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Multipoint Green's functions in 1 + 1 dimensional integrable quantum field theories

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Abstract

We calculate the multipoint Green's functions in 1+1 dimensional integrable quantum field theories. We use the crossing formula for general models and calculate the 3 and 4 point functions taking in to account only the lower nontrivial intermediate states contributions. Then we apply the general results to the examples of the scaling Z_2 Ising model, sinh-Gordon model and Z_3 scaling Potts model. We demonstrate this calculations explicitly. The results can be applied to physical phenomena as for example to the Raman scattering.

1. Introduction

A complete set of dynamical correlation functions contains the entire information about a given system. Unfortunately, in practice only few such functions can be measured by available experimental techniques. Usually experiments, such as neutron scattering measurements, probe

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two point functions. However, there are exceptions and there are several experimental techniques such as resonance Raman and resonance X-ray scattering which measure four-point functions or even something more complicated [1–4]. These higher order correlation functions carry information about the nonlinear dynamics which is especially important and interesting in strongly correlated models. It is also an interesting theoretical problem since such models usually require some special non-perturbative approaches. The latter fact brings us to (1+1)-dimensional models where such approaches are available.

The problem becomes especially interesting for massive quantum field theories where almost nothing is known about multipoint correlation functions. Meanwhile, as will be demonstrated in this paper, it is possible to calculate them by using the results for matrix elements or form factors of various operators. In the present paper we will obtain three- and four-point functions for massive integrable models in (1+1)-dimensions. For a low particle intermediate state approximation we apply the general results to three models. We calculate correlation functions of the order parameter fields for the off-critical Z_2 Ising model and the Z_3 Potts model perturbed by the thermal operator and for the fundamental field in the sinh-Gordon model. The models are chosen in a sequence of increasing complexity: the Ising model is equivalent to the model of non-interacting massive Majorana fermions with a trivial S-matrix, the sinh-Gordon model is very similar to the Ising one, but has the simplest possible nontrivial S-matrix (a diagonal one without poles), and the Z_3 model takes the complexity one step further having a diagonal S-matrix with one pole on the physical sheet corresponding to a bound state of the fundamental particles. In this article we will explore the crossing formula [5,6] in order to start calculation of the multipoint Green's functions or Wightman functions in 1+1 dimensional integrable quantum field theories. By means of the form factor program multi-point functions have been discussed previously. In [7] four-point Green's functions were investigated for the scaling Z_2 Ising model, the XY-model and the $O(3)$ nonlinear sigma-model. In [8] the three-point function for the Z_3 Potts model was constructed.

It is well known that the n-particle form factor of the local field $\phi(x)$

$$\langle 0|\phi(0)|\theta_1, ...\theta_n\rangle$$

is an analytic function of the variables $\theta_1, .., \theta_n$. More general form factors as

$$\langle \theta_1, ..., \theta_n|\varphi(0)|\theta_{n+1}, ..., \theta_{n+k}\rangle$$

already are not functions but distributions or generalized functions [5,6,9]. In fact the crossing formula is defining the generalized form factors in the language of simple form factors. For example, in the case of the 3-particle form factor we have[1]

$$\begin{aligned}\langle \theta_1|\varphi(0)|\theta_2, \theta_3\rangle &= \langle 0|\varphi(0)|\theta_1 + i\pi - i\epsilon, \theta_2, \theta_3\rangle \\ &\quad + \delta_{\theta_1\theta_2}\langle 0|\varphi(0)|\theta_3\rangle + \delta_{\theta_1\theta_3}\langle 0|\varphi(0)|\theta_2\rangle S(\theta_{23})\end{aligned}$$

The ϵ-prescription and the δ-functions make the left hand side a distribution. In more complicated cases we can define the generalized form factors as explained in [5,9]. In this article we will consider 3 and 4 point Green's functions. Using this definition we will evaluate the multipoint correlators or Green's functions defined as time order products of operators:

$$\langle 0|T\varphi_1(x_1)\varphi_2(x_2)...\varphi_n(x_n)|0\rangle$$

We will transform these correlators into sums of products of matrix elements inserting between the fields the identity

[1] See (A.1).

$$\sum |\theta_1,...\theta_n\rangle\langle\theta_n,...\theta_1| = 1$$

and then using the crossing formula we will step by step calculate the Wightman and Green's functions.

The results can be applied to physical phenomena as for example to Raman scattering and nonlinear susceptibility [10].

2. Green's functions

Below in this Section we will do our calculations in the most general form valid for all integrable models. In the next sections we will apply the results to several concrete examples. We will concentrate on the most difficult case of the four-point function, the calculations of the three-point one are comparatively straightforward.

The Green's functions are time ordered n-point functions, written as a sum over all permutations of the fields φ_i and variables x_i

$$\tau_{\underline{\varphi}}(\underline{x}) = \langle\, 0\,|\, T\varphi_1(x_1)\,\ldots\,\varphi_n(x_n)|\, 0\,\rangle = \sum_{\pi\in S_n} \Theta(\underline{\pi x^0})\, w_{\underline{\pi\varphi}}(\underline{\pi x}) \tag{1}$$

where $w_{\underline{\pi\varphi}}(\underline{\pi x}) = \langle\, 0\,|\,\varphi_{\pi 1}(x_{\pi 1})\ldots\varphi_{\pi n}(x_{\pi n})|\, 0\,\rangle$ is the Wightman function and $\Theta(\underline{\pi t}) = \Theta(t_{\pi 1} - t_{\pi 2})\ldots\,\Theta(t_{\pi(n-1)} - t_{\pi n})$. The Fourier transform is the Green's function in momentum space

$$\begin{aligned}\tilde{\tau}_{\underline{\varphi}}(\underline{k}) &= \sum_{\pi\in S_n}\int \underline{d^2x}e^{ix_ik_i}\,\Theta(\underline{\pi x^0})\,\langle\, 0\,|\,\varphi_{\pi 1}(x_{\pi 1})\ldots\varphi_{\pi n}(x_{\pi n})|\, 0\,\rangle && (2)\\ &= (2\pi)^2\,\delta^{(2)}\left(\sum k_i\right)\,\tilde{\Xi}_{\underline{\varphi}}(\underline{k}) && (3)\end{aligned}$$

where we have used translation invariance and split off the energy momentum δ-function defining $\tilde{\Xi}(\underline{k})$. The full Green's function may be decomposed into the connected ones

$$\tilde{\tau}_{\underline{\varphi}}(\underline{k}) = \sum_{\underline{k}_1\cup\cdots\cup\underline{k}_m=\underline{k}} \tilde{\tau}_c(\underline{k}_1)\ldots\tilde{\tau}_c(\underline{k}_m)\,.$$

2.1. The Green's functions in low particle approximation

Inserting sets of intermediate states $|\underline{p^{(j)}}\rangle = |p_1^{(j)},\ldots,p_{n_j}^{(j)}\rangle$ in (1) we obtain (see Appendix B)

$$\begin{aligned}\tilde{\Xi}_{\underline{\varphi}}(\underline{k}) = &\sum_{\pi\in S_n}\sum\frac{1}{\underline{n}!}\int_{\underline{p^{(1)}}}\cdots\int_{\underline{p^{(n-1)}}}\langle\varphi_{\pi 1}(0)|\underline{p^{(1)}}\rangle\langle\underline{p^{(1)}}|\ldots|\underline{p^{(n-1)}}\rangle\langle\underline{p^{(n-1)}}|\,\varphi_{\pi n}(0)\rangle\\ &\times 2\pi\,\delta\left(k_{\pi 2}^1 - \sum\left(p_j^{(2)}\right)^1 + \sum\left(p_j^{(1)}\right)^1\right)\ldots 2\pi\,\delta\left(k_{\pi n}^1 + \sum\left(p_j^{(n-1)}\right)^1\right)\\ &\times\frac{-i}{\sum_{i=2}^n k_{\pi i}^0 + \sum\omega_j^{(1)} - i\epsilon}\,\frac{-i}{\sum_{i=3}^n k_{\pi i}^0 + \sum\omega_j^{(2)} - i\epsilon}\cdots\frac{-i}{k_{\pi n}^0 + \sum\omega_j^{(n-1)} - i\epsilon}\end{aligned} \tag{4}$$

with $\int_{\underline{p^{(j)}}} = \int_{p_1^{(j)}}\ldots\int_{p_{n_j}^{(j)}}$, $\int_p = \int\frac{d^2p}{2\pi 2\omega}$, $\omega = \sqrt{m^2 + \left(p^1\right)^2}$, $\underline{n}! = \prod_j \underline{n^{(j)}}!$, $\underline{n^{(j)}}! = \prod_j n_k^{(j)}!$ and $n_k^{(j)} =$ the number of particles of type k in the state $|\underline{p^{(j)}}\rangle$. For explicit calculation it is convenient to take the limit $k_i^1 \to 0$, then the δ-functions in (4) simplify to

$$2\pi\delta\left(\sum\left(p_j^{(1)}\right)^1\right)\dots 2\pi\delta\left(\sum\left(p_j^{(n-1)}\right)^1\right).$$

2.1.1. S-matrix and form factors

For integrable quantum field theories the n-particle S-matrix factorizes into $n(n-1)/2$ two-particle ones

$$S^{(n)}(\theta_1,\dots,\theta_n)=\prod_{i<j}S(\theta_{ij}),$$

where the product on the right hand side has to be taken in a specific order (see e.g. [11]). The numbers θ_{ij} are the rapidity differences $\theta_{ij}=\theta_i-\theta_j$, which are related to the momenta of the particles by $p=m(\cosh\theta,\sinh\theta)$. To simplify the calculations we will consider only theories with diagonal scattering and only one type of particles. The generalization to more types of particles is straightforward and will be used for the Z_3-model.

The form factors of a local bosonic field $\varphi(x)$ are the matrix elements

$$F^\varphi(\underline{\theta})=\langle\,0\,|\,\varphi(0)\,|\,\theta_1,\dots,\theta_n\,\rangle\,. \tag{5}$$

They satisfy the form factor equations (i)–(v) (see e.g. [12]). As a generalization we write

$$F^\varphi(\underline{\theta}';\underline{\theta})=\langle\,\theta'_{n'},\dots,\theta'_1\,|\,\varphi(0)\,|\,\theta_1,\dots,\theta_n\,\rangle$$

which is related to (5) by crossing. In particular (see Appendix A)

$$F^\varphi(\theta_1;\theta_2,\theta_3)=F^\varphi(\theta_1,\theta_2-i\pi_-,\theta_3-i\pi_+)+\delta_{\theta_{12}}+\delta_{\theta_{13}} \tag{6}$$

$$F^\varphi(\theta_2,\theta_3;\theta_4)=F^\varphi(\theta_3+i\pi_+,\theta_2+i\pi_-,\theta_4)+\delta_{\theta_{24}}+\delta_{\theta_{34}} \tag{7}$$

with $i\pi_\pm=i\pi\pm i\epsilon$ and $\delta_{\theta_{12}}=4\pi\delta(\theta_1-\theta_2)$. The form factors $F^\varphi(\underline{\theta})$ are meromorphic functions whereas the $F^\varphi(\underline{\theta}';\underline{\theta})$ are distributions.

2.1.2. Examples

The 2-point function Let $\varphi(x)$ be a scalar chargeless bosonic field with the normalization $\langle 0|\varphi(x)|p\rangle=1$. The 2-point Wightman function in the 1-particle intermediate state approximation is

$$w^1(x_1-x_2)=\int\frac{dp}{2\pi 2\omega}\langle\,0\,|\,\varphi(x_1)\,|\,p\,\rangle\langle\,p\,|\,\varphi(x_2)\,|\,0\rangle=i\Delta_+(x_1-x_2)$$

and the 2-point Green's function in this approximation is

$$\tau^1(x_1-x_2)=\Theta(x_1^0-x_2^0)w^1(x_1-x_2)+\Theta(x_2^0-x_1^0)w^1(x_2-x_1)=i\Delta_F(x_1-x_2)$$

or in momentum space

$$\tilde\tau^1(k_1,k_2)=(2\pi)^2\,\delta^{(2)}(k_1+k_2)\,\tilde\Xi(\underline{k}),\quad \tilde\Xi(\underline{k})=\frac{i}{k_1^2-m^2+i\epsilon}\,.$$

The 3-point function We consider $\tilde\Xi_{\underline{\varphi}}(\underline{k})$ for $n=3$. For the detailed calculations see Appendix C.1. Let $\underline{\varphi}=(\varphi,\varphi,\epsilon)$ with $\langle 0|\epsilon|\theta_1,\theta_2\rangle=F^\epsilon(\theta_1,\theta_2)$. We calculate $\tilde\Xi_{\varphi\varphi\epsilon}(\underline{k})$ in the limit $k_i^1\to 0$. For the various permutations in (3) we obtain:

a) For the permutation $\pi=123$ we use the intermediate states approximation $\langle 0|\varphi|\theta_1\rangle\langle\theta_1|\varphi|\theta_2,\theta_3\rangle\langle\theta_3,\theta_2|\epsilon|0\rangle$ then

$$\tilde{\Xi}^{12}_{\varphi\varphi\epsilon}(k_1,k_2,k_3) = -\frac{1}{64\pi m^4}\frac{m}{-k_1^0+m-i\epsilon}\int d\theta \frac{m}{\omega}\frac{2m}{k_3^0+2\omega-i\epsilon} \times F^{\varphi}(i\pi,\theta,-\theta)F^{\epsilon}(-\theta+i\pi,\theta+i\pi) \tag{8}$$

b) For the permutation $\pi=321$ we use the intermediate states approximation $\langle 0|\epsilon|\theta_1,\theta_2\rangle\langle\theta_2,\theta_1|\varphi|\theta_3\rangle\langle\theta_3|\varphi|0\rangle$ then

$$\tilde{\Xi}^{21}_{\epsilon\varphi\varphi}(k_3,k_2,k_1) = -\frac{1}{64\pi m^4}\frac{m}{k_1^0+m-i\epsilon}\int d\theta \frac{m}{\omega}\frac{2m}{-k_3^0+2\omega-i\epsilon} \times F^{\epsilon}(\theta,-\theta)F^{\varphi}(-\theta+i\pi,\theta+i\pi,0) \tag{9}$$

c) For the permutation $\pi=132$ we use three intermediate states approximations
i) $\langle 0|\varphi|\theta_1\rangle\langle\theta_1|\epsilon|\theta_2\rangle\langle\theta_2|\varphi|0\rangle$:

$$\tilde{\Xi}^{11}_{\varphi\epsilon\varphi}(k_1,k_3,k_2) = \frac{-1}{4m^4}\frac{m}{-k_1^0+m-i\epsilon}\frac{m}{k_2^0+m-i\epsilon}F^{\epsilon}(i\pi,0) \tag{10}$$

ii) $\langle 0|\varphi|\theta_1\rangle\langle\theta_1|\epsilon|\theta_2,\theta_3,\theta_4\rangle\langle\theta_4,\theta_3,\theta_2|\varphi|0\rangle$:

$$\tilde{\Xi}^{13}_{\varphi\epsilon\varphi}(k_1,k_3,k_2) = -\frac{1}{64\pi m^4}\frac{m}{-k_1^0+m-i\epsilon}\int d\theta \frac{m}{\omega}\frac{2m}{k_2^0+2\omega+m-i\epsilon} \times F^{\epsilon}(\theta,-\theta)F^{\varphi}(-\theta+i\pi,\theta+i\pi,i\pi) \tag{11}$$

iii) $\langle 0\,|\,\varphi\,|\,\theta_1,\theta_2,\theta_3\rangle\,\langle\theta_3,\theta_2,\theta_1|\,\epsilon\,|\,\theta_4\rangle\,\langle\theta_4\,|\,\varphi\,|\,0\,\rangle$:

$$\tilde{\Xi}^{31}_{\varphi\epsilon\varphi}(k_1,k_3,k_2) = -\frac{1}{64\pi m^4}\frac{m}{k_2^0+m-i\epsilon}\int d\theta \frac{m}{\omega}\frac{2m}{-k_1^0+2\omega+m-i\epsilon} \times F^{\varphi}(0,\theta,-\theta)F^{\epsilon}(-\theta+i\pi,\theta+i\pi)$$

