(-\frac{\theta}{2\pi i}\right)\Gamma\left(1 - \frac{1}{N} + \frac{\theta}{2\pi i}\right)}{\Gamma\left(\frac{\theta}{2\pi i}\right)\Gamma\left(1 - \frac{1}{N} - \frac{\theta}{2\pi i}\right)} \\ \tilde{b}(\theta) = \frac{b(\theta)}{a(\theta)} &= \frac{\theta}{\theta - i\eta},\ \tilde{c}(\theta) = \frac{c(\theta)}{a(\theta)} = \frac{-i\eta}{\theta - i\eta},\ \eta = \frac{2\pi}{N}. \end{aligned}$$

2.2 SU(N) form factors

Minimal form factor function $F(\theta)$, ϕ- and τ-function. To construct the form factors we need the “minimal form factor function $F(\theta)$” for two particles [9, 26]

$$\begin{aligned} F(\theta) &= c \exp \int_0^\infty \frac{dt}{t \sinh^2 t} e^{\frac{t}{N}} \sinh t\,(1 - 1/N)\,(1 - \cosh t\,(1 - \theta/(i\pi))) \\ &= \frac{G\left(\frac{1}{2\pi i}\theta\right) G\left(1 - \frac{1}{2\pi i}\theta\right)}{G\left(1 - \frac{1}{N} + \frac{1}{2\pi i}\theta\right) G\left(2 - \frac{1}{N} - \frac{1}{2\pi i}\theta\right)},\quad c = F(i\pi) = \frac{G^2\left(\frac{1}{2}\right)}{G^2\left(\frac{3}{2} - \frac{1}{N}\right)} \end{aligned} \tag{2.3}$$

where $G(z)$ is Barnes G-function. It is the minimal solution of the equations

$$F(\theta) = F(-\theta) a(\theta)\,,\quad F(i\pi - \theta) = F(i\pi + \theta)$$

where $a(\theta)$ is the highest weight amplitude of the corresponding channel of the S-matrix (2.1).

The ϕ-function satisfies [9, 26]

$$\prod_{k=0}^{N-2} \tilde{\phi}\,(-\theta - ki\eta) \prod_{k=0}^{N-1} F\,(\theta + ki\eta) = 1$$

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with the solution

$$\tilde{\phi}(\theta) = \Big(F(-\theta)\,\bar{F}(i\pi+\theta)\Big)^{-1} = \Gamma\left(-\frac{\theta}{2\pi i}\right)\Gamma\left(1-\frac{1}{N}+\frac{\theta}{2\pi i}\right) \tag{2.4}$$

where

$$\begin{aligned}\bar{F}(\theta) &= \bar{c}\exp\int_0^\infty \frac{dt}{t\sinh^2 t} e^{\frac{t}{N}}\sinh t/N\left(1-\cosh t\left(1-\theta/(i\pi)\right)\right) \\ &= \frac{G\left(\frac{1}{2}-\frac{1}{N}+\frac{1}{2}\frac{\theta}{i\pi}\right)G\left(\frac{3}{2}-\frac{1}{N}-\frac{1}{2}\frac{\theta}{i\pi}\right)}{G\left(\frac{1}{2}+\frac{1}{2}\frac{\theta}{i\pi}\right)G\left(\frac{3}{2}-\frac{1}{2}\frac{\theta}{i\pi}\right)}\,,\quad \bar{c} = \bar{F}(i\pi) = G^2\left(1-\frac{1}{N}\right)\end{aligned} \tag{2.5}$$

is the minimal F-function for a particle and an anti-particle satisfying

$$\bar{F}(\theta) = -\bar{F}(-\theta)\,b(i\pi-\theta). \tag{2.6}$$

The τ-function is

$$\tau(z) = \Big(\tilde{\phi}(z)\tilde{\phi}(-z)\Big)^{-1} = \frac{1}{2\pi^2}\frac{z\sinh\frac{1}{2}z}{\Gamma\left(1-\frac{1}{N}+\frac{1}{2}\frac{z}{i\pi}\right)\Gamma\left(1-\frac{1}{N}-\frac{1}{2}\frac{z}{i\pi}\right)}. \tag{2.7}$$

n particle form factors. The matrix element of a local operator $\mathcal{O}(x)$ for a state of n particles of kind α_i with rapidities θ_i

$$\langle\, 0\,|\,\mathcal{O}(x)\,|\,\theta_1,\dots,\theta_n\,\rangle^{\text{in}}_{\underline{\alpha}} = e^{-ix(p_1+\dots+p_n)}F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta}) \tag{2.8}$$

defines the generalized form factor $F^{\mathcal{O}}_{1\dots n}(\underline{\theta})$, which is a co-vector valued function with components $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$. The form factors satisfy the *form factor equations* (i)–(v) (see appendix D). Solutions of these equations can be written as follows:

As usual we split off the minimal part [6]

$$F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta}) = N_n F(\underline{\theta})K_{\underline{\alpha}}(\underline{\theta}),\quad F(\underline{\theta}) = \prod_{1\le i<j\le n} F(\theta_{ij}) \tag{2.9}$$

where $\underline{\alpha} = (\alpha_1,\dots,\alpha_n)$, $\underline{\theta} = (\theta_1,\dots,\theta_n)$ and $F(\theta)$ is defined by (2.3). The K-function is given by *an 'off-shell' Bethe ansatz* in terms of the multiple contour integral

$$K^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta}) = \int_{\mathcal{C}_{\underline{\theta}}} d\underline{z}\,\tilde{h}(\underline{\theta},\underline{z})\,p^{\mathcal{O}}(\underline{\theta},\underline{z})\,\tilde{\Psi}_{\underline{\alpha}}(\underline{\theta},\underline{z}) \tag{2.10}$$

with $\underline{z} = (z_1,\dots,z_m)$ and $\int_{\mathcal{C}_{\underline{\theta}}} d\underline{z} = \frac{1}{m!}\int_{\mathcal{C}_{\underline{\theta}}} dz_1\cdots\int_{\mathcal{C}_{\underline{\theta}}} dz_m$. The integration contour $\mathcal{C}_{\underline{\theta}}$ (see figure 1) and the scalar function $h(\underline{\theta},\underline{z})$ depend only on the S-matrix and not on the specific operator $\mathcal{O}(x)$

$$\tilde{h}(\underline{\theta},\underline{z}) = \prod_{i=1}^n\prod_{j=1}^m \tilde{\phi}(\theta_i - z_j)\prod_{1\le i<j\le m}\tau(z_i-z_j)\,,\ \tau(z) = \frac{1}{\tilde{\phi}(-z)\tilde{\phi}(z)}\,. \tag{2.11}$$

The dependence on the specific operator $\mathcal{O}(x)$ is encoded in the scalar p-function $p^{\mathcal{O}}(\underline{\theta},\underline{z})$ which is in general a simple function of e^{θ_i} and e^{z_j}.

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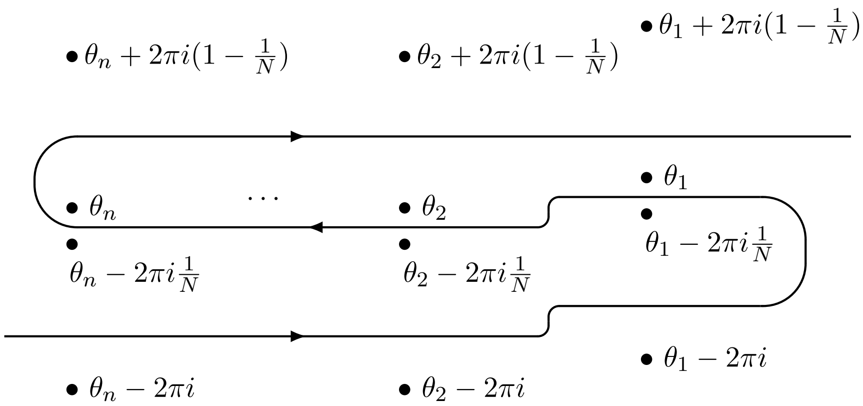


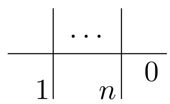
Figure 1. The integration contour $\mathcal{C}_{\underline{\theta}}$. The bullets refer to poles of the integrand in (2.10).

Bethe state. The state $\tilde{\Psi}_{\underline{\alpha}}$ in (2.10) is a linear combination of the basic Bethe ansatz co-vectors

$$\tilde{\Psi}_{\underline{\alpha}}(\underline{\theta}, \underline{z}) = L_{\underline{\beta}}(\underline{z}) \tilde{\Phi}^{\underline{\beta}}_{\underline{\alpha}}(\underline{\theta}, \underline{z}), \quad \text{with } 1 < \beta_i \le N. \tag{2.12}$$

As usual in the context of the algebraic Bethe ansatz [27, 28] the basic Bethe ansatz co-vectors are obtained from the monodromy matrix

$$\tilde{T}_{1\dots n,0}(\underline{\theta}, z) = \tilde{S}_{10}(\theta_1 - z) \cdots \tilde{S}_{n0}(\theta_n - z) = \tag{2.13}$$



$$\equiv \begin{pmatrix} \tilde{A}_{1\dots n}(\underline{\theta}, z) & \tilde{B}_{1\dots n,\beta}(\underline{\theta}, z) \\ \tilde{C}^{\beta'}_{1\dots n}(\underline{\theta}, z) & \tilde{D}^{\beta'}_{1\dots n,\beta}(\underline{\theta}, z) \end{pmatrix}, \quad 2 \le \beta, \beta' \le N.$$

where the S-matrix $\tilde{S}_{i0}$ is given by (2.2). The matrices $\tilde{A}_{1\dots n}$, $\tilde{B}_{1\dots n,\beta}$, $\tilde{C}^{\beta'}_{1\dots n}$ and $\tilde{D}^{\beta'}_{1\dots n,\beta}$ act in the N^n-dimensional "quantum space" denoted by the indices $1\dots n$ and in the N-dimensional "auxiliary space" [27, 28]. The indices β' and β with $2 \le \beta', \beta \le N$ correspond to an $N-1$-dimensional sub-space of the "auxiliary space" and in that space $\tilde{B}_{1\dots n,\beta}$ is a co-vector, $\tilde{C}^{\beta'}_{1\dots n}$ a vector and $\tilde{D}^{\beta'}_{1\dots n,\beta}$ a matrix.

The reference co-vector is defined as usual by $\Omega_{1\dots n}\tilde{B}_{1\dots n,\beta} = 0$ (for $\beta = 2, \dots, N$) which implies for the components of $\Omega_{1\dots n}$

$$\Omega_{\underline{\alpha}} = \delta^1_{\alpha_1} \dots \delta^1_{\alpha_n}.$$

It is an eigenstate of $\tilde{A}_{1\dots n}$ and $\tilde{D}^{\beta'}_{1\dots n,\beta}$

$$\Omega_{1\dots n}\, \tilde{A}_{1\dots n}(\underline{\theta}, z) = \Omega_{1\dots n}, \quad \Omega_{1\dots n}\tilde{D}^{\beta'}_{1\dots n,\beta}(\underline{\theta}, z) = \delta^{\beta'}_{\beta} \prod_{i=1}^{n} \tilde{b}(\theta_i - z)\Omega_{1\dots n}.$$

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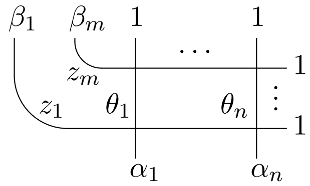
The basic Bethe ansatz co-vectors in (2.12) are defined as

$$\tilde{\Phi}^{\underline{\beta}}_{\underline{\alpha}}(\underline{\theta},\underline{z}) = \left(\Omega_{1\dots n}\tilde{C}^{\beta_m}_{1\dots n}(\underline{\theta},z_m)\cdots\tilde{C}^{\beta_1}_{1\dots n}(\underline{\theta},z_1)\right)_{\underline{\alpha}} = \tag{2.14}$$



where $1 < \beta_i \le N$.

The technique of the '*nested Bethe ansatz*' means that for the coefficients $L_{\underline{\beta}}(\underline{z})$ in (2.12) one makes the analogous construction as for $K_{\underline{\alpha}}(\underline{\theta})$ in (2.10), where now the indices $\underline{\beta}$ take only the values $2 \le \beta_i \le N$. This nesting is repeated until the space of the coefficients becomes one dimensional. The final result is

$$K^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta}) = \int d\underline{\underline{z}}\,\tilde{h}(\underline{\theta},\underline{\underline{z}})\,p^{\mathcal{O}}(\underline{\theta},\underline{\underline{z}})\,\tilde{\Phi}_{\underline{\alpha}}(\underline{\theta},\underline{\underline{z}}) \tag{2.15}$$

with the complete h-function

$$\tilde{h}\,(\underline{\theta},\underline{\underline{z}}) = \prod_{j=0}^{N-2}\tilde{h}(\underline{z}_j,\underline{z}_{j+1})\,, \quad \underline{z}_0 = \underline{\theta} \tag{2.16}$$

and the complete Bethe ansatz state

$$\tilde{\Phi}_{\underline{\alpha}}(\underline{\theta},\underline{\underline{z}}) = (\tilde{\Phi}^{(N-2)})^{\underline{\alpha}_{N-1}}_{\underline{\alpha}_{N-2}}(\underline{z}^{(N-2)},\underline{z}^{(N-1)})\dots(\tilde{\Phi}^{(1)})^{\underline{\alpha}_2}_{\underline{\alpha}_1}(\underline{z}^{(1)},\underline{z}^{(2)})\tilde{\Phi}^{\underline{\alpha}_1}_{\underline{\alpha}}(\underline{\theta},\underline{z}^{(1)}) \tag{2.17}$$

where $\underline{\underline{z}} = (\underline{z}^{(1)},\dots,\underline{z}^{(N-1)})$, $\underline{z}^{(j)} = (z^{(j)}_1,\dots,z^{(j)}_{n_j})$ and $\underline{\alpha}_{N-1} = (N,\dots,N)$.

It is well known (see [29]) that the 'off-shell' Bethe ansatz states are highest weight states if they satisfy certain matrix difference equations. If there are n particles the SU(N) weights are [26]

$$\begin{aligned} w &= (n-n_1, n_1-n_2,\dots,n_{N-2}-n_{N-1},n_{N-1}) \\ &= w^{\mathcal{O}} + L(1,\dots,1) \end{aligned} \tag{2.18}$$

where $n_1 = m, n_2,\dots$ are the numbers of C operators in the various levels of the nesting, $w^{\mathcal{O}}$ is the weight vector of the operator $\mathcal{O}$ and $L = 0,1,2,\dots$; note that $w = (1,\dots,1)$ correspond to the vacuum sector. The number of particles n depend on the charge $Q^{\mathcal{O}}$ of operator $\mathcal{O}$, i.e. $n = Q^{\mathcal{O}} \bmod N$, the other numbers n_j depend on the weights $w^{\mathcal{O}}_j$ as expressed by (2.18). For some examples of operators these numbers are given explicitly below in section 3.

3 Rapidity space clustering

We shift k of the n rapidities in the form factor $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ (2.9) to ∞ and define

$\underline{\theta}_W = (\theta_1+W,\dots,\theta_k+W,\theta_{k+1},\dots,\theta_{k+l}) = (\underline{\hat{\theta}}+W,\underline{\check{\theta}})$.

We investigate the behavior of $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta}_W)$ for $W\to\infty$: The result is of the form

$$F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta}_W) \overset{W\to\infty}{\to} c^{\mathcal{O}}_{\hat{\mathcal{O}}\check{\mathcal{O}}}(k,l,W)F^{\hat{\mathcal{O}}}_{\underline{\hat{\alpha}}}(\underline{\hat{\theta}})F^{\check{\mathcal{O}}}_{\underline{\check{\alpha}}}(\underline{\check{\theta}}) \tag{3.1}$$

where $\underline{\alpha} = (\alpha_1,\dots,\alpha_n)$, $\underline{\hat{\alpha}} = (\alpha_1,\dots,\alpha_k)$ and $\underline{\check{\alpha}} = (\alpha_{k+1},\dots,\alpha_n)$. We calculate the functions $c^{\mathcal{O}}_{\hat{\mathcal{O}}\check{\mathcal{O}}}(W)$ for several operators.

3.1 Examples of local fields

In this article we consider the following fields:

The SU(N) Noether current.

$$J_a^\mu = \bar\psi_\beta \gamma^\mu \left(T_a\right)_\alpha^\beta \psi^\alpha$$

transforms as the adjoint representation with highest weights $w^J = (2,1,\ldots,1,0)$. The N^2-1 generators of SU(N) satisfy

$$[T_a, T_b] = i f_{abc} T_c, \quad \operatorname{Tr} T_a = 0, \quad \operatorname{Tr}(T_a T_b) = \tfrac{1}{2}\delta_{ab}\,.$$

The conservation law $\partial_\mu J_a^\mu(x) = 0$ implies that $J_a^\mu(x)$ may be written in terms of the pseudo potential $J_a(x)$ as

$$J_a^\mu(x) = \epsilon^{\mu\nu}\partial_\nu J_a(x) \tag{3.2}$$

with the quantum numbers

$$\begin{array}{ll} \text{charge} & Q^J = 0 \\ \text{weight vector} & w^J = (2,1,\ldots,1,0) \\ \text{statistics factor} & \sigma^J = 1 \\ \text{spin} & s^J = 0. \end{array} \tag{3.3}$$

Due the Swieca et al. [30] the bound state of $N-1$ particles is to be identified with the anti-particle. This means that the anti-particle $\bar\alpha$ of a fundamental particle α of rank 1 is a bound state of rank $N-1$

$$\bar\alpha = (\rho) = (\rho_1 \ldots \rho_{N-1}), \text{ with } \rho_1 < \cdots < \rho_{N-1},\ \rho_i \neq \alpha\,. \tag{3.4}$$

The charge conjugation matrix is given by

$$\mathbf{C}_{\beta\bar\alpha} = \mathbf{C}_{\beta(\rho_1 \ldots \rho_{N-1})} = \mathbf{C}^{\bar\alpha\beta} = \epsilon_{\beta\rho_1\ldots\rho_{N-1}} \tag{3.5}$$

with $\mathbf{C}_{\beta\bar\alpha}\mathbf{C}^{\bar\alpha\gamma} = \delta_\beta^\gamma$. In terms of fields this means $\bar\psi_\beta = \mathbf{C}_{\beta(\rho)}\bar\psi^{(\rho)} = \mathbf{C}_{\beta(\rho)}\psi^{\rho_1}\ldots\psi^{\rho_{N-1}}$.

For the Bethe ansatz the formulation of the Noether current given by

$$J_\mu^{\alpha(\rho)} = \bar\psi^{(\rho)}\gamma_\mu\psi^\alpha - \mathbf{C}^{\alpha(\rho)}\mathbf{C}_{(\sigma)\beta}\bar\psi^{(\sigma)}\gamma_\mu\psi^\beta/N$$

with $\mathbf{C}_{\alpha(\rho)}J_\mu^{\alpha(\rho)} = 0$ is more convenient, which means for the pseudo potentials

$$J_a = \mathbf{C}_{\beta(\rho)}\left(T_a\right)_\alpha^\beta J^{\alpha(\rho)}. \tag{3.6}$$

Because the Bethe ansatz yields highest weight states we obtain the matrix elements of the highest weight component $J(x) = J^{1\bar N}(x) = J^{1(12\ldots N-1)}(x)$. The form factor is given by (2.9) and (2.10) with the p-function for the operator $J(x)$ [9]

$$p^J(\underline{\theta},\underline{\underline{z}}) = e^{i\pi\frac{1}{N}n_1}\left(\prod_{i=1}^n e^{-\frac{1}{2}\theta_i}\right)\left(\prod_{i=1}^{n_1} e^{\frac{1}{2}z_i^{(1)}}\right)\left(\prod_{i=1}^{n_{N-1}} e^{\frac{1}{2}z_i^{(N-1)}}\right)\Big/\left(\sum_{i=1}^n e^{-\theta_i}\right) \tag{3.7}$$

for $n = 0 \bmod N$. The general weight formula of the Bethe states (2.18) implies that the numbers of integrations in (2.15) satisfy

$$n_j = n\,(1 - j/N) - 1,\ \ j = 1, \dots, N-1. \tag{3.8}$$

In particular the one particle and one anti-particle form factor is [9]

$$\begin{aligned} F^{J_a}_{\alpha\bar\beta}(\theta,\omega) &= (T_a)_{\alpha\bar\beta}\,\frac{1}{\cosh\frac{1}{2}(\theta-\omega)}\bar F(\theta-\omega)/\bar F(i\pi) \\ F^{J^{\gamma\bar\delta}}_{\alpha\bar\beta}(\theta,\omega) &= \left(\delta^\gamma_\alpha\delta^{\bar\delta}_{\bar\beta} - \mathbf{C}^{\gamma\bar\delta}\mathbf{C}_{\bar\beta\alpha}/N\right)\frac{1}{\cosh\frac{1}{2}(\theta-\omega)}\bar F(\theta-\omega)/\bar F(i\pi) \end{aligned} \tag{3.9}$$

where $(T_a)_{\alpha\bar\beta} = \mathbf{C}_{\delta\bar\beta}\,(T_a)^\delta_\alpha$ and $\bar F(\theta)$ defined in (2.5) is the "minimal form factor function" for one particle and one anti-particle.

Energy momentum $T^{\mu\nu}$. We write the energy momentum tensor in terms of an energy momentum potential

$$T^{\mu\nu}(x) = R^{\mu\nu}(i\partial_x)T(x),\quad R^{\mu\nu}(P) = -P^\mu P^\nu + g^{\mu\nu}P^2 \tag{3.10}$$

with

$$\begin{array}{ll} \text{charge} & Q^T = 0 \\ \text{weight vector} & w^T = (0,\dots,0) \\ \text{statistics factor} & \sigma^T = 1 \\ \text{spin} & s^T = 0. \end{array} \tag{3.11}$$

We propose the p-function of the potential

$$p^T(\underline\theta,\underline{\underline z}) = \frac{\sum e^{z_j^{(1)}}}{\sum e^{\theta_j}} - \frac{\sum e^{-z_j^{(1)}}}{\sum e^{-\theta_j}} = p^{T_+}(\underline\theta,\underline z) + p^{T_-}(\underline\theta,\underline z).$$

The general weight formula of Bethe states (2.18) implies that the numbers of integrations in (2.15) satisfy

$$n_j = n\,(1 - j/N)\,,\ \ j = 1, \dots, N-1. \tag{3.12}$$

The one particle and one anti-particle form factors are [9]

$$\begin{aligned} F^{T}_{\alpha\bar\beta}(\theta,\omega) &= \mathbf{C}_{\alpha\bar\beta}\frac{-i}{\cosh\frac{1}{2}(\theta-\omega)}\frac{1}{\theta-\omega-i\pi}\bar F(\theta-\omega)/\bar F(i\pi) \\ F^{T^{\rho\sigma}}_{\alpha\bar\beta}(\theta,\omega) &= 4m^2\mathbf{C}_{\alpha\bar\beta}e^{\frac{1}{2}(\rho+\sigma)(\theta+\omega+i\pi)}\frac{\sinh\frac{1}{2}(\theta-\omega-i\pi)}{\theta-\omega-i\pi}\bar F(\theta-\omega)/\bar F(i\pi),\ \ \rho,\sigma=\pm \end{aligned} \tag{3.13}$$

The iso-scalar field $\phi(x)$: with the quantum numbers

$$\begin{array}{ll} \text{charge} & Q^\phi = 0 \\ \text{weight vector} & w^\phi = (0,\dots,0) \\ \text{statistics factor} & \sigma^\phi = e^{-i\eta} \\ \text{spin} & s^\phi = 0, \end{array}$$

and the p-function

$$p^{\phi}(\underline{\theta},\underline{\underline{z}}) = e^{i\frac{\pi}{N}n_1}\left(\prod_{i=1}^{n} e^{-\left(1-\frac{1}{N}\right)\theta_i}\right)\left(\prod_{i=1}^{n_1} e^{z_j^{(1)}}\right) \tag{3.14}$$

for $n = 0 \bmod N$. The general weight formula of Bethe states (2.18) implies that the numbers of integrations in (2.15) satisfy

$$n_j = n\left(1 - j/N\right),\ j = 1,\ldots,N-1. \tag{3.15}$$

The one particle and one anti-particle form factor is

$$F_{\alpha\bar{\delta}}^{\phi}(\theta,\omega) = \mathbf{C}_{\alpha\bar{\delta}} 2i\left(1-\sigma^{\phi}\right)\frac{e^{-\left(\frac{1}{2}-\frac{1}{N}\right)(\theta-\omega-i\pi)}}{\theta-\omega-i\pi}\frac{\bar{F}(\theta-\omega)}{\bar{F}(i\pi)}$$

if we normalize the field by $\langle 0|\phi(x)|0\rangle = 1$.

