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Bethe ansatz and exact form factors of the $O(6)$ Gross Neveu-model*

Hrachya M Babujian[1], **Angela Foerster**[2] **and Michael Karowski**

Institut für Theoretische Physik, Freie Universität Berlin, Arnimallee 14, 14195 Berlin, Germany

E-mail: babujian@yerphi.am, angela@if.ufrgs.br and karowski@physik.fu-berlin.de

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Abstract

The isomorphism $SU(4) \simeq O(6)$ is used to construct the form factors of the $O(6)$ Gross–Neveu model as bound state form factors of the $SU(4)$ chiral Gross–Neveu model. This technique is generalized and is then applied to use the $O(6)$ as the starting point of the nesting procedure to obtain the $O(N)$ form factors for general even N.

Keywords: integrability, form factors, Bethe ansatz, correlation functions

(Some figures may appear in colour only in the online journal)

1. Introduction

In previous decades integrable quantum field theories in $1+1$ dimensions have been investigated very intensively. One of the pioneers was Petr Kulish: an infinite set of conservation laws for the sine-Gordon and the massive Thirring model was derived by Kulish and Nissimov in [1] (see also [2]). In [3], Kulish has shown that these conservation laws imply the factorization of the S-matrix. He also made a seminal contribution in the algebraic formulation of the nested Bethe ansatz: in [4] Kulish and Reshetikhin constructed the nested version of the algebraic Bethe ansatz for a $GL(N)$ invariant model. The 'off-shell' version of this nested algebraic Bethe ansatz was later developed in [5] to solve matrix difference equations. This technique was applied in [6–8] to construct form factors for the $SU(N)$ chiral Gross–Neveu model.

In a previous paper [9] we constructed the $O(N)$ nested Bethe ansatz, which needs deeper investigations. We introduced an intertwiner, which connects two different S-matrices in the nesting procedure $S(\theta, N)$ and $S(\theta, N-2)$. Then we applied this technique in [10] and [11] to the $O(N)$ nonlinear σ-model and the $O(N)$ Gross–Neveu model with even N, respectively. In

* Dedicated to the memory of Petr Petrovich Kulish.
[1] Yerevan Physics Institute, Alikhanian Brothers 2, Yerevan, 375036, Armenia
[2] Instituto de Física da UFRGS, Av. BentoGonçalves 9500, Porto Alegre—RS, Brazil

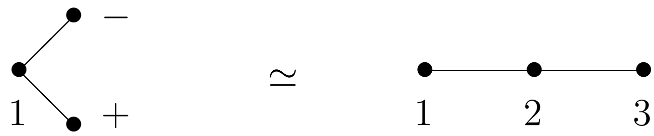


Figure 1. The isomorphism $0(6) \simeq SU(4)$ in terms of the Dynkin diagrams.

the present article we will consider the form factors of the $O(6)$ Gross–Neveu model which will be the starting model for the nesting procedure for the $O(N)$ Gross–Neveu model. The $O(4)$ Gross–Neveu-model will be considered in forthcoming papers.

Our results are related to the $N = 4$ supersymmetric Yang-Mills (SYM) theory. It is known that the $O(6)$ or $SU(4)$ Bethe ansatz structure is connected to the $N = 4$ SYM theory, which, in turn, is equivalent by the AdS/CFT conjecture to the super-string theory on the product space $\mathrm{AdS}_5 \times S_5$. This equivalence means that there is a one-to-one correspondence between all aspects of the theories including the global symmetry observables and the field content with correlation functions. In the $N = 4$ SYM theory there is an automorphism symmetry group of the supersymmetry algebra known as R-symmetry, which causes the supercharges to change by a phase rotation. Thus for the $N = 4$ SYM theory the R-symmetry group is $SU(4) \simeq O(6)$. This group is part of the full group of symmetry of the theory known as superconformal group and is given by $S(2, 2\,|\,4)$ which also includes the conformal subgroup $SO(2, 4)$ and Poincare supersymmetry [12, 13]. Therefore all integrable structures associated with $SU(4) \simeq O(6)$ group are interesting tools for this big AdS/CFT correspondence conjecture.

In [14] was shown that the isomorphism $O(6) \simeq SU(4)$, see figure 1, leads to an identity between the $O(6)$ Gross–Neveu model and the $SU(4)$ chiral Gross–Neveu model. The four right-handed (left-handed) $O(6)$ kinks correspond to the four fundamental $SU(4)$ particles (antiparticles). The six fundamental $O(6)$ particles correspond to the six $SU(4)$ bound states. In [14] the isomorphism was shown for the *S*-matrices. In this article we demonstrate the isomorphism for the form factors.

In [10] and [11] we constructed form factors for the $O(N)$ σ-model and the $O(N)$ Gross–Neveu model (for N even), respectively. For these constructions we used the nested Bethe ansatz, which means that for the level N one needs the results from level $N - 2$, etc. In [10] we used the isomorphism $O(4) \simeq SU(2) \times SU(2)$ as the starting point of the nesting procedure for the $O(N)$ σ-model. The $SU(N)$ form factors were constructed in [6–8, 15, 16]. The results of the present article, which rely on the isomorphism $O(6) \simeq SU(4)$ may serve as the starting point of the nesting procedure for the $O(N)$ Gross–Neveu model.