Finally we obtain

$$\begin{aligned}\tilde{\Xi}_{\varphi\varphi\epsilon}(k_1,k_2,k_3) &= \tilde{\Xi}^{12}_{\varphi\varphi\epsilon}(k_1,k_2,k_3)+\tilde{\Xi}^{21}_{\epsilon\varphi\varphi}(k_3,k_1,k_2)+\tilde{\Xi}^{11}_{\varphi\epsilon\varphi}(k_1,k_3,k_2)\\ &+\tilde{\Xi}^{13}_{\varphi\epsilon\varphi}(k_1,k_3,k_2)+\tilde{\Xi}^{31}_{\varphi\epsilon\varphi}(k_1,k_3,k_2)+(k_1\leftrightarrow k_2)\,.\end{aligned} \tag{12}$$

The 4-point function We consider $\tilde{\Xi}_{\underline{\varphi}}(\underline{k})$ for $n=4$. For the detailed calculations see again Appendix B. Let $\underline{\varphi}=(\varphi,\varphi,\varphi,\varphi)$. We use the intermediate states approximation $\langle 0|\varphi|\theta_1\rangle\langle\theta_1|\varphi|\theta_2,\theta_3\rangle\langle\theta_3,\theta_2|\varphi|\theta_4\rangle\langle\theta_4|\varphi|0\rangle$ then the connected part yields for $k_i^1=0$

$$\tilde{\Xi}_{\underline{\varphi}}(\underline{k}) = -\frac{1}{32}\frac{i}{m^6\pi}\sum_{perm(k)}\frac{m}{-k_1^0+m-i\epsilon}\frac{m}{k_4^0+m-i\epsilon}g\left(-\left(k_3^0+k_4^0\right)/(2m)+i\epsilon\right) \tag{13}$$

$$g(x) = \frac{-1}{4}\int d\theta\frac{1}{\cosh\theta}\frac{1}{\cosh\theta-x}I_{\underline{\varphi}}(0,\theta,-\theta,0)$$

with

$$I_{\underline{\varphi}}(\theta_1,\theta_2,\theta_3,\theta_4) = F^{\varphi}(\theta_1;\theta_2,\theta_3)F^{\varphi}(\theta_2,\theta_3;\theta_4)) = I_1(\underline{\theta})+I_2(\underline{\theta})$$

and

$$
\begin{aligned}
I_1(\underline{\theta}) &= \tfrac{1}{2} F^{\varphi}(\theta_1, \theta_2 - i\pi_+, \theta_3 - i\pi_-) F^{\varphi}(\theta_3 + i\pi_+, \theta_2 + i\pi_-, \theta_4) \\
&\quad + \tfrac{1}{2} F^{\varphi}(\theta_1, \theta_2 - i\pi_-, \theta_3 - i\pi_+) F^{\varphi}(\theta_3 + i\pi_-, \theta_2 + i\pi_+, \theta_4) \\
I_2(\underline{\theta}) &= \tfrac{1}{2} \left(\delta_{\theta_{12}}\left(1 + S(\theta_{23})\right) + \delta_{\theta_{13}}\left(1 + S(\theta_{23})\right)\right) F^{\varphi}(\theta_3 + i\pi_+, \theta_2 + i\pi_-, \theta_4) \\
&\quad + \tfrac{1}{2} F^{\varphi}(\theta_1, \theta_2 - i\pi_-, \theta_3 - i\pi_+) \left(\delta_{\theta_{24}}\left(1 + S(\theta_{32})\right) + \delta_{\theta_{34}}\left(1 + S(\theta_{32})\right)\right)
\end{aligned}
$$

see (B.4).

3. Models

3.1. The scaling Z_2 Ising model

In the scaling limit this model may be described by an interacting Bose field $\sigma_n^z = C m^{1/8} \sigma(x)$, where C is a numerical constant and $m = h - J$. The excitations are non-interacting Majorana fermions with the 2-particle S-matrix $S(\theta) = -1$. The field $\epsilon(x)$ is defined by $\sigma^x = (m/J)^{1/2} \epsilon(x) \sim \bar{\psi}\psi(x)$, where ψ is a free Majorana spinor field. The n-particle form factors for the order parameter $\sigma(x)$ were proposed in [13,14] as

$$
F^{Z(2)}(\underline{\theta}) = \langle\, 0 |\, \sigma(0)\, |\, \theta_1, \ldots, \theta_n \,\rangle = (2i)^{\frac{n-1}{2}} \prod_{i<j} \tanh \tfrac{1}{2}\theta_{ij} . \tag{14}
$$

3.1.1. The 3-point function

We investigate the Fourier transform of the Green's function

$$
\tau_{\varphi\varphi\epsilon}(\underline{x}) = \langle 0 | T \varphi(x_1) \varphi(x_2) \epsilon(x_3) | 0 \rangle
$$

where $\varphi(x)$ is the order parameter $\sigma(x)$ and $\epsilon(x) \sim \bar{\psi}\psi(x)$. For a free Majorana spinor field $\psi(x)$ we have (up to a constant)

$$
\langle 0 | \epsilon(0) | \theta_1, \theta_2 \rangle = \sinh \tfrac{1}{2}\theta_{12} \,.
$$

We apply the general results (8) – (12) and obtain (for details see Appendix C.1)

$$
\begin{aligned}
\tilde{\Xi}^{12}_{\varphi\varphi\epsilon}(k_1, k_2, k_3) &= -\frac{i}{32\pi m^4} \frac{m}{-k_1^0 + m - i\epsilon} h_+^{Z2}(-k_3^0/(2m) + i\epsilon) \\
h_+^{Z2}(x) &= \frac{1}{2i} \int d\theta \frac{1}{\cosh\theta} \frac{1}{\cosh\theta - x} F^{\varphi}(i\pi, \theta, -\theta) F^{\epsilon}(-\theta + i\pi, \theta + i\pi)
\end{aligned}
$$

Similarly we get

$$
\tilde{\Xi}^{21}_{\epsilon\varphi\varphi}(k_3, k_2, k_1) = -\frac{i}{32\pi m^4} \frac{m}{k_1^0 + m - i\epsilon} h_+^{Z2}(k_3^0/(2m) + i\epsilon)
$$

with (using (14))

$$
\begin{aligned}
h_+^{Z2}(x) &= \int_{-\infty}^{\infty} \frac{1}{\cosh\theta} \frac{1}{\cosh\theta - x} \frac{(\cosh\theta + 1)^2}{\cosh\theta} d\theta \\
&= -\frac{2}{x} - \frac{2}{x}\pi - \frac{1}{x^2}\pi - 4\frac{(1+x)^2}{x^2\sqrt{x^2 - 1}} \operatorname{arctanh} \frac{1+x}{\sqrt{x^2 - 1}} .
\end{aligned}
$$

For the function $h_+^{Z2}(x)$ see Fig. 1

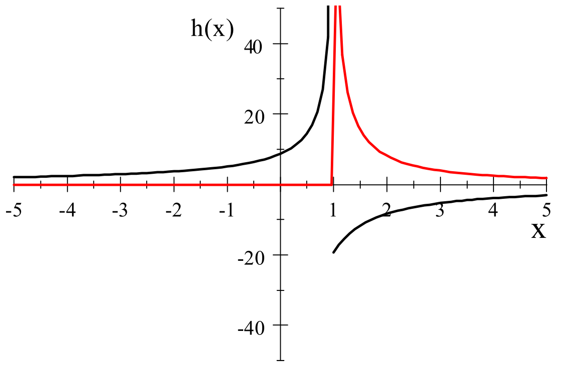

Fig. 1. Plot of $\mathrm{Re}\, h_+^{Z2}(x)$ (black) and $\mathrm{Im}\, h_+^{Z2}(x)$ (red). (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

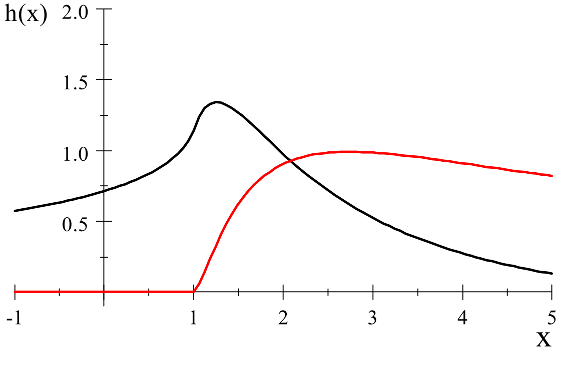

Fig. 2. Plot of $\mathrm{Re}\, h_-^{Z2}(x)$ (black) and $\mathrm{Im}\, h_-^{Z2}(x)$ (red). (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

With (11) we obtain

$$\tilde{\Xi}^{13}_{\varphi\epsilon\varphi}(k_1,k_3,k_2) = -\frac{i}{32\pi m^4}\frac{m}{-k_1^0+m-i\epsilon} h_-^{Z2}(-k_2^0/(2m)-\tfrac{1}{2}+i\epsilon)$$

$$h_-^{Z2}(x) = \frac{1}{2i}\int d\theta \frac{1}{\cosh\theta}\frac{1}{\cosh\theta - x} F^\epsilon(\theta,-\theta)F^\varphi(-\theta+i\pi,\theta+i\pi,i\pi)$$

and

$$\tilde{\Xi}^{31}_{\varphi\epsilon\varphi}(k_1,k_3,k_2) = -\frac{i}{32\pi m^4}\frac{m}{k_2^0+m-i\epsilon} h_-^{Z2}(k_1^0/(2m)-\tfrac{1}{2}+i\epsilon)$$

with

$$h_-^{Z2}(x) = \int_{-\infty}^{\infty} d\theta \frac{(\cosh\theta-1)^2}{\cosh^2\theta}\frac{1}{\cosh\theta - x}$$

$$= -\frac{2}{x}+\frac{2}{x}\pi - \frac{1}{x^2}\pi - 4\frac{(x-1)^2}{x^2\sqrt{x^2-1}}\operatorname{arctanh}\frac{1+x}{\sqrt{x^2-1}}\,.$$

For the function $h_-^{Z2}(x)$ see Fig. 2

Finally

$$\begin{aligned}\tilde{\Xi}_{\varphi\varphi\epsilon}(k_1,k_2,k_3) &= \tilde{\Xi}^{12}_{\varphi\varphi\epsilon}(k_1,k_2,k_3)+\tilde{\Xi}^{21}_{\epsilon\varphi\varphi}(k_3,k_1,k_2)+\tilde{\Xi}^{11}_{\varphi\epsilon\varphi}(k_1,k_3,k_2)\\
&\quad+\tilde{\Xi}^{13}_{\varphi\epsilon\varphi}(k_1,k_3,k_2)+\tilde{\Xi}^{31}_{\varphi\epsilon\varphi}(k_1,k_3,k_2)+(k_1\leftrightarrow k_2)\\
&=-\frac{i}{32\pi m^4}\frac{m}{-k_1^0+m-i\epsilon}h_+(-k_3^0/(2m)+i\epsilon)\\
&\quad-\frac{i}{32\pi m^4}\frac{m}{k_1^0+m-i\epsilon}h_+(k_3^0/(2m)+i\epsilon)\\
&\quad+\frac{-i}{4m^4}\frac{m}{-k_1^0+m-i\epsilon}\frac{m}{k_2^0+m-i\epsilon}\\
&\quad-\frac{i}{32\pi m^4}\frac{m}{-k_1^0+m-i\epsilon}h_-(-k_2^0/(2m)-\tfrac{1}{2}+i\epsilon)\\
&\quad-\frac{i}{32\pi m^4}\frac{m}{k_2^0+m-i\epsilon}h_-(k_1^0/(2m)-\tfrac{1}{2}+i\epsilon)+(k_1\leftrightarrow k_2)\,.\end{aligned}\tag{15}$$

This result can be applied to nonlinear susceptibility [10].

3.1.2. The 4-point function

We investigate the Fourier transform of the Green's function

$$\tau_{\varphi\varphi\varphi\varphi}(\underline{x})=\langle 0|T\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|0\rangle$$

for the order parameter $\varphi(x)=\sigma(x)$. From (13) for $k_i=(k_i^0,0)$ in momentum space the contribution from I_2 in (B.4) vanishes, because $S(0)=-1$ and we get

$$\tilde{\Xi}_{\underline{\varphi}}(\underline{k})=-\frac{i}{32\pi m^6}\sum_{perm(k)}\frac{m}{-k_1^0+m-i\epsilon}\frac{m}{k_4^0+m-i\epsilon}g^{Z2}\left(\frac{-1}{2m}\left(k_3^0+k_4^0\right)+i\epsilon\right)\tag{16}$$

$$g^{Z2}(x)=-\frac{1}{4}\int d\theta\frac{1}{\cosh\theta}\frac{1}{\cosh\theta-x}I^{Z_2}_{\underline{\varphi}}(0,\theta,-\theta,0)$$

From (B.4) and (14) we obtain (see Appendix C.1) for the contribution of I_1

$$I^{Z_2}_{\underline{\varphi}}(0,\theta,-\theta,0)=2\tanh^2\theta\coth^4\tfrac{1}{2}(\theta-i\epsilon)+(\epsilon\to-\epsilon)\tag{17}$$

and

$$g^{Z2}(x)=\frac{16}{1-x}-\frac{15\pi}{2x}-\frac{8}{x}-\frac{4\pi+2}{x^2}-\frac{\pi}{x^3}-\frac{(x+1)^2\sqrt{x^2-1}}{x^3(x-1)^2}2\ln\left(-x+\sqrt{x^2-1}\right).$$

For the function $g^{Z2}(x)$ see Fig. 3

This result can be applied to Raman scattering [10].

3.2. The sinh-Gordon model

The classical field equation[2] is

[2] For details see Appendix C.2.

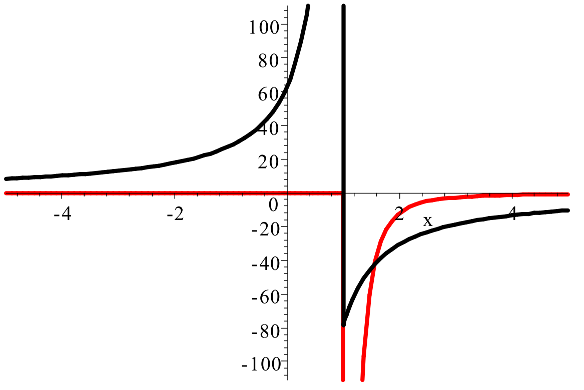

Fig. 3. Plot of $\operatorname{Re} g^{Z2}(x)$ (black), $\operatorname{Im} g^{Z2}(x+i\epsilon)$ (red) for the scaling Ising model. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

$$\Box\varphi(t,x) + \frac{\alpha}{\beta}\sinh\beta\varphi(t,x) = 0. \tag{18}$$

The sinh-Gordon S-matrix was derived in [11,15][3]

$$S^{SG}(\theta) = \frac{\sinh\theta - i\sin\pi\mu}{\sinh\theta + i\sin\pi\mu} = -\exp\left(-2\int_0^\infty \frac{dt}{t}\,\frac{\cosh\left(\frac{1}{2}-\mu\right)t}{\cosh\frac{1}{2}t}\sinh t\,\frac{\theta}{i\pi}\right)$$

where μ is related to the coupling constant by

$$0 < \mu = \frac{\beta^2}{8\pi + \beta^2} < 1.$$

The sinh-Gordon minimal form factor is [16,17]

$$F^{SG}(\theta) = \exp\int_0^\infty \frac{dt}{t\sinh t}\left(\frac{\cosh\left(\frac{1}{2}-\mu\right)t}{\cosh\frac{1}{2}t} - 1\right)\cosh t\left(1-\frac{\theta}{i\pi}\right).$$

3.2.1. *The 4-point function*

We consider the 4-point Green's function

$$\tau_{\varphi\varphi\varphi\varphi}(\underline{x}) = \langle 0 \,|\, T\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4) \,|\, 0\rangle$$

and calculate the function $\tilde{\Xi}_{\underline{\varphi}}(\underline{k})$ given by (13) (for details see Appendix C.2)

$$\tilde{\Xi}^{SG}_{\underline{\varphi}}(\underline{k}) = \frac{-i}{32\pi m^6}\sum_{perm(k)} \frac{m}{-k_1^0 + m - i\epsilon}\,\frac{m}{k_4^0 + m - i\epsilon}\, g^{SG}\left(\frac{-1}{2m}\left(k_3^0 + k_4^0\right)\right)$$

with $g^{SG}(x) = g_1^{SG}(x) + g_2^{SG}(x)$ and

[3] The sinh-Gordon S-matrix is obtained from the sine-Gordon one by analytic continuation of the coupling constant: $\beta \to i\beta$.