The fundamental field $\psi^{\alpha}(x)$: of the chiral SU(N) Gross-Neveu model with the quantum numbers

$$\begin{array}{lll} \text{charge} & Q^{\psi} & = 1 \\ \text{weight vector} & w^{\psi} & = (1,0,\ldots,0) \\ \text{statistics factor} & \sigma^{\psi} & = e^{\left(1-\frac{1}{N}\right)i\pi} \\ \text{spin} & s^{\psi} & = -\frac{1}{2}\left(1-\frac{1}{N}\right) \end{array} \tag{3.16}$$

The p-function of the highest weight component $\psi = \psi^1$ for $n = 1 \bmod N$ is [9]

$$p^{\psi}(\underline{\theta},\underline{\underline{z}}) = e^{\frac{1}{2}n_1 i\eta}\left(\prod_{i=1}^{n} e^{-\frac{1}{2}\left(1-\frac{1}{N}\right)\theta_i}\right)\left(\prod_{i=1}^{n_1} e^{\frac{1}{2}z_i^{(1)}}\right) \tag{3.17}$$

and the 1-particle matrix element is

$$F_{\alpha}^{\psi}(\theta) = \delta_{\alpha}^{1} e^{-\frac{1}{2}\left(1-\frac{1}{N}\right)\theta}. \tag{3.18}$$

The general weight formula of Bethe states (2.18) with $w^{\psi} = (1,0,\ldots,0)$ implies that the numbers of integrations in (2.15) satisfy

$$n_j = (n-1)\left(1 - j/N\right),\ j = 1,\ldots,N-1. \tag{3.19}$$

The field $\chi^{\bar{\alpha}}(x)$: with the quantum numbers

$$\begin{array}{lll} \text{charge} & Q^{\chi} & = N-1 \\ \text{weight vector} & w^{\chi} & = (1,1,\ldots 1,0) \\ \text{statistics factor} & \sigma_1^{\chi} & = e^{i\pi\left(N-\frac{1}{N}\right)} \\ \text{spin} & s^{\chi} & = \frac{1}{2}\left(1-\frac{1}{N}\right). \end{array}$$

The p-function of the highest weight component $\chi = \chi^{\bar{N}}$ for $n = (N-1) \bmod N$ is

$$p^{\chi}(\underline{\theta},\underline{\underline{z}}) = e^{\left(n_1+\frac{1}{2}n_{N-1}\right)i\eta}\left(\prod_{j=1}^{n} e^{-\left(1-\frac{1}{2N}\right)\theta_j}\right)\left(\prod_{j=1}^{n_1} e^{z_j^{(1)}}\right)\left(\prod_{j=1}^{n_{N-1}} e^{\frac{1}{2}z_j^{(N-1)}}\right)/\sum e^{-\theta_i} \tag{3.20}$$

with $n_j = (n+1)\left(1-j/N\right) - 1$ and the 1-anti-particle matrix element is (see [26])

$$F_{\bar{\alpha}}^{\chi^{\bar{\beta}}}(\omega) = \delta_{\bar{\alpha}}^{\bar{\beta}} e^{\frac{1}{2}\left(1-\frac{1}{N}\right)\omega}. \tag{3.21}$$

3.2 Results

As examples of the general formula (3.1) we obtain:

1. Particle number $n = 0 \bmod N$ and $k = 0 \bmod N$

$$F^{J_a}_{\underline{\alpha}}(\underline{\theta}_W) \overset{W\to\infty}{\to} -2\eta W^{-1} f_{abc} F^{J_b}_{\underline{\hat{\alpha}}}(\underline{\hat{\theta}}) F^{J_c}_{\underline{\check{\alpha}}}(\underline{\check{\theta}}) , \qquad \text{see Theorem 1} \tag{3.22}$$

$$F^{\phi}_{\underline{\alpha}}(\underline{\theta}_W) \overset{W\to\infty}{\to} F^{\phi}_{\underline{\hat{\alpha}}}(\underline{\hat{\theta}}) F^{\phi}_{\underline{\check{\alpha}}}(\underline{\check{\theta}}) , \qquad \text{see Theorem 2} \tag{3.23}$$

$$F^{T}_{\underline{\alpha}}(\underline{\theta}_W) \overset{W\to\infty}{\to} 2\eta W^{-2} F^{J_a}_{\underline{\hat{\alpha}}}(\underline{\hat{\theta}}) F^{J_a}_{\underline{\check{\alpha}}}(\underline{\check{\theta}}) , \qquad \text{see Theorem 3.} \tag{3.24}$$

2. Particle number $n = 0 \bmod N$ and $k = 1 \bmod N$

$$F^{J}_{\underline{\alpha}}(\underline{\theta}_W) \overset{W\to\infty}{\to} c^J_{\psi\chi}(k,l,W) F^{\psi}_{\underline{\hat{\alpha}}}(\underline{\hat{\theta}}) F^{\chi}_{\underline{\check{\alpha}}}(\underline{\check{\theta}}) , \qquad \text{see Theorem 4} \tag{3.25}$$

$$c^J_{\psi\chi}(k,l,W) = e^{i\pi l_1} d\, W^{\frac{1}{N^2}} e^{-\frac{1}{2}(1-\frac{1}{N})W}$$

$$F^{T}_{\underline{\alpha}}(\underline{\theta}_W) \overset{W\to\infty}{\to} c^T_{\psi\chi}(k,l,W) \mathbf{C}_{\alpha\bar{\beta}} F^{\psi^\alpha}_{\underline{\hat{\alpha}}}(\underline{\hat{\theta}}) F^{\chi^{\bar{\beta}}}_{\underline{\check{\alpha}}}(\underline{\check{\theta}}) , \qquad \text{see Conjecture 2} \tag{3.26}$$

$$c^T_{\psi\chi}(k,l,W) = -ie^{i\pi l_1} d\, W^{\frac{1}{N^2}-1} e^{-\frac{1}{2}(1-\frac{1}{N})W}$$

with the constant $d = 2\,(2\pi)^{-\frac{1+N}{N^2}}\, e^{-i\pi\left(N+\frac{1}{2N}\right)}/\bar{F}(i\pi)$.

3. Particle number $n = 1 \bmod N$ and $k = 0 \bmod N$

$$\begin{aligned} F^{\psi^\beta}_{\underline{\alpha}}(\underline{\theta}_W) &\overset{W\to\infty}{\to} i\eta W^{-1} \mathbf{C}_{\gamma\bar{\delta}} F^{J^{\beta\bar{\delta}}}_{\underline{\hat{\alpha}}}(\underline{\hat{\theta}}) F^{\psi^\gamma}_{\underline{\check{\alpha}}}(\underline{\check{\theta}}) \\ &= 2i\eta W^{-1} F^{J_a}_{\underline{\hat{\alpha}}}(\underline{\hat{\theta}})\, (T_a)^\beta_\delta\, F^{\psi^\delta}_{\underline{\check{\alpha}}}(\underline{\check{\theta}}) , \quad \text{see Theorem 5.} \end{aligned} \tag{3.27}$$

4. Particle number $n = 1 \bmod N$ and $k = 1 \bmod N$

$$F^{\psi^\alpha}_{\underline{\alpha}}(\underline{\theta}_W) \overset{W\to\infty}{\to} e^{i\pi l_1} e^{-\frac{1}{2}(1-\frac{1}{N})W} F^{\psi^\alpha}_{\underline{\hat{\alpha}}}(\underline{\hat{\theta}}) F^{\phi}_{\underline{\check{\alpha}}}(\underline{\check{\theta}}) , \quad \text{see Theorem 6.} \tag{3.28}$$

4 Proofs

We use the short notation $\underline{\theta}_W$ of section 3 and in addition $\underline{\underline{z}}_W = (\underline{z}^{(1)}_W, \dots, \underline{z}^{(N-1)}_W)$ where we shift k_j of the $z^{(j)}_i$ and define
$\underline{z}^{(j)}_W = (z^{(j)}_1 + W, \dots, z^{(j)}_{k_j} + W, z^{(j)}_{k_j+1}, \dots, z^{(j)}_{n_j}) = (\underline{\hat{z}}^{(j)} + W, \underline{\check{z}}^{(j)}),\ (j = 1, \dots, N-1)$.
The choice of the k_j integrations out of the n_j ones in (2.10) is arbitrary therefore there is a factor of $\binom{n_j}{k_j}$ such that $\binom{n_j}{k_j}\frac{1}{n_j!} = \frac{1}{k_j!}\frac{1}{l_j!}$, $(l_j = n_j - k_j)$ and there is the replacement

$$\int d\underline{z}_j \dots \to \int d\underline{\hat{z}}_j \dots \int d\underline{\check{z}}_j \dots$$

The asymptotic behavior of the form factors given by (2.9) and (2.10) with $\underline{\theta} = \underline{\theta}_W$ for $W \to \infty$ is obtained from the asymptotic behavior of $F(\underline{\theta}_W)$, $\tilde{h}(\underline{\theta}_W, \underline{\underline{z}}_W)$, $\tilde{\Psi}(\underline{\theta}_W, \underline{\underline{z}}_W)$ and the p-functions (see appendix E). In the following, some equations are written for simplicity up to constant factors. Constant factors in eq. (3.1) are finally obtained by form factor equation (iii) of appendix D.

4.1 Theorem 1

Theorem 1 *The form factor of the pseudo-potential of the current for particle number $n = 0 \bmod N$ and $k = 0 \bmod N$ shows the cluster behavior*

$$F_{\underline{\alpha}}^{J^{\beta(\sigma)}}(\underline{\theta}_W) \overset{W\to\infty}{\to} i\eta\frac{1}{W}\mathbf{C}_{\gamma(\lambda)}\left(F_{\underline{\hat{\alpha}}}^{J^{\beta(\lambda)}}(\underline{\hat{\theta}})F_{\underline{\check{\alpha}}}^{J^{\gamma(\sigma)}}(\underline{\check{\theta}}) - F_{\underline{\hat{\alpha}}}^{J^{\gamma(\sigma)}}(\underline{\hat{\theta}})F_{\underline{\check{\alpha}}}^{J^{\beta(\lambda)}}(\underline{\check{\theta}})\right)$$

which is equivalent to[3]

$$F_{\underline{\alpha}}^{J_a}(\underline{\theta}_W) \overset{W\to\infty}{\to} -2\eta\frac{1}{W}f_{abc}F_{\underline{\hat{\alpha}}}^{J_b}(\underline{\hat{\theta}})F_{\underline{\check{\alpha}}}^{J_c}(\underline{\check{\theta}}).$$

Proof. We use the short notations of (2.9) ... (2.17) and investigate (for $J = J^{1\bar{N}}$)

$$F_{\underline{\alpha}}^{J}(\underline{\theta}_W) = N_n^J F(\underline{\theta}_W)\int d\underline{\underline{z}}\,\tilde{h}\,(\underline{\theta}_W,\underline{\underline{z}}_W)p^J(\underline{\theta}_W,\underline{\underline{z}}_W)\tilde{\Phi}_{\underline{\alpha}}(\underline{\theta}_W,\underline{\underline{z}}_W).$$

From the asymptotic behavior of $F(\underline{\theta}_W)\tilde{h}\,(\underline{\theta}_W,\underline{\underline{z}}_W)$ and $p^J(\underline{\theta}_W,\underline{z}_W^{(1)},\underline{z}_W^{(N-1)})$ in (E.18), (E.7) and (F.3) we derive for $W\to\infty$ the exponential behavior

$$F(\underline{\theta}_W)\tilde{h}\,(\underline{\theta}_W,\underline{\underline{z}}_W)p^J(\underline{\theta}_W,\underline{\underline{z}}_W) \propto \left(e^{-\frac{1}{2}W}\right)^{\tilde{k}_1^2+\tilde{k}_{N-1}^2+\sum_{j=1}^{N-2}(\tilde{k}_j-\tilde{k}_{j+1})^2} \tag{4.1}$$

where $\tilde{k}_j = k_j - k\,(1-j/N)$. For $k = 0 \bmod N$ the leading behavior $\left(e^{-\frac{1}{2}W}\right)^0$ is obtained for $\tilde{k}_j = 0$, therefore

$$k_j = k\,(1-j/N)\,,\;\; l_j = l\,(1-j/N)-1,\;\; j=1,\ldots,N-1.$$

For these values of k_j and l_j we obtain, more precisely, with (E.18), (E.7) and (E.22) in leading order the asymptotic behavior (up to a constant factor)

$$\begin{aligned}&F(\underline{\theta}_W)\tilde{h}\,(\underline{\theta}_W,\underline{\underline{z}}_W)p^J(\underline{\theta}_W,\underline{\underline{z}}_W)\tilde{\Phi}_{\underline{\alpha}}(\underline{\theta}_W,\underline{\underline{z}}_W)\\ &\overset{W\to\infty}{\to}\left(F(\underline{\hat{\theta}})\tilde{h}\,(\underline{\hat{\theta}},\underline{\underline{\hat{z}}})\tilde{\Phi}_{\underline{\alpha}}(\underline{\hat{\theta}},\underline{\underline{\hat{z}}})\right)\left(F(\underline{\check{\theta}})\tilde{h}\,(\underline{\check{\theta}},\underline{\underline{\check{z}}})p^J(\underline{\check{\theta}},\underline{\underline{\check{z}}})\tilde{\Phi}_{\underline{\alpha}}(\underline{\check{\theta}},\underline{\underline{\check{z}}})\right).\end{aligned} \tag{4.2}$$

The $\underline{\underline{\hat{z}}}$-integral vanishes because of Lemma 1 and therefore in leading order

$$F_{\underline{\alpha}}^{J}(\underline{\theta}_W) \to 0.$$

Order $\frac{1}{W}$: we have to apply the asymptotic behavior of the h-function (E.12), (E.13) and (E.14) and the Bethe state (E.23) and (E.20).

We present a complete proof of this $\frac{1}{W}$-term for SU(2) and for general N the example of appendix B for one particle and one anti-particle. In addition we show consistency of the general clustering formula with the form factor equation (iii) (see Remark 1).

We have to consider the 2 contributions:

[3] For SU(2) this result (3.22) was obtained previously by Smirnov [7].

A) From the *h-function*: Note that because of Lemma 1 in $\tilde h_1(\underline\theta,\underline z)$ of (E.14) only the $\hat z_j$-dependent terms contribute. Therefore we get on the r.h.s. of (4.2) from $\tilde h_1$ for $k_1 = k\,(1-1/N)\,,\ l_1 = l\,(1-1/N)-1$

$$\Big(F(\hat{\underline\theta})\tilde h(\hat{\underline\theta},\hat{\underline z})\,(-\textstyle\sum\hat z_j)\,\tilde\Phi_{\hat{\underline\alpha}}(\hat{\underline\theta},\hat{\underline z})\Big)\Big(F(\check{\underline\theta})\tilde h(\check{\underline\theta},\check{\underline z})p^J(\check{\underline\theta},\check{\underline z})\tilde\Phi_{\check{\underline\alpha}}(\check{\underline\theta},\check{\underline z})\Big)$$

and (up to a constant factor)

$$F^J_{\underline\alpha}(\underline\theta_W)_A \to \frac{1}{W}\Big(F^J(\hat{\underline\theta})M_1^2\Big)_{\hat{\underline\alpha}}\Big(F^J_{\check{\underline\alpha}}(\check{\underline\theta})\Big) \tag{4.3}$$

where (A.2) and the definition (2.10) for $\mathcal O = J$ have been used.

B) From the *Bethe state*: Again because of Lemma 1 we may take in (E.20) only the first term and write with $\tilde\Phi^{D_j}_{\hat{\underline\alpha}}(\hat{\underline\theta},\hat{\underline z}) = \Big(\Omega C(\hat{\underline\theta},\hat z_k)\dots D(\hat{\underline\theta},\hat z_j)\dots C(\hat{\underline\theta},\hat z_1)\Big)_{\hat{\underline\alpha}}$

$$\tilde\Phi_{\underline\alpha 1}(\hat{\underline\theta},\check{\underline\theta},\hat{\underline z},\check{\underline z}) \to \sum_j\Big(\tilde\Phi^{D_j}_{\hat{\underline\alpha}}(\hat{\underline\theta},\hat{\underline z})\Big)\Big(\tilde\Phi(\check{\underline\theta},\check{\underline z})M_1^2\Big)_{\check{\underline\alpha}}$$

and we get

$$\sum_j\Big(F(\hat{\underline\theta})\tilde h(\hat{\underline\theta},\hat{\underline z})\tilde\Phi^{D_j}(\hat{\underline\theta},\hat{\underline z})\Big)_{\hat{\underline\alpha}}\Big(F(\check{\underline\theta})\tilde h(\check{\underline\theta},\check{\underline z})p^J(\check{\underline\theta},\check{\underline z})\tilde\Phi(\check{\underline\theta},\check{\underline z})M_1^2\Big)_{\check{\underline\alpha}}$$

and (up to a constant factor)

$$F^J_{\underline\alpha}(\underline\theta_W)_B \to \frac{1}{W}\Big(F^J(\hat{\underline\theta})\Big)_{\hat{\underline\alpha}}\Big(F^J(\check{\underline\theta})M_1^2\Big)_{\check{\underline\alpha}}. \tag{4.4}$$

where (A.3) has been used. The final result is

$$F^J_{\underline\alpha}(\underline\theta_W) \to i\eta\frac{1}{W}\left(\Big(F^J(\hat{\underline\theta})M_1^2\Big)_{\hat{\underline\alpha}}\Big(F^J_{\check{\underline\alpha}}(\check{\underline\theta})\Big)-\Big(F^J_{\hat{\underline\alpha}}(\hat{\underline\theta})\Big)\Big(F^J(\check{\underline\theta})M_1^2\Big)_{\check{\underline\alpha}}\right)$$

which is for SU(2) the component $(\beta,(\sigma)) = (1,1)$ of (3.22) because $F^J_{\underline\alpha}(\underline\theta) = F^{J^{11}}_{\underline\alpha}(\underline\theta)$ and $\Big(F^J(\underline\theta)M_1^2\Big)_{\underline\alpha} = F^{J^{12}}_{\underline\alpha}(\underline\theta)+F^{J^{21}}_{\underline\alpha}(\underline\theta)$. The other components are obtained by SU(2)-transformations. The constant factor is calculated below and the minus sign is due to SU(2) invariance. In terms of the components J_a (3.6) this can be written as in (3.22) (see (4.11)).

■

Calculation of the functions $c^J_{JJ}(k,l,W)$: defined by

$$F^{J^{\beta(\sigma)}}_{\underline\alpha}(\underline\theta_W)\overset{W\to\infty}{\to} c^J_{JJ}(k,l,W)\Big(\mathbf C_{\gamma(\lambda)}F^{J^{\beta(\lambda)}}_{\hat{\underline\alpha}}(\hat{\underline\theta})F^{J^{\gamma(\sigma)}}_{\check{\underline\alpha}}(\check{\underline\theta})-\mathbf C_{\gamma(\lambda)}F^{J^{\gamma(\sigma)}}_{\hat{\underline\alpha}}(\hat{\underline\theta})F^{J^{\beta(\lambda)}}_{\check{\underline\alpha}}(\check{\underline\theta})\Big)$$

for general N. Here and in the following we use the short notation

$$S^{\underline\alpha'\alpha'}_{\alpha\underline\alpha}(\theta,\underline\theta) = S^{\alpha'_n\alpha'}_{\gamma_n\alpha_n}(\theta-\theta_n)\dots S^{\alpha'_1\gamma_2}_{\alpha\alpha_1}(\theta-\theta_1)\,. \tag{4.5}$$

We also use the satistics factor $\dot\sigma^{\mathcal{O}}_\alpha$, which is related to the "physical" statistics by[4]

$$\dot\sigma^{\mathcal{O}}_\alpha = \sigma^{\mathcal{O}}_\alpha (-1)^{(N-1)+(1-1/N)(n-Q^{\mathcal{O}})} \tag{4.6}$$

where $Q^{\mathcal{O}}$ is the charge of $\mathcal{O}$.