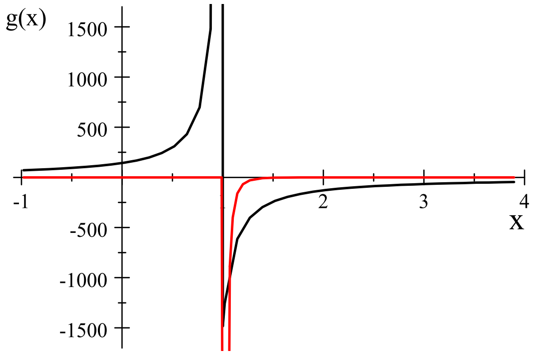

Fig. 4. Plot of $\mathrm{Re}\, g_1^{SG}(x)$ (black) and $\mathrm{Im}\, g_1^{SG}(x)$ (red) for $\mu = 0.3$. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

$$g_i^{SG}(x) = -\frac{1}{4} \int \frac{1}{\cosh\theta} \frac{1}{\cosh\theta - x} I_{\underline{\varphi} i}^{SG}(0, \theta, -\theta, 0) d\theta$$

From (B.4) and (C.5) we obtain using $I_{\underline{\varphi}}^{Z_2}$ as defined in (17)

$$\begin{aligned} & I_{\underline{\varphi} 1}^{SG}(0, \theta, -\theta, 0) \\ & = \tfrac{1}{2} Z^{\varphi} F^{SG}(0, \theta - i\pi_+, -\theta - i\pi_-) F^{SG}(-\theta + i\pi_+, \theta + i\pi_-, 0) + (\epsilon \to -\epsilon) \\ & = f^{SG}(\theta) I_{\underline{\varphi}}^{Z_2}(0, \theta, -\theta, 0) \end{aligned}$$

where[4]

$$f^{SG}(\theta) = -\frac{(Z^{\varphi})^2 \sin^2 \pi\mu}{F^2(i\pi)(2i)^2} \left(F_0(\theta + i\pi)\right)^4 F_0(2\theta) F_0(-2\theta)$$

and $F_0(\theta) = F^{SG}(\theta) / \left(-i \sinh \frac{1}{2}\theta\right)$. Therefore as in (16) we obtain

$$g_1^{SG}(x) = -\int_{-\infty}^{\infty} f^{SG}(\theta) \left(\frac{\coth^4 \frac{1}{2}\theta \tanh^2 \theta}{\cosh\theta} \frac{1}{\cosh\theta - x} - \frac{16}{\theta^2} \frac{1}{1 - x} \right) d\theta.$$

The functions $g_1^{SG}(x)$ for $\mu = 0.3$ and $\mu = 0.5$ are plotted in Fig. 4 and 5. The function $g_2^{SG}(x)$ given by $I_{\underline{\varphi} 2}(\underline{\theta})$ as defined in (B.4) follows from (B.7)

$$g_2^{SG}(x) = -32 \frac{\pi \sqrt{Z^{\varphi}}}{\sin \pi\mu} \frac{1}{1 - x}. \tag{19}$$

3.3. The Z_3-model

The model we consider is the Z_N-symmetric CFT perturbed by the thermal operator

$$S = S[Z_N] + \lambda \int d\tau dx \epsilon(\tau, x) \tag{20}$$

[4] As usual, in the context of the sine-Gordon model, the normalization of the field is given by $\langle 0 | \varphi(0) | p \rangle = \sqrt{Z^{\varphi}}$ (see Appendix C.2).

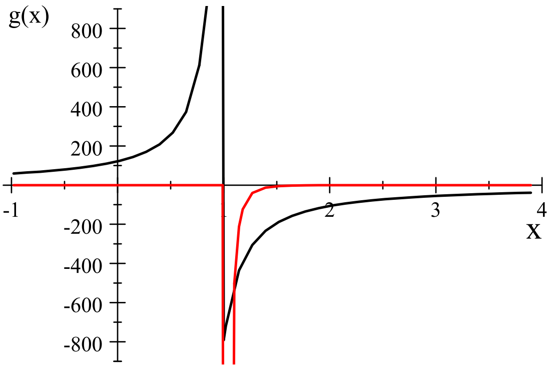

Fig. 5. Plot of $\mathrm{Re}\, g_1^{SG}(x)$ (black) and $\mathrm{Im}\, g_1^{SG}(x)$ (red) for $\mu = 0.5$. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

for a particular value $N = 3$. Such model appears as the continuum limit of the lattice model describing the integrable anti-ferromagnetic chain of spins $S = N/2$ in an applied magnetic field [18]

$$H = \sum_n \Big[J P_N(\mathbf{S}_n \mathbf{S}_{n+1}) + H S_n^z \Big], \tag{21}$$

where $P_N(x)$ is the polynomial of the N-th degree [19,20]. The continuum limit of this model at $H = 0$ is the $SU_N(2)$ Wess–Zumino–Novikov–Witten (WZNW) model perturbed by the irrelevant operator

$$H = W[SU_N(2)] + \eta \int dx\, J^a \bar{J}^b \Phi^{ab}_{adj},$$

where Φ_{adj} is the primary field in the adjoint representation and $J, \bar{J}$ are the holomorphic and antiholomorphic currents of the $su_N(2)$ Kac–Moody algebra. Whence the magnetic field is applied along the z-axis the z-components of the currents acquire finite expectation values

$$\langle J^z \rangle = \langle \bar{J}^z \rangle = \frac{1}{2} \chi H,$$

where $\chi \sim 1/J$ is the uniform magnetic susceptibility, and the irrelevant operator becomes relevant [21]

$$J^a \bar{J}^b \Phi^{ab}_{adj} \to \frac{1}{4} (\chi H)^2 \Phi^{zz}_{adj}.$$

The conformal embedding $SU_N(2) = U(1) \times Z_N$ establishes the equivalence between the diagonal component of the adjoint primary field and the thermal operator ϵ and hence the equivalence between the massive sector of model (21) and model (20).

The Z_3 CFT and the exact solution of the massive theory (20) for $N = 3$ suggest that in the disordered phase there are 2 types of particles 1 and 2 and two corresponding fields (order parameters $\sigma_1,\ \sigma_2 = \sigma_1^*$) with

$$\langle 0 \,|\, \sigma_1(x) \,|\, p \rangle_1 = 1\,, \quad \langle 0 \,|\, \sigma_2(x) \,|\, p \rangle_2 = 1\,.$$

where the indices correspond to the emission of particle 1 and 2 (the latter is a bound state of two 1-particles and simultaneously the anti-particle of particle 1).

The two-particle S-matrix for the Z_3-Potts model perturbed by the thermal operator has been proposed by Köberle and Swieca [22]. It coincides with the one derived from the Bethe ansatz solution of model (21) [18]. The scattering matrix of two particles of type 1 is

$$S^{Z_3}(\theta) = \frac{\sinh\frac{1}{2}(\theta + \frac{2}{3}i\pi)}{\sinh\frac{1}{2}(\theta - \frac{2}{3}i\pi)} .$$

This S-matrix is consistent with the picture that the bound state of two particles of type 1 is the particle 2 which is the anti-particle of 1.

The form factors of the Z_N-model (20) have been proposed in [23,24,12]. The minimal solution of the Watson's and the crossing equations

$$F(\theta) = F(-\theta)S(\theta), \quad F(i\pi - \theta) = F(i\pi + \theta)$$

for the $Z(3)$ model is

$$\begin{aligned} F^{Z3}(i\pi x) &= \sin\tfrac{1}{2}\pi x \, \exp\int_0^\infty \left(\frac{\sinh\frac{1}{3}t}{t\sinh^2 t}\,(1-\cosh t\,(1-x))\right)dt \\ &= \sin\tfrac{1}{2}\pi x \, \frac{G\left(\frac{1}{3}+\frac{1}{2}x\right)G\left(\frac{4}{3}-\frac{1}{2}x\right)}{G\left(\frac{2}{3}+\frac{1}{2}x\right)G\left(\frac{5}{3}-\frac{1}{2}x\right)} \end{aligned}$$

where $G(x)$ is the Barnes G-function [25] with the defining relation

$$G(x+1) = G(x)\,\Gamma(x) .$$

The form factor of the order parameter field σ_1 and two particles of type 2 is

$$F_{22}^{\sigma_1}(\underline{\theta}) = \langle 0 \,|\, \sigma_1(0) \,|\, p_1, p_2 \rangle_{22} = c_2 \frac{F(\theta_{12})}{\sinh\frac{1}{2}(\theta_{12} - \frac{2}{3}i\pi)\sinh\frac{1}{2}\left(\theta_{12} + \frac{2}{3}i\pi\right)} \tag{22}$$

and for the 3 particles of type 1, 1 and 2

$$\begin{aligned} F_{112}^{\sigma_1}(\underline{\theta}) &= \langle 0 \,|\, \sigma_1(0) \,|\, p_1, p_2, p_3 \rangle_{112} \\ &= c_3 \frac{F(\theta_{12})\cosh\frac{1}{2}\theta_{12}}{\sinh\frac{1}{2}(\theta_{12} - \frac{2}{3}i\pi)\sinh\frac{1}{2}\left(\theta_{12} + \frac{2}{3}i\pi\right)} \prod_{i=1}^{2} \frac{F_{12}^{\min}(\theta_{i3})}{\cosh\frac{1}{2}\theta_{i3}} \end{aligned} \tag{23}$$

where

$$\begin{aligned} F_{(12)}^{\min}(i\pi x) &= c\exp\int_0^\infty \frac{dt}{t\sinh^2 t}\sinh\frac{2}{3}t\,\left(1-\cosh t\,(1-x)\right) \\ &= \frac{G\left(\frac{1}{6}+\frac{1}{2}x\right)G\left(\frac{7}{6}-\frac{1}{2}x\right)}{G\left(\frac{5}{6}+\frac{1}{2}x\right)G\left(\frac{11}{6}-\frac{1}{2}x\right)} \end{aligned}$$

is the minimal form factor of the particles 1 and 2.

3.3.1. The 3-point function

We consider the Green's function $\tau_{\sigma_1\sigma_1\sigma_1}(\underline{x}) = \langle 0 | T\sigma_1(x_1)\sigma_1(x_2)\sigma_1(x_3) | 0 \rangle$, this 3-point function was also investigated in [8]. As in (10) we have the simple contribution

$$\begin{aligned}\tilde{\Xi}^{11}_{\sigma_1\sigma_1\sigma_1}(k_1,k_2,k_3) &= \int_{p_1}\int_{p_2} \langle 0|\sigma_1(0)|p_1\rangle\langle p_1|\sigma_1(0)|\bar{p}_2\rangle\langle\bar{p}_2|\sigma_1(0)|0\rangle \\ &\times 2\pi\delta(p_1)\, 2\pi\delta(\bar{p}_2)\, \frac{-i}{k_2^0+k_3^0+\omega_1-i\epsilon}\,\frac{-i}{k_3^0+\omega_2-i\epsilon} \\ &= \frac{-1}{4m^4}\,\frac{m}{-k_1^0+m-i\epsilon}\,\frac{m}{k_3^0+m-i\epsilon}\,F^{\sigma_1}_{22}(i\pi,0)\end{aligned}$$

and as in (8) we calculate for the intermediate states
$\langle 0|\sigma_1(0)|p_1\rangle\langle p_1|\sigma_1(0)|p_2,p_3\rangle\langle p_3,p_2|\sigma_1(0)|0\rangle$ (for details see Appendix C.3)

$$\begin{aligned}\tilde{\Xi}^{12}_{\sigma_1\sigma_1\sigma_1}(k_1,k_2,k_3) &= -\frac{1}{64\pi m^4}\,\frac{m}{-k_1^0+m-i\epsilon}\int d\theta\,\frac{m}{\omega}\,\frac{2m}{k_3^0+2\omega-i\epsilon}\, I^{12}_{\sigma_1\sigma_1\sigma_1}(\theta) \\ I^{12}_{\sigma_1\sigma_1\sigma_1}(\theta) &= F^{\sigma_1}_{211}(i\pi,\theta,-\theta)\,F^{\sigma_1}_{22}(-\theta+i\pi,\theta+i\pi)\end{aligned}$$

where we have used the crossing relation

$$\langle p_1|\sigma_1(0)|p_2,p_3\rangle = F^{\sigma_1}_{211}(\theta_1+i\pi,\theta_2,\theta_3)+\delta_{\theta_1\theta_2}+\delta_{\theta_1\theta_3}S(\theta_{23}).$$

The δ-function terms do not contribute because $F^{\sigma_1}_{22}(0,0)=0$. Inserting the form factor functions we get (up to constant factors)

$$\begin{aligned}\tilde{\Xi}^{12}_{\sigma_1\sigma_1\sigma_1}(k_1,k_2,k_3) &= -\frac{1}{64\pi m^4}\,\frac{m}{-k_1^0+m-i\epsilon}\,h^{Z3}\left(-\frac{k_3^0}{2m}+i\epsilon\right) \\ h^{Z3}(x) &= \int_{-\infty}^{\infty} d\theta\,\frac{1}{\cosh\theta}\,\frac{1}{\cosh\theta-x}\,\frac{F(2\theta)F(-2\theta)F^{\min}_{(12)}(\theta+i\pi)F^{\min}_{(12)}(-\theta+i\pi)}{\left(\sinh(\theta-\frac{1}{3}i\pi)\,\sinh(\theta+\frac{1}{3}i\pi)\,\sinh\frac{1}{2}\theta\right)^2}.\end{aligned}$$

For the intermediate states $\langle 0|\sigma_1(0)|\bar{p}_1,\bar{p}_2\rangle\langle\bar{p}_2,\bar{p}_1|\sigma_1(0)|\bar{p}_3\rangle\langle\bar{p}_3|\sigma_1(0)|0\rangle$ we get

$$\tilde{\Xi}^{\bar{2}\bar{1}}_{\sigma_1\sigma_1\sigma_1}(k_1,k_2,k_3) = -\frac{1}{64\pi m^4}\,\frac{m}{k_3^0+m-i\epsilon}\,h^{Z3}\left(\frac{k_1^0}{2m}+i\epsilon\right)$$

and as in (12)

$$\begin{aligned}\tilde{\Xi}_{\sigma_1\sigma_1\sigma_1}(k_1,k_2,k_3) &= const.\sum_{perm(k)}\frac{m}{-k_1^0+m-i\epsilon}\,\frac{m}{k_3^0+m-i\epsilon} \\ &+ const.'\sum_{perm(k)}\frac{m}{-k_1^0+m-i\epsilon}\,h^{Z(3)}\left(-\frac{k_3^0}{2m}+i\epsilon\right)+(k_i\to -k_i)\ .\end{aligned}$$

As expected, there is a threshold singularity at $k^0=2m$.

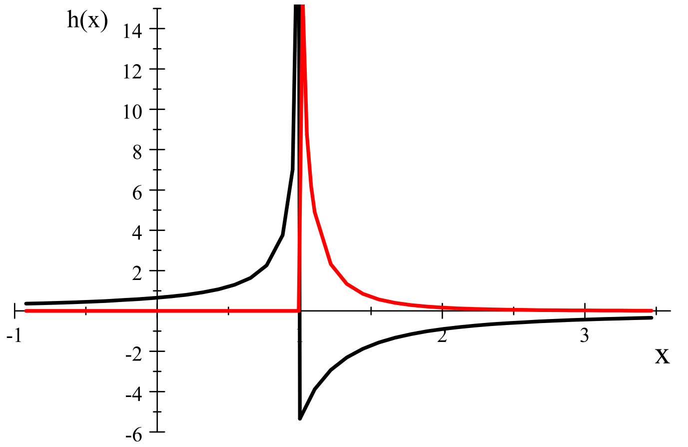

Fig. 6. Plot of $\mathrm{Re}\, h^{Z3}(x)$ (black) and $\mathrm{Im}\, h^{Z3}(x)$ (red) for the Z_3 model. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

3.3.2. The 4-point function

Due to (B.4) there are contributions to the four point Green's function from I_1, I_2 and I_3. The one from I_3 belongs to the disconnected part and the one from I_2 is trivial as in (19) and (B.7). We restrict here to the contribution from I_1. We consider the Green's function $\tau_{\sigma_1\sigma_2\sigma_1\sigma_2}(\underline{x}) = \langle 0 \,|\, T\sigma_1(x_1)\sigma_2(x_2)\sigma_1(x_3)\sigma_2(x_4) \,|\, 0 \rangle$ and as in (13) we obtain (for details see Appendix C.3)

$$\begin{aligned}\tilde{\Xi}_{\sigma_1\sigma_2\sigma_3\sigma_4}(\underline{k}) &= \sum_{\pi\in S_4} \tilde{\Xi}^{121}_{\sigma_{\pi 1}\sigma_{\pi 2}\sigma_{\pi 3}\sigma_{\pi 4}}(k_{\pi 1}, k_{\pi 2}, k_{\pi 3}, k_{\pi 4}), \quad (\sigma_3 = \sigma_1, \sigma_4 = \sigma_2) \\ &= -\frac{1}{32}\frac{i}{m^6\pi} \sum_{\pi\in S_4} \frac{m}{-k^0_{\pi 1} + m - i\epsilon} \frac{m}{k^0_{\pi 4} + m - i\epsilon} g^{Z3}_\pi\left(-\frac{k^0_{\pi 3} + k^0_{\pi 4}}{2m} + i\epsilon\right).\end{aligned}$$

Obviously, if $\sigma_3 = \sigma_1$ and $\sigma_4 = \sigma_2$ there are three functions $g^{Z3}_\pi(x)$

$$g^{Z3}_\pi(x) = \begin{cases} g^{Z3}_I(x) & \text{if } \sigma_{\pi 1}\sigma_{\pi 2}\sigma_{\pi 3}\sigma_{\pi 4} = \sigma_1\sigma_2\sigma_1\sigma_2 \text{ or } (\sigma_1 \leftrightarrow \sigma_2) \\ g^{Z3}_{II}(x) & \text{if } \sigma_{\pi 1}\sigma_{\pi 2}\sigma_{\pi 3}\sigma_{\pi 4} = \sigma_1\sigma_1\sigma_2\sigma_2 \text{ or } (\sigma_1 \leftrightarrow \sigma_2) \\ g^{Z3}_{III}(x) & \text{if } \sigma_{\pi 1}\sigma_{\pi 2}\sigma_{\pi 3}\sigma_{\pi 4} = \sigma_1\sigma_2\sigma_2\sigma_1 \text{ or } (\sigma_1 \leftrightarrow \sigma_2)\,. \end{cases}$$

It turns out that $g^{Z3}_{III}(x) = g^{Z3}_I(x)$. For plots of the functions $g^{Z3}_I(x)$ and $g^{Z3}_{II}(x)$ see Figs. 7 and 8.

4. Conclusion

In this paper we develop a technique to calculate multipoint Wightman or Green's functions in integrable quantum field theories in 1+1 dimension. We insert intermediate states between the fields and use the crossing formula to write the Wightman function in terms of form factors in a model independent way. We expect good approximations for low number of particles in the intermediate states. In the present article we demonstrate this technique explicitly for 3- and 4-point functions of simple models with no backward scattering: the scaling Z_2 Ising, the scaling Z_3 Potts and the sinh-Gordon model. The results can be applied to physical phenomena, for example to Raman scattering [10]. In a forthcoming article we will generalize the technique to models with backward scattering, as the $O(N)$ σ- and the $O(N)$ Gross–Neveu model.

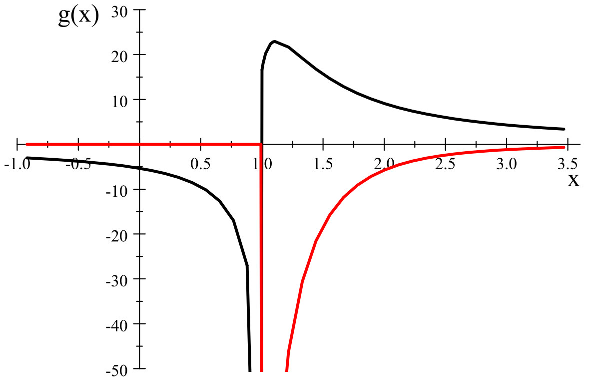

Fig. 7. Plot of $\mathrm{Re}\, g_I^{Z3}(x)$ (black) and $\mathrm{Im}\, g_I^{Z3}(x)$ (red) for the Z_3 model. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

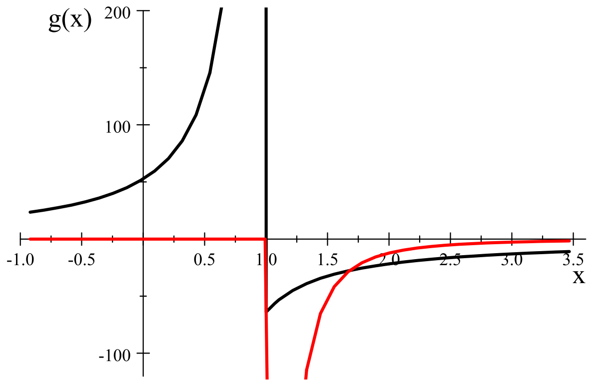

Fig. 8. Plot of $\mathrm{Re}\, g_{II}^{Z3}(x)$ (black) and $\mathrm{Im}\, g_{II}^{Z3}(x)$ (red) for the Z_3 model. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

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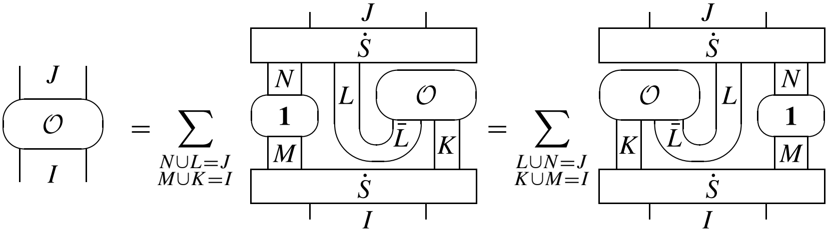
Appendix A. Crossing

The general crossing formula (31) in [9]

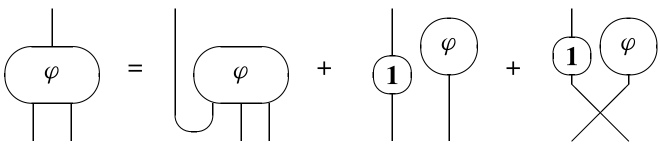
$$
\begin{aligned}
&F^{\mathcal{O}\,J}_{\;\;\,I}(\underline{\theta}'_J;\underline{\theta}_I) \\
&= \sigma_{\mathcal{O}J} \sum_{\substack{L\cup N=J\\ K\cup M=I}} \dot{S}^J_{NL}(\underline{\theta}'_N,\underline{\theta}'_L)\,\mathbf{1}^N_M(\underline{\theta}'_N,\underline{\theta}_M)\,\mathbf{C}^{L\bar{L}}\,F^{\mathcal{O}}_{\bar{L}K}(\underline{\theta}'_{\bar{L}}+i\pi_-,\underline{\theta}_K)\,\dot{S}^{MK}_I(\underline{\theta}_I)
\end{aligned}
$$

$$= \sum_{\substack{L\cup N=J \\ K\cup M=I}} \dot{S}^J_{LN}(\underline{\theta}'_L, \underline{\theta}'_N)\, F^{\mathcal{O}}_{K\bar{L}}(\underline{\theta}_K, \underline{\theta}'_{\bar{L}} - i\pi_-) \mathbf{C}^{\bar{L}L}\, \mathbf{1}^N_M(\underline{\theta}'_N, \underline{\theta}_M)\, \dot{S}^{KM}_I(\underline{\theta}_I)$$



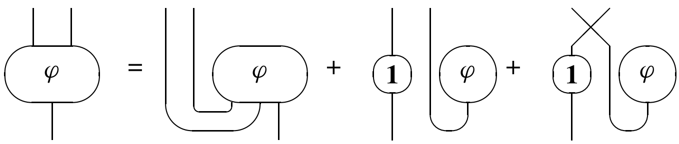
For a scalar bosonic field $\varphi(x)$ the matrix element $\langle \theta_1 \,|\, \varphi(0) \,|\, \theta_2, \theta_3 \rangle$ is

$$F^{\varphi}(\theta_1; \theta_2, \theta_3) = F^{\varphi}(\theta_1 + i\pi_-, \theta_2, \theta_3) + \delta_{\theta_{12}} + \delta_{\theta_{13}} S(\theta_{23}) \tag{A.1}$$



and for $\langle \theta_3, \theta_2 \,|\, \varphi(0) \,|\, \theta_4 \rangle$ we have

$$F^{\varphi}(\theta_2, \theta_3; \theta_4) = F^{\varphi}(\theta_3 + i\pi_-, \theta_2 + i\pi_-, \theta_4) + \delta_{\theta_{24}} + S(\theta_{32})\delta_{\theta_{34}}$$



with $i\pi_\pm = i\pi \pm i\epsilon$ and $\delta_{\theta_{12}} = 4\pi\delta(\theta_1 - \theta_2)$. Using the form factor equation (iii) and Lorentz invariance (see e.g. [12])

$$\operatorname*{Res}_{\theta_{12}=i\pi} F(\theta_1, \theta_2, \theta_3) = 2i\, (\mathbf{1} - S(\theta_{23}))$$

$$F(\theta_1, \theta_2, \theta_3) = F(\theta_1 + \mu, \theta_2 + \mu, \theta_3 + \mu)$$

we can rewrite these equations as (6) and (7). And further one derives

$$\begin{aligned} &\tfrac{1}{2} F(\theta_1, \theta_2 - i\pi_-, \theta_3 - i\pi_+) + \delta_{\theta_{12}} + \delta_{\theta_{13}} \\ &= \tfrac{1}{2}\left(F(\theta_1, \theta_2 - i\pi_+, \theta_3 - i\pi_-) + \delta_{\theta_{12}}\,(1 + S(\theta_{23})) + \delta_{\theta_{13}}\,(1 + S(\theta_{23}))\right) \end{aligned} \tag{A.2}$$

$$\begin{aligned} &\tfrac{1}{2} F(\theta_3 + i\pi_+, \theta_2 + i\pi_-, \theta_4) + \delta_{\theta_{24}} + \delta_{\theta_{34}} \\ &= \tfrac{1}{2}\left(F(\theta_3 + i\pi_-, \theta_2 + i\pi_+, \theta_4) + \delta_{\theta_{24}}\,(1 + S(\theta_{32})) + \delta_{\theta_{34}}\,(1 + S(\theta_{32}))\right)\,. \end{aligned} \tag{A.3}$$

Appendix B. Green's function and intermediate states in low particle approximation

Let $\varphi(x)$ a scalar charge-less bosonic field with the normalization $\langle 0 \,|\, \varphi(x) \,|\, \theta \rangle = 1$.

Simple examples of Wigthman and Green's functions

w1: The 2-point Wigthman function in 1-intermediate particle approximation is

$$w^1(x_1-x_2)=\int_\theta \langle 0\,|\,\varphi(x_1)\,|\,\theta\,\rangle\langle\,\theta\,|\,\varphi(x_2)\,|\,0\rangle = i\Delta_+(x_1-x_2)$$

and the Green's function in this approximation is the free Feynman propagator

$$\tau^1(x)=\Theta(t)w^1(x)+\Theta(-t)w^1(-x)=\Delta_F(x_1-x_2)=\int_p e^{-ixp}\frac{i}{p^2-m^2+i\epsilon}$$

w101: The 4-point Wightman function in 1-0-1-intermediate particle approximation is (with $\int_\theta=\frac{1}{4\pi}\int d\theta$)

$$\begin{aligned} w^{101}(\underline{x}) &= \int_{\theta_1}\langle 0\,|\,\varphi(x_1)\,|\,\theta_1\,\rangle\langle\,\theta_1\,|\,\varphi(x_2)\,|\,0\rangle \int_{\theta_4}\langle 0\,|\,\varphi(x_3)\,|\,\theta_4\rangle\langle\,\theta_4\,|\,\varphi(x_4)\,|\,0\,\rangle \\ &= w^1(x_1-x_2)w^1(x_3-x_4) \end{aligned} \tag{B.1}$$

which implies that also

$$\tau^{101}(\underline{x})=\tau^1(x_1-x_2)\tau^1(x_3-x_4).$$

w121: The 4-point Wightman function in 1-2-1-intermediate particle approximation is (with $\int_{\underline{\theta}}=\int_{\theta_1}\cdots\int_{\theta_3}$)

$$\begin{aligned} w^{121}(\underline{x}) &= \frac{1}{2}\int_{\underline{\theta}}\langle 0\,|\,\varphi(x_1)\,|\,\theta_1\,\rangle\,\langle\,\theta_1\,|\,\varphi(x_2)\,|\,\theta_2,\theta_3\rangle\,\langle\,\theta_3,\theta_2\,|\,\varphi(x_3)\,|\,\theta_4\rangle\,\langle\,\theta_4\,|\,\varphi(x_4)\,|\,0\,\rangle \\ &= \frac{1}{2}\int_{\underline{\theta}} e^{-ix_1p_1-ix_2(p_2+p_3-p_1)-ix_3(p_4-p_2-p_3)+ix_4p_4}F(\theta_1;\theta_2,\theta_3)F(\theta_2,\theta_3;\theta_4). \end{aligned} \tag{B.2}$$