We apply the general procedure of appendix C: using $a(W) \to e^{-i\pi\left(1-\frac{1}{N}\right)}$ of (E.1) and $\sigma^J_1 = 1$, $Q^J = 0$ we check (C.4) and (C.7) for this case

$$\dot\sigma^J_1(n) S^{\breve\alpha' 1}_{1\underline{\breve\alpha}}(\theta + W, \underline{\breve\theta}) \overset{W\to\infty}{\to} (-1)^{(N-1)+(1-1/N)(k+l)} (a(W))^l \mathbf{1}^{\underline{\breve\alpha}'}_{\underline{\breve\alpha}} \to \dot\sigma^J_1(k) \mathbf{1}^{\underline{\breve\alpha}'}_{\underline{\breve\alpha}}$$
$$\dot\sigma^J_1(n) S^{\bar 1\underline{\hat\alpha}'}_{\underline{\hat\alpha}\bar 1}(\underline{\hat\theta} + W, \omega) \overset{W\to\infty}{\to} (-1)^{(N-1)+(1-1/N)(k+l)} (a(W))^{k(N-1)} \mathbf{1}^{\underline{\hat\alpha}'}_{\underline{\hat\alpha}} \to (-1)^{(N-1)k} \dot\sigma^J_1(l) \mathbf{1}^{\underline{\hat\alpha}'}_{\underline{\hat\alpha}}$$

Therefore, as proofed in appendix C, $c^J_{JJ}(k, l, W)$ is independent of k and l, because $(-1)^{(N-1)k} = 1$ for $k = 0 \bmod N$. It is convenient to consider the special case $c^J_{JJ}(N, N, W)$:

1) We take the bound states $\bar 1 = (\hat\alpha_2 \dots \hat\alpha_N)$ and $\bar N = (\breve\alpha_1 \dots \breve\alpha_{N-1})$ and calculate for $\underline{\theta}_W = (\hat\theta + W, \hat\omega + W, \breve\omega, \breve\theta)$

$$\operatorname*{Res}_{\hat\theta = i\pi + \hat\omega} F^{J 1\bar N}_{1\bar 1 \bar N 1}(\underline{\theta}_W) = 2i\, \mathbf{C}_{1\bar 1}\, F^{J 1\bar N}_{\bar N 1}(\breve\omega, \breve\theta) \left(1 - \dot\sigma^J_{\bar 1}(2N) S^{\bar N 1, \bar 1}_{\bar 1, \bar N 1}(\hat\omega + W; \breve\omega, \breve\theta)\right)$$
$$\overset{W\to\infty}{\to} -2i\, \mathbf{C}_{1\bar 1}\, i\eta \frac{1}{W} F^{J 1\bar N}_{\bar N 1}(\breve\omega, \breve\theta). \tag{4.7}$$

It was used that (2.1), (B.1), (E.1) including $1/W$ terms and $a(\theta)a(-\theta) = 1$ imply

$$\dot\sigma^J_{\bar 1}(2N) S^{\bar N 1, \bar 1}_{\bar 1, \bar N 1}(\hat\omega + W; \breve\omega, \breve\theta)$$
$$\overset{W\to\infty}{\to} (-1)^{(N-1)+(1-1/N)2N} \left(a(W)\tilde b(W)\right) \left((-1)^{N-1} a(-W)\right) \to 1 + i\eta \frac{1}{W}.$$

2) Taking first $W \to \infty$ and then the Res means

$$\operatorname*{Res}_{\hat\theta = i\pi + \hat\omega} \left(F^{J 1\bar N}_{1\bar 1 \bar N 1}(\underline{\theta}_W\right) \overset{W\to\infty}{\to} c^J_{JJ}(N, N, W) \mathbf{C}_{\gamma(\lambda)} \left(F^{J 1^{(\lambda)}}_{1\bar 1}(\underline{\hat\theta}) F^{J\gamma\bar N}_{\bar N 1}(\underline{\breve\theta}) - F^{J\gamma\bar N}_{1\bar 1}(\underline{\hat\theta}) F^{J 1^{(\lambda)}}_{\bar N 1}(\underline{\breve\theta})\right)\Big)$$
$$= c^J_{JJ}(N, N, W)\, (-2i)\, \mathbf{C}_{1\bar 1} F^{J 1\bar N}_{\bar N 1}(\breve\omega, \breve\theta). \tag{4.8}$$

where (3.9) was used. As result we obtain from (4.7) and (4.8)

$$c^J_{JJ}(k, l, W) = i\eta \frac{1}{W}.$$

Remark 1 *Note that this also proves consistency of the clustering formula (3.22) for general N with the form factor equation* (iii).

[4] See eqs. (27) and (28) in [9].

Equivalence. We prove that

$$F_{\underline{\alpha}}^{J_a}(\underline{\theta}_W) \to -2\eta \frac{1}{W} f_{abc} F_{\underline{\hat{\alpha}}}^{J_b}(\underline{\hat{\theta}}) F_{\underline{\check{\alpha}}}^{J_c}(\underline{\check{\theta}}) \tag{4.9}$$

is equivalent to

$$F_{\underline{\alpha}}^{J^{\alpha(\rho)}}(\underline{\theta}_W) \to \frac{1}{W} i\eta \mathbf{C}_{\gamma(\sigma)} \left(F_{\underline{\hat{\alpha}}}^{J^{\alpha(\sigma)}}(\underline{\hat{\theta}}) F_{\underline{\check{\alpha}}}^{J^{\gamma(\rho)}}(\underline{\check{\theta}}) - F_{\underline{\hat{\alpha}}}^{J^{\gamma(\rho)}}(\underline{\hat{\theta}}) F_{\underline{\check{\alpha}}}^{J^{\alpha(\sigma)}}(\underline{\check{\theta}}) \right).$$

We have the general relations [31, 32]

$$[T_a, T_b]_\alpha^\beta = i f_{abc} (T_c)_\alpha^\beta, \quad (T_b)_\alpha^\beta (T_b)_\gamma^\delta = \frac{1}{2} \left(\delta_\alpha^\delta \delta_\gamma^\beta - \frac{1}{N} \delta_\alpha^\beta \delta_\gamma^\delta \right). \tag{4.10}$$

By (3.6) and (4.9) we obtain for $W \to \infty$

$$\begin{aligned} F_{\underline{\alpha}}^{J_a}(\underline{\theta}_W) &= \mathbf{C}_{\gamma(\rho)} (T_a)_\alpha^\gamma F_{\underline{\alpha}}^{J^{\alpha(\rho)}}(\underline{\theta}_W) \\ &\to -2\eta \frac{1}{W} f_{abc} F_{\underline{\hat{\alpha}}}^{J_b}(\underline{\hat{\theta}}) F_{\underline{\check{\alpha}}}^{J_c}(\underline{\check{\theta}}) \\ &= i\eta \frac{1}{W} \mathbf{C}_{\gamma(\rho)} (T_a)_\alpha^\gamma \left(\mathbf{C}_{\gamma'(\rho)'} \left(F_{\underline{\hat{\alpha}}}^{J^{\alpha(\rho)'}} \right) \left(F_{\underline{\check{\alpha}}}^{J^{\gamma'(\rho)}} \right) - \mathbf{C}_{\gamma'(\rho)'} \left(F_{\underline{\hat{\alpha}}}^{J^{\gamma'(\rho)}} \right) \left(F_{\underline{\check{\alpha}}}^{J^{\alpha(\rho)'}} \right) \right) \end{aligned} \tag{4.11}$$

where the relations (4.10) have been used. This proves the equivalency.

4.2 Theorem 2

Theorem 2 *The form factor of the field $\phi(x)$ for particle number $n = 0 \bmod N$ and $k = 0 \bmod N$ shows the cluster behavior*

$$F_{\underline{\alpha}}^{\phi}(\underline{\theta}_W) \overset{W\to\infty}{\to} F_{\underline{\hat{\alpha}}}^{\phi}(\underline{\hat{\theta}}) F_{\underline{\check{\alpha}}}^{\phi}(\underline{\check{\theta}}).$$

Remark 2 *Note that this is the typical behavior of an exponential of a bosonic field (see [33]).*

Proof. We investigate

$$F_{\underline{\alpha}}^{\phi}(\underline{\theta}_W) = N_n^{\phi} F(\underline{\theta}_W) \int d\underline{\underline{z}} \tilde{h}(\underline{\theta}_W, \underline{\underline{z}}_W) p^{\phi}(\underline{\theta}_W, \underline{\underline{z}}_W) \tilde{\Phi}_{\underline{\alpha}}(\underline{\theta}_W, \underline{\underline{z}}_W)$$

From the asymptotic behavior of $F(\underline{\theta}_W)\tilde{h}(\underline{\theta}_W, \underline{\underline{z}}_W)$ and $p^{\phi}(\underline{\theta}_W, \underline{\underline{z}}_W)$ in (E.18), (E.9) and (F.2) we derive for $W \to \infty$ the exponential behavior

$$F(\underline{\theta}_W)\tilde{h}(\underline{\theta}_W, \underline{\underline{z}}_W) p^{\phi}(\underline{\theta}_W, \underline{\underline{z}}_W) \propto \left(e^{-\frac{1}{2}W} \right)^{\tilde{k}_1^2 + \tilde{k}_{N-1}^2 + \sum_{j=1}^{N-2} (\tilde{k}_j - \tilde{k}_{j+1})^2 - \tilde{k}_1} \tag{4.12}$$

where $\tilde{k}_j = k_j - k(1 - j/N)$. For $k = 0 \bmod N$ the leading behavior $\left(e^{-\frac{1}{2}W} \right)^0$ is obtained for $\tilde{k}_j = 0 \Rightarrow$

$$k_j = k(1 - j/N), \; l_j = l(1 - j/N), \; j = 1, \ldots, N-1$$

For these values of k_j and l_j we obtain, more precisely, with (E.18), (E.9) and (E.22) in leading order the asymptotic behavior (up to a constant factor)

$$\begin{aligned}&F(\underline{\theta}_W)p^\phi(\underline{\theta}_W,\underline{z}_W)\tilde{\Phi}_{\underline{\alpha}}(\underline{\theta}_W,\underline{z}_W)\\ &\to\left(F(\underline{\hat{\theta}})\tilde{h}(\underline{\hat{\theta}},\underline{\hat{z}})p^\phi(\underline{\hat{\theta}},\underline{\hat{z}})\tilde{\Phi}_{\underline{\hat{\alpha}}}(\underline{\hat{\theta}},\underline{\hat{z}})\right)\left(F(\underline{\check{\theta}})\tilde{h}(\underline{\check{\theta}},\underline{\check{z}})p^\phi(\underline{\check{\theta}},\underline{\check{z}})\tilde{\Phi}_{\underline{\check{\alpha}}}(\underline{\check{\theta}},\underline{\check{z}})\right)\end{aligned}$$

such that

$$F^\phi_{\underline{\alpha}}(\underline{\theta}_W)\to F^\phi_{\underline{\hat{\alpha}}}(\underline{\hat{\theta}})F^\phi_{\underline{\check{\alpha}}}(\underline{\check{\theta}})$$

The constant factor is again calculated using the form factor equation (iii). ■

Calculation of the function $c^\phi_{\phi\phi}(k,l,W)$ *:* defined by

$$F^\phi_{\underline{\alpha}}(\underline{\theta}_W)\to c^\phi_{\phi\phi}(k,l,W)F^\phi_{\underline{\hat{\alpha}}}(\underline{\hat{\theta}})F^\phi_{\underline{\check{\alpha}}}(\underline{\check{\theta}})\,.$$

We apply the general procedure of appendix C: using $a(W)\to e^{-i\pi\left(1-\frac{1}{N}\right)}$ of (E.1) and $\sigma^\phi=e^{i\eta}$, $Q^\phi=0$ we check (C.4) and (C.7) for this case

$$\begin{aligned}\dot{\sigma}^\phi_1(n)S^{\underline{\check{\alpha}}'1}_{1\underline{\check{\alpha}}}(\theta+W,\underline{\check{\theta}})&=e^{i\eta}(-1)^{(N-1)+(1-1/N)(k+l)}(a(W))^l\mathbf{1}^{\underline{\check{\alpha}}'}_{\underline{\check{\alpha}}}\to\dot{\sigma}^\phi_1(k)\mathbf{1}^{\underline{\check{\alpha}}'}_{\underline{\check{\alpha}}}\\ \dot{\sigma}^\phi_1(n)S^{\bar{1}\underline{\hat{\alpha}}'}_{\underline{\hat{\alpha}}\bar{1}}(\underline{\hat{\theta}}+W,\omega)&=e^{i\eta}(-1)^{(N-1)+(1-1/N)(k+l)}(a(W))^{(N-1)k}\mathbf{1}^{\underline{\hat{\alpha}}'}_{\underline{\hat{\alpha}}}\to\dot{\sigma}^\phi_1(l)\mathbf{1}^{\underline{\hat{\alpha}}'}_{\underline{\hat{\alpha}}}\end{aligned}$$

Therefore, as proofed in appendix C, $c^\phi_{\phi\phi}(k,l,W)$ is independent of k and l, because $(-1)^{(N-1)k}=1$ for $k=0\bmod N$. The special case $c^\phi_{\phi\phi}(k,0,W)$ is obtained by the form factor equation (v) with $s^\phi=0$ and (3.23) for $\underline{\check{\alpha}}=\emptyset$

$$\begin{aligned}F^\phi_{\underline{\hat{\alpha}}\emptyset}(\underline{\theta}_W)&\to e^{Ws^\phi}F^\phi_{\underline{\hat{\alpha}}\emptyset}(\underline{\theta})=F^\phi_{\underline{\hat{\alpha}}}(\underline{\theta})\\ F^\phi_{\underline{\hat{\alpha}}\emptyset}(\underline{\theta}_W)&\to c^\phi_{\phi\phi}(k,0,W)F^\phi_{\underline{\hat{\alpha}}}(\underline{\theta})F^\phi_\emptyset\end{aligned}$$

which implies

$$c^\phi_{\phi\phi}(k,l,W)=1$$

if we normalize the field $\phi(x)$ by $F^\phi_\emptyset=\langle 0|\phi(x)|0\rangle=1$.

4.3 Theorem 3

Theorem 3 *The form factor of the energy momentum potential for particle number* $n=0\bmod N$ *and* $k=0\bmod N$ *satisfies*

$$F^T_{\underline{\alpha}}(\underline{\theta}_W)=O(W^{-2})\ \textit{for}\ W\to\infty\,. \tag{4.13}$$

More precisely

Conjecture 1 *The cluster behavior of form factor of* T *for* $k=0\bmod N$ *reads as*

$$F^T_{\underline{\alpha}}(\underline{\theta}_W)\overset{W\to\infty}{\to}2\eta W^{-2}F^{J_a}_{\underline{\hat{\alpha}}}(\underline{\hat{\theta}})F^{J_a}_{\underline{\check{\alpha}}}(\underline{\check{\theta}})=\eta W^{-2}\mathbf{C}_{\alpha\bar{\delta}}\mathbf{C}_{\beta\bar{\gamma}}F^{J^{\alpha\bar{\gamma}}}_{\underline{\hat{\alpha}}}(\underline{\hat{\theta}})F^{J^{\beta\bar{\delta}}}_{\underline{\check{\alpha}}}(\underline{\check{\theta}}) \tag{4.14}$$

We have no general proof of this conjecture. The problem is that the expansion for large W of the integrand in the contour integral representation in (2.10) must not be interchanged with the integration, this is only allowed up to the $1/W$-term.

However, we have checked consistency with the form factor equation (iii), which also yields the function $c^T_{JJ}(k,l,W) = \eta W^{-2}$.

Proof. To prove (4.13) we investigate for $n = k + l,\ m = k_1 + l_1,\ m = n/2$

$$F^T_{\underline{\alpha}}(\underline{\theta}_W) = N^T_n \int d\underline{z} F(\underline{\theta}_W)\tilde{h}(\underline{\theta}_W, \underline{z}_W) p^T(\underline{\theta}_W, \underline{z}_W)\Psi_{\underline{\alpha}}(\underline{\theta}_W, \underline{z}_W)\,.$$

From the asymptotic behavior of $F(\underline{\theta}_W)\tilde{h}\,(\underline{\theta}_W, \underline{\underline{z}}_W)$ and $p^T(\underline{\theta}_W, \underline{z}_W)$ in (E.18), (E.8) and (F.2) we derive for $W \to \infty$ the exponential behavior

$$F(\underline{\theta}_W)\tilde{h}\,(\underline{\theta}_W, \underline{\underline{z}}_W)p^T(\underline{\theta}_W, \underline{\underline{z}}_W) \propto \left(e^{-\frac{1}{2}W}\right)^{\tilde{k}_1^2+\tilde{k}_{N-1}^2+\sum_{j=1}^{N-2}(\tilde{k}_j-\tilde{k}_{j+1})^2} \tag{4.15}$$

where $\tilde{k}_j = k_j - k\,(1 - j/N)$. For $k = 0 \bmod N$ the leading behavior $\left(e^{-\frac{1}{2}W}\right)^0$ is obtained for $\tilde{k}_j = 0$. Therefore by (3.12)

$$k_j = k\,(1 - j/N)\,,\ l_j = l\,(1 - j/N)\,,\ j = 1, \dots, N - 1.$$

For these values of k_j and l_j we obtain, more precisely, with (E.18), (E.8) and (E.22) in leading order the asymptotic behavior (up to a constant factor)

$$\begin{aligned}&F(\underline{\theta}_W)\tilde{h}\,(\underline{\theta}_W, \underline{\underline{z}}_W)p^T(\underline{\theta}_W, \underline{\underline{z}}_W)\tilde{\Phi}_{\underline{\alpha}}(\underline{\theta}_W, \underline{\underline{z}}_W)\\ &\overset{W\to\infty}{\to} \left(F(\hat{\underline{\theta}})\tilde{h}\,(\hat{\underline{\theta}}, \hat{\underline{\underline{z}}})\tilde{\Phi}_{\underline{\alpha}}(\hat{\underline{\theta}}, \hat{\underline{\underline{z}}})\right)\left(F(\check{\underline{\theta}})\tilde{h}\,(\check{\underline{\theta}}, \check{\underline{\underline{z}}})\tilde{\Psi}_{\underline{\alpha}}(\check{\underline{\theta}}, \check{\underline{\underline{z}}})\right)\left(p^{T+}(\hat{\underline{\theta}}, \hat{\underline{z}}) + p^{T-}(\check{\underline{\theta}}, \check{\underline{z}})\right)\end{aligned}$$

However, this means that in leading order

$$F^T_{\underline{\alpha}}(\underline{\theta}_W) \to 0$$

because of Lemma 1.

Order $\frac{1}{W}$: similarly, as in the proof of Theorem 1 we discuss the contribution from $\tilde{h}_1(\underline{\theta}, \underline{z})$ of (E.14), however, for $k_1 = k\,(1 - 1/N)\,,\ l_1 = l\,(1 - 1/N)$ there are no the $\hat{z}_j$-dependent terms and therefore this contribution vanishes by Lemma 1.

From the Bethe state $\tilde{\Psi}_{\underline{\alpha}1}(\hat{\underline{\theta}}, \check{\underline{\theta}}, \hat{\underline{z}}, \check{\underline{z}})$ of (E.20) and (E.8) we obtain contributions of the type

$$\int d\underline{z}\,\tilde{h}(\underline{\theta}, \underline{z})\left(\tilde{\Psi}(\underline{\theta}, \underline{z})M_1^2\right)_{\underline{\alpha}} \text{ and } \int d\underline{z}\,\tilde{h}(\underline{\theta}, \underline{z})p^{T\pm}(\underline{\theta}, \underline{z})\left(\tilde{\Psi}(\underline{\theta}, \underline{z})M_1^2\right)_{\underline{\alpha}}$$

where both are $= 0$, the first one because of Lemma 1 and the second one because $T(x)$ is an iso-scalar operator. Therefore there are no contributions of order W^{-1} and

$$F^T_{\underline{\alpha}}(\underline{\theta}_W) \to O(W^{-2}) \text{ for } W \to \infty\,.$$

■

Calculation of the functions $c^T_{JJ}(k,l,W)$: defined by

$$F^T_{\underline{\alpha}}(\underline{\theta}_W) \overset{W\to\infty}{\to} c^T_{JJ}(k,l,W)\mathbf{C}_{\alpha\bar\delta}\mathbf{C}_{\beta\bar\gamma}F^{J^{\alpha\bar\gamma}}_{\underline{\hat\alpha}}F^{J^{\beta\bar\delta}}_{\underline{\check\alpha}}(\underline{\check\theta})\,.$$

We apply the general procedure of appendix C: using $a(W)\to e^{-i\pi\left(1-\frac{1}{N}\right)}$ of (E.1) and $\sigma^T_1=1,\ Q^T=0$ we check (C.4) and (C.7) for this case

$$\dot\sigma^T_1(n)S^{\underline{\check\alpha}'1}_{1\underline{\check\alpha}}(\theta,\underline{\check\theta}) \overset{W\to\infty}{\to} (-1)^{(N-1)+(1-1/N)(k+l)}(a(W))^l 1^{\underline{\check\alpha}'}_{\underline{\check\alpha}} \to \dot\sigma^T_1(k)1^{\underline{\check\alpha}'}_{\underline{\check\alpha}}$$

$$\dot\sigma^T_{\bar 1}(n)S^{\bar 1\underline{\hat\alpha}'}_{\underline{\hat\alpha}\bar 1}(\underline{\hat\theta}+W,\omega) \overset{W\to\infty}{\to} (-1)^{(N-1)+(1-1/N)(k+l)}(a(W))^{k(N-1)}1^{\underline{\hat\alpha}'}_{\underline{\hat\alpha}} \to (-1)^{(N-1)k}\dot\sigma^J_1(l)1^{\underline{\hat\alpha}'}_{\underline{\hat\alpha}}$$

Therefore, as proofed in appendix C, $c^T_{JJ}(k,l,W)$ is independent of k and l, because $(-1)^{(N-1)k}=1$ for $k=0 \bmod N$. It is convenient to consider the special case $c^T_{JJ}(N,N,W)$:

1) We take the bound states $\bar 1=(\hat\alpha_2\dots\hat\alpha_N)$ and $\bar 1=(\check\alpha_1\dots\check\alpha_{N-1})$ and calculate for $\underline{\theta}_W=(\hat\theta+W,\hat\omega+W,\check\omega,\check\theta)$

$$\operatorname*{Res}_{\hat\theta=i\pi+\hat\omega}F^T_{1\bar 1\bar 1 1}(\underline{\theta}_W)=2i\,\mathbf{C}_{1\bar 1}\,F^T_{\bar 1 1}(\check\omega,\check\theta)\left(1-\dot\sigma^T_{\bar 1}(2N)S^{\bar 1 1,\bar 1}_{1,\bar 1 1}(\hat\omega+W;\check\omega,\check\theta)\right)$$
$$\overset{W\to\infty}{\to} -2i\,W^{-2}i\eta\,(1-1/N)\left(\check\theta-\hat\omega+i\pi\right)F^T_{\bar 1 1}(\check\omega,\check\theta)$$

It was used that $\mathbf{C}_{1\bar 1}=1$ and that (2.1), (B.1) and (E.1) imply

$$\begin{aligned}&\dot\sigma^T_{\bar 1}(2N)S^{\bar 1 1,\bar 1}_{1,1\bar 1}(\hat\omega+W;\check\omega,\check\theta)\\&=(-1)^{(N-1)+(1-1/N)2N}\left(a(\hat\omega+W-\check\omega)\right)\left((-1)^{N-1}a(i\pi-\left(\hat\omega+W-\check\theta\right))\right)\\&\overset{W\to\infty}{\to}\exp\left(-i\eta\,(1-1/N)\left((\hat\omega+W-\check\omega)^{-1}+\left(i\pi-\left(\hat\omega+W-\check\theta\right)\right)^{-1}\right)\right)\\&\overset{W\to\infty}{\to} 1+W^{-2}i\eta\,(1-1/N)\left(\check\theta-\hat\omega+i\pi\right).\end{aligned}$$

2) Taking first $W\to\infty$ and then the Res means

$$\operatorname*{Res}_{\hat\theta=i\pi+\hat\omega}\left(F^T_{1\bar 1\bar 1 1}(\underline{\theta}_W)\overset{W\to\infty}{\to}c^T_{JJ}(N,N,W)\mathbf{C}_{\alpha\bar\delta}\mathbf{C}_{\beta\bar\gamma}F^{J^{\alpha\bar\gamma}}_{1\bar 1}(\hat\theta,\hat\omega)F^{J^{\beta\bar\delta}}_{\bar 1 1}(\check\omega,\check\theta)\right)$$
$$=-2ic^T_{JJ}(N,N,W)F^{J^{1\bar 1}}_{\bar 1 1}(\check\omega,\check\theta)$$

where (3.9) has been used. The particle anti-particle form factors (3.9) and (3.13) satisfy applying the form factor equation (ii)

$$F^{J^{1\bar 1}}_{\bar 1 1}(\theta,\omega)=(1-1/N)\,i\,(\theta-\omega+i\pi)\,F^T_{\bar 1 1}(\theta,\omega) \tag{4.16}$$

therefore

$$c^T_{JJ}(k,l,W)=\eta W^{-2}$$

which supports (3.24).