Using equations (6), (7) and the identity

$(a+b+c)\,(d+e+f)=\left(\frac{1}{2}a+b+c\right)d+a\left(\frac{1}{2}d+e+f\right)+(b+c)\,(e+f)$ we derive

$$\begin{aligned} & F(\theta_1;\theta_2,\theta_3)F(\theta_2,\theta_3;\theta_4) \\ &= \left(F(\theta_1,\theta_2-i\pi_-,\theta_3-i\pi_+)+\delta_{\theta_{12}}+\delta_{\theta_{13}}\right) \\ &\times\left(F(\theta_3+i\pi_+,\theta_2+i\pi_-,\theta_4)+\delta_{\theta_{24}}+\delta_{\theta_{34}}\right) \\ &= \left(\tfrac{1}{2}F(\theta_1,\theta_2-i\pi_-,\theta_3-i\pi_+)+\delta_{\theta_{12}}+\delta_{\theta_{13}}\right)F(\theta_3+i\pi_+,\theta_2+i\pi_-,\theta_4) \\ &+ F(\theta_1,\theta_2-i\pi_-,\theta_3-i\pi_+)\left(\tfrac{1}{2}F(\theta_3+i\pi_+,\theta_2+i\pi_-,\theta_4)+\delta_{\theta_{24}}+\delta_{\theta_{34}}\right) \\ &+ \left(\delta_{\theta_1\theta_2}+\delta_{\theta_1\theta_3}\right)\left(\delta_{\theta_4\theta_2}+\delta_{\theta_4\theta_3}\right) \end{aligned}$$

which is using (A.2) and (A.3) equal to

$$\begin{aligned} &= \tfrac{1}{2}\left(F(\theta_1,\theta_2-i\pi_+,\theta_3-i\pi_-)+\delta_{\theta_{12}}\,(1+S(\theta_{23}))+\delta_{\theta_{13}}\,(1+S(\theta_{23}))\right) \\ &\times F(\theta_3+i\pi_+,\theta_2+i\pi_-,\theta_4) \\ &+ F(\theta_1,\theta_2-i\pi_-,\theta_3-i\pi_+) \end{aligned}$$

$$\times \tfrac{1}{2}\left(F(\theta_3+i\pi_-,\theta_2+i\pi_+,\theta_4)+\delta_{\theta_{24}}(1+S(\theta_{32}))+\delta_{\theta_{34}}(1+S(\theta_{32}))\right)$$
$$+\left(\delta_{\theta_1\theta_2}+\delta_{\theta_1\theta_3}\right)\left(\delta_{\theta_4\theta_2}+\delta_{\theta_4\theta_3}\right)$$
$$=\tfrac{1}{2}F(\theta_1,\theta_2-i\pi_+,\theta_3-i\pi_-)F(\theta_3+i\pi_+,\theta_2+i\pi_-,\theta_4) \tag{B.3}$$
$$+F(\theta_1,\theta_2-i\pi_-,\theta_3-i\pi_+)\tfrac{1}{2}F(\theta_3+i\pi_-,\theta_2+i\pi_+,\theta_4)$$
$$+\tfrac{1}{2}\left(\delta_{\theta_{12}}(1+S(\theta_{23}))+\delta_{\theta_{13}}(1+S(\theta_{23}))\right)F(\theta_3+i\pi_+,\theta_2+i\pi_-,\theta_4)$$
$$+F(\theta_1,\theta_2-i\pi_-,\theta_3-i\pi_+)\tfrac{1}{2}\left(\delta_{\theta_{24}}(1+S(\theta_{32}))+\delta_{\theta_{34}}(1+S(\theta_{32}))\right)$$
$$+\left(\delta_{\theta_1\theta_2}+\delta_{\theta_1\theta_3}\right)\left(\delta_{\theta_4\theta_2}+\delta_{\theta_4\theta_3}\right)$$
$$=I_1(\underline{\theta})+I_2(\underline{\theta})+I_3(\underline{\theta})$$

where we have introduced

$$\begin{aligned}
I_1(\underline{\theta}) &= \tfrac{1}{2}F(\theta_1,\theta_2-i\pi_+,\theta_3-i\pi_-)F(\theta_3+i\pi_+,\theta_2+i\pi_-,\theta_4)\\
&\quad+\tfrac{1}{2}F(\theta_1,\theta_2-i\pi_-,\theta_3-i\pi_+)F(\theta_3+i\pi_-,\theta_2+i\pi_+,\theta_4)\\
I_2(\underline{\theta}) &= \tfrac{1}{2}\left(\delta_{\theta_{12}}(1+S(\theta_{23}))+\delta_{\theta_{13}}(1+S(\theta_{23}))\right)F(\theta_3+i\pi_+,\theta_2+i\pi_-,\theta_4)\\
&\quad+\tfrac{1}{2}F(\theta_1,\theta_2-i\pi_-,\theta_3-i\pi_+)\left(\delta_{\theta_{24}}(1+S(\theta_{32}))+\delta_{\theta_{34}}(1+S(\theta_{32}))\right)\\
I_3(\underline{\theta}) &= \left(\delta_{\theta_{12}}+\delta_{\theta_{13}}\right)\left(\delta_{\theta_{24}}+\delta_{\theta_{34}}\right)
\end{aligned} \tag{B.4}$$

From I_3 we calculate

$$w_3^{121}(\underline{x})=w^1(x_1-x_4)\,w^1(x_2-x_3)+w^1(x_1-x_3)\,w^1(x_2-x_4) \tag{B.5}$$

which together with (B.1) yields the disconnected part of $\tilde{\tau}(\underline{k})=\tilde{\tau}_{disc}(\underline{k})+\tilde{\tau}_c(\underline{k})$. This means the connected part of the Green's function is given by $I_1(\underline{\theta})+I_2(\underline{\theta})$.

General case We start with (1), insert sets of intermediate states and write $y_i=x_{\pi i}$

$$\tilde{\tau}_{\underline{\varphi}}(\underline{k})=\sum_{\pi\in S_n}\sum\frac{1}{\underline{n}!}\int \underline{d^2y}e^{iy_ik_{\pi i}}\Theta_{1\dots n}(\underline{y})$$
$$\times\int\limits_{\underline{p^{(1)}}}\dots\int\limits_{\underline{p^{(n-1)}}}\langle 0|\varphi_{\pi 1}(y_1)|\underline{p^{(1)}}\rangle\langle\underline{p^{(1)}}|\dots|\underline{p^{(n-1)}}\rangle\langle\underline{p^{(n-1)}}|\varphi_{\pi n}(y_n)|0\rangle$$

with the notation of (4). We perform the y-integrations and obtain for $\tilde{\Xi}$ defined in (3)

$$\tilde{\Xi}_{\underline{\varphi}}(\underline{k})=\sum_{\pi\in S_n}\sum\frac{1}{\underline{n}!}\int\limits_{\underline{p^{(1)}}}\dots\int\limits_{\underline{p^{(n-1)}}}\langle 0|\varphi_{\pi 1}(0)|\underline{p^{(1)}}\rangle\langle\underline{p^{(1)}}|\dots|\underline{p^{(n-1)}}\rangle\langle\underline{p^{(n-1)}}|\varphi_{\pi n}(0)|0\rangle$$
$$\times 2\pi\delta\left(k_{\pi 2}-\sum p_j^{(2)}+\sum p_j^{(1)}\right)^1\dots 2\pi\delta\left(k_{\pi n}+\sum p_j^{(n-1)}\right)^1$$
$$\times\frac{-i}{\sum_{i=2}^n k_{\pi i}^0+\sum\omega_j^{(1)}-i\epsilon}\,\frac{-i}{\sum_{i=3}^n k_{\pi i}^0+\sum\omega_j^{(2)}-i\epsilon}\cdots\frac{-i}{k_{\pi n}^0+\sum\omega_j^{(n-1)}-i\epsilon}$$

because

$$\int\underline{dy^1}e^{-iy_i^1\left(k_{\pi i}-\sum p_j^{(i)}+\sum p_j^{(i-1)}\right)^1}$$
$$=2\pi\delta\left(\sum k_i^1\right)2\pi\delta\left(k_{\pi 2}-\sum p_j^{(2)}+\sum p_j^{(1)}\right)^1\dots 2\pi\delta\left(k_{\pi n}+\sum p_j^{(n-1)}\right)^1$$

and

$$\int \underline{dy}^0 \Theta_{1\dots n}(\underline{y}) e^{iy_i^0\left(k_{\pi i} - \sum p_j^{(i)} + \sum p_j^{(i-1)}\right)^0} = 2\pi\delta\left(\sum k_i^0\right)$$
$$\times \frac{-i}{k_{\pi 2}^0 + \dots + k_{\pi n}^0 + \sum \omega_j^{(1)} - i\epsilon} \frac{-i}{k_{\pi 3}^0 + \dots + k_{\pi n}^0 + \sum \omega_j^{(2)} - i\epsilon} \cdots \frac{-i}{k_{\pi n}^0 + \sum \omega_j^{(n-1)} - i\epsilon}$$

which proves (4). For $k_i^1 \to 0$ the result is

$$\tilde{\Xi}_{\underline{\varphi}}(\underline{k}) = \sum_{\pi \in S_n} \sum \frac{1}{\underline{n}!} \int_{\underline{p}^{(1)}} \dots \int_{\underline{p}^{(n-1)}} \langle \varphi_{\pi 1}(0) | \underline{p}^{(1)} \rangle \langle \underline{p}^{(1)} | \dots | \underline{p}^{(n-1)} \rangle \langle \underline{p}^{(n-1)} | \varphi_{\pi n}(0) \rangle \tag{B.6}$$
$$\times 2\pi\delta\left(\sum p_j^{(1)}\right)^1 \dots 2\pi\delta\left(\sum p_j^{(n-1)}\right)^1$$
$$\times \frac{-i}{\sum_{i=2}^n k_{\pi i}^0 + \sum \omega_j^{(1)} - i\epsilon} \frac{-i}{\sum_{i=3}^n k_{\pi i}^0 + \sum \omega_j^{(2)} - i\epsilon} \cdots \frac{-i}{k_{\pi n}^0 + \sum \omega_j^{(n-1)} - i\epsilon}$$

Example: the 4-point function Let $\underline{\varphi} = \varphi\varphi\varphi\varphi$. We use the intermediate states $\langle 0|\varphi(0)|p_1\rangle\langle p_1|\varphi(0)|p_2, p_3\rangle\langle p_3, \overline{p_2}|\varphi(0)|p_4\rangle\langle p_4|\varphi(0)|0\rangle$ and as in (B.2) we obtain in this approximation

$$\tilde{\Xi}_{\underline{\varphi}}(\underline{k}) = \sum_{\pi \in S_n} \int_{p_1} \int_{p_2} \int_{p_3} \int_{p_4} 2\pi\delta(p_1) 2\pi\delta(p_2 + p_3) 2\pi\delta(p_4)$$
$$\times \frac{1}{2!} \langle 0|\varphi(0)|p_1\rangle\langle p_1|\varphi(0)|p_2, p_3\rangle\langle p_3, p_2|\varphi(0)|p_4\rangle\langle p_4|\varphi(0)|0\rangle$$
$$\times \frac{-i}{-k_{\pi 1}^0 + \omega_1 - i\epsilon} \frac{-i}{k_{\pi 3}^0 + k_{\pi 4}^0 + \omega_2 + \omega_3 - i\epsilon} \frac{-i}{k_{\pi 4}^0 + \omega_4 - i\epsilon}$$
$$= -\frac{1}{32} \frac{i}{m^6 \pi} \sum_{perm(k)} \frac{m}{-k_1^0 + m - i\epsilon} \frac{m}{k_4^0 + m - i\epsilon} g\left(-\left(k_3^0 + k_4^0\right)/(2m) + i\epsilon\right)$$
$$g(x) = \frac{-1}{4} \int d\theta \frac{1}{\cosh\theta} \frac{1}{\cosh\theta - x} F^\varphi(0; \theta, -\theta) F^\varphi(\theta, -\theta; 0)$$

which is (13) and $F(0; \theta, -\theta)F(\theta, -\theta; 0)$ is given by (see (B.4))

$$F(\theta_1; \theta_2, \theta_3)F(\theta_2, \theta_3; \theta_4) = I_1(\underline{\theta}) + I_2(\underline{\theta}) + I_3(\underline{\theta}).$$

As mentioned in the context of (B.5) the connected part of $\tilde{\Xi}_{\underline{\varphi}}(\underline{k})$ is obtained by

$$g_1(x) + g_2(x) = \frac{-1}{4} \int d\theta \frac{1}{\cosh\theta} \frac{1}{\cosh\theta - x} \left(I_1(\underline{\theta}) + I_2(\underline{\theta})\right).$$

The contribution from I_1 will be calculated for the $Z(2)$-, the $Z(3)$-scaling Ising and the sinh-Gordon models. The contribution from I_2 leads to $0/0$, therefore the limit $k_i^1 \to 0$ has to be taken more carefully.

Contribution of I_2: For $k_i = (k_i^0, m\sinh\kappa_i)$ this is equal to

$$\tilde{\Xi}_{\varphi 2}(\underline{k}) = \frac{1}{2}\sum_{perm(k)}\frac{1}{(2m)^2}\int_{p_2}\int_{p_3}(2\pi)\,\delta(k_1^1+k_2^1-p_2^1-p_3^1)I_2(\kappa_1,\theta_2,\theta_3,-\kappa_4)$$
$$\times\frac{-i}{-k_1^0+\omega_1-i\epsilon}\frac{-i}{k_3^0+k_4^0+\omega_2+\omega_3-i\epsilon}\frac{-i}{k_4^0+\omega_4-i\epsilon}$$

with $I_2(\kappa_1,\theta_2,\theta_3,-\kappa_4)$ given by (B.4). Taking first the term with $\delta_{\kappa_1\theta_2}$ we get (because $\theta_3\to\kappa_2$)

$$\frac{1}{2}\frac{1}{(2m)^2}\sum_{perm(k)}\frac{1}{2m\cosh\kappa_2}\tfrac{1}{2}\left(1+S(\kappa_{12})\right)F^{\varphi}(\kappa_2+i\pi_+,\kappa_1+i\pi_-,-\kappa_4)$$
$$\times\frac{-i}{-k_1^0+m\cosh\kappa_1}\frac{-i}{k_3^0+k_4^0+m\cosh\kappa_1+m\cosh\kappa_2}\frac{-i}{k_4^0+m\cosh\kappa_4}.$$

We write

$$F^{\varphi}(\theta_1,\theta_2,\theta_3)=\prod_{i<j}\tanh\tfrac{1}{2}\theta_{ij}\,\tilde{F}^{\varphi}(\theta_1,\theta_2,\theta_3)$$

then for small κ_i (using $S(0)=-1$)

$$\tfrac{1}{2}\left(1+S(\kappa_{12})\right)F^{\varphi}(\kappa_2+i\pi_+,\kappa_1+i\pi_-,-\kappa_4)$$
$$\to\tfrac{1}{2}\kappa_{12}S'(0)\tanh\frac{1}{2}\kappa_{21}\coth\frac{1}{2}(\kappa_2+\kappa_4)\coth\frac{1}{2}(\kappa_1+\kappa_4)\,\tilde{F}^{\varphi}(i\pi,i\pi,0)$$
$$\to\frac{-\kappa_{21}^2}{(\kappa_2+\kappa_4)(\kappa_1+\kappa_4)}S'(0)\tilde{F}^{\varphi}(i\pi,i\pi,0)$$

Similarly we get the other contribution from I_2 and calculate

$$\frac{(\kappa_1-\kappa_2)^2}{(\kappa_2+\kappa_4)(\kappa_1+\kappa_4)}+\frac{(\kappa_1-\kappa_3)^2}{(\kappa_3+\kappa_4)(\kappa_1+\kappa_4)}$$
$$+\frac{(\kappa_3-\kappa_4)^2}{(\kappa_1+\kappa_4)(\kappa_1+\kappa_3)}+\frac{(\kappa_2-\kappa_4)^2}{(\kappa_1+\kappa_4)(\kappa_1+\kappa_2)}=-8$$

up to terms proportional to $\kappa_1+\kappa_2+\kappa_3+\kappa_4$, which do not contribute because of the δ-function $\delta\left(k_1^1+k_2^1+k_3^1+k_4^1\right)$. Therefore in the limit $\kappa_i\to 0$

$$\tilde{\Xi}_{\varphi 2}(\underline{k})=-\frac{1}{32}\frac{i}{m^6\pi}\sum_{perm(k)}\frac{m}{-k_1^0+m-i\epsilon}\frac{m}{k_4^0+m-i\epsilon}g_2\left(-\left(k_3^0+k_4^0\right)/(2m)+i\epsilon\right)$$
$$g_2(x)=-8\pi\,S'(0)\tilde{F}^{\varphi}(i\pi,i\pi,0)\frac{1}{1-x}. \tag{B.7}$$

Appendix C. Models

C.1. The scaling Z_2 Ising model

The Quantum Ising model is described by the Hamiltonian

$$H=\sum_n\Big(-J\sigma_n^z\sigma_{n+1}^z+h\sigma_n^x\Big), \tag{C.1}$$

where σ^a are the Pauli matrices. This model has numerous condensed matter realizations being one of the most popular models of condensed matter theory. It describes a sequence of coupled two level systems. They may represent spins; then the first term describes an anisotropic exchange interaction. In this case σ^z directly couples to external magnetic field: $\mu_B B_n^z \sigma_n^z$.

States of the two level systems may also correspond to positions of electric charges in a double well potential. Then the first term is the dipole–dipole interaction and the transverse field describes the quantum tunneling between the wells. Then σ^a would be the dipole moment operators. Their interaction with the electric field is given by $pE_n^z \sigma_n^z$ with p being the dipole moment.

Since the dominant interaction is ferromagnetic, the strongest fluctuations take place at zero wave vectors which guarantees a direct coupling to the electromagnetic field creating optimal resonance conditions. The Ising model (C.1) has two phases depending on the sign of $m = h - J$. The resonance occurs in the paramagnetic phase $m > 0$ when the ground state average of the order parameter $\langle \sigma^z \rangle = 0$. In that case the electromagnetic field has a nonzero matrix element between the ground state and single magnon state.

In the scaling limit model (C.1) can be described by an interacting Bose field $\sigma_n^z = Cm^{1/8}\sigma(x)$, where C is a numerical constant and $m = h - J$. The excitations are non-interacting Majorana fermions with the 2-particle S-matrix $S^{Z(2)}(\theta) = -1$. The field $\sigma^x = (m/J)^{1/2}\epsilon(x) \sim \bar{\psi}\psi(x)$, where ψ is a free Majorana spinor field. The n-particle form factors for the order parameter $\sigma(x)$ is given by (14). From $\epsilon(x) \sim \bar{\psi}\psi(x)$ one has for a free Majorana spinor field (up to a constant)

$$\langle 0|\epsilon(0)|\theta_1, \theta_2\rangle = \sinh \tfrac{1}{2}\theta_{12}\,. \tag{C.2}$$

C.1.1. The 3-point function

We calculate $\tilde{\Xi}_{\varphi\varphi\epsilon}(\underline{k})$ in the limit $k_i^1 \to 0$. For the various permutations in (4) we obtain:

a) For the permutation $\pi = 123$ and $n_1 = 1,\ n_2 = 2$

$$\tilde{\Xi}^{12}_{\varphi\varphi\epsilon}(k_1,k_2,k_3) = \frac{1}{2!}\int\limits_{p_1}\int\limits_{p_2}\int\limits_{p_3} 2\pi\delta(p_1)\,2\pi\delta(p_2+p_3)\,\langle 0|\varphi(0)|p_1\rangle$$
$$\times\langle p_1|\varphi(0)|p_2,p_3\rangle\langle p_3,p_2|\epsilon(0)|0\rangle \frac{-i}{-k_1^0+m-i\epsilon}\frac{-i}{k_3^0+\omega_2+\omega_3-i\epsilon}$$
$$= -\frac{1}{64m^4\pi}\frac{m}{-k_1^0+m-i\epsilon}\int d\theta\,\frac{2mF^{\varphi}(i\pi,\theta,-\theta)F^{\epsilon}(-\theta+i\pi,\theta+i\pi)}{k_3^0+2\omega-i\epsilon}$$

which is (8). Equations (14) and (C.2) imply

$$F^{\varphi}(i\pi,\theta,-\theta)F^{\epsilon}(-\theta+i\pi,\theta+i\pi)$$
$$= 2i\tanh\tfrac{1}{2}(i\pi-\theta)\tanh\tfrac{1}{2}(i\pi+\theta)\tanh\theta\,\sinh(-\theta) = 2i\frac{(\cosh\theta+1)^2}{\cosh\theta}$$

and therefore

$$\tilde{\Xi}^{12}_{\varphi\varphi\epsilon}(k_1,k_2,k_3) = -\frac{i}{32\pi m^4}\frac{m}{-k_1^0+m-i\epsilon}h_+^{Z(2)}(-k_3^0/(2m)+i\epsilon)$$
$$h_+^{Z(2)}(x) = \int\limits_{-\infty}^{\infty} d\theta\,\frac{(\cosh\theta+1)^2}{\cosh^2\theta}\frac{1}{\cosh\theta-x}$$

$$= -\frac{2}{x} - \frac{2}{x}\pi - \frac{1}{x^2}\pi - 4\frac{(1+x)^2}{x^2\sqrt{x^2-1}} \operatorname{arctanh} \frac{1+x}{\sqrt{x^2-1}} .$$

b) For the permutation $\pi = 321$ and $n_1 = 2,\ n_2 = 1$

$$\begin{aligned}
&\tilde{\Xi}^{21}_{\epsilon\varphi\varphi}(k_3, k_2, k_1) \\
&= \frac{1}{2!} \int_{p_1} \int_{p_2} \int_{p_3} 2\pi\delta(p_1 + p_2)\, 2\pi\delta(p_3)\, \langle 0|\epsilon(0)|p_1, p_2\rangle \langle p_2, p_1|\varphi(0)|p_3\rangle \\
&\times \langle p_3|\varphi(0)|0\rangle \frac{-i}{k_3^0 + k_2^0 + \omega_1 + \omega_2 - i\epsilon} \frac{-i}{k_1^0 + \omega_3 - i\epsilon} \\
&= \frac{1}{2!} \frac{-i}{k_1^0 + m - i\epsilon} \frac{1}{2m} \int_\theta \frac{1}{2\omega} F^\epsilon(\theta, -\theta)\, F^\varphi(-\theta + i\pi, \theta + i\pi, 0) \frac{-i}{-k_3^0 + 2\omega - i\epsilon} \\
&= -\frac{1}{64m^4\pi} \frac{m}{k_1^0 + m - i\epsilon} \int d\theta \frac{m}{\omega} \frac{2m F^\epsilon(\theta, -\theta)\, F^\varphi(-\theta + i\pi, \theta + i\pi, 0)}{-k_3^0 + 2\omega - i\epsilon}
\end{aligned}$$

which is (9) and

$$F^\epsilon(\theta, -\theta)\, F^\varphi(-\theta + i\pi, \theta + i\pi, 0) = 2i \frac{(\cosh\theta + 1)^2}{\cosh\theta}$$

imply again

$$\tilde{\Xi}^{21}_{\epsilon\varphi\varphi}(k_3, k_2, k_1) = -\frac{i}{32\pi m^4} \frac{m}{k_1^0 + m - i\epsilon} h_+^{Z(2)}(k_3^0/(2m) + i\epsilon) .$$

c) For the permutation $\pi = 132$ and $n_1 = 1,\ n_2 = 1$

$$\begin{aligned}
\tilde{\Xi}^{11}_{\varphi\epsilon\varphi}(k_1, k_3, k_2) &= \int_{p_1} \int_{p_2} 2\pi\delta(p_1)\, 2\pi\delta(p_2)\, \langle 0|\varphi(0)|p_1\rangle \langle p_1|\epsilon(0)|p_2\rangle \langle p_2|\varphi(0)|0\rangle \\
&\times \frac{-i}{k_3^0 + k_2^0 + \omega_1 - i\epsilon} \frac{-i}{k_2^0 + \omega_2 - i\epsilon} \\
&= -\frac{1}{4m^2} \frac{m}{-k_1^0 + m - i\epsilon} \frac{m}{k_2^0 + m - i\epsilon} F^\epsilon(i\pi, 0) \\
&= -\frac{i}{4m^2} \frac{m}{-k_1^0 + m - i\epsilon} \frac{m}{k_2^0 + m - i\epsilon}
\end{aligned}$$

which is (10) because $F^\epsilon(i\pi, 0) = \sinh \frac{1}{2} i\pi = i$.

d) For the permutation $\pi = 132$ and $n_1 = 3,\ n_2 = 1$

$$\begin{aligned}
&\tilde{\Xi}^{31}_{\varphi\epsilon\varphi}(k_1, k_3, k_2) \\
&= \frac{1}{3!} \int_{p_1} \int_{p_2} \int_{p_3} \int_{p_4} \langle 0|\varphi(0)|p_1, p_2, p_3\rangle \langle p_3, p_2, p_1|\epsilon(0)|p_4\rangle \langle p_4|\varphi(0)|0\rangle \\
&\times 2\pi\delta\left(p_1^1 + p_2^1 + p_3^1\right) 2\pi\delta\left(p_4^1\right) \frac{-i}{-k_1^0 + \omega_1 + \omega_2 + \omega_3 - i\epsilon} \frac{-i}{k_3^0 + \omega_4 - i\epsilon}
\end{aligned}$$

$$= \frac{1}{3!}\frac{1}{2m}\int_{p_1}\int_{p_2}\int_{p_3} \frac{-i}{-k_1^0+\omega_1+\omega_2+\omega_3-i\epsilon}\frac{-i}{k_3^0+m-i\epsilon}$$
$$\times 2\pi\delta\left(p_1^1+p_2^1+p_3^1\right)F^{\varphi}(\theta_1,\theta_2,\theta_3)F^{\epsilon}(\theta_1,\theta_2,\theta_3;\theta_4)$$

There are 3 contributions from

$$F^{\epsilon}(\theta_1,\theta_2,\theta_3;\theta_4) = \delta_{\theta_1\theta_4}F^{\epsilon}(\theta_3+i\pi,\theta_2+i\pi) - \delta_{\theta_2\theta_4}F^{\epsilon}(\theta_3+i\pi,\theta_1+i\pi)+\delta_{\theta_3\theta_4}F^{\epsilon}(\theta_2+i\pi,\theta_1+i\pi)$$

It turns out that all 3 give the same result, therefore

$$\begin{aligned}
&\tilde{\Xi}^{31}_{\varphi\epsilon\varphi}(k_1,k_3,k_2)\\
&= 3\frac{1}{3!}\frac{1}{2m}\frac{-i}{k_3^0+m-i\epsilon}\int_{p_1}\int_{p_2}\int_{p_3}\frac{-i}{-k_1^0+\omega_1+\omega_2+\omega_3-i\epsilon}\\
&\times 2\pi\delta\left(p_1^1+p_2^1+p_3^1\right)F(\theta_1,\theta_2,\theta_3)\delta_{\theta_1\theta_4}F^{\epsilon}(\theta_3+i\pi,\theta_2+i\pi)\\
&= \frac{1}{2}\frac{(-i)^2}{(2m)^3 m}\frac{1}{4\pi}\frac{m}{k_3^0+m-i\epsilon}\int d\theta\frac{2m}{2\omega}\frac{2mF(0,\theta,-\theta)F^{\epsilon}(-\theta+i\pi,\theta+i\pi)}{-k_1^0+m+2\omega-i\epsilon}\\
&= -\frac{1}{64m^4\pi}\frac{m}{k_3^0+m-i\epsilon}\int d\theta\frac{m}{\omega}\frac{2mF(0,\theta,-\theta)F^{\epsilon}(-\theta+i\pi,\theta+i\pi)}{-k_1^0+m+2\omega-i\epsilon}
\end{aligned}$$

and

$$F^{\varphi}(0,\theta,-\theta)F^{\epsilon}(-\theta+i\pi,\theta+i\pi) = 2i\frac{(\cosh\theta-1)^2}{\cosh\theta}$$

$$\tilde{\Xi}^{31}_{\varphi\epsilon\varphi}(k_1,k_3,k_2) = \frac{-1}{32m^4\pi}\frac{m}{k_3^0+m-i\epsilon}h_-^{Z(2)}(k_1^0/(2m)-\tfrac{1}{2})$$

$$\begin{aligned}
h_-^{Z(2)}(x) &= \int_{-\infty}^{\infty} d\theta\frac{(\cosh\theta-1)^2}{\cosh^2\theta}\frac{1}{\cosh\theta-x}\\
&= -\frac{2}{x}+\frac{2}{x}\pi-\frac{1}{x^2}\pi-4\frac{(x-1)^2}{x^2\sqrt{x^2-1}}\operatorname{arctanh}\frac{1+x}{\sqrt{x^2-1}}
\end{aligned}$$

d) For the permutation $\pi = 132$ and $n_1 = 1,\ n_2 = 3$ we find, similarly

$$\tilde{\Xi}^{13}_{\varphi\epsilon\varphi}(k_1,k_3,k_2) = \frac{-1}{32m^4\pi}\frac{m}{-k_1^0+m-i\epsilon}h_-^{Z(2)}(-k_2^0/(2m)-\tfrac{1}{2}+i\epsilon).$$

Finally with (12) we obtain (15).

C.1.2. The 4-point function

From (13) for $k_i = (k_i^0, 0)$ in momentum space the contribution from I_2 in (B.4) vanishes, because $S(0) = -1$ and we get

$$\tilde{\Xi}_{\underline{\varphi}}(\underline{k}) = -\frac{i}{32\pi m^6} \sum_{perm(k)} \frac{m}{-k_1^0 + m - i\epsilon} \frac{m}{k_4^0 + m - i\epsilon} g^{Z2}\left(-\frac{k_3^0 + k_4^0}{2m} + i\epsilon\right)$$

$$g^{Z2}(x) = -\frac{1}{4}\int d\theta \frac{1}{\cosh\theta} \frac{1}{\cosh\theta - x} I_{\underline{\varphi}}^{Z_2}(0,\theta,-\theta,0)$$

From (B.4) and (14) we obtain

$$\begin{aligned}
&I_{\underline{\varphi}}^{Z_2}(0,\theta,-\theta,0)\\
&= \tfrac{1}{2} F(0,\theta - i\pi_+, -\theta - i\pi_-) F(-\theta + i\pi_+, \theta + i\pi_-, 0) + (\epsilon \to -\epsilon)\\
&= \tfrac{1}{2}(2i)^2 \tanh\frac{1}{2}(-\theta + i\pi + i\epsilon)\tanh\frac{1}{2}(\theta + i\pi - i\epsilon)\tanh\frac{1}{2}(2\theta)\\
&\times \tanh\frac{1}{2}(-2\theta)\tanh\frac{1}{2}(-\theta + i\pi + i\epsilon)\tanh\frac{1}{2}(\theta + i\pi - i\epsilon) + (\epsilon \to -\epsilon)\\
&= 2\tanh^2\theta \coth^4 \tfrac{1}{2}(\theta - i\epsilon) + (\epsilon \to -\epsilon)\,.
\end{aligned}$$

and

$$\begin{aligned}
g^{Z(2)}(x) &= -\frac{1}{2}\int_{-\infty}^{\infty} d\theta \left(\frac{\coth^4 \frac{1}{2}(\theta - i\epsilon)\tanh^2\theta + (\epsilon \to -\epsilon)}{\cosh\theta\,(\cosh\theta - x)}\right)\\
&= -\int_{-\infty}^{\infty}\left(\frac{\coth^4 \frac{1}{2}\theta\,\tanh^2\theta}{\cosh\theta}\frac{1}{\cosh\theta - x} - \frac{16}{\theta^2}\frac{1}{1-x}\right) d\theta
\end{aligned}$$

which can be calculated:
for $\operatorname{Re} x < -1$

$$g^{Z(2)}(x) = \frac{16}{1-x} - \frac{15\pi}{2x} - \frac{8}{x} - \frac{4\pi + 2}{x^2} - \frac{\pi}{x^3} - \frac{(x+1)^2\sqrt{x^2-1}}{x^3(x-1)^2} 2\ln\left(-x + \sqrt{x^2-1}\right)$$

for $\operatorname{Re} x > 1$

$$\begin{aligned}
g^{Z(2)}(x \pm i\epsilon) &= \frac{16}{1-x} - \frac{15\pi}{2x} - \frac{8}{x} - \frac{4\pi + 2}{x^2} - \frac{\pi}{x^3}\\
&\quad - \frac{(x+1)^2\sqrt{x^2-1}}{x^3(x-1)^2} 2\left(\pm i\pi + \ln\left(x + \sqrt{x^2-1}\right)\right)
\end{aligned}$$

$$\operatorname{Im} g^{Z(2)}(x \pm i\epsilon) = \mp\Theta(x-1)2\pi\frac{(x+1)^2\sqrt{x^2-1}}{x^3(x-1)^2}$$

for $-1 < x < 1$

$$\begin{aligned}
g^{Z(2)}(x) &= \frac{16}{1-x} - \frac{15\pi}{2x} - \frac{8}{x} - \frac{4\pi + 2}{x^2} - \frac{\pi}{x^3}\\
&\quad - \frac{(x+1)^2 i\sqrt{1-x^2}}{x^3(x-1)^2} 2\ln\left(-x + i\sqrt{1-x^2}\right)\\
&= \left(\frac{94}{3} + 10\pi\right) + O(x)
\end{aligned}$$