Remark 3 *Repeating the last discussion for the more general case* $k = N$, $l = LN$, $\underline{\theta}_W = (\hat{\omega} + W, \hat{\theta} + W, \check{\underline{\theta}}, \check{\underline{\omega}})$ *and* $\check{\alpha} = (\alpha \dots \alpha \bar{\alpha} \dots \bar{\alpha})$ *for a fixed* $\alpha = 1, \dots, N$ *we obtain as an generalization of (4.16) the interesting relation of energy momentum and current form factors*

$$\left(1 - \frac{1}{N}\right) \sum_{j=1}^{L} i \left(\check{\theta}_j - \check{\omega}_j - i\pi\right) F^T_{\alpha\dots\alpha\bar{\alpha}\dots\bar{\alpha}}(\check{\underline{\theta}}, \check{\underline{\omega}}) = \mathbf{C}_{\alpha\bar{\alpha}} \left(F^{J^{\alpha\bar{\alpha}}}_{\alpha\dots\alpha\bar{\alpha}\dots\bar{\alpha}}(\check{\underline{\theta}}, \check{\underline{\omega}})\right) \tag{4.17}$$
$$= 2\,(T_a)^{\alpha}_{\alpha}\, F^{J_a}_{\alpha\dots\alpha\bar{\alpha}\dots\bar{\alpha}}(\check{\underline{\theta}}, \check{\underline{\omega}}).$$

Equivalence. Using the general relations (4.10), (3.6) and $\mathbf{C}_{\alpha\bar{\gamma}} F^{J^{\alpha\bar{\gamma}}}_{\underline{\check{\alpha}}} = 0$ we obtain

$$2F^{J_a}_{\underline{\hat{\alpha}}}(\underline{\hat{\theta}}) F^{J_a}_{\underline{\check{\alpha}}}(\underline{\check{\theta}}) = 2\,(T_a)_{\alpha\bar{\gamma}}\, F^{J^{\alpha\bar{\gamma}}}_{\underline{\hat{\alpha}}}(\underline{\hat{\theta}})\,(T_a)_{\beta\bar{\delta}}\, F^{J^{\beta\bar{\delta}}}_{\underline{\check{\alpha}}}(\underline{\check{\theta}}) = \mathbf{C}_{\beta\bar{\gamma}} \mathbf{C}_{\alpha\bar{\delta}}\, F^{J^{\alpha\bar{\gamma}}}_{\underline{\hat{\alpha}}}(\underline{\hat{\theta}}) F^{J^{\beta\bar{\delta}}}_{\underline{\check{\alpha}}}(\underline{\check{\theta}}).$$

with $(T_a)_{\alpha\bar{\beta}} = \mathbf{C}_{\delta\bar{\beta}}\,(T_a)^{\delta}_{\alpha}$.

4.4 Theorem 4

Theorem 4 *The cluster behavior of the form factor of the pseudo-potential of the current for particle number* $n = 0 \bmod N$ *and* $k = 1 \bmod N$ *reads as*

$$F^J_{\underline{\alpha}}(\underline{\theta}_W) \overset{W\to\infty}{\to} c^J_{\psi\chi}(k, l, W) F^{\psi}_{\underline{\hat{\alpha}}}(\underline{\hat{\theta}}) F^{\chi}_{\underline{\check{\alpha}}}(\underline{\check{\theta}})$$
$$c^J_{\psi\chi}(k, l, W) = e^{i\pi l_1} 2\,(2\pi)^{-\frac{1+N}{N^2}}\, e^{-i\pi\left(N + \frac{1}{2N}\right)} / \bar{F}(i\pi)\, W^{\frac{1}{N^2}} e^{-\frac{1}{2}\left(1 - \frac{1}{N}\right)W}$$

with $l_1 = (l+1)(1 - 1/N) - 1$.

Proof. We investigate

$$F^J_{\underline{\alpha}}(\underline{\theta}_W) = N^J_n F(\underline{\theta}_W) \int d\underline{z}\tilde{h}\,(\underline{\theta}_W, \underline{z}_W) p^J(\underline{\theta}_W, \underline{z}_W) \tilde{\Phi}_{\underline{\alpha}}(\underline{\theta}_W, \underline{z}_W)$$

The exponential behavior of the integrand is again given by (4.1). For $k = 1 \bmod N$ the leading asymptotic behavior $\left(e^{-\frac{1}{2}W}\right)^{1-\frac{1}{N}}$ is obtained for $\tilde{k}_j = j/N - 1$ which implies

$$k_j = (k-1)(1 - j/N)\,, \quad l_j = (l+1)(1 - j/N) - 1,\ j = 1, \dots, N-1$$

For these values of k_j and l_j we obtain, more precisely

$$F(\underline{\theta}_W)\tilde{h}\,(\underline{\theta}_W, \underline{z}_W) p^J(\underline{\theta}_W, \underline{z}_W) \tilde{\Phi}_{\underline{\alpha}}(\underline{\theta}_W, \underline{z}_W)$$
$$\to W^{\frac{1}{N^2}} e^{-\frac{1}{2}\left(1 - \frac{1}{N}\right)W} \left(F(\underline{\hat{\theta}})\tilde{h}(\underline{\hat{\theta}}, \underline{\hat{z}}) p^{\psi}(\underline{\hat{\theta}}, \underline{\hat{z}}) \tilde{\Psi}_{\underline{\hat{\alpha}}}(\underline{\hat{\theta}}, \underline{\hat{z}})\right) \left(F(\underline{\check{\theta}})\tilde{h}(\underline{\check{\theta}}, \underline{\check{z}}) p^{\chi}(\underline{\check{\theta}}, \underline{\check{z}}) \tilde{\Phi}_{\underline{\check{\alpha}}}(\underline{\check{\theta}}, \underline{\check{z}})\right),$$

which implies (up to const.)

$$F^J_{\underline{\alpha}}(\underline{\theta}_W) \to W^{\frac{1}{N^2}} e^{-\frac{1}{2}\left(1 - \frac{1}{N}\right)W} F^{\psi}_{\underline{\hat{\alpha}}}(\underline{\hat{\theta}}) F^{\chi}_{\underline{\check{\alpha}}}(\underline{\check{\theta}}).$$

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Calculation of the function $c^J_{\psi\chi}(k,l,W)$ *:* defined by

$$F^J_{\underline{\alpha}}(\underline{\theta}_W) \overset{W\to\infty}{\to} c^J_{\psi\chi}(k,l,W) F^\psi_{\underline{\hat\alpha}}(\underline{\hat\theta}) F^\chi_{\underline{\check\alpha}}(\underline{\check\theta}).$$

We apply the procedure of appendix C: using $a(W) \to e^{-i\pi\left(1-\frac{1}{N}\right)}$ of (E.1) we check (C.4) and (C.7) with $\sigma^J = 1,\ Q^J = 0,\ \sigma^\psi = e^{i\pi\left(1-\frac{1}{N}\right)},\ Q^\psi = 1$ and $\sigma^\chi_1 = e^{i\pi\left(N-\frac{1}{N}\right)},\ Q^\chi = N-1$

$$\dot\sigma^J_1(n) S^{\underline{\check\alpha}'1}_{1\underline{\check\alpha}}(\theta+W,\underline{\check\theta}) \overset{W\to\infty}{\to} (-1)^{(N-1)+(1-1/N)(k+l)} (a(W))^l \mathbf{1}^{\underline{\check\alpha}'}_{\underline{\check\alpha}} \to \dot\sigma^\psi_1(k) \mathbf{1}^{\underline{\check\alpha}'}_{\underline{\check\alpha}}$$
$$\dot\sigma^J_1(n) S^{\bar1\underline{\hat\alpha}'}_{\underline{\hat\alpha}1}(\underline{\hat\theta}+W,\omega) \overset{W\to\infty}{\to} (-1)^{(N-1)+(1-1/N)(k+l)} (a(W))^{(N-1)k} \mathbf{1}^{\underline{\hat\alpha}'}_{\underline{\hat\alpha}} \to (-1)^{(N-1)k} \dot\sigma^\chi_1(l) \mathbf{1}^{\underline{\hat\alpha}'}_{\underline{\hat\alpha}}$$

Therefore $c^J_{\psi\chi}(k,l,W)$ is independent of k and for $k = 1 \bmod N$ (see C)

$$c^J_{\psi\chi}(k,l,W) = c^J_{\psi\chi}(k_0,l_0,W)(-1)^{(N-1)(l-l_0)/N}$$

The special case $c^J_{\psi\chi}(1,N-1,W)$ is calculated by the following example, which implies

$$c^J_{\psi\chi}(k,l,W) = e^{i\pi l_1} 2\,(2\pi)^{-\frac{1+N}{N^2}} e^{-i\pi\left(N+\frac{1}{2N}\right)} / \bar F(i\pi)\, W^{\frac{1}{N^2}} e^{-\frac{1}{2}\left(1-\frac{1}{N}\right)W} \tag{4.18}$$

because $l_1 = (l+1)(1-1/N) - 1$.

Example. The particle anti-particle of (3.9) and asymptotic behavior of the particle anti-particle minimal form factor function (E.5) imply

$$F^J_{1\bar N}(\underline{\theta}_W) \overset{W\to\infty}{\to} 2e^{-\frac{1}{2}(\theta-\omega+W)} \left((2\pi)^{-1-\frac{1}{N}} W^{\frac{1}{N}} e^{\frac{1}{2}W} e^{\frac{1}{2}(\theta-\omega-i\pi)}\right)^{\frac{1}{N}} / \bar F(i\pi).$$

The asymptotic relation (3.25), (3.18) and (3.21) give

$$F^J_{1\bar N}(\underline{\theta}_W) \overset{W\to\infty}{\to} c^J_{\psi\chi}(1,N-1,W) F^\psi_1(\theta) F^\chi_{\bar N}(0) = c^J_{\psi\chi}(1,N-1,W) e^{-\frac{1}{2}\left(1-\frac{1}{N}\right)\theta} e^{\frac{1}{2}\left(1-\frac{1}{N}\right)\omega}$$

which means

$$c^J_{\psi\chi}(1,N-1,W) = 2e^{-\frac{1}{2}W}\left((2\pi)^{-1-\frac{1}{N}} W^{\frac{1}{N}} e^{\frac{1}{2}W} e^{-\frac{1}{2}i\pi}\right)^{\frac{1}{N}} / \bar F(i\pi)$$

and (4.18).

4.5 Conjecture 2

Conjecture 2 *The form factor of the energy momentum potential for particle number* $n = 0 \bmod N$ *and* $k = 1 \bmod N$ *shows the cluster behavior (3.26)*

$$F^T_{\underline{\alpha}}(\underline{\theta}_W) \overset{W\to\infty}{\to} c^T_{\psi\chi}(k,l,W) \mathbf{C}_{\alpha\bar\beta} F^{\psi^\alpha}_{\underline{\hat\alpha}}(\underline{\hat\theta}) F^{\chi^{\bar\beta}}_{\underline{\check\alpha}}(\underline{\check\theta}).$$

We have no general proof of this conjecture. The problem is the same as in Conjecture 1, that the expansion for large W of the integrand in the multiple contour integral representation in (2.10) must not be interchanged with the integration. However, the relation (4.17) implies the cluster relation (3.26) and we have again checked consistency with the form factor equation (iii), which also yields the function $c^T_{\psi\chi}(k,l,W)$ of (3.26).

Calculation of the function $c^T_{\psi\chi}(k,l,W)$. In the same way as above for $c^J_{\psi\chi}$ we prove that $c^T_{\psi\chi}(k,l,W)$ is independent of k and for $k = 1 \bmod N$

$$c^T_{\psi\chi}(k,l,W) = c^T_{\psi\chi}(k_0,l_0,W)(-1)^{(N-1)(l-l_0)/N}$$

The special case $c^T_{\psi\chi}(1,N-1,W)$ is calculated by the following example, which implies

$$c^T_{\psi\chi}(k,l,W) = ie^{i\pi l_1}\, W^{\frac{1}{N^2}-1} e^{-\frac{1}{2}\left(1-\frac{1}{N}\right)W} 2\,(2\pi)^{-\frac{1+N}{N^2}}\, e^{-i\pi\left(N+\frac{1}{2N}\right)} / \bar{F}(i\pi) \tag{4.19}$$

Example. The particle anti-particle of (3.13) and asymptotic behavior of the particle anti-particle minimal form factor function (E.5) imply

$$F^T_{1\bar{N}}(\underline{\theta}_W) \overset{W\to\infty}{\to} 2e^{-\frac{1}{2}(\theta-\omega+W)} \frac{1}{iW}\left((2\pi)^{-1-\frac{1}{N}}\, W^{\frac{1}{N}} e^{\frac{1}{2}W} e^{\frac{1}{2}(\theta-\omega-i\pi)}\right)^{\frac{1}{N}} / \bar{F}(i\pi).$$

The asymptotic relation (3.26), (3.18) and (3.21) give

$$F^T_{1\bar{N}}(\underline{\theta}_W) \overset{W\to\infty}{\to} c^T_{\psi\chi}(1,N-1,W) F^{\psi}_1(\theta) F^{\chi}_{\bar{N}}(\omega) = c^T_{\psi\chi}(1,N-1,W) e^{-\frac{1}{2}\left(1-\frac{1}{N}\right)\theta} e^{\frac{1}{2}\left(1-\frac{1}{N}\right)\omega}$$

which means

$$c^T_{\psi\chi}(1,N-1,W) = \frac{1}{iW} 2e^{-\frac{1}{2}W}\left((2\pi)^{-1-\frac{1}{N}}\, W^{\frac{1}{N}} e^{\frac{1}{2}W} e^{-\frac{1}{2}i\pi}\right)^{\frac{1}{N}} / \bar{F}(i\pi)$$

and (4.19).

4.6 Theorem 5

Theorem 5 *The cluster behavior of form factor of the fundamental field for the number particles* $n = 1 \bmod N$ *and* $k = 0 \bmod N$ *reads as*

$$F^{\psi^\beta}_{\underline{\alpha}}(\underline{\theta}_W) \overset{W\to\infty}{\to} \frac{1}{W} i\eta \mathbf{C}_{\gamma\bar{\delta}} F^{J^{\beta\bar{\delta}}}_{\underline{\hat{\alpha}}}(\underline{\hat{\theta}}) F^{\psi^\gamma}_{\underline{\check{\alpha}}}(\underline{\check{\theta}}) = \frac{1}{W} 2i\eta F^{J_a}_{\underline{\hat{\alpha}}}(\underline{\hat{\theta}})\,(T_a)^\beta_\delta\, F^{\psi^\delta}_{\underline{\check{\alpha}}}(\underline{\check{\theta}}). \tag{4.20}$$

Proof. We investigate

$$F^{\psi}_{\underline{\alpha}}(\underline{\theta}_W) = N^{\psi}_n F(\underline{\theta}_W) \int d\underline{\underline{z}}\, \tilde{h}\,(\underline{\theta}_W, \underline{\underline{z}}_W) p^{\psi}(\underline{\theta}_W, \underline{z}_W) \tilde{\Phi}_{\underline{\alpha}}(\underline{\theta}_W, \underline{\underline{z}}_W)$$

From the asymptotic behavior of $F(\underline{\theta}_W, \underline{\omega}_W)\tilde{h}\,(\underline{\theta}_W, \underline{\omega}_W, \underline{\underline{z}}_W)$ and $p^{\psi}(\underline{\theta}_W, \underline{z}_W)$ in (E.18), (E.10) and (F.4) we derive for $W \to \infty$ the exponential behavior

$$F(\underline{\theta}_W)\tilde{h}\,(\underline{\theta}_W, \underline{\underline{z}}_W) p^{\psi}(\underline{\theta}_W, \underline{\underline{z}}_W) \propto \left(e^{-\frac{1}{2}W}\right)^{\tilde{k}_1^2 + \tilde{k}_{N-1}^2 + \sum_{j=1}^{N-2}\left(\tilde{k}_j - \tilde{k}_{j+1}\right)^2}, \quad \tilde{k}_j = k_j - k\,(1 - j/N) \tag{4.21}$$

For $k = 0 \bmod N$ and $l = 1 \bmod N$ the leading asymptotic behavior $\propto \left(e^{-\frac{1}{2}W}\right)^0$ is obtained for $\tilde{k}_j = 0$ i.e. $k_j = k\,(1 - j/N)$ and $l_j = (l-1)\,(1 - j/N)$ by (3.19), which implies (up to const.)

$$\begin{aligned} &F(\underline{\theta}_W)\tilde{h}(\underline{\theta}_W, \underline{\underline{z}}_W) p^{\psi}(\underline{\theta}_W, \underline{z}_W) \tilde{\Phi}_{\underline{\alpha}}(\underline{\theta}_W, \underline{\underline{z}}_W) \\ &\to \left(F(\underline{\hat{\theta}})\tilde{h}(\underline{\hat{\theta}}, \underline{\underline{\hat{z}}}) \tilde{\Phi}_{\underline{\hat{\alpha}}}(\underline{\hat{\theta}}, \underline{\underline{\hat{z}}})\right)\left(F(\underline{\check{\theta}})\tilde{h}(\underline{\check{\theta}}, \underline{\underline{\check{z}}}) p^{\psi}(\underline{\check{\theta}}, \underline{\check{z}}) \tilde{\Phi}_{\underline{\check{\alpha}}}(\underline{\check{\theta}}, \underline{\underline{\check{z}}})\right) \end{aligned} \tag{4.22}$$

However, this means that in leading order

$$F^{\psi}_{\underline{\alpha}}(\underline{\theta}_W) \to 0$$

because of Lemma 1. The proof of the $\frac{1}{W}$ contribution is similar to that one of Theorem 1.

Order $\frac{1}{W}$: we have to apply the asymptotic behavior of the h-function (E.12) and the Bethe state (E.21).

The result for the contribution of h_1 is (up to a constant)

$$F^{\psi 1}_{\underline{\alpha},h_1}(\underline{\theta}_W) \to i\eta W^{-1}\left(\mathbf{C}_{1\bar{1}} F^{J^{1\bar{1}}}_{\underline{\hat{\alpha}}}(\underline{\hat{\theta}})\right) F^{\psi^1}_{\underline{\check{\alpha}}}(\underline{\check{\theta}}) \tag{4.23}$$

and the result for the contribution of Φ_1 is

$$F^{\psi 1}_{\underline{\alpha},\Phi_1}(\underline{\theta}_W) \to i\eta W^{-1}\left(\mathbf{C}^{(1)}_{\gamma\bar{\delta}} F^{J^{1\bar{\delta}}}_{\underline{\hat{\alpha}}}(\underline{\hat{\theta}}) F^{\psi^\gamma}_{\underline{\check{\alpha}}}(\underline{\check{\theta}})\right)$$

Because $\mathbf{C}^{(1)}_{\gamma\bar{\delta}} + \mathbf{C}_{1\bar{1}}\delta^1_\gamma\delta^{\bar{1}}_{\bar{\delta}} = \mathbf{C}_{\gamma\bar{\delta}}$ the claim (3.27) is proved. ∎

Calculation of the function $c^\psi_{J\psi}(k,l,W)$: defined by

$$F^{\psi^\beta}_{\underline{\alpha}}(\underline{\theta}_W) \overset{W\to\infty}{\to} c^\psi_{J\psi}(k,l,W)\mathbf{C}_{\gamma\bar{\delta}} F^{J^{\beta\bar{\delta}}}_{\underline{\hat{\alpha}}}(\underline{\hat{\theta}}) F^{\psi^\gamma}_{\underline{\check{\alpha}}}(\underline{\check{\theta}})$$

We apply the procedure of appendix C: using $a(W) \to e^{-i\pi\left(1-\frac{1}{N}\right)}$ of (E.1) we check (C.4) and (C.7) with $\sigma^\psi_1 = e^{\left(1-\frac{1}{N}\right)i\pi}$ and $\sigma^J_1 = 1$ for $k = 0 \bmod N$

$$\begin{aligned}
\dot{\sigma}^\psi_1(n) S^{\underline{\check{\alpha}}'1}_{1\underline{\check{\alpha}}}(\theta + W, \underline{\check{\theta}}) &\overset{W\to\infty}{\to} e^{\left(1-\frac{1}{N}\right)i\pi}(-1)^{(N-1)+(1-1/N)(k+l-1)}(a(W))^l \mathbf{1}^{\underline{\check{\alpha}}'}_{\underline{\check{\alpha}}} \\
&\to (-1)^{(N-1)+(1-1/N)k}\mathbf{1}^{\underline{\check{\alpha}}'}_{\underline{\check{\alpha}}} = \dot{\sigma}^J_1(k)\mathbf{1}^{\underline{\check{\alpha}}'}_{\underline{\check{\alpha}}}\,.
\end{aligned}$$

$$\begin{aligned}
\dot{\sigma}^\psi_1(n) S^{\bar{1}\underline{\hat{\alpha}}'}_{\underline{\hat{\alpha}}\bar{1}}(\underline{\hat{\theta}} + W, \omega) &\overset{W\to\infty}{\to} e^{\left(1-\frac{1}{N}\right)i\pi}(-1)^{(N-1)+(1-1/N)(k+l)}(a(W))^{(N-1)k}\mathbf{1}^{\underline{\hat{\alpha}}'}_{\underline{\hat{\alpha}}} \\
&\overset{W\to\infty}{\to} (-1)^{(N-1)k}\dot{\sigma}^\psi_1(l)\mathbf{1}^{\underline{\hat{\alpha}}'}_{\underline{\hat{\alpha}}}\,.
\end{aligned}$$

Therefore $c^\psi_{J\psi}(k,l,W)$ is independent of k and l. It is convenient to consider the special case $c^\psi_{J\psi}(N,1,W)$:

1) We take the bound states $\bar{1} = (\hat{\alpha}_2 \dots \hat{\alpha}_N)$ and calculate for $\underline{\theta}_W = (\hat{\theta} + W, \hat{\omega} + W, \check{\theta})$

$$\begin{aligned}
\operatorname*{Res}_{\hat{\theta}=i\pi+\hat{\omega}} F^\psi_{1\bar{1}1}(\underline{\theta}_W) &= 2i\,\mathbf{C}_{1\bar{1}}\,F^\psi_1(\check{\theta})\left(1 - \dot{\sigma}^\psi_{\bar{1}}(N+1)S^{1,\bar{1}}_{\bar{1},1}(\hat{\omega} + W, \check{\theta})\right) \\
&\overset{W\to\infty}{\to} -2i\,\mathbf{C}_{1\bar{1}}\,i\eta\,(1 - 1/N)\,\frac{1}{W}F^\psi_1(\check{\theta}).
\end{aligned} \tag{4.24}$$

It was used that $\sigma^\psi_1 = e^{i\pi\left(1-\frac{1}{N}\right)}$, $\sigma^\psi_1\sigma^\psi_{\bar{1}} = (-1)^{(N-1)}$ (see [9]), (B.1), (E.1) and $a(\theta)a(-\theta) = 1$ imply

$$\begin{aligned}
&\dot{\sigma}^\psi_{\bar{1}}(N+1)S^{1,\bar{1}}_{\bar{1},1}(\hat{\omega} + W, \check{\theta}) \overset{W\to\infty}{\to} e^{-i\pi\left(1-\frac{1}{N}\right)}(-1)^{(1-1/N)N}(-1)^{(N-1)}a(-W) \\
&\to e^{-i\pi\left(1-\frac{1}{N}\right)}(-1)^{(1-1/N)N}(-1)^{(N-1)}e^{i\pi\left(1-\frac{1}{N}\right)}e^{i\eta\left(1-\frac{1}{N}\right)\frac{1}{W}} \to 1 + i\eta\,(1 - 1/N)\,\frac{1}{W}
\end{aligned}$$

2) Taking first $W \to \infty$ and then the Res means

$$\begin{aligned}
\operatorname*{Res}_{\hat{\theta}=i\pi+\hat{\omega}}\left(F^\psi_{1\bar{1}1}(\underline{\theta}_W) \overset{W\to\infty}{\to} c^\psi_{J\psi}(N,1,W)\left(\mathbf{C}_{\gamma\bar{\delta}}F^{J^{1\bar{\delta}}}_{1\bar{1}}(\omega,\theta)F^{\psi^\gamma}_1(\check{\theta})\right)\right) \\
= c^\psi_{J\psi}(N,1,W)\,(1 - 1/N)\,(-2i)\,\mathbf{C}_{1\bar{1}}F^{\psi^\gamma}_1(\check{\theta}).
\end{aligned}$$

As result we obtain $c^\psi_{J\psi}(k,l,W) = i\eta\frac{1}{W}$ which proves (3.27).