The intrinsic coupling g_R [7], defined by $\tilde{\Xi}(0) = -\frac{i}{m^6} g_R$ is

$$g_R = \frac{47}{2\pi} + \frac{15}{2} = 14.98028$$

C.2. The sinh-Gordon model

The classical sinh-Gordon Lagrangian is

$$\mathcal{L}^{SG} = \tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{\alpha}{\beta^2}\left(\cosh\beta\varphi - 1\right) = \tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}\alpha\varphi^2 + \beta^2\alpha\frac{1}{24}\varphi^4 + O\left(\beta^3\right) \quad \text{(C.3)}$$

and the field equation

$$\Box\varphi(t,x) + \frac{\alpha}{\beta}\sinh\beta\varphi(t,x) = 0.$$

with

$$0 < \mu = \frac{\beta^2}{8\pi + \beta^2} < 1$$

The model is super-renormalizable, therefore after introducing normal products in (C.3) there are only two finite renormalization constants. The wave function and the mass renormalization constants are given by [16,26]

$$\langle 0 \,|\, \varphi(0) \,|\, p\rangle = \sqrt{Z^\varphi}\,, \quad \alpha = m^2\frac{\pi\mu}{\sin\pi\mu}\,.$$

with [16]

$$Z^\varphi = (1-\mu)\frac{\frac{\pi}{2}\mu}{\sin\frac{\pi}{2}\mu}E(-\mu)\,, \quad E(x) = \exp\left(-\pi\int_0^x \frac{t}{\sin\pi t}dt\right).$$

The **S-matrix** can be obtained by analytic continuation (from $\beta \to i\beta$) of the sine-Gordon S-matrix which was derived in [11,15]

$$S^{SG}(x) = \frac{\sinh\theta - i\sin\pi\mu}{\sinh\theta + i\sin\pi\mu} = -\exp\left(-2\int_0^\infty \frac{dt}{t}\,\frac{\cosh\left(\frac{1}{2}-\mu\right)t}{\cosh\frac{1}{2}t}\sinh t\frac{\theta}{i\pi}\right).$$

The minimal **sinh-Gordon form factor** is [16,17]

$$\begin{aligned} F^{SG}(\theta) &= \exp\int_0^\infty \frac{dt}{t\sinh t}\left(\frac{\cosh\left(\frac{1}{2}-\mu\right)t}{\cosh\frac{1}{2}t} - 1\right)\cosh t\left(1 - \frac{\theta}{i\pi}\right) \\ &= -i\sinh\tfrac{1}{2}\theta\,\xi\left(\mu + (1-\theta/(i\pi))\right)\xi\left(\mu - (1-\theta/(i\pi))\right) \end{aligned} \quad \text{(C.4)}$$

where the meromorphic function[5]

[5] The function $E(x)$ was introduced in [16] and also used in [6] and [9].

$$\xi(x) = \sqrt{\frac{1}{\cos\frac{1}{2}\pi x}E(x)} = \prod_{k=0}^{\infty} \frac{\Gamma\left(1+k-\frac{1}{2}x\right)\Gamma\left(\frac{1}{2}+k+\frac{1}{2}x\right)}{\Gamma\left(\frac{3}{2}+k-\frac{1}{2}x\right)\Gamma\left(1+k+\frac{1}{2}x\right)} \frac{\Gamma\left(\frac{3}{2}+k\right)}{\Gamma\left(\frac{1}{2}+k\right)}$$

has been introduced, for more details see Appendix C.2.2. The 3-particle form factor is [16,17]

$$F^{SG}(\theta_1,\theta_2,\theta_3) = -\sqrt{Z^\varphi}\frac{\sin\pi\mu}{F(i\pi)} \frac{F(\theta_{12})F(\theta_{13})F(\theta_{23})}{\cosh\frac{1}{2}\theta_{12}\,\cosh\frac{1}{2}\theta_{13}\,\cosh\frac{1}{2}\theta_{23}} \tag{C.5}$$

where the normalization follows from the form factor equation (iii) and (C.4)

$$\operatorname*{Res}_{\theta_{12}=i\pi} F^{SG}(\theta_1,\theta_2,\theta_3) = 2i\,(\mathbf{1}-S(\theta_{23}))\sqrt{Z^\varphi}$$

$$F^{SG}(\theta+i\pi)F^{SG}(\theta) = \frac{\sinh\theta}{\sinh\theta + i\sin\pi\mu}.$$

C.2.1. The 4-point function

From (13) for $k_i = (k_i^0, 0)$ we get with $\underline{\varphi} = \varphi\varphi\varphi\varphi$

$$\tilde{\Xi}^{SG}_{\underline{\varphi}}(\underline{k}) = \frac{-i}{32\pi m^6}\sum_{perm(k)} \frac{m}{-k_1^0+m-i\epsilon}\frac{m}{k_4^0+m-i\epsilon} g^{SG}\left(\frac{-1}{2m}\left(k_3^0+k_4^0\right)\right)$$

where $g^{SG}(x) = g_1^{SG}(x) + g_2^{SG}(x)$ and

$$g_i^{SG}(x) = -\frac{1}{4}\int \frac{1}{\cosh\theta}\frac{1}{\cosh\theta - x} I^{SG}_{\underline{\varphi} i}(0,\theta,-\theta,0)d\theta.$$

From (B.4) and (C.5) we obtain

$$\begin{aligned} I^{SG}_{\underline{\varphi}1}(0,\theta,-\theta,0) &= \tfrac{1}{2}Z^\varphi F^{SG}(0,\theta-i\pi_+,-\theta-i\pi_-)F^{SG}(-\theta+i\pi_+,\theta+i\pi_-,0)\\ &\quad + (\epsilon\to-\epsilon)\\ &= f^{SG}(\theta) I^{Z_2}_{\underline{\varphi}}(0,\theta,-\theta,0) \end{aligned}$$

where $I^{Z_2}_{\underline{\varphi}}$ as defined in (17). We have introduced

$$f^{SG}(\theta) = -\frac{(Z^\varphi)^2\sin^2\pi\mu}{F^2(i\pi)\,(2i)^2}\left(F_0(\theta+i\pi)\right)^4 F_0(2\theta)F_0(-2\theta)$$

with

$$F_0(\theta) = F^{SG}(\theta)/\left(-i\sinh\tfrac{1}{2}\theta\right) = \xi\left(\mu+(1-\theta/(i\pi))\right)\xi\left(\mu-(1-\theta/(i\pi))\right).$$

Therefore as in (16) and (17) we obtain

$$\begin{aligned} g_1^{SG}(x) &= -\frac{1}{4}\int\frac{1}{\cosh\theta}\frac{1}{\cosh\theta-x}I_1^{SG}(0,\theta,-\theta,0)d\theta\\ &= -\int_{-\infty}^{\infty}\left(f^{SG}(\theta)\frac{\coth^4\frac{1}{2}\theta\,\tanh^2\theta}{\cosh\theta}\frac{1}{\cosh\theta-x} - f^{SG}(0)\frac{16}{\theta^2}\frac{1}{1-x}\right)d\theta. \end{aligned}$$

The functions $g_1^{SG}(x)$ for $\mu = 0.3$ and $\mu = 0.5$ are plotted in Fig. 4 and 5.

The contribution from I_2 follows from (B.7) as

$$g_2^{SG}(x) = -16\pi i\frac{dS(\theta)}{d\theta}\sqrt{Z^\varphi}\frac{1}{1-x-i\epsilon} = \frac{-32\pi}{\sin\pi\mu}\sqrt{Z^\varphi}\frac{1}{1-x}.$$

C.2.2. Properties of $\xi(x)$

Representations

$$\xi(x) = \exp \frac{1}{2} \int_0^\infty \frac{dt}{t \sinh t} \left(\frac{\cosh\left(\frac{1}{2} - x\right) t}{\cosh \frac{1}{2} t} - 1 \right)$$
$$= \exp \left\{ \frac{1}{2} \left(\frac{i}{\pi} \left(\mathrm{Li}_2(e^{ix\pi}) - \mathrm{Li}_2(-e^{ix\pi}) \right) \right.\right.$$
$$\left.\left. - \, x \ln\left(1 - e^{ix\pi}\right) - (1-x) \ln\left(1 + e^{ix\pi}\right) + \ln 2 + \tfrac{1}{2} i\pi \left(x - \tfrac{1}{2}\right) \right) \right\}$$
$$= \frac{1}{\sqrt{\pi}} \frac{G\left(1 + \frac{1}{2}x\right) G\left(\frac{3}{2} - \frac{1}{2}x\right)}{G\left(1 - \frac{1}{2}x\right) G\left(\frac{1}{2} + \frac{1}{2}x\right)}$$

where G is Barnes G-function and $\mathrm{Li}_2(x)$ the dilogarithm.[6] The function $\xi(x)$ is meromorphic and satisfies

$$\xi(1-x) = \xi(x)\,, \quad \xi(x)\xi(-x) \cos \tfrac{1}{2}\pi x = \sqrt{E(x)E(-x)} = 1$$

which imply the form factor equation (i) $F^{SG}(x) = F^{SG}(-x) S^{SG}(x)$.

C.3. The Z_3-model

The two-particle S-matrix for the Z_N-Ising model has been proposed by Köberle and Swieca [22]. The scattering of two particles of type 1 is given by

$$S(\theta) = \frac{\sinh \frac{1}{2}(\theta + \frac{2\pi i}{N})}{\sinh \frac{1}{2}(\theta - \frac{2\pi i}{N})}\,. \tag{C.6}$$

This S-matrix is consistent with the picture that the bound state of $N-1$ particles of type 1 is the anti-particle of 1. The form factors of the Z_N-model have been proposed in [23,12]. The minimal solution of Watson's and the crossing equations

$$F(\theta) = F(-\theta)S(\theta)\,, \quad F(i\pi - \theta) = F(i\pi + \theta)$$

for the Z_3 model is

$$F^{Z3}(i\pi x) = c_1 \sin \tfrac{1}{2}\pi x \, \exp \int_0^\infty \left(\frac{\sinh \frac{1}{3} t}{t \sinh^2 t} \left(1 - \cosh t\,(1-x)\right) \right) dt$$
$$= \pi^{1/3} \sin \tfrac{1}{2}\pi x \, \frac{G\left(\frac{1}{3} + \frac{1}{2}x\right) G\left(\frac{4}{3} - \frac{1}{2}x\right)}{G\left(\frac{2}{3} + \frac{1}{2}x\right) G\left(\frac{5}{3} - \frac{1}{2}x\right)}$$

where $G(x)$ is Barnes G-function.

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6 In Mathematica: $\mathrm{Li}_2(x) = \mathrm{PolyLog}[2, x]$.

The form factor of the order parameter $\sigma_1(x)$ and two particles of type 2 is given by (22) [23,12] where $c_2 = -\frac{1}{2}\sqrt{2}\sqrt[4]{3}\pi^{-\frac{1}{3}}G\left(\frac{4}{3}\right)/G\left(\frac{2}{3}\right)$ is determined by the form factor equation (iv) $\operatorname{Res}_{\theta_{12}=\frac{2}{3}i\pi} F_{22}^{\sigma_1}(\underline{\theta}) = \sqrt{2}F_1^{\sigma_1}\Gamma = \sqrt{2}\Gamma$. The intertwiner Γ defined by is defined by

$$i \operatorname*{Res}_{\theta=\frac{2}{3}i\pi} \frac{\sinh\frac{1}{2}(i\pi x + \frac{2\pi i}{3})}{\sinh\frac{1}{2}(i\pi x - \frac{2\pi i}{3})} = -\sqrt{3} = \Gamma_2^{11}\Gamma_{11}^2\,, \quad \Gamma_2^{11} = \Gamma_{11}^2 = \Gamma = i3^{\frac{1}{4}}.$$

The form factor of $\sigma_1(x)$ for the 3 particles of type 112 is given by (23) [23,12], where $c_3 = \sqrt{3}\pi^{\frac{2}{3}}\left(G\left(\frac{4}{3}\right)\right)^2/\left(G\left(\frac{2}{3}\right)\right)^2$ is determined by $\operatorname{Res}_{\theta_{12}=\frac{2}{3}i\pi} F_{112}^{\sigma_1}(\underline{\theta}) = \sqrt{2}F_{22}^{\sigma_1}(\underline{\theta})\Gamma$. The minimal form factor of the particles 1 and 2

$$F_{(12)}^{\min}(i\pi x) = c\exp\int_0^\infty \frac{dt}{t\sinh^2 t}\sinh\frac{2}{3}t\,\left(1-\cosh t\,(1-x)\right) = \frac{G\left(\frac{1}{6}+\frac{1}{2}x\right)G\left(\frac{7}{6}-\frac{1}{2}x\right)}{G\left(\frac{5}{6}+\frac{1}{2}x\right)G\left(\frac{11}{6}-\frac{1}{2}x\right)}$$

satisfies $F_{(12)}^{\min}(\theta) = F_{(12)}^{\min}(-\theta)S_{(12)}(\theta)$, where $S_{(12)}(\theta) = -\frac{\sinh\frac{1}{2}\left(\theta+\frac{1}{3}i\pi\right)}{\sinh\frac{1}{2}\left(\theta-\frac{1}{3}i\pi\right)}$ is the S-matrix for the particles 1 and 2.