Equivalence. Using the general relations (4.10) and (3.6) we obtain

$$\begin{aligned}
&2F^{J_a}_{\underline{\hat\alpha}}(\underline{\hat\theta})\,(T_a)^\beta_\delta\,F^{\psi^\delta}_{\underline{\check\alpha}}(\underline{\check\theta})\\
&=2\mathbf{C}_{\gamma\bar\epsilon}\,(T_a)^\gamma_\alpha\,F^{J^{\alpha\bar\epsilon}}_{\underline{\hat\alpha}}(\underline{\hat\theta})\,(T_a)^\beta_\delta\,F^{\psi^\delta}_{\underline{\check\alpha}}(\underline{\check\theta})\\
&=2\mathbf{C}_{\gamma\bar\epsilon}\left(\frac{1}{2}\left(\delta^\beta_\alpha\delta^\gamma_\delta-1/N\,\delta^\gamma_\alpha\delta^\beta_\delta\right)\right)F^{J^{\alpha\bar\epsilon}}_{\underline{\hat\alpha}}(\underline{\hat\theta})F^{\psi^\delta}_{\underline{\check\alpha}}(\underline{\check\theta})\\
&=\mathbf{C}_{\gamma\bar\epsilon}F^{J^{\beta\bar\epsilon}}_{\underline{\hat\alpha}}(\underline{\hat\theta})F^{\psi^\gamma}_{\underline{\check\alpha}}(\underline{\check\theta})
\end{aligned}$$

because of $\mathbf{C}_{\alpha\bar\epsilon}F^{J^{\alpha\bar\epsilon}}_{\underline{\hat\alpha}}(\underline{\hat\theta})=0$.

4.7 Theorem 6

Theorem 6 *The cluster behavior of form factor of the fundamental field for particle number $n=1\,\mathrm{mod}\,N$ and $k=1\,\mathrm{mod}\,N$ reads as*

$$F^{\psi^\alpha}_{\underline{\alpha}}(\underline{\theta}_W)\overset{W\to\infty}{\to}(-1)^{l_1}e^{-\frac{1}{2}\left(1-\frac{1}{N}\right)W}F^{\psi^\alpha}_{\underline{\hat\alpha}}(\underline{\hat\theta})F^{\phi}_{\underline{\check\alpha}}(\underline{\check\theta})\,.$$

Proof. We investigate

$$F^{\psi}_{\underline{\alpha}}(\underline{\theta}_W)=N^\psi_nF(\underline{\theta}_W)\int d\underline{\underline{z}}\tilde h\,(\underline{\theta}_W,\underline{\underline{z}}_W)p^\psi(\underline{\theta}_W,\underline{\underline{z}}_W)\tilde\Psi_{\underline{\alpha}}(\underline{\theta}_W,\underline{\underline{z}}_W)$$

and obtain as above the exponential behavior (4.21). The leading behavior $\propto\left(e^{-\frac{1}{2}W}\right)^{1-\frac{1}{N}}$ is obtained for $\tilde k_j=j/N-1$ which means

$$k_j=(k-1)\,(1-j/N)\,,\;\;l_j=l\,(1-j/N)\;,\quad\text{for}\;\;j=1,\dots,N-1$$

and (up to const.)

$$\begin{aligned}
&F(\underline{\theta}_W)p^\psi(\underline{\theta}_W,\underline{\underline{z}}_W)\tilde\Psi_{\underline{\alpha}}(\underline{\theta}_W,\underline{\underline{z}}_W)\\
&\to\left(e^{-\frac{1}{2}W}\right)^{1-\frac{1}{N}}\left(F(\underline{\hat\theta})\tilde h(\underline{\hat\theta},\underline{\underline{\hat z}})p^\psi(\underline{\hat\theta},\underline{\underline{\hat z}})\tilde\Psi_{\underline{\hat\alpha}}(\underline{\hat\theta},\underline{\underline{\hat z}})\right)\left(F(\underline{\check\theta})\tilde h(\underline{\check\theta},\underline{\underline{\check z}})p^\phi(\underline{\check\theta},\underline{\underline{\check z}})\tilde\Psi_{\underline{\check\alpha}}(\underline{\check\theta},\underline{\underline{\check z}})\right).
\end{aligned}$$

proving (3.28). ■

Calculation of the function $c^\psi_{\psi\phi}(k,l,W)$: defined by

$$F^{\psi}_{\underline{\alpha}}(\underline{\theta}_W)\overset{W\to\infty}{\to}c^\psi_{\psi\phi}(k,l,W)F^{\psi}_{\underline{\hat\alpha}}(\underline{\hat\theta})F^{\phi}_{\underline{\check\alpha}}(\underline{\check\theta}).\tag{4.25}$$

We apply the procedure of appendix C: using $a(W)\overset{W\to\infty}{\to}e^{-i\pi\left(1-\frac{1}{N}\right)}$ of (E.1) we check (C.4) and (C.7) with $\sigma^\psi_1=e^{i\pi\left(1-\frac{1}{N}\right)}$, $Q^\psi=1$ and $\sigma^\phi_1=e^{-i\eta}$, $Q^\phi=0$

$$\begin{aligned}
&\dot\sigma^\psi_1(n)S^{\underline{\check\alpha}'1}_{1\underline{\check\alpha}}(\theta+W,\underline{\check\theta})\to e^{i\pi\left(1-\frac{1}{N}\right)}(-1)^{(N-1)+(1-1/N)(k+l-1)}(a(W))^l\mathbf{1}^{\underline{\check\alpha}'}_{\underline{\check\alpha}}\to\dot\sigma^\psi_1(k)\mathbf{1}^{\underline{\check\alpha}'}_{\underline{\check\alpha}}\\
&\dot\sigma^\psi_1(n)S^{\bar1\underline{\hat\alpha}'}_{\underline{\hat\alpha}\bar1}(\underline{\hat\theta}+W,\omega)\to e^{i\pi\left(1-\frac{1}{N}\right)}(-1)^{(N-1)+(1-1/N)(k+l-1)}(a(W))^{(N-1)k}\mathbf{1}^{\underline{\hat\alpha}'}_{\underline{\hat\alpha}}\\
&\qquad\qquad\to(-1)^{(N-1)k}\dot\sigma^\phi_1(l)\mathbf{1}^{\underline{\hat\alpha}'}_{\underline{\hat\alpha}}
\end{aligned}$$

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Therefore (C.3) and (C.5) imply that $c^{\psi}_{\psi\phi}(k,l,W)$ is independent of k and (C.6) and (C.8) that for $k = 1 \bmod N$

$$c^{\psi}_{\psi\phi}(k,l,W) = c^{\psi}_{\psi\phi}(k_0,l_0,W)(-1)^{(N-1)(l-l_0)/N}$$

The special case $c^{\psi}_{\psi\chi}(1,0,W)$ is obtained by (4.25) for $\underline{\check{\alpha}} = \emptyset$ and the form factor equation (v) with spin $s^{\psi} = -\frac{1}{2}\left(1-\frac{1}{N}\right)$

$$\begin{aligned} F^{\psi}_{\underline{\alpha}}(\underline{\theta}_W) &\to c^{J}_{\psi\chi}(1,0,W)F^{\psi}_{\underline{\hat{\alpha}}}(\underline{\hat{\theta}})F^{\phi}_{\emptyset} \\ F^{\psi}_{\underline{\alpha}}(\underline{\theta}_W) &\to e^{s^{\psi}W} = e^{-\frac{1}{2}\left(1-\frac{1}{N}\right)W}F^{\psi}_{\underline{\hat{\alpha}}}(\underline{\hat{\theta}}) \end{aligned}$$

if we normalize the field $\phi(x)$ by $F^{\phi}_{\emptyset} = \langle 0|\phi(x)|0\rangle = 1$, this gives the result

$$c^{\psi}_{\psi\phi}(k,l,W) = (-1)^{l_1}e^{-\frac{1}{2}\left(1-\frac{1}{N}\right)W}$$

because $l_1 = l(1-1/N)$.

5 Summary

In this article we investigate the rapidity clustering of exact multi-particle form factors of the SU(N) chiral Gross-Neveu model. For some examples of local fields, in particular, the Noether current, the energy momentum tensor, the fundamental spinor field etc, we explicitly demonstrate the clustering or factorization phenomena. In a forthcoming paper we will consider the form factor of the Noether current in a special form, in order to connect the asymptotic clustering with Bjorken scattering.

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A Some lemmata

Lemma 1 *For $n = 0 \bmod N$, $n_j = n\left(1-\frac{j}{N}\right)$ and $p(\underline{\theta},\underline{z})$ = independent of $z_i^{(j)}$ the K-function vanishes*

$$K_{\underline{\alpha}}(\underline{\theta}) = \int d\underline{z}\tilde{h}\,(\underline{\theta},\underline{z})\tilde{\Phi}_{\underline{\alpha}}(\underline{\theta},\underline{z}) = 0 \tag{A.1}$$

For SU(2) the proof of this lemma is quite analog to that for the Sine-Gordon model in [34]. For general N we present an example (see Proposition 1).

Lemma 2 *For* SU(2) *and* $m = n/2$

$$K_{\underline{\alpha}}(\underline{\theta}) = \frac{1}{m!}\int_{\mathcal{C}_{\underline{\theta}}} dz_1 \cdots \int_{\mathcal{C}_{\underline{\theta}}} dz_m \, \tilde{h}(\underline{\theta},\underline{z}) \left(-\sum_{i=1}^{m} z_i\right) \tilde{\Psi}_{\underline{\alpha}}(\underline{\theta},\underline{z}) = (-1)^m \, 8\pi^5 i \left(K^J(\underline{\theta}) M_1^2\right)_{\underline{\alpha}} \tag{A.2}$$

which is a non-highest weight K-function and

$$K_{\underline{\alpha}}(\underline{\theta}) = i\pi \frac{1}{m!}\int_{\mathcal{C}_{\underline{\theta}}} dz_1 \cdots \int_{\mathcal{C}_{\underline{\theta}}} dz_m \, \tilde{h}(\underline{\theta},\underline{z}) \sum_{j=1}^{m} \tilde{\Phi}_{\underline{\alpha}}^{D_j}(\underline{\theta},\underline{z}) = -(-1)^m \, 8\pi^5 i K_{\underline{\alpha}}^J(\underline{\theta}) \tag{A.3}$$

where $\tilde{\Phi}_{\hat{\underline{\alpha}}}^{D_j}(\hat{\underline{\theta}},\hat{\underline{z}}) = \left(\Omega C(\hat{\underline{\theta}},\hat{z}_k) \dots D(\hat{\underline{\theta}},\hat{z}_j) \dots C(\hat{\underline{\theta}},\hat{z}_1)\right)_{\hat{\underline{\alpha}}}$.

For SU(2) the proofs are similar to the one of Lemma 1. For general N see the proofs of Propositions 2 and 3.

B Examples of particle anti-particle form factors

B.1 Bound states — anti-particles

The following is taken from [9, 25, 26].[5]

B.1.1 Bound state S-matrix

The S-matrix of a particle and an anti-particle (which is a bound state of $N-1$ particles (3.4) [30]) is

$$S_{\bar{\alpha}\beta}^{\delta\bar{\gamma}}(\theta) = (-1)^{N-1}\left(\delta_{\bar{\alpha}}^{\bar{\gamma}}\delta_{\beta}^{\delta}\, b(\pi i - \theta) + \mathbf{C}^{\delta\bar{\gamma}}\mathbf{C}_{\bar{\alpha}\beta}\, c(\pi i - \theta)\right) \tag{B.1}$$

where the charge conjugation matrices are given by (3.5).

B.1.2 Bound state form factors

The general form factor formula for n particles $\underline{\alpha}$ and $\bar{n}$ anti-particles $\bar{\underline{\delta}}$ is

$$F_{\underline{\alpha}\bar{\underline{\delta}}}(\underline{\theta},\underline{\omega}) = N_{n\bar{n}} F(\underline{\theta},\underline{\omega}) K_{\underline{\alpha}\bar{\underline{\delta}}}(\underline{\theta},\underline{\omega})$$

where

$$\begin{aligned}
F(\underline{\theta},\underline{\omega}) &= \left(\prod_{1\le i<j\le n} F(\theta_{ij})\right)\left(\prod_{1\le i\le n}\prod_{1\le j\le \bar{n}} \bar{F}(\theta_i - \omega_j)\right)\left(\prod_{1\le i<j\le \bar{n}} F(\omega_{ij})\right)\\
K_{\underline{\alpha}\bar{\underline{\delta}}}(\underline{\theta},\underline{\omega}) &= \int_{\mathcal{C}_{\underline{\theta},\underline{\omega}}} \underline{dz}\,\tilde{h}(\underline{\theta},\underline{z}) p(\underline{\theta},\underline{\omega},\underline{z}) \tilde{\Psi}_{\underline{\alpha}\bar{\underline{\delta}}}(\underline{\theta},\underline{\omega},\underline{z})\\
\tilde{h}(\underline{\theta},\underline{z}) &= \prod_{i=1}^{n}\prod_{j=1}^{m} \tilde{\phi}(\theta_i - z_j) \prod_{1\le i<j\le m} \tau(z_{ij})\\
\tilde{\phi}(\theta) &= \Gamma\left(-\frac{\theta}{2\pi i}\right)\Gamma\left(1 - \frac{1}{N} + \frac{\theta}{2\pi i}\right)
\end{aligned}$$

[5] See also [29] for U(N) Bethe ansatz.

with $\int_{\mathcal{C}_{\underline{\theta},\underline{\omega}}} d\underline{z} = \frac{1}{m!}\int_{\mathcal{C}_{\underline{\theta},\underline{\omega}}} dz_1 \cdots \int_{\mathcal{C}_{\underline{\theta},\underline{\omega}}} dz_m$. The minimal F-function for a particle and an anti-particle $\bar{F}(\theta)$ is defined in (2.5) and satisfies (2.6) and the asymptotic behavior (E.5). The 0-level Bethe ansatz state writes in terms of the basic states as

$$\tilde{\Psi}_{\underline{\alpha}\underline{\bar{\delta}}}(\underline{\theta},\underline{\omega},\underline{z}) = L_{\underline{\beta}\underline{\bar{\epsilon}}}(\underline{z},\underline{\omega})\tilde{\Phi}^{\underline{\beta}\underline{\bar{\epsilon}}}_{\underline{\alpha}\underline{\bar{\delta}}}(\underline{\theta},\underline{\omega},\underline{z})$$

The function $L_{\underline{\beta}(\sigma)}(\underline{z},\underline{\omega})$, $(\sigma) = (1,\sigma_2,\ldots,\sigma_{N-1})$ is given by the 1-level off-shell Bethe ansatz, etc. The final formula is

$$K^{\mathcal{O}}_{\underline{\alpha}\underline{\bar{\delta}}}(\underline{\theta},\underline{\omega}) = \int d\underline{z}_1 \cdots \int d\underline{z}_{N-1} \tilde{h}(\underline{\theta},\underline{\omega},\underline{\underline{z}}) p^{\mathcal{O}}(\underline{\theta},\underline{\omega},\underline{\underline{z}})\tilde{\Phi}_{\underline{\alpha}\underline{\bar{\delta}}}(\underline{\theta},\underline{\omega},\underline{\underline{z}}) \tag{B.2}$$

$$\tilde{h}(\underline{\theta},\underline{\omega},\underline{\underline{z}}) = \prod_{j=0}^{N-2}\tilde{h}(\underline{z}_j,\underline{z}_{j+1})\prod_{i=1}^{\bar{n}}\prod_{j=1}^{n_{N-1}}\tilde{\chi}(\omega_i - z_j^{(N-1)}) \tag{B.3}$$

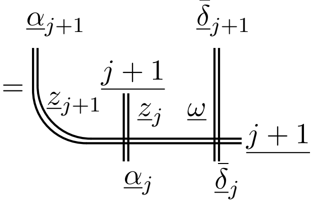
$$\tilde{\chi}(\omega) = \Gamma\left(\frac{1}{2}+\frac{\omega}{2\pi i}\right)\Gamma\left(\frac{1}{2}-\frac{1}{N}-\frac{\omega}{2\pi i}\right). \tag{B.4}$$

The complete Bethe ansatz state is

$$\tilde{\Phi}_{\underline{\alpha}\underline{\bar{\delta}}}(\underline{\theta},\underline{\omega},\underline{\underline{z}}) = \tilde{\Phi}^{(N-2)}{}^{\underline{\alpha}_{N-1}\underline{\bar{\delta}}_{N-1}}_{\underline{\alpha}_{N-2}\underline{\bar{\delta}}_{N-2}}(\underline{z}_{N-2},\underline{\omega},\underline{z}_{N-1})\ldots\tilde{\Phi}^{(1)}{}^{\underline{\alpha}_2\underline{\bar{\delta}}_2}_{\underline{\alpha}_1\underline{\bar{\delta}}_1}(\underline{z}_1,\underline{\omega},\underline{z}_2)\tilde{\Phi}^{\underline{\alpha}_1\underline{\bar{\delta}}_1}_{\underline{\alpha}\underline{\bar{\delta}}}(\underline{\theta},\underline{\omega},\underline{z}_1) \tag{B.5}$$

where $\underline{\alpha}_{N-1} = (N,\ldots,N)$ and $\underline{\bar{\delta}}_{N-1} = (\bar{N},\ldots,\bar{N})$ consists of highest weight bound states $\bar{N} = (1,2,\ldots,N-1)$. The state of level j is given by monodromy matrices as

$$\tilde{\Phi}^{(j)}{}^{\underline{\alpha}_{j+1}\underline{\bar{\delta}}_{j+1}}_{\underline{\alpha}_j\underline{\bar{\delta}}_j}(\underline{z}_j,\underline{\omega},\underline{z}_{j+1}) = \tilde{T}^{(j)}{}^{\underline{\alpha}_{j+1},\underline{j+1}\underline{\bar{\delta}}_{j+1}}_{\underline{\alpha}_j\underline{\bar{\delta}}_j,\underline{j+1}}(\underline{z}_j,\underline{\omega},\underline{z}_{j+1}) =$$



If there are n particles and $\bar{n}$ anti-particles the SU(N) weights are [26, 29]

$$\begin{aligned} w &= (n-n_1, n_1-n_2, \ldots, n_{N-2}-n_{N-1}, n_{N-1}-\bar{n}) + \bar{n}(1,\ldots,1) \\ &= w^O + L(1,\ldots,1) \end{aligned} \tag{B.6}$$

where $n_1 = m, n_2, \ldots$ are the numbers of C operators in the various levels of the nesting, w^O is the weight vector of the operator $\mathcal{O}$ and $L = 0,1,2,\ldots$.

B.2 Lemma 1 for general N and $n = \bar{n} = 1$

Proposition 1 *The K-function given by (2.10) with p-function* $=1$

$$K_{\alpha\bar{\delta}}(\theta,\omega) = \int d\underline{z}\tilde{h}(\underline{\theta},\underline{\omega},\underline{\underline{z}})\tilde{\Phi}_{\alpha\bar{\delta}}(\theta,\omega,\underline{\underline{z}}) = 0 \tag{B.7}$$

for $n = \bar{n} = 1$.

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Proof. The weight formula (B.6) implies that $n_j = 1$ for $j = 1, \dots, N-1$ and the L-function of level j is

$$L^{(j)}_{\beta\bar\gamma}(z,\omega) = \int_{\mathcal{C}} du\, \tilde\phi\,(z-u)\, L^{(j+1)}_{\beta'\bar\gamma'}(u,\omega)\left(T^{\beta',j+1\bar\gamma'}_{\beta\bar\gamma,j+1}(z,\omega,u)\right) = \mathbf{C}^{(j)}_{\beta\bar\gamma} L^{(j)}_{ca}(z,\omega)$$

where (B.20)–(B.29) have been used. For $j = 0$

$$K_{\alpha\bar\delta}(\underline{\theta},\underline{\omega}) = \mathbf{C}_{\alpha\bar\delta} L^{(0)}_{ca}(\theta,\omega) = 0$$

by (B.21). ■

B.3 Theorem 1 for general N and $n = \bar n = 2,\ k = \bar k = 1$

We consider form factors of the pseudo potential $J(x)$ for particles and anti-particles. Formula (B.6) means, generalizing (3.8)

$$n_j = n\,(1 - j/N) + \bar n j/N - 1\,. \tag{B.8}$$

and the p-function is [26]

$$p^J(\underline{\theta},\underline{\omega},\underline{z},\underline{z}^{(N-1)}) = \frac{\left(\prod e^{\frac12 z_i^{(1)}}\right)\left(\prod e^{\frac12 z_i^{(N-1)}}\right)\left(\prod e^{-\frac12\theta_i}\right)\left(\prod e^{-\frac12\omega_i}\right)}{\sum e^{-\theta_i} + \sum e^{-\omega_i}} \tag{B.9}$$

with the asymptotic behavior

$$p^J(\underline{\theta}_W,\underline{\omega}_W,\underline{\underline{z}}_W) \to e^{-\frac12 W(k-k_1-k_{N-1})}\left(\prod e^{-\frac12\hat\theta}\right)\left(\prod e^{\frac12\hat z^{(1)}}\right)\left(\prod e^{\frac12\hat z^{(N-1)}}\right)\left(\prod e^{-\frac12\hat\omega}\right) p^J(\underline{\check\theta},\underline{\underline{\check z}}). \tag{B.10}$$

In particular for $n = \bar n = 2$ and $k = \bar k = 1$ we prove the proposition:

Proposition 2 *The form factor of the current for $n = \bar n = 2$ and $k = \bar k = 1$ satisfies the clustering formula (3.22) in the form*

$$\begin{aligned} F^{J^{\beta\bar\epsilon}}_{\underline{\alpha\bar\delta}}(\underline{\theta}_W,\underline{\omega}_W) &\overset{W\to\infty}{\to} i\eta W^{-1}\left(\mathbf{C}_{\gamma\bar\kappa} F^{J^{\beta\bar\kappa}}_{\hat\alpha\hat{\bar\delta}}(\hat\theta,\hat\omega) F^{J^{\gamma\bar\epsilon}}_{\check\alpha\check{\bar\delta}}(\check\theta,\check\omega) - \mathbf{C}_{\gamma\bar\kappa} F^{J^{\gamma\bar\epsilon}}_{\hat\alpha\hat{\bar\delta}}(\hat\theta,\hat\omega)) F^{J^{\beta\bar\kappa}}_{\check\alpha\check{\bar\delta}}(\check\theta,\check\omega)\right) \\ &= -2\eta W^{-1} f_{abc} F^{J_b}_{\hat\alpha\hat{\bar\delta}}(\hat\theta,\hat\omega) F^{J_c}_{\check\alpha\check{\bar\delta}}(\check\theta,\check\omega). \end{aligned} \tag{B.11}$$

Proof. The exponential behavior (4.1) implies for $n = \bar n = 2$ and $k = \bar k = 1$ that $k_j = 1$ and $l_j = 0$ for $j = 1, \dots, N-1$. We investigate for $J = J^{1\bar N}$ ($\bar N$ = bound state $(1 \dots N-1)$)

$$K^J_{\underline{\alpha\bar\delta}}(\underline{\theta},\underline{\omega}) = \int d\underline{\underline{z}}\,\tilde h\,(\underline{\theta},\underline{\omega},\underline{\underline{z}}) p^J(\underline{\theta},\underline{\omega},\underline{\underline{z}}) \tilde\Phi_{\underline{\alpha\bar\delta}}(\underline{\theta},\underline{\omega},\underline{\underline{z}}) \tag{B.12}$$

We have proved in theorem 1 that in leading order

$$F^J_{\underline{\alpha}}(\underline{\theta}_W) \to 0$$

Order $\frac{1}{W}$: we have to apply the asymptotic behavior of the h-function (E.12) and the Bethe state (E.21).

The *result for the contribution of* h_1 is

$$F^{J^{1\bar N}}_{\underline{\alpha\bar\delta},h_1}(\underline{\theta}_W,\underline{\omega}_W)\to i\eta W^{-1}\left(\mathbf{C}_{1\bar 1}F^{J^{1\bar 1}}_{\hat\alpha\hat{\bar\delta}}(\hat\theta,\hat\omega)-\mathbf{C}_{N\bar N}F^{J^{N\bar N}}_{\hat\alpha\hat{\bar\delta}}(\hat\theta,\hat\omega)\right)F^{J^{1\bar N}}_{\check\alpha\check{\bar\delta}}(\check\theta,\check\omega)$$

and the *result for the contribution of* Φ_1 is

$$\begin{aligned}F^{J^{1\bar N}}_{\underline{\alpha\bar\delta},\Phi_1}(\underline{\theta}_W,\underline{\omega}_W)\to i\eta W^{-1}\Big(&\mathbf{C}^{(1)}_{\delta\bar\kappa}F^{J^{1\bar\kappa}}_{\hat\alpha\hat{\bar\delta}}(\hat\theta,\hat\omega)F^{J^{\delta\bar N}}_{\check\alpha\check{\bar\delta}}(\check\theta,\check\omega)-\mathbf{C}_{\delta\bar\kappa}F^{J^{\delta\bar N}}_{\hat\alpha\hat{\bar\delta}}(\hat\theta,\hat\omega)F^{J^{1\bar\kappa}}_{\check\alpha\check{\bar\delta}}(\check\theta,\check\omega)\\&+\mathbf{C}_{N\bar N}F^{J^{N\bar N}}_{\hat\alpha\hat{\bar\delta}}(\hat\theta,\hat\omega)F^{J^{1\bar N}}_{\check\alpha\check{\bar\delta}}(\check\theta,\check\omega)\Big)\,.\end{aligned}$$

Because $\mathbf{C}^{(1)}_{\delta\bar\kappa}+\mathbf{C}_{1\bar 1}\delta^1_\delta\delta^{\bar 1}_{\bar\kappa}=\mathbf{C}_{\delta\bar\kappa}$ (see (B.30)) the relation (B.11) is proved. ■

B.4 Theorem 5 for general N and $n=2,\ \bar n=1,\ k=\bar k=1$

We consider form factors of the fundamental field $\psi(x)$ for particles and anti-particles. Formula (B.6) means, generalizing (3.19)

$$n_j=(n-1)\,(1-j/N)+\bar nj/N,\ \ j=1,\dots,N-1. \tag{B.13}$$

and the p-function is

$$p^\psi(\underline{\theta},\underline{\omega},\underline{z})=e^{\frac12 n_1 i\eta}e^{i\pi\bar n(1-\frac2N)}\left(\prod_{i=1}^n e^{-\frac12(1-\frac1N)\theta_i}\right)\prod_{i=1}^{\bar n}\left(e^{-\frac12\frac{\omega}{N}}\right)\left(\prod_{i=1}^{n_1}e^{\frac12 z_i}\right) \tag{B.14}$$

with the asymptotic behavior

$$p^\psi(\underline{\theta}_W,\underline{\omega}_W,\underline{z}_W)\to e^{-\frac12 W\left((1-\frac1N)k+\frac1N\bar k-k_1\right)}p^\psi(\hat{\underline{\theta}},\hat{\underline{\omega}},\hat{\underline{z}})p^\psi(\check{\underline{\theta}},\check{\underline{\omega}},\check{\underline{z}}). \tag{B.15}$$

In particular for $n=\bar n=2$ and $k=\bar k=1$ we prove the proposition:

Proposition 3 *The form factor of the current for $n=2,\ \bar n=1$ and $k=\bar k=1$ satisfies the clustering formula (3.27) in the form*

$$F^{\psi^\beta}_{\underline{\alpha\bar\delta}}(\underline{\theta}_W,\omega_W)\to i\eta W^{-1}\left(\mathbf{C}_{\gamma\bar\kappa}F^{J^{\beta\bar\kappa}}_{\hat\alpha\hat{\bar\delta}}(\hat\theta,\hat\omega)F^{\psi^\gamma}_{\check\alpha}(\check\theta)\right). \tag{B.16}$$

Proof. The exponential behavior (4.21) implies for $n=\bar n=2$ and $k=\bar k=1$ that $k_j=1$ and $l_j=0$ for $j=1,\dots,N-1$. We investigate for $\psi=\psi^1$

$$K^\psi_{\underline{\alpha\bar\delta}}(\underline{\theta},\omega)=\int d\underline{z}\tilde h\,(\underline{\theta},\underline{\omega},\underline{z})p^\psi(\underline{\theta},\underline{\omega},\underline{z})\tilde\Phi_{\underline{\alpha\bar\delta}}(\underline{\theta},\omega,\underline{z}) \tag{B.17}$$

We have proved in theorem 5 that in leading order

$$F^\psi_{\underline\alpha}(\underline{\theta}_W)\to 0$$

Order $\frac{1}{W}$: we have to apply the asymptotic behavior of the h-function (E.12) and the Bethe state (E.21).

The *result for the contribution of* h_1 is

$$F^{\psi^1}_{\underline{\alpha\bar\delta},h_1}(\underline{\theta}_W,\omega_W)\to i\eta W^{-1}\left(\mathbf{C}_{1\bar 1}F^{J^{1\bar 1}}_{\hat\alpha\hat{\bar\delta}}(\hat\theta,\hat\omega)\right)F^{\psi^1}_{\check\alpha}(\check\theta)$$

and the *result for the contribution of from* Φ_1 is

$$F^{\psi^1}_{\underline{\alpha\bar\delta},\tilde\Phi_1}(\underline{\theta}_W,\omega_W)\to i\eta W^{-1}\left(\mathbf{C}^{(1)}_{\gamma\bar\kappa}F^{J^{1\bar\kappa}}_{\hat\alpha\hat{\bar\delta}}(\hat\theta,\hat\omega)\right)F^{\psi^\gamma}_{\check\alpha}(\check\theta)$$

Because $\mathbf{C}^{(1)}_{\delta\bar\kappa}+\mathbf{C}_{1\bar 1}\delta^1_\delta\delta^{\bar 1}_{\bar\kappa}=\mathbf{C}_{\delta\bar\kappa}$ (see (B.30)) the relation (B.16) is proved. ■

B.5 Formulas

Definition 1 *We define (for $0 \leq j \leq N-2$) iteratively*

$$L^{(j)}_{xy}(z,\omega) = \int du\tilde{\phi}(z-u)L^{(j+1)}_{ca}(u,\omega)\tilde{x}(z-u)\tilde{y}(\omega-z) \tag{B.18}$$

$$L^{(j)}_{uxy}(z,\omega) = \int du\tilde{\phi}(z-u)L^{(j+1)}_{ca}(u,\omega)u\tilde{x}(z-u)\tilde{y}(\omega-z) \tag{B.19}$$

with

$$\tilde{x}(z),\tilde{y}(z) = \tilde{a} = 1,\ \tilde{b}(z) = \frac{z}{z-i\eta},\ \tilde{c}(z) = \frac{-i\eta}{z-i\eta},\ \tilde{d}(z) = \frac{-i\eta}{i\pi - z},\ \eta = \frac{2\pi}{N}$$
$$L^{(N-1)}_{ca}(z,\omega) = (-1)^{N-1}\tilde{\chi}_{N-1}(\omega - z)$$

Proposition 4

1. *If* $\tilde{\chi}_{N-1}(\omega) = \tilde{\chi}(\omega) = \Gamma\left(\frac{1}{2}+\frac{\omega}{2\pi i}\right)\Gamma\left(\frac{1}{2}-\frac{1}{N}-\frac{\omega}{2\pi i}\right)$ *then*

$$L^{(j)}_{ca}(z,\omega) = (-1)^{N-1}c_{N-2}\cdots c_j\tilde{\chi}_j(\omega - z)$$

with

$$\tilde{\chi}_j(\omega) = \Gamma\left(-\tfrac{1}{2}+j/N-\omega/(2\pi i)\right)\Gamma\left(\tfrac{1}{2}+\omega/(2\pi i)\right) \tag{B.20}$$

$$c_j = 4\pi^2\frac{\Gamma\left(1-\frac{1}{N}\right)\Gamma\left(\frac{j+1}{N}\right)}{\Gamma\left(\frac{1}{N}j\right)},\ 0<j<N-1$$

$$c_{N-2}\cdots c_j = (4\pi^2)^{N-1-j}\frac{\left(\Gamma\left(1-\frac{1}{N}\right)\right)^{N-j}}{\Gamma\left(\frac{1}{N}j\right)}$$

$$c_0 = 0 \Rightarrow L^{(0)}_{ca}(z,\omega) = 0. \tag{B.21}$$

2.

$$L^{(j)}_{bd}(z,\omega) = L^{(j)}_{ca}(z,\omega)/(N-j-1) \tag{B.22}$$

$$L^{(j)}_{aa}(z,\omega) = L^{(j)}_{ca}(z,\omega)\left(1+N\left(z-\omega-i\pi\right)/(2i\pi j)\right) \tag{B.23}$$

$$K_{aa}(\theta,\omega) = L^{(0)}_{aa}(\theta,\omega) = (-1)^{N-1}c_{N-2}\cdots c_1\frac{4\pi^4}{\sin\frac{\pi}{N}}\frac{1}{\cosh\frac{1}{2}(\theta-\omega)}. \tag{B.24}$$

3.

$$L^{(j)}_{uca}(z,\omega) = \frac{1}{j}\left(\left(1+j\right)z-\omega-i\pi\right)L^{(j)}_{ca}(z,\omega) \tag{B.25}$$

$$L^{(j)}_{ubd}(z,\omega) = -\left(\frac{1}{j}\left(z-\omega-i\pi\right)+\frac{1}{N-j-1}\left(i\pi-\omega\right)\right)L^{(j)}_{ca}(z,\omega) \tag{B.26}$$

in particular

$$L^{(0)}_{uca}(\theta,\omega) = \frac{2i\pi}{N}K_{aa}(\theta,\omega) \tag{B.27}$$

$$L^{(0)}_{ubd}(\theta,\omega) = -\frac{2i\pi}{N}K_{aa}(\theta,\omega). \tag{B.28}$$

4. *If* $L^{(N-1)}_{\beta'(\mu')}(z,\omega) = \mathbf{C}^{(N-1)}_{\beta(\mu)} L^{(j)}_{ca}(z,\omega) = \delta^N_\beta \delta^{(1\dots N-1)}_{(\mu)} (-1)^{N-1}\tilde{\chi}(\omega - z)$ *then*

$$L^{(j)}_{\beta(\mu)}(z,\omega) = \int_{\mathcal{C}} du\tilde{\phi}\,(z-u)\,L^{(j+1)}_{\beta'(\mu')}(u,\omega)\left(T^{\beta',j+1(\mu')}_{\beta(\mu),j+1}(z,\omega,u)\right) = \mathbf{C}^{(j)}_{\beta(\mu)} L^{(j)}_{ca}(z,\omega) \tag{B.29}$$

where

$$\mathbf{C}^{(j)}_{\beta(\mu)} = \mathbf{C}_{\beta(\mu)} \text{ for } \beta > j \text{ else } = 0. \tag{B.30}$$

5.

$$\begin{aligned} L^{(j)}_{u\beta(\mu)}(z,\omega) &= \int_{\mathcal{C}} du\tilde{\phi}\,(z-u)\,L^{(j+1)}_{u\beta'(\mu')}(u,\omega)u\left(T^{\beta',j+1(\mu')}_{\beta(\mu),j+1}(z,\omega,u)\right) \\ &= \frac{1}{j}\left(\left(N\delta^N_\beta \delta^{\bar{N}}_{(\mu)} - \mathbf{C}^{(j)}_{\beta(\mu)}\right)\right) zL^{(j)}_{ca}(z,\omega) + const.L^{(j)}_{ca}(z,\omega) \end{aligned} \tag{B.31}$$

in particular

$$L^{(N-1)}_{u\beta(\mu)}(z,\omega) = \delta^N_\beta \delta^{\bar{N}}_{(\mu)} zL^{(N-1)}_{ca}(z,\omega).$$

6. *for* $j+1 < \alpha' < N$

$$\int_{\mathcal{C}} du\tilde{\phi}\,(z-u)\,L^{(j+1)}_{aa}(u,\omega)\left(T^{\alpha',j+1\bar{N}}_{\alpha(\rho)j+1}(z,\omega,u)\right) = \delta^{\alpha'}_{\alpha}\delta^{\bar{N}}_{(\rho)} L^{(j)}_{aa}(z,\omega). \tag{B.32}$$

Proof. We use

$$\begin{aligned} &\frac{1}{2\pi i}\left(\int_{\mathcal{C}_a} + \int_{\mathcal{C}_b}\right) dz\Gamma(a-z)\Gamma(b-z)\Gamma(c+z)\Gamma(d+z) \\ &= -\frac{\Gamma(c+a)\,\Gamma(d+a)\,\Gamma(c+b)\,\Gamma(d+b)}{\Gamma(c+d+a+b)} \end{aligned}$$

where $\mathcal{C}_a$ encircles the poles of $\Gamma(a-z)$ clockwise.

1. With $\tilde{\phi}(z)\tilde{c}(z) = -\frac{1}{N}\Gamma\left(-\frac{z}{2\pi i}\right)\Gamma\left(-\frac{1}{N}+\frac{z}{2\pi i}\right)$ and $\tilde{\chi}_{j+1}(\omega)$ of (B.20) follows

$$\begin{aligned} &\int_{\mathcal{C}_{\underline{\theta}}} du\tilde{\phi}(z-u)\tilde{c}(z-u)\tilde{\chi}_{j+1}(\omega-u) \\ &= -\frac{1}{N}\int_{\mathcal{C}_{\underline{\theta}}} du\Gamma\left(-\tfrac{z-u}{2\pi i}\right)\Gamma\left(-\tfrac{1}{N}+\tfrac{z-u}{2\pi i}\right)\Gamma\left(-\tfrac{1}{2}+\tfrac{j+1}{N}-\tfrac{\omega-u}{2\pi i}\right)\Gamma\left(\tfrac{1}{2}+\tfrac{\omega-u}{2\pi i}\right) \\ &= -\frac{4}{N}\pi^2\Gamma\left(-\frac{1}{N}\right)\frac{\Gamma\left(\frac{j+1}{N}\right)}{\Gamma\left(\frac{1}{N}j\right)}\Gamma\left(\frac{1}{2}-\frac{1}{2}\frac{z-\omega}{i\pi}\right)\Gamma\left(-\frac{1}{2}+\frac{1}{N}j+\frac{1}{2}\frac{z-\omega}{i\pi}\right) \\ &= c_j\tilde{\chi}_j(\omega-u) \end{aligned}$$

and iterating this result $\Rightarrow$ 1.

2. With $\tilde{\phi}(z)\tilde{b}(z) = -\Gamma(1-\frac{z}{2\pi i})\,\Gamma\left(-\frac{1}{N}+\frac{z}{2\pi i}\right)$ and $\tilde{d}(\omega)\tilde{\chi}_{j+1}(\omega) = \frac{1}{N}\Gamma\left(-\frac{1}{2}+\frac{j+1}{N}-\frac{\omega}{2\pi i}\right)\Gamma\left(-\frac{1}{2}+\frac{\omega}{2\pi i}\right) \Rightarrow$

$$\begin{aligned}
&\int_{\mathcal{C}_{\underline{\theta}}} du\tilde{\phi}(z-u)\tilde{b}(z-u)\tilde{d}(\omega-u)\tilde{\chi}_{j+1}(\omega-u)\\
&= -\frac{1}{N}\int_{\mathcal{C}_{\underline{\theta}}} du\Gamma\left(-\tfrac{z-u}{2\pi i}\right)\Gamma\left(-\tfrac{1}{N}+\tfrac{z-u}{2\pi i}\right)\Gamma\left(-\tfrac{1}{2}+\tfrac{j+1}{N}-\tfrac{\omega-u}{2\pi i}\right)\Gamma\left(-\tfrac{1}{2}+\tfrac{\omega-u}{2\pi i}\right)\\
&= \frac{1}{N-j-1}c_j\tilde{\chi}_j(\omega-u)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathcal{C}_{\underline{\theta}}} du\tilde{\phi}(z-u)\tilde{\chi}_{j+1}(\omega-u)\\
&= \int_{\mathcal{C}_{\underline{\theta}}} du\Gamma\left(-\tfrac{z-u}{2\pi i}\right)\Gamma\left(1-\tfrac{1}{N}+\tfrac{z-u}{2\pi i}\right)\Gamma\left(-\tfrac{1}{2}+\tfrac{j+1}{N}-\tfrac{\omega-u}{2\pi i}\right)\Gamma\left(\tfrac{1}{2}+\tfrac{\omega-u}{2\pi i}\right)\\
&= c_j\tilde{\chi}_j(\omega-u)\left(1+\frac{N}{2i\pi j}(z-\omega-i\pi)\right)
\end{aligned}$$

and $\tilde{\chi}_0(\omega) = \Gamma\left(-\frac{1}{2}-\frac{\omega}{2\pi i}\right)\Gamma\left(\frac{1}{2}+\frac{\omega}{2\pi i}\right) = \frac{-2i\pi^2}{(i\pi+\omega)\cosh\frac{1}{2}\omega} \Rightarrow$ 2.

3. With $\tilde{c}(z-u)u = \frac{2i\pi}{N}+\tilde{c}(z-u)\left(z-\frac{2i\pi}{N}\right) \Rightarrow$

$$\begin{aligned}
L^{(j)}_{uca}(z,\omega) &= 2i\pi/N\,L^{(j)}_{aa}(z,\omega)+(z-2i\pi/N)\,L^{(j)}_{ca}(z,\omega)\\
&= (2i\pi\,(1+N\,(z-\omega-i\pi)\,/(2i\pi j))\,/N+(z-2i\pi/N))\,L^{(j)}_{ca}(z,\omega)\\
&= ((1+j)\,z-\omega-i\pi)\,/j\,L^{(j)}_{ca}(z,\omega).
\end{aligned}$$

and using $\tilde{b} = 1-\tilde{c},\ \tilde{d}(\omega-u)u = -\frac{2i\pi}{N}-\tilde{d}(\omega-u)\,(i\pi-\omega) \Rightarrow$

$$\begin{aligned}
L^{(j)}_{ubd}(z,\omega) &= -2i\pi/N\,L^{(j)}_{ba}-(i\pi-\omega)\,L^{(j)}_{bd}\\
&= -\left(2i\pi/N\,L^{(j)}_{aa}-L^{(j)}_{ca}\right)-(i\pi-\omega)\,/(N-j-1)\,L^{(j)}_{ca}\\
&= -\left((z-\omega-i\pi)\,/j+(i\pi-\omega)\,/(N-j-1)\right)L^{(j)}_{ca}(z,\omega).
\end{aligned}$$

By (B.23)

$$L^{(0)}_{uca}(z,\omega) = \lim_{j\to 0}\frac{\frac{1}{j}\left((1+j)\,z-\omega-i\pi\right)}{1+\frac{N}{2i\pi j}(z-\omega-i\pi)}L^{(0)}_{aa}(z,\omega) = \frac{2i\pi}{N}K_{aa}(z,\omega)$$

$$L^{(0)}_{ubd}(z,\omega) = \lim_{j\to 0}\frac{-\left(\frac{1}{j}(z-\omega-i\pi)+\frac{1}{N-j-1}(i\pi-\omega)\right)}{1+\frac{N}{2i\pi j}(z-\omega-i\pi)}L^{(0)}_{aa}(z,\omega) = -\frac{2i\pi}{N}K_{aa}(z,\omega).$$

$\Rightarrow$ 3.