C.3.1. The 3-point function

The three point function Green's function (see also [8]) $\tau_{\sigma_1\sigma_1\sigma_1}(\underline{x}) = \langle 0\,|\,T\sigma_1(x_1)\sigma_1(x_2)\sigma_1(x_3)\,|\,0\rangle$ of the field $\sigma_1(x)$ is different from zero, because $\sigma_1\sigma_1\sigma_1$ is in the vacuum sector. We have the contributions

$$\tilde{\Xi}_{\sigma_1\sigma_1\sigma_1}(\underline{k}) = \sum_{\pi\in S_3}\Big(\tilde{\Xi}_{\sigma_1\sigma_1\sigma_1}^{11}(k_{\pi 1},k_{\pi 2},k_{\pi 3}) + \tilde{\Xi}_{\sigma_1\sigma_1\sigma_1}^{12}(k_{\pi 1},k_{\pi 2},k_{\pi 3}) + \tilde{\Xi}_{\sigma_1\sigma_1\sigma_1}^{\bar{2}\bar{1}}(k_{\pi 1},k_{\pi 2},k_{\pi 3})\Big)\,.$$

For the permutation $\pi = (1,2,3)$ and the intermediate states $\langle 0|\sigma_1(0)|p_1\rangle\langle p_1|\sigma_1(0)|\bar{p}_2\rangle\langle\bar{p}_2|\,\sigma_1(0)|0\rangle$ the $\tilde{\Xi}$-function is as in (10)

$$\begin{aligned}\tilde{\Xi}_{\sigma_1\sigma_1\sigma_1}^{11}(k_1,k_2,k_3) &= \int_{p_1}\int_{p_2}\langle 0|\sigma_1(0)|p_1\rangle\langle p_1|\sigma_1(0)|\bar{p}_2\rangle\langle\bar{p}_2|\,\sigma_1(0)|0\rangle \\ &\quad\times 2\pi\delta(p_1)\,2\pi\delta(\bar{p}_2)\,\frac{-i}{k_2^0+k_3^0+\omega_1-i\epsilon}\,\frac{-i}{k_3^0+\omega_2-i\epsilon} \\ &= \frac{-1}{4m^4}\,\frac{m}{-k_1^0+m-i\epsilon}\,\frac{m}{k_3^0+m-i\epsilon}F_{22}^{\sigma_1}(i\pi,0)\end{aligned}$$

For the permutation $\pi = (1,2,3)$ and the intermediate states $\langle 0|\sigma_1(0)|p_1\rangle\langle p_1|\sigma_1(0)|p_2,p_3\rangle\langle p_3,p_2|\,\sigma_1(0)|0\rangle$ the $\tilde{\Xi}$-function is as in (8)

$$\begin{aligned}\tilde{\Xi}_{\sigma_1\sigma_1}^{12}(k_1,k_2,k_3) &= \frac{1}{2!}\int_{p_1}\int_{p_2}\int_{p_3}\langle 0|\sigma_1(0)|p_1\rangle\langle p_1|\sigma_1(0)|p_2,p_3\rangle\langle p_3,p_2|\,\sigma_1(0)|0\rangle \\ &\quad\times 2\pi\delta(p_1)\,2\pi\delta(p_2+p_3)\,\frac{-i}{k_2^0+k_3^0+\omega_1-i\epsilon}\,\frac{-i}{k_3^0+\omega_2+\omega_3-i\epsilon}\end{aligned}$$

$$= -\frac{1}{64\pi m^4}\frac{m}{-k_1^0+m-i\epsilon}\int d\theta \frac{m}{\omega}\frac{2m}{k_3^0+2\omega-i\epsilon} I^{12}_{\sigma_1\sigma_1\sigma_1}(\theta)$$

$$I^{12}_{\sigma_1\sigma_1\sigma_1}(\theta) = F^{\sigma_1}_{211}(i\pi,\theta,-\theta)F^{\sigma_1}_{22}(-\theta+i\pi,\theta+i\pi)$$

where the crossing relation

$$\langle p_1|\sigma_1(0)|p_2,p_3\rangle = F^{\sigma_1}_{211}(\theta_1+i\pi,\theta_2,\theta_3)+\delta_{\theta_1\theta_2}+\delta_{\theta_1\theta_3}S(\theta_{23})$$

has been used. The δ-terms do not contribute because $F^{\sigma_1}_{22}(0,0)=0$. Inserting the form factor functions we get

$$\tilde\Xi^{12}_{\sigma_1\sigma_1\sigma_1}(k_1,k_2,k_3) = -\frac{1}{64\pi m^4}\frac{m}{-k_1^0+m-i\epsilon}h^{Z3}\left(-\frac{k_3^0}{2m}+i\epsilon\right)$$

$$h^{Z3}(x) = \int_{-\infty}^{\infty} d\theta \frac{1}{\cosh\theta}\frac{1}{\cosh\theta-x}\frac{F(2\theta)F(-2\theta)F^{\min}_{(12)}(i\pi+\theta)F^{\min}_{(12)}(i\pi-\theta)}{\left(\sinh(\theta-\frac{1}{3}i\pi)\sinh(\theta+\frac{1}{3}i\pi)\sinh\frac{1}{2}\theta\right)^2}.$$

For the intermediate states $\langle 0|\sigma_1(0)|\bar p_1,\bar p_2\rangle\langle\bar p_2,\bar p_1|\sigma_1(0)|\bar p_3\rangle\langle\bar p_3|\sigma_1(0)|0\rangle$, where $|\bar p\rangle$ is a particle state of type 2 which is the anti-particle of 1, the $\tilde\Xi$-function is as in (9)

$$\tilde\Xi^{\bar2\bar1}_{\sigma_1\sigma_1\sigma_1}(\underline{k}) = \frac{1}{2!}\int_{p_1}\int_{p_2}\int_{p_3}\langle 0|\sigma_1(0)|\bar p_1,\bar p_2\rangle\langle\bar p_2,\bar p_1|\sigma_1(0)|\bar p_3\rangle\langle\bar p_3|\sigma_1(0)|0\rangle$$

$$\times 2\pi\delta(p_1)\,2\pi\delta(p_2+p_3)\frac{-i}{\pi k_2^0+\pi k_3^0+\omega_1+\omega_2-i\epsilon}\frac{-i}{\pi k_3^0+\omega_3-i\epsilon}$$

$$= -\frac{1}{64\pi m^4}\frac{m}{k_2^0+k_3^0+m-i\epsilon}\int d\theta\frac{m}{\omega}\frac{2m}{-k_1^0+2\omega-i\epsilon}I^{\bar2\bar1}_{\sigma_1\sigma_1\sigma_1}(\theta)$$

$$I^{\bar2\bar1}_{\sigma_1\sigma_1\sigma_1}(\theta) = F^{\sigma_1}_{22}(\theta,-\theta)F^{\sigma_1}_{112}(-\theta+i\pi,\theta+i\pi,0)$$

where (22), (23) and the crossing relation

$$\langle\bar p_2,\bar p_1|\sigma_1(0)|\bar p_3\rangle = \left(F^{\sigma_1}_{112}(\theta_2+i\pi,\theta_1+i\pi,\theta_3)+\delta_{\theta_1\theta_3}+S(\theta_{21})\delta_{\theta_2\theta_3}\right)$$

have been used. Again the δ-terms do not contribute because $F^{\sigma_1}_{22}(0,0)=0$. Inserting the form factor functions we get

$$\tilde\Xi^{\bar2\bar1}_{\sigma_1\sigma_1\sigma_1}(k_1,k_2,k_3) = -\frac{1}{64\pi m^4}\frac{m}{k_3^0+m-i\epsilon}h^{Z3}\left(\frac{k_1^0}{2m}+i\epsilon\right)$$

and as in (12)

$$\tilde\Xi_{\sigma_1\sigma_1\sigma_1}(k_1,k_2,k_3) = const.\sum_{perm(k)}\frac{m}{-k_1^0+m-i\epsilon}\frac{m}{k_3^0+m-i\epsilon}$$

$$+\,const.'\sum_{perm(k)}\frac{m}{-k_1^0+m-i\epsilon}h^{Z(3)}\left(-\frac{k_3^0}{2m}+i\epsilon\right)+(k_i\to -k_i)\,.$$

For $h^{Z3}(x)$ see Fig. 6.

C.3.2. The 4-point function

We consider the Green's function $\tau_{\sigma_1\sigma_2\sigma_1\sigma_2}(\underline{x}) = \langle 0 \,|\, T\sigma_1(x_1)\sigma_2(x_2)\sigma_1(x_3)\sigma_2(x_4) \,|\, 0 \rangle$ and calculate for $\sigma_3 = \sigma_1$ and $\sigma_4 = \sigma_2$

$$\tilde{\Xi}_{\sigma_1\sigma_2\sigma_3\sigma_4}(k_1,k_2,k_3,k_4) = \sum_{\pi\in S_4} \tilde{\Xi}^{121}_{\sigma_{\pi 1}\sigma_{\pi 2}\sigma_{\pi 3}\sigma_{\pi 4}}(k_{\pi 1},k_{\pi 2},k_{\pi 3},k_{\pi 4})$$

where as in (B.3) the result is expressed by 3 terms I_1, I_2, I_3. The I_3 term again contributes to the disconnected part and the I_2 is as in (19) given by $g_2 = const./(1-x)$. We calculate here the more interesting contribution from I_1. For the various permutations we have to calculate as in (B.6):

I) for the permutation $\pi = (1,2,3,4)$ and the intermediate states $\langle 0|\sigma_1(0)|p_1\rangle\langle p_1|\sigma_2(0)|\bar{p}_2, p_3\rangle\langle p_3, \bar{p}_2|\sigma_1(0)|p_4\rangle\langle p_4|\sigma_2(0)|0\rangle$ we obtain

$$\tilde{\Xi}^{121}_{\sigma_1\sigma_2\sigma_1\sigma_2}(k_1,k_2,k_3,k_4) = -\frac{1}{32}\frac{i}{m^6\pi}\frac{m}{-k_1^0+m-i\epsilon}\frac{m}{k_4^0+m-i\epsilon}g_I^{Z3}\left(-\frac{k_3^0+k_4^0}{2m}+i\epsilon\right)$$

$$g_I^{Z3}(x) = \frac{-1}{2}\int d\theta \frac{1}{\cosh\theta}\frac{1}{\cosh\theta - x} I^{121}_{\sigma_1\sigma_2\sigma_1\sigma_2}(0,\theta,-\theta,0)$$

$$I^{121}_{\sigma_1\sigma_2\sigma_1\sigma_2}(\theta_1,\theta_2,\theta_3,\theta_4) = \langle p_1|\sigma_2(0)|\bar{p}_2, p_3\rangle\langle p_3, \bar{p}_2|\sigma_1(0)|p_4\rangle_1$$

where $\langle\dots\rangle_1$ means that we only take into account the term from I_1 and as in (B.4)

$$\begin{aligned} &I^{121}_{\sigma_1\sigma_2\sigma_1\sigma_2}(0,\theta,-\theta,0) \\ &\quad= \tfrac{1}{2}F^{\sigma_2}_{221}(0,\theta-i\pi_+,-\theta-i\pi_-)F^{\sigma_1}_{211}(-\theta+i\pi_+,\theta+i\pi_-,0) + (\pi_+ \leftrightarrow \pi_-) \\ &\quad= \tfrac{1}{2}\frac{c_3^2\left(F^{Z3}(i\pi-\theta)\,F^{\min}_{12}(i\pi-\theta)\right)^2 F^{\min}_{12}(2\theta)\,F^{\min}_{12}(-2\theta)}{\left(\sinh\tfrac{1}{2}\left(\theta-\tfrac{1}{3}i\pi\right)\sinh\tfrac{1}{2}\left(\theta+\tfrac{1}{3}i\pi\right)\cosh\theta\right)^2} + (\epsilon\to-\epsilon) \end{aligned}$$

where (23) and charge conjugation invariance $F^{\sigma_2}_{221}(\underline{\theta}) = F^{\sigma_1}_{112}(\underline{\theta})$ have been used. For $g_I^{Z3}(x)$ see Fig. 7. For the permutations $\pi = (3,2,1,4),\ (1,4,3,2),\ (3,4,1,2)$ the result is, similarly, expressed by $g_I^{Z3}(x)$.

II) for $\pi = (1,3,2,4)$ and the intermediate states $\langle 0|\sigma_1(0)|p_1\rangle\langle p_1|\sigma_1(0)|p_2, p_3\rangle\langle p_3, p_2|\sigma_2(0)|p_4\rangle\langle p_4|\sigma_2(0)|0\rangle$ we obtain

$$\tilde{\Xi}^{121}_{\sigma_1\sigma_1\sigma_2\sigma_2}(k_1,k_3,k_2,k_4) = -\frac{1}{32}\frac{i}{m^6\pi}\frac{m}{-k_1^0+m-i\epsilon}\frac{m}{k_4^0+m-i\epsilon}g_{II}^{Z3}\left(-\frac{k_2^0+k_4^0}{2m}+i\epsilon\right)$$

$$g_{II}^{Z3}(x) = \frac{-1}{4}\int d\theta \frac{1}{\cosh\theta}\frac{1}{\cosh\theta - x} I^{121}_{\sigma_1\sigma_1\sigma_2\sigma_2}(0,\theta,-\theta,0)$$

$$I^{121}_{\sigma_1\sigma_1\sigma_2\sigma_2}(\theta_1,\theta_2,\theta_3,\theta_4) = \langle p_1|\sigma_1(0)|p_2, p_3\rangle\langle p_3, p_2|\sigma_2(0)|p_4\rangle_1$$

where $\langle\dots\rangle_1$ means that we only consider the term from I_1 and as in (B.4)

$$\begin{aligned} &I^{121}_{\sigma_1\sigma_1\sigma_2\sigma_2}(0,\theta,-\theta,0) \\ &= \tfrac{1}{2}F^{\sigma_1}_{211}(0,\theta-i\pi_+,-\theta-i\pi_-)F^{\sigma_2}_{221}(-\theta+i\pi_+,\theta+i\pi_-,0) + (\pi_+ \leftrightarrow \pi_-)\,. \end{aligned}$$

For $g_{II}^{Z3}(x)$ see Fig. 8. For the permutations
$\pi = (3,1,2,4),\ (1,3,4,2),\ (3,1,4,2)$ the result is, similarly, expressed by $g_{II}^{Z3}(x)$.

III) for $\pi = (1,2,4,3)$ and the intermediate states
$\langle 0|\sigma_1(0)|p_1\rangle\langle p_1|\sigma_2(0)|\bar p_2, p_3\rangle\langle p_3, \bar p_2|\sigma_2(0)|\bar p_4\rangle\langle \bar p_4|\sigma_1(0)|0\rangle$ we obtain

$$\tilde\Xi^{121}_{\sigma_1\sigma_2\sigma_2\sigma_1}(k_1,k_2,k_4,k_3) = -\frac{1}{32}\frac{i}{m^6\pi}\frac{m}{-k_1^0+m-i\epsilon}\frac{m}{k_3^0+m-i\epsilon}g_{III}^{Z3}\left(-\frac{k_4^0+k_3^0}{2m}+i\epsilon\right)$$

$$g_{III}^{Z3}(x) = \frac{-1}{4}\int d\theta\frac{1}{\cosh\theta}\frac{1}{\cosh\theta - x}I^{121}_{\sigma_1\sigma_2\sigma_2\sigma_1}(0,\theta,-\theta,0)$$

$$I^{121}_{\sigma_1\sigma_2\sigma_2\sigma_1}(\theta_1,\theta_2,\theta_3,\theta_4) = \langle p_1|\sigma_2(0)|\bar p_2, p_3\rangle\langle p_3,\bar p_2|\sigma_2(0)|\bar p_4\rangle_1$$

where $\langle\dots\rangle_1$ means that we only consider the term from I_1 and as in (B.4)

$$I^{121}_{\sigma_1\sigma_2\sigma_2\sigma_1}(0,\theta,-\theta,0)$$
$$= \tfrac{1}{2}F^{\sigma_2}_{221}(0,\theta-i\pi_+,-\theta-i\pi_-)F^{\sigma_2}_{212}(-\theta+i\pi_+,\theta+i\pi_-,0) + (\pi_+\leftrightarrow\pi_-)\,.$$

It turns out that $I^{121}_{\sigma_1\sigma_2\sigma_2\sigma_1}(0,\theta,-\theta,0) = I^{121}_{\sigma_1\sigma_2\sigma_1\sigma_2}(0,\theta,-\theta,0)$ which follows from charge conjugation invariance $F^{\sigma_2}_{221}(\theta_1,\theta_2,\theta_3) = F^{\sigma_1}_{112}(\theta_1,\theta_2,\theta_3)$, therefore

$$g_{III}^{Z3}(x) = g_I^{Z3}(x)\,.$$

For the permutations $\pi = (3,2,4,1),\ (1,4,2,3),\ (3,4,2,1)$ the result is, similarly, expressed by $g_I^{Z3}(x)$.

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