4.

$$\begin{aligned}
&\mathbf{C}^{(j+1)}_{\beta'(\mu')}T^{\beta',j+1(\mu')}_{\beta(\mu),j+1}\\
&= \mathbf{C}^{(j+1)}_{\beta'(\mu')}\left(\tilde{b}\delta^{\beta'}_{j+1}\delta^{j+1}_{\beta}\delta^{(\mu')}_{(\mu)}+\tilde{c}\delta^{\beta'}_{\beta}\delta^{(\mu')}_{(\mu)}+\tilde{b}\tilde{d}\delta^{j+1}_{\beta}\mathbf{C}^{\beta'(\mu')}\mathbf{C}_{(\mu)j+1}+\tilde{c}\tilde{d}\delta^{\beta'}_{\beta}\mathbf{C}^{j+1(\mu')}\mathbf{C}_{(\mu)j+1}\right)\\
&= \tilde{b}\delta^{j+1}_{\beta}\mathbf{C}^{(j+1)}_{j+1(\mu)}+\tilde{c}\mathbf{C}^{(j+1)}_{\beta(\mu)}+\tilde{b}\tilde{d}\mathbf{C}^{(j+1)}_{\beta(\mu')}\mathbf{C}^{j+1(\mu')}\mathbf{C}_{(\mu)j+1}+\tilde{c}\tilde{d}\delta^{j+1}_{\beta}\mathbf{C}^{(j+1)}_{(\mu)j+1}
\end{aligned}$$

$\mathbf{C}^{(j+1)}_{(\mu)j+1} = 0$ and $\mathbf{C}^{(j+1)}_{\beta(\mu')}\mathbf{C}^{j+1(\mu')}\mathbf{C}_{(\mu)j+1} = (N-j-1)\delta^{j+1}_{\beta}\mathbf{C}_{j+1(\mu)} \Rightarrow$

$$\begin{aligned}L^{(j)}_{\beta(\mu)}(z,\omega) &= \int_{\mathcal{C}} du\tilde{\phi}\,(z-u)\,L^{(j+1)}_{ca}(u,\omega)\mathbf{C}^{(j+1)}_{\beta'(\mu')}\left(T^{\beta',j+1(\mu')}_{\beta(\mu),j+1}(z,\omega,u)\right)\\ &= L^{(j)}_{ca}\mathbf{C}^{(j+1)}_{\beta(\mu)} + L^{(j)}_{bd}(N-j-1)\delta^{j+1}_{\beta}\mathbf{C}_{j+1(\mu)}\\ &= L^{(j)}_{ca}\left(\mathbf{C}^{(j+1)}_{\beta(\mu)} + \delta^{j+1}_{\beta}\mathbf{C}_{j+1(\mu)}\right) = L^{(j)}_{ca}\mathbf{C}^{(j)}_{\beta(\mu)}\end{aligned}$$

5. By induction: let $L^{(N-1)}_{u\beta(\mu)}(z,\omega) = \delta^{N}_{\beta}\delta^{\bar{N}}_{(\mu)} z L^{(N-1)}_{ca}(z,\omega)$ and $L^{(j+1)}_{u\beta'(\mu')}(u,\omega) = \frac{1}{j+1}\left(\left(N\delta^{N}_{\beta'}\delta^{(1\ldots N-1)}_{(\mu')} - \mathbf{C}^{(j+1)}_{\beta'(\mu')}\right)\right)L^{(j+1)}_{ca} \Rightarrow$

$$\begin{aligned}L^{(j)}_{u\beta(\mu)}(z,\omega) &= \int_{\mathcal{C}} du\tilde{\phi}\,(z-u)\,L^{(j+1)}_{u\beta'(\mu')}(u,\omega)u\left(T^{\beta',j+1(\mu')}_{\beta(\mu),j+1}(z,\omega,u)\right)\\ &= \frac{1}{j+1}\int_{\mathcal{C}} du\tilde{\phi}\,(z-u)\left(\left(N\delta^{N}_{\beta'}\delta^{(1\ldots N-1)}_{(\mu')} - \mathbf{C}^{(j+1)}_{\beta'(\mu')}\right)L^{(j+1)}_{ca}(u,\omega)\right)\\ &\quad\times u\left(\tilde{c}\delta^{\beta'}_{\beta}\delta^{(\mu')}_{(\mu)} + \tilde{b}\tilde{d}\delta^{j+1}_{\beta}\mathbf{C}^{\beta'(\mu')}\mathbf{C}_{(\mu)j+1}\right)\\ &= \frac{1}{j+1}\left(L^{(j)}_{uca}(z,\omega)\left(N\delta^{N}_{\beta}\delta^{(1\ldots N-1)}_{(\mu)} - \mathbf{C}^{(j+1)}_{\beta(\mu)}\right) + L^{(j)}_{ubd}(z,\omega)\delta^{j+1}_{\beta}\,(j+1)\,\mathbf{C}_{j+1(\mu)}\right)\\ &= \frac{L^{(j)}_{ca}(z,\omega)}{j+1}\left(\left(\frac{1}{j}\,(1+j)\left(N\delta^{N}_{\beta}\delta^{(1\ldots N-1)}_{(\mu)} - \mathbf{C}^{(j+1)}_{\beta(\mu)}\right) - \frac{1}{j}\delta^{j+1}_{\beta}\,(j+1)\,\mathbf{C}_{j+1(\mu)}\right)z\right.\\ &\quad\left. + \,\text{const}\right)\\ &= \left(N\delta^{N}_{\beta}\delta^{(1\ldots N-1)}_{(\mu)} - \mathbf{C}^{(j)}_{\beta(\mu)}\right)/j\; zL^{(j)}_{ca}(z,\omega) + \text{const}L^{(j)}_{ca}.\end{aligned}$$

$\Rightarrow$ 5. because

$(N\delta^{N}_{\beta'}\delta^{(1\ldots N-1)}_{(\mu')}\mathbf{C}^{\beta'(\mu')} - \mathbf{C}^{(j+1)}_{\beta'(\mu')}\mathbf{C}^{\beta'(\mu')})\mathbf{C}_{(\mu)j+1} = \left(N\mathbf{C}_{j+1(\mu)} - (N-j-1)\mathbf{C}_{j+1(\mu)}\right) = (j+1)\,\mathbf{C}_{j+1(\mu)}$.

6. $T^{\alpha',j+1\bar{N}}_{\alpha(\rho)j+1} = \tilde{c}\delta^{\alpha'}_{\alpha}\delta^{\bar{N}}_{(\rho)}$ holds for $j+1 < \alpha' < N$ and with (B.23) for $j+1$ follows

$$\begin{aligned}&\int_{\mathcal{C}} du\tilde{\phi}\,(z-u)\,\tilde{c}\,(z-u)\,L^{(j+1)}_{aa}(u,\omega)\\ &= \int_{\mathcal{C}} du\tilde{\phi}\,(z-u)\,\tilde{c}\,(z-u)\,L^{(j+1)}_{ca}(u,\omega)\,(1 + N\,(u-\omega-i\pi)\,/(2i\pi\,(j+1)))\\ &= L^{(j)}_{ca}(z,\omega)\,(1 + N\,(-\omega-i\pi)\,/(2i\pi\,(j+1))) + L^{(j)}_{uca}N/(2i\pi\,(j+1))\\ &= L^{(j)}_{ca}(z,\omega)\left(1 + \frac{N\,(-\omega-i\pi)}{2i\pi\,(j+1)} + \frac{N}{2i\pi\,(j+1)}\frac{1}{j}\,((1+j)\,z-\omega-i\pi)\right) = L^{(j)}_{aa}(z,\omega)\end{aligned}$$

by (B.23) $\Rightarrow$ 6. ■

C The functions $c(k,l,W)$

The functions $c^{\mathcal{O}}_{\hat{\mathcal{O}}\check{\mathcal{O}}}(k,l,W)$ in (3.1)

$$F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta}_W) \to c^{\mathcal{O}}_{\hat{\mathcal{O}}\check{\mathcal{O}}}(k,l,W) F^{\hat{\mathcal{O}}}_{\underline{\hat{\alpha}}}(\underline{\hat{\theta}}) F^{\check{\mathcal{O}}}_{\underline{\check{\alpha}}}(\underline{\check{\theta}})$$

are calculated using the form factor equation (iii) (see (D.3)), by taking for $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta}_W)$ first the Res and then the limit $W \to \infty$ or exchanging the procedures. We use two special cases of the form factor equation (iii):

I As in (3.4) we take the bound state $(\alpha_1 \dots \alpha_{N-1}) = (2 \dots N) = \bar{1}$ with rapidity ω and (iii) reads as

$$\operatorname*{Res}_{\omega=i\pi+\theta} F^{\mathcal{O}}_{\bar{1}1\underline{\hat{\alpha}}\underline{\check{\alpha}}}(\omega,\theta,\underline{\hat{\theta}},\underline{\check{\theta}}) = 2i\,\mathbf{C}_{\bar{1}1}\, F^{\mathcal{O}}_{\underline{\hat{\alpha}}'\underline{\check{\alpha}}'}(\underline{\hat{\theta}},\underline{\check{\theta}}) \left(\mathbf{1}^{\underline{\hat{\alpha}}'\underline{\check{\alpha}}'}_{\underline{\hat{\alpha}}\underline{\check{\alpha}}} - \dot{\sigma}^{\mathcal{O}}_1(n) S^{\underline{\hat{\alpha}}'1}_{1\underline{\hat{\alpha}}}(\theta,\underline{\hat{\theta}}) S^{\underline{\check{\alpha}}'1}_{1\underline{\check{\alpha}}}(\theta,\underline{\check{\theta}})\right) \quad \text{(C.1)}$$

where we use the short notation of (4.5) and (4.6) for the statistics factor $\dot{\sigma}^{\mathcal{O}}_1(n)$.

II The form factor equation (i)

$$F^{\psi}_{\underline{\hat{\alpha}}\bar{\beta}\gamma\underline{\check{\alpha}}}(\underline{\hat{\theta}},\omega,\theta,\underline{\check{\theta}}) = F^{\psi}_{\bar{\beta}'\gamma'\underline{\hat{\alpha}}''\underline{\check{\alpha}}}(\omega,\theta,\underline{\hat{\theta}},\underline{\check{\theta}}) S^{\gamma'\underline{\hat{\alpha}}''}_{\underline{\hat{\alpha}}'\gamma}(\underline{\hat{\theta}},\theta) S^{\bar{\beta}'\underline{\hat{\alpha}}'}_{\underline{\hat{\alpha}}\bar{\beta}}(\underline{\hat{\theta}},\omega)$$

implies that

$$\operatorname*{Res}_{\omega=i\pi+\theta} F^{\mathcal{O}}_{\underline{\hat{\alpha}}\bar{1}1\underline{\check{\alpha}}}(\underline{\hat{\theta}},\omega,\theta,\underline{\check{\theta}}) = 2i\mathbf{C}_{\bar{1}1} F^{\mathcal{O}}_{\underline{\hat{\alpha}}'\underline{\check{\alpha}}'}(\underline{\hat{\theta}},\underline{\check{\theta}}) \quad \text{(C.2)}$$
$$\times \left((-1)^{(N-1)k}\mathbf{1}^{\underline{\hat{\alpha}}'\underline{\check{\alpha}}'}_{\underline{\hat{\alpha}}\underline{\check{\alpha}}} - \dot{\sigma}^{\mathcal{O}}_1(n) S^{\bar{1}\underline{\hat{\alpha}}'}_{\underline{\hat{\alpha}}\bar{1}}(\underline{\hat{\theta}},\omega) S^{\underline{\check{\alpha}}'1}_{1\underline{\check{\alpha}}}(\theta,\underline{\check{\theta}})\right)$$

where $k = |\underline{\hat{\alpha}}|$. It has been used that crossing [9] $S^{\bar{1}\beta}_{\alpha\bar{1}}(\theta) = (-1)^{(N-1)} S^{\beta 1}_{1\alpha}(i\pi-\theta)$ implies

$$S^{1\alpha''}_{\alpha'1}(\theta) S^{\bar{1}\alpha'}_{\alpha\bar{1}}(\theta - i\pi) = (-1)^{(N-1)} S^{1\alpha''}_{\alpha'1}(\theta) S^{\alpha'1}_{1\alpha}(-\theta) = (-1)^{(N-1)} \mathbf{1}^{\alpha''}_{\alpha}.$$

We consider 4 procedures:

1. Let $\underline{\theta}_W = (\omega+W, \theta+W, \underline{\hat{\theta}}+W, \underline{\check{\theta}})$, $k = N + |\underline{\hat{\alpha}}| > N$, $l = |\underline{\check{\alpha}}|$ then by (C.1) and (3.1)

$$\operatorname*{Res}_{\omega=i\pi+\theta} F^{\mathcal{O}}_{\bar{1}1\underline{\hat{\alpha}}\underline{\check{\alpha}}}(\underline{\theta}_W) = 2i\mathbf{C}_{\bar{1}1} F^{\mathcal{O}}_{\underline{\hat{\alpha}}'\underline{\check{\alpha}}'}(\underline{\hat{\theta}}+W,\underline{\check{\theta}}) \left(\mathbf{1}^{\underline{\hat{\alpha}}'\underline{\check{\alpha}}'}_{\underline{\hat{\alpha}}\underline{\check{\alpha}}} - \dot{\sigma}^{\mathcal{O}}_1(n) S^{\underline{\hat{\alpha}}'1}_{1\underline{\hat{\alpha}}}(\theta,\underline{\hat{\theta}}) S^{\underline{\check{\alpha}}'1}_{1\underline{\check{\alpha}}}(\theta+W,\underline{\check{\theta}})\right) \quad \text{(C.3)}$$
$$\overset{W\to\infty}{\to} 2ic^{\mathcal{O}}_{\hat{\mathcal{O}}\check{\mathcal{O}}}(k-N,l,W) F^{\hat{\mathcal{O}}}_{\underline{\hat{\alpha}}'}(\underline{\hat{\theta}}) \left(\mathbf{1}^{\underline{\hat{\alpha}}'}_{\underline{\hat{\alpha}}} - \dot{\sigma}^{\hat{\mathcal{O}}}_1(k) S^{\underline{\hat{\alpha}}'1}_{1\underline{\hat{\alpha}}}(\theta,\underline{\hat{\theta}})\right) F^{\check{\mathcal{O}}}_{\underline{\check{\alpha}}}(\underline{\check{\theta}})$$

if

$$\dot{\sigma}^{\mathcal{O}}_1(n) S^{\underline{\check{\alpha}}'1}_{1\underline{\check{\alpha}}}(\theta+W,\underline{\check{\theta}}) \overset{W\to\infty}{\to} \dot{\sigma}^{\hat{\mathcal{O}}}_1(k)\mathbf{1}^{\underline{\check{\alpha}}'}_{\underline{\check{\alpha}}}. \quad \text{(C.4)}$$

2. Inverting the procedures

$$\operatorname*{Res}_{\omega=i\pi+\theta} \left\{F^{\mathcal{O}}_{\bar{1}1\underline{\hat{\alpha}}\underline{\check{\alpha}}}(\underline{\theta}_W) \overset{W\to\infty}{\to} c^{\mathcal{O}}_{\hat{\mathcal{O}}\check{\mathcal{O}}}(k,l,W)\, F^{\hat{\mathcal{O}}}_{\bar{1}1\underline{\hat{\alpha}}}(\omega,\theta,\underline{\hat{\theta}}) F^{\check{\mathcal{O}}}_{\underline{\check{\alpha}}}(\underline{\check{\theta}})\right\}$$
$$= c^{\mathcal{O}}_{\hat{\mathcal{O}}\check{\mathcal{O}}}(k,l,W)\, 2iF^{\hat{\mathcal{O}}}_{\underline{\hat{\alpha}}'}(\underline{\hat{\theta}})\mathbf{C}_{\bar{1}1} \left(\mathbf{1}^{\underline{\hat{\alpha}}'}_{\underline{\hat{\alpha}}} - \dot{\sigma}^{\hat{\mathcal{O}}}_1(k) S^{\underline{\hat{\alpha}}'1}_{1\underline{\hat{\alpha}}}\right) F^{\check{\mathcal{O}}}_{\underline{\check{\alpha}}}(\underline{\check{\theta}}) \quad \text{(C.5)}$$

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3. Let $\underline{\theta}_W = (\hat{\underline{\theta}} + W, \omega, \theta, \check{\underline{\theta}})$, $k = |\hat{\underline{\alpha}}|$, $l = N + |\check{\underline{\alpha}}| > N$ then by (C.2) and (3.1)

$$\begin{aligned} \operatorname*{Res}_{\omega=i\pi+\theta} F^{\mathcal{O}}_{\hat{\underline{\alpha}}\bar{1}1\check{\underline{\alpha}}}(\underline{\theta}_W) &= 2i\,\mathbf{C}_{1\bar{1}}\, F^{\mathcal{O}}_{\hat{\underline{\alpha}}'\check{\underline{\alpha}}'}(\hat{\underline{\theta}} + W, \check{\underline{\theta}}) \\ &\quad \times \left((-1)^{(N-1)k} 1^{\hat{\underline{\alpha}}'}_{\hat{\underline{\alpha}}} 1^{\check{\underline{\alpha}}'}_{\check{\underline{\alpha}}} - \dot{\sigma}^{\mathcal{O}}_1(n) S^{\bar{1}\hat{\underline{\alpha}}'}_{\hat{\underline{\alpha}}\bar{1}}(\hat{\underline{\theta}} + W, \omega) S^{\check{\underline{\alpha}}'1}_{1\check{\underline{\alpha}}}(\theta, \check{\underline{\theta}}) \right) \\ \overset{W\to\infty}{\to} 2i\mathbf{C}_{\bar{1}1} c^{\mathcal{O}}_{\hat{\mathcal{O}}\check{\mathcal{O}}}(k, l-N, W) F^{\hat{\mathcal{O}}}_{\hat{\underline{\alpha}}}(\hat{\underline{\theta}}) F^{\check{\mathcal{O}}}_{\check{\underline{\alpha}}'}(\check{\underline{\theta}})\, &(-1)^{(N-1)k} \left(1^{\check{\underline{\alpha}}'}_{\check{\underline{\alpha}}} - \dot{\sigma}^{\check{\mathcal{O}}}_1(l) S^{\check{\underline{\alpha}}'1}_{1\check{\underline{\alpha}}}(\theta, \check{\underline{\theta}}) \right) \end{aligned} \tag{C.6}$$

if

$$\dot{\sigma}^{\mathcal{O}}_1(n) S^{\bar{1}\hat{\underline{\alpha}}'}_{\hat{\underline{\alpha}}\bar{1}}(\hat{\underline{\theta}} + W, \omega) \overset{W\to\infty}{\to} (-1)^{(N-1)k} \dot{\sigma}^{\hat{\mathcal{O}}}_1(l) 1^{\hat{\underline{\alpha}}'}_{\hat{\underline{\alpha}}}\,. \tag{C.7}$$

4. Taking first $W \to \infty$ and then the Res means

$$\begin{aligned} \operatorname*{Res}_{\omega=i\pi+\theta} \left\{ F^{\mathcal{O}}_{\hat{\underline{\alpha}}\bar{1}1\check{\underline{\alpha}}}(\underline{\theta}_W) \overset{W\to\infty}{\to} c^{\mathcal{O}}_{\hat{\mathcal{O}}\check{\mathcal{O}}}(k, l, W) F^{\hat{\mathcal{O}}}_{\hat{\underline{\alpha}}}(\hat{\underline{\theta}}) F^{\check{\mathcal{O}}}_{\bar{1}1\check{\underline{\alpha}}}(\omega, \theta, \check{\underline{\theta}}) \right\} \\ = c^{\mathcal{O}}_{\hat{\mathcal{O}}\check{\mathcal{O}}}(k, l, W) F^{\hat{\mathcal{O}}}_{\hat{\underline{\alpha}}}(\hat{\underline{\theta}}) 2i\mathbf{C}_{\bar{1}1} F^{\check{\mathcal{O}}}_{\check{\underline{\alpha}}'}(\check{\underline{\theta}}) \left(1^{\check{\underline{\alpha}}'}_{\check{\underline{\alpha}}} - \dot{\sigma}^{\check{\mathcal{O}}}_1(l) S^{\check{\underline{\alpha}}'1}_{1\check{\underline{\alpha}}}(\theta, \check{\underline{\theta}}) \right). \end{aligned} \tag{C.8}$$

1. and 2. prove that $c^{\mathcal{O}}_{\hat{\mathcal{O}}\check{\mathcal{O}}}(k, l, W)$ is independent of k

$$c^{\mathcal{O}}_{\hat{\mathcal{O}}\check{\mathcal{O}}}(k-N, l, W) = c^{\mathcal{O}}_{\hat{\mathcal{O}}\check{\mathcal{O}}}(k, l, W)\,.$$

3. and 4. imply that $c^{\mathcal{O}}_{\hat{\mathcal{O}}\check{\mathcal{O}}}(k, l, W)$ depends on l as

$$c^{\mathcal{O}}_{\hat{\mathcal{O}}\check{\mathcal{O}}}(k, l-N, W) = c^{\mathcal{O}}_{\hat{\mathcal{O}}\check{\mathcal{O}}}(k, l, W)(-1)^{(N-1)k}$$

which means that $c^{\mathcal{O}}_{\hat{\mathcal{O}}\check{\mathcal{O}}}(k, l, W)$ is independent of l if $k = 0 \bmod N$ and in general

$$c^{\mathcal{O}}_{\hat{\mathcal{O}}\check{\mathcal{O}}}(k, l, W) = c^{\mathcal{O}}_{\hat{\mathcal{O}}\check{\mathcal{O}}}(k_0, l_0, W)(-1)^{(N-1)k(l-l_0)/N}$$

where $(l - l_0) = 0 \bmod N$ and $c^{\mathcal{O}}_{\hat{\mathcal{O}}\check{\mathcal{O}}}(k_0, l_0, W)$ is obtained by a simple example.

D Form factor equations

The co-vector valued function $F^{\mathcal{O}}_{1\ldots n}(\underline{\theta})$ is meromorphic in all variables $\theta_1, \ldots, \theta_n$ and satisfies the following relations [6, 7]:

(i) The Watson's equations describe the symmetry property under the permutation of both, the variables θ_i, θ_j and the spaces $i, j = i+1$ at the same time

$$F^{\mathcal{O}}_{\ldots ij\ldots}(\ldots, \theta_i, \theta_j, \ldots) = F^{\mathcal{O}}_{\ldots ji\ldots}(\ldots, \theta_j, \theta_i, \ldots)\, S_{ij}(\theta_{ij}) \tag{D.1}$$

for all possible arrangements of the θ's.

(ii) The crossing relation implies a periodicity property under the cyclic permutation of the rapidity variables and spaces

$$\begin{aligned} &{}^{\text{out},\bar{1}}\langle\, p_1 \,|\, \mathcal{O}(0) \,|\, p_2, \ldots, p_n \,\rangle^{\text{in,conn.}}_{2\ldots n} \\ &= F^{\mathcal{O}}_{1\ldots n}(\theta_1 + i\pi, \theta_2, \ldots, \theta_n) \dot{\sigma}^{\mathcal{O}}_1 \mathbf{C}^{\bar{1}1} = F^{\mathcal{O}}_{2\ldots n1}(\theta_2, \ldots, \theta_n, \theta_1 - i\pi) \mathbf{C}^{1\bar{1}}. \end{aligned} \tag{D.2}$$

The components of the vector $\dot{\sigma}_1^{\mathcal{O}}$ are given by $\dot{\sigma}_\alpha^{\mathcal{O}} = \sigma_\alpha^{\mathcal{O}}(-1)^{(N-1)+(1-1/N)(n-Q^{\mathcal{O}})}$ [9], where the statistics factor $\sigma_\alpha^{\mathcal{O}}$ is determined by the space-like commutation rule of the operator $\mathcal{O}$ and the field which creates the particle α. The charge conjugation matrix $\mathbf{C}^{\bar{1}1}$ is given by (3.5).

(iii) There are poles determined by one-particle states in each sub-channel given by a subset of particles of the state. In particular the function $F_{\underline{\alpha}}^{\mathcal{O}}(\underline{\theta})$ has a pole at $\theta_{12} = i\pi$ such that

$$\operatorname*{Res}_{\theta_{12}=i\pi} F_{1\dots n}^{\mathcal{O}}(\theta_1,\dots,\theta_n) = 2i\,\mathbf{C}_{12}\,F_{3\dots n}^{\mathcal{O}}(\theta_3,\dots,\theta_n)\left(\mathbf{1} - \dot{\sigma}^{\mathcal{O}} S_{2n}\dots S_{23}\right). \tag{D.3}$$

(iv) If there are also bound states in the model the function $F_{\underline{\alpha}}^{\mathcal{O}}(\underline{\theta})$ has additional poles. If for instance the particles 1 and 2 form a bound state (12), there is a pole at $\theta_{12} = i\eta$ such that

$$\operatorname*{Res}_{\theta_{12}=i\eta} F_{12\dots n}^{\mathcal{O}}(\theta_1,\theta_2,\dots,\theta_n) = F_{(12)\dots n}^{\mathcal{O}}(\theta_{(12)},\dots,\theta_n)\,\sqrt{2}\Gamma_{12}^{(12)} \tag{D.4}$$

where the bound state intertwiner $\Gamma_{12}^{(12)}$ and the values of θ_1, θ_2, $\theta_{(12)}$ are given in general in [34–36].

(v) Naturally, since we are dealing with relativistic quantum field theories we finally have

$$F_{1\dots n}^{\mathcal{O}}(\theta_1+\mu,\dots,\theta_n+\mu) = e^{s\mu}\,F_{1\dots n}^{\mathcal{O}}(\theta_1,\dots,\theta_n) \tag{D.5}$$

if the local operator transforms under Lorentz transformations as $\mathcal{O} \to e^{s\mu}\mathcal{O}$ where s is the "spin" of $\mathcal{O}$.

There exist bound states of r fundamental particles $(\rho_1\dots\rho_r)$ (with $\rho_1 < \dots < \rho_r$) which transform as the anti-symmetric SU(N) tensor representation of rank r, $(0 < r < N)$.

E Asymptotic behavior for $W \to \infty$

We use the short notations of section 4.

S-matrix. For $W \to \infty$ (up to higher order)

$$\begin{aligned} a(\theta+W) &\to e^{-i\pi\left(1-\frac{1}{N}\right)} e^{-i\eta\left(1-\frac{1}{N}\right)\frac{1}{\theta+W}} \\ \tilde{b}(\theta+W) &\to 1 + i\eta\frac{1}{\theta+W} - \eta^2\frac{1}{(\theta+W)^2} \\ \tilde{c}(\theta+W) &\to -i\eta\frac{1}{W}. \end{aligned} \tag{E.1}$$

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Minimal form factor function F and $\tilde{\phi}$-, τ-function. The functions defined in (2.3)–(2.7) satisfy the asymptotic behavior for $W \to \infty$ (up to higher order)

$$F(\theta+W) \to X(W)^{-\left(1-\frac{1}{N}\right)} \left(e^{\frac{1}{2}(\theta-i\pi)}\right)^{1-\frac{1}{N}} \tag{E.2}$$

$$\tilde{\phi}(\theta \pm W) \to X(W) e^{\mp\frac{1}{2}\theta} e^{\mp i\pi\frac{1}{2}\left(1-\frac{1}{N}\right)} \left(1 \mp \frac{1}{NW}\left(\theta + i\pi\left(1-\frac{1}{N}\right)\right)\right) \tag{E.3}$$

$$\tau(\theta+W) \to X(W)^{-2} e^{\theta}\left(1+\frac{2\theta}{NW}\right) \tag{E.4}$$

$$\bar{F}(\theta+W) \to X(W)^{-\frac{1}{N}} \left(e^{\frac{1}{2}(\theta-i\pi)}\right)^{\frac{1}{N}} = \left((2\pi)^{-1-\frac{1}{N}} W^{\frac{1}{N}} e^{\frac{1}{2}W} e^{\frac{1}{2}(\theta-i\pi)}\right)^{\frac{1}{N}} \tag{E.5}$$

$$X(W) = (2\pi)^{1+\frac{1}{N}} W^{-\frac{1}{N}} e^{-\frac{1}{2}W}. \tag{E.6}$$

p-functions. The asymptotic behavior for $W \to \infty$ of the p-functions (3.7), (3.14) and (3.17) are given by

$$p^J(\underline{\theta}_W, \underline{\underline{z}}_W) \to e^{-\frac{1}{2}W(k-k_1-k_{N-1})} \left(\sum_{i=1}^k e^{-\hat{\theta}_i}\right) p^J(\underline{\hat{\theta}}, \underline{\underline{\hat{z}}}) p^J(\underline{\check{\theta}}, \underline{\underline{\check{z}}}) \tag{E.7}$$

$$p^T(\underline{\theta}_W, \underline{z}_W) \to \frac{\sum e^{\hat{z}_i}}{\sum e^{\hat{\theta}_i}} - \frac{\sum e^{-\check{z}_i}}{\sum e^{-\check{\theta}_i}} = p^{T+}(\underline{\hat{\theta}}, \underline{\hat{z}}) + p^{T-}(\underline{\check{\theta}}, \underline{\check{z}}) \tag{E.8}$$

$$p^\phi(\underline{\theta}_W, \underline{z}_W) \to e^{-W\left(\left(1-\frac{1}{N}\right)k-k_1\right)} p^\phi(\underline{\hat{\theta}}, \underline{\hat{z}}) p^\phi(\underline{\check{\theta}}, \underline{\check{z}}) \tag{E.9}$$

$$p^\psi(\underline{\theta}_W, \underline{z}_W) \to e^{-W\frac{1}{2}\left(\left(1-\frac{1}{N}\right)k-k_1\right)} p^\psi(\underline{\hat{\theta}}, \underline{\hat{z}}) p^\psi(\underline{\check{\theta}}, \underline{\check{z}}). \tag{E.10}$$

F-function. Using (E.2) we calculate for $F(\underline{\theta})$ defined in (2.9) for $W \to \infty$ (up to a constant factor)

$$F(\underline{\theta}_W) \to F_0(\underline{\theta}, W) = X(W)^{-\left(1-\frac{1}{N}\right)kl} F(\underline{\hat{\theta}}) F(\underline{\check{\theta}}) \left(\prod_{i=1}^k e^{\frac{1}{2}l\left(1-\frac{1}{N}\right)\hat{\theta}_i}\right) \left(\prod_{i=1}^l e^{-\frac{1}{2}k\left(1-\frac{1}{N}\right)\check{\theta}_i}\right) \tag{E.11}$$

with $X(W)$ defined in (E.6).

The h-funktion: defined in (2.11) satisfies for $W \to \infty$ (up to a constant factor)

$$\tilde{h}(\underline{\theta}_W, \underline{z}_W) \to \tilde{h}_0(\underline{\theta}, \underline{z}, W)\left(1 + \frac{1}{W}\tilde{h}_1(\underline{\theta}, \underline{z}) + O(W^{-2})\right) \tag{E.12}$$

where

$$\tilde{h}_0(\underline{\theta}, \underline{z}, W) = X(W)^{lk_1+kl_1-2k_1l_1} \tilde{h}(\underline{\hat{\theta}}, \underline{\hat{z}}) \tilde{h}(\underline{\check{\theta}}, \underline{\check{z}}) \\ \times \left(\prod_{i=1}^k e^{-\frac{1}{2}l_1\hat{\theta}_i}\right) \left(\prod_{i=1}^l e^{\frac{1}{2}k_1\check{\theta}_i}\right) \left(\prod_{i=1}^{k_1} e^{\left(l_1-\frac{1}{2}l\right)\hat{z}_i}\right) \left(\prod_{i=1}^{l_1} e^{-\left(k_1-\frac{1}{2}k\right)\check{z}_i}\right) \tag{E.13}$$

and (up to a constant[6] which will not contribute to our results)

$$\tilde{h}_1(\underline{\theta}, \underline{z}) = \frac{1}{N}\left(k_1 \sum \check{\theta}_i - l_1 \sum \hat{\theta}_i - (l-2l_1)\sum \hat{z}_i + (k-2k_1)\sum \check{z}_i\right). \tag{E.14}$$

[6] $\frac{1}{N} i\pi\left(1-\frac{1}{N}\right)(lk_1+kl_1)$.

The complete h-function: defined by (2.16) satisfies

$$\tilde{h}\left(\underline{\theta}_W, \underline{\underline{z}}_W\right) = \prod_{j=0}^{N-2} \tilde{h}(\underline{z}_W^{(j)}, \underline{z}_W^{(j+1)}) \to \tilde{h}_0\left(\underline{\theta}, \underline{\underline{z}}, W\right)\left(1 + \frac{1}{W}\tilde{h}_1\left(\underline{\theta}, \underline{\underline{z}}\right) + O(W^{-2})\right) \tag{E.15}$$

$$\tilde{h}_0\left(\underline{\theta}, \underline{\underline{z}}, W\right) = \prod_{j=0}^{N-2} \tilde{h}_0(\underline{z}^{(j)}, \underline{z}^{(j+1)}, W) \tag{E.16}$$

$$\tilde{h}_1\left(\underline{\theta}, \underline{\underline{z}}\right) = \sum_{j=0}^{N-2} \tilde{h}_1\left(\underline{z}^{(j)}, \underline{z}^{(j+1)}\right) \tag{E.17}$$

where $\tilde{h}_0$ and $\tilde{h}_1$ given by (E.13) and (E.14) (with $\underline{z}^{(0)} = \underline{\theta}$ and $k, k_1 \to k_j, k_{j+1}$ etc.). This means that in leading order

$$\begin{aligned}\tilde{h}\left(\underline{\theta}_W, \underline{\underline{z}}_W\right) \to\ & \tilde{h}\left(\underline{\hat{\theta}}, \underline{\underline{\hat{z}}}\right)\tilde{h}\left(\underline{\check{\theta}}, \underline{\underline{\check{z}}}\right)\\ & \times X(W)^{\sum_{j=0}^{N-2}(l_j k_{j+1} + k_j l_{j+1} - 2k_{j+1}l_{j+1})}\left(\prod_{i=1}^{k} e^{-\frac{1}{2}l_1\hat{\theta}_i}\right)\left(\prod_{i=1}^{l} e^{\frac{1}{2}k_1\check{\theta}_i}\right)\\ & \times \prod_{j=1}^{N-2}\left(\left(\prod_{i=1}^{k_j} e^{-\frac{1}{2}(l_{j-1} - 2l_j + l_{j+1})\hat{z}_i^{(j)}}\right)\left(\prod_{i=1}^{l_j} e^{\frac{1}{2}(k_{j-1} - 2k_j + k_{j+1})\check{z}_j^{(j)}}\right)\right)\\ & \times \left(\prod_{i=1}^{k_{N-1}} e^{-\frac{1}{2}(l_{N-2} - 2l_{N-1})\hat{z}_i^{(N-1)}}\right)\left(\prod_{i=1}^{l_{N-1}} e^{\frac{1}{2}(k_{N-2} - 2k_{N-1})\check{z}_i^{(N-1)}}\right).\end{aligned}$$

Function $F * h$: in leading order

$$\begin{aligned}& F(\underline{\theta}_W)\tilde{h}\left(\underline{\theta}_W, \underline{\underline{z}}_W\right)\\ & \to F(\underline{\hat{\theta}})\tilde{h}\left(\underline{\hat{\theta}}, \underline{\underline{\hat{z}}}\right)F(\underline{\check{\theta}})\tilde{h}\left(\underline{\check{\theta}}, \underline{\underline{\check{z}}}\right)X^{-\left(1-\frac{1}{N}\right)kl + \sum_{j=0}^{N-2}(l_j k_{j+1} + k_j l_{j+1} - 2k_{j+1}l_{j+1})}\\ & \quad \times \left(\prod_{i=1}^{k} e^{\frac{1}{2}\left(l\left(1-\frac{1}{N}\right) - l_1\right)\hat{\theta}_i}\right)\left(\prod_{i=1}^{l} e^{-\frac{1}{2}\left(k\left(1-\frac{1}{N}\right) - k_1\right)\check{\theta}_i}\right)\\ & \quad \times \prod_{j=1}^{N-2}\left(\left(\prod_{i=1}^{k_j} e^{-\frac{1}{2}(l_{j-1} - 2l_j + l_{j+1})\hat{z}_i^{(j)}}\right)\left(\prod_{i=1}^{l_j} e^{\frac{1}{2}(k_{j-1} - 2k_j + k_{j+1})\check{z}_j^{(j)}}\right)\right)\\ & \quad \times \left(\prod_{i=1}^{k_{N-1}} e^{-\frac{1}{2}(l_{N-2} - 2l_{N-1})\hat{z}_i^{(N-1)}}\right)\left(\prod_{i=1}^{l_{N-1}} e^{\frac{1}{2}(k_{N-2} - 2k_{N-1})\check{z}_i^{(N-1)}}\right).\end{aligned} \tag{E.18}$$

Bethe state. By the asymptotic expansion of the S-matrix (2.2)

$$\begin{aligned}\tilde{S}_{\alpha\beta}^{\delta\gamma}(\theta) &= \mathbf{1}_{\alpha\beta}^{\delta\gamma}\tilde{b}(\theta) + \mathbf{P}_{\alpha\beta}^{\delta\gamma}\tilde{c}(\theta) = \tilde{b}(\theta)\left(\mathbf{1}_{\alpha\beta}^{\delta\gamma} - \frac{i\eta}{\theta}\mathbf{M}_{\alpha\beta}^{\delta\gamma} + O(\theta^{-2})\right)\\ \mathbf{1}_{\alpha\beta}^{\delta\gamma} &= \delta_\alpha^\gamma\delta_\beta^\delta,\ \mathbf{M}_{\alpha\beta}^{\delta\gamma} = \mathbf{P}_{\alpha\beta}^{\delta\gamma} = \delta_\alpha^\delta\delta_\beta^\gamma\end{aligned}$$

we obtain [37] for the monodromy matrix (2.13)[7]

$$\begin{aligned}\tilde{T}_{\underline{\alpha}\beta}^{\beta'\underline{\alpha}'}(\underline{\theta} + W, z) &= \left(\mathbf{1} - W^{-1}\,i\eta\,\mathbf{M}\right)_{\underline{\alpha}\beta}^{\beta'\underline{\alpha}'} + O(W^{-2})\\ \tilde{T}_{\underline{\alpha}\beta}^{\beta'\underline{\alpha}'}(\underline{\theta}, z + W) &= \left(\mathbf{1} + W^{-1}\,i\eta\,\mathbf{M}\right)_{\underline{\alpha}\beta}^{\beta'\underline{\alpha}'} + O(W^{-2}).\end{aligned} \tag{E.19}$$

[7] Up to $\frac{1}{w}$ terms from $\tilde{b}$, which will not contribute to our calculations, so we will skip them in the following.

The matrix elements of $\mathbf{M}_{\underline{\alpha\beta}}^{\underline{\beta'\alpha'}} = (\sum_{i=1}^{n} \mathbf{1} \cdots \mathbf{P}_i \cdots \mathbf{1})_{\underline{\alpha\beta}}^{\underline{\beta'\alpha'}}$, as a matrix in the auxiliary space, yields the $su(N)$ Lie algebra generators.

More general, the product $\tilde{T}_{\underline{\alpha\beta}}^{\underline{\beta'\alpha'}}(\underline{\theta}, \underline{z}) = \left(\tilde{T}(\underline{\theta}, z_m) \ldots \tilde{T}(\underline{\theta}, z_1)\right)_{\underline{\alpha\beta}}^{\underline{\beta'\alpha'}}$ satisfies

$$\tilde{T}_{\underline{\alpha\beta}}^{\underline{\beta'\alpha'}}(\underline{\theta} + W, \underline{z}) \to (\mathbf{1} - W^{-1}\, i\eta\, \mathbf{M})_{\underline{\alpha\beta}}^{\underline{\beta'\alpha'}}, \quad \tilde{T}_{\underline{\alpha\beta}}^{\underline{\beta'\alpha'}}(\underline{\theta}, \underline{z} + W) \to (\mathbf{1} + W^{-1}\, i\eta\, \mathbf{M})_{\underline{\alpha\beta}}^{\underline{\beta'\alpha'}}$$

$$\mathbf{M}_{\underline{\alpha\beta}}^{\underline{\beta'\alpha'}} = \sum_{i'=1}^{m} (\mathbf{1} + \cdots + \mathbf{M}_{i'} + \cdots + \mathbf{1})_{\underline{\alpha\beta}}^{\underline{\beta'\alpha'}} .$$

The basic Bethe ansatz state (2.14) for level 0 may be written as

$$\tilde{\Phi}_{\underline{\alpha}}^{\underline{\beta}}(\underline{\theta}, \underline{z}) = \left(\Omega \tilde{C}^{\beta_m}(\underline{\theta}, z_m) \cdots \tilde{C}^{\beta_1}(\underline{\theta}, z_1)\right)_{\underline{\alpha}} = \tilde{T}_{\underline{\alpha},\underline{1}}^{\underline{\beta},\underline{1}}(\underline{\theta}, \underline{z})$$

and

$$\tilde{\Phi}_{\underline{\alpha}}^{\underline{\beta}}(\underline{\theta}_W, \underline{z}_W) = \tilde{T}_{\underline{\alpha},\underline{1}}^{\underline{\beta},\underline{1}}(\underline{\theta}_W, \underline{z}_W) \to \tilde{\Phi}_0{}_{\underline{\alpha}}^{\underline{\beta}}(\underline{\theta}, \underline{z}) + \frac{1}{W} \tilde{\Phi}_1{}_{\underline{\alpha}}^{\underline{\beta}}(\underline{\theta}, \underline{z})$$

where

$$\begin{aligned}
\tilde{\Phi}_0{}_{\underline{\alpha}}^{\underline{\beta}}(\underline{\theta}, \underline{z}) &= \tilde{\Phi}_{\underline{\hat{\alpha}}}^{\underline{\hat{\beta}}}(\underline{\hat{\theta}}, \underline{\hat{z}}) \tilde{\Phi}_{\underline{\check{\alpha}}}^{\underline{\check{\beta}}}(\underline{\check{\theta}}, \underline{\check{z}}) \\
\tilde{\Phi}_1{}_{\underline{\alpha}}^{\underline{\beta}}(\underline{\theta}, \underline{z}) &= i\eta \left(\left(\tilde{T}_{\underline{\hat{\alpha}},\underline{\hat{\beta}}'}^{\underline{\hat{\beta}},\underline{\hat{\alpha}}'}(\underline{\hat{\theta}}, \underline{\hat{z}}) \right) \left(\tilde{T}_{\underline{\check{\alpha}}',\underline{\check{1}}}^{\underline{\check{\beta}}',\underline{\check{1}}}(\underline{\check{\theta}}, \underline{\check{z}}) \check{\mathbf{M}}_{\underline{\check{\alpha}},\underline{\hat{1}}}^{\underline{\hat{\beta}}',\underline{\check{\alpha}}'} \right) - \left(\hat{\mathbf{M}}_{\underline{\hat{\alpha}}',\underline{\check{\beta}}'}^{\underline{\check{\beta}},\underline{\hat{1}}} \tilde{T}_{\underline{\hat{\alpha}},\underline{\hat{1}}}^{\underline{\hat{\beta}},\underline{\hat{\alpha}}'}(\underline{\hat{\theta}}, \underline{\hat{z}}) \right) \left(\tilde{T}_{\underline{\check{\alpha}},\underline{\check{1}}}^{\underline{\check{\beta}}',\underline{\check{1}}}(\underline{\check{\theta}}, \underline{\check{z}}) \right) \right) .
\end{aligned} \tag{E.20}$$

Similarly, for the higher levels of the Bethe ansatz. The complete Bethe ansatz state (2.17) satisfies

$$\tilde{\Phi}_{\underline{\alpha}}(\underline{\theta}_W, \underline{\underline{z}}_W) = \tilde{\Phi}_0{}_{\underline{\alpha}}(\underline{\theta}, \underline{\underline{z}}) + \frac{1}{W} \tilde{\Phi}_1{}_{\underline{\alpha}}(\underline{\theta}, \underline{\underline{z}}) + O(W^{-2}) \tag{E.21}$$

where

$$\tilde{\Phi}_0{}_{\underline{\alpha}}(\underline{\theta}, \underline{\underline{z}}) = \tilde{\Phi}_{\underline{\hat{\alpha}}}(\underline{\hat{\theta}}, \underline{\underline{\hat{z}}}) \tilde{\Phi}_{\underline{\check{\alpha}}}(\underline{\check{\theta}}, \underline{\underline{\check{z}}}) \tag{E.22}$$

$$\tilde{\Phi}_1{}_{\underline{\alpha}}(\underline{\theta}, \underline{\underline{z}}) = \sum_{j=0}^{N-2} \tilde{\Phi}_0^{(N-2)}{}_{\underline{\alpha}_{N-2}}^{\underline{\alpha}_{N-1}}\left(\underline{z}^{(N-2)}, \underline{z}^{(N-1)}\right) \ldots \tilde{\Phi}_1^{(j)}{}_{\underline{\alpha}_j}^{\underline{\alpha}_{j+1}}\left(\underline{z}^{(j)}, \underline{z}^{(j+1)}\right) \ldots \tilde{\Phi}_0{}_{\underline{\alpha}}^{\underline{\alpha}_1}\left(\underline{\theta}, \underline{z}^{(1)}\right) . \tag{E.23}$$

F Exponential behavior

Iso-scalar. Let $\mathcal{O}$ be an iso-scalar operator, then (2.18) implies (3.15)

$$n_j = n\,(1 - j/N) \Rightarrow l_j = n\,(1 - j/N) - k_j$$

The asymptotic relation (E.18) gives the exponential behavior

$$F(\underline{\theta}_W) \tilde{h}\,(\underline{\theta}_W, \underline{\underline{z}}_W) \propto X(W)^{-\left(1-\frac{1}{N}\right)kl + \sum_{j=0}^{N-2}(l_j k_{j+1} + k_j l_{j+1} - 2k_{j+1} l_{j+1})} \tag{F.1}$$

with $X(W)$ given in (E.6). By elementary calculations one shows that the exponent of $X(W)$ can be written as

$$-\,(1 - 1/N)\,kl + \sum_{j=0}^{N-2} \left(l_j k_{j+1} + (k_j - 2k_{j+1})\, l_{j+1}\right) = \tilde{k}_1^2 + \tilde{k}_{N-1}^2 + \sum_{j=1}^{N-2} \left(\tilde{k}_{j+1} - \tilde{k}_j\right)^2 \tag{F.2}$$

where $\tilde{k}_j = k_j - k\,(1 - j/N)$.

Adjoint representation. Let $\mathcal{O}$ transform as the adjoint representation, then (2.18) implies (3.8)

$$n_j = n\,(1 - j/N) - 1 \Rightarrow l_j = n\,(1 - j/N) - k_j - 1$$

Therefore in (F.2) there is in addition

$$-k + 2k_1 - \sum_{j=1}^{N-2} (-k_{j+1} + k_j) = -k + k_1 + k_{N-1} = \tilde{k}_1 + \tilde{k}_{N-1}$$

and the exponent of $X(W)$ in (F.1) is

$$\begin{aligned} &-(1 - 1/N)\,kl + \sum_{j=0}^{N-2} (l_j k_{j+1} + k_j l_{j+1} - 2k_{j+1} l_{j+1}) \\ &= \tilde{k}_1^2 + \tilde{k}_{N-1}^2 + \sum_{j=1}^{N-2} \left(\tilde{k}_{j+1} - \tilde{k}_j\right)^2 + \tilde{k}_1 + \tilde{k}_{N-1}. \end{aligned} \tag{F.3}$$

Iso-vector. Let $\mathcal{O}$ be an iso-vector operator, then (2.18) implies (3.19)

$$n_j = (n - 1)\,(1 - j/N) \Rightarrow l_j = n\,(1 - j/N) - k_j - (1 - j/N)\,.$$

Therefore in (F.2) there is in addition

$$\begin{aligned} &-(k - 2k_1)\,(1 - 1/N) - \sum_{j=1}^{N-2} \left((1 - j/N)\,k_{j+1} + (k_j - 2k_{j+1})\,(1 - (j+1)/N)\right) \\ &= k_1 - k\,(1 - 1/N) = \tilde{k}_1 \end{aligned}$$

and the exponent of $X(W)$ in (F.1) is

$$\begin{aligned} &-(1 - 1/N)\,kl + \sum_{j=0}^{N-2} (l_j k_{j+1} + k_j l_{j+1} - 2k_{j+1} l_{j+1}) \\ &= \tilde{k}_1^2 + \tilde{k}_{N-1}^2 + \sum_{j=1}^{N-2} \left(\tilde{k}_{j+1} - \tilde{k}_j\right)^2 + \tilde{k}_1. \end{aligned} \tag{F.4}$$

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