The article is organized as follows. In section 2 we recall some results on the needed *S*-matrices, in particular the bound state procedure. In section 3 we recall results on the $SU(4)$ and $O(6)$ form factors. We show that the form factors for $O(6)$ vector particles are to be identified with $SU(4)$ bound state form factors. In section 4 we apply these results to some examples. In section 5 we generalize the results to the so called 'modified form factors'. We prove that they can be used to start the nested 'off-shell' Bethe ansatz to solve the $O(N)$ form factor equations. The appendix provides the more complicated proofs of the results we have obtained and further explicit calculations.

2. *S*-matrix

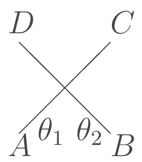
2.1. The $SU(4)$ S-matrix

The *S*-matrix of the $SU(4)$ chiral Gross–Neveu model for the scattering of two fundamental particles (transforming as the $SU(4)$ vector representation) is given by [6, 14, 17–19]

$$S^{SU(4)}(\theta) = b^{SU(4)}(\theta)\,\mathbf{1} + c^{SU(4)}(\theta)\,\mathbf{P} \tag{1}$$

or in terms of the components

$$\left(S^{SU(4)}\right)_{AB}^{DC}(\theta) = b^{SU(4)}(\theta)\,\delta_A^C\delta_B^D + c^{SU(4)}(\theta)\,\delta_A^D\delta_B^C =$$



with the rapidity difference of the particles $\theta = \theta_{12} = \theta_1 - \theta_2$. The two S-matrix eigenvalues are $S_\pm^{SU(4)} = b^{SU(4)} \pm c^{SU(4)}$ with

$$\left(S_+^{SU(4)}, S_-^{SU(4)}\right) = \left(\frac{\theta - \frac{1}{2}\mathrm{i}\pi}{\theta + \frac{1}{2}\mathrm{i}\pi}, 1\right) S_-^{SU(4)}. \tag{2}$$

Unitarity can be written as

$$S_{+,-}^{SU(4)}(-\theta)S_{+,-}^{SU(4)}(\theta) = 1.$$

The highest weight amplitude

$$a^{SU(4)}(\theta) = S_+^{SU(4)}(\theta) = -\frac{\Gamma\left(1 - \frac{1}{2}\frac{\theta}{\mathrm{i}\pi}\right)\Gamma\left(\frac{3}{4} + \frac{1}{2}\frac{\theta}{\mathrm{i}\pi}\right)}{\Gamma\left(1 + \frac{1}{2}\frac{\theta}{\mathrm{i}\pi}\right)\Gamma\left(\frac{3}{4} - \frac{1}{2}\frac{\theta}{\mathrm{i}\pi}\right)} \tag{3}$$

is essential for the Bethe ansatz which will be used to construct the form factors. In order to simplify the formulae we extract the factor $a^{SU(4)}(\theta)$ from the S-matrix and define

$$\tilde{S}^{SU(4)}(\theta) = S^{SU(4)}(\theta)/a^{SU(4)}(\theta) = \tilde{b}^{SU(4)}(\theta)\mathbf{1} + \tilde{c}^{SU(4)}(\theta)\mathbf{P} \tag{4}$$

with

$$\tilde{b}^{SU(4)}(\theta) = \frac{\theta}{\theta - \frac{1}{2}\mathrm{i}\pi},\ \tilde{c}^{SU(4)}(\theta) = \frac{-\frac{1}{2}\mathrm{i}\pi}{\theta - \frac{1}{2}\mathrm{i}\pi}.$$

The S-matrix eigenvalue $S_-^{SU(4)}(\theta)$ has a pole at $\theta = \frac{1}{2}\mathrm{i}\pi$ which means that there exist a bound state of 2 fundamental particles, which transforms as an $SU(4)$ anti-symmetric tensor. This have to be identified with a fundamental particle of the $O(6)$ model (see below). The bound states of 3 fundamental particles (ABC) (with $1 \leqslant A < B < C \leqslant 4$) is to be identified with an anti-particle of a fundamental particle $D : (ABC) = \bar{D}$ [6, 19, 20]. The charge conjugation matrix is

$$\mathbf{C}_{(ABC)D} = \epsilon_{ABCD} \tag{5}$$

where ϵ_{ABCD} is total anti-symmetric and $\epsilon_{1234} = 1$.

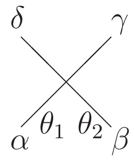
2.2. The $O(6)$ S-matrix

The $O(6)$ Gross–Neveu S-matrix for the scattering of two fundamental particles (transforming as the $O(6)$ vector representation) can be written as [21]

$$S^{O(6)}(\theta) = b^{O(6)}(\theta)\mathbf{1} + c^{O(6)}(\theta)\mathbf{P} + d^{O(6)}(\theta)\mathbf{K}, \tag{6}$$

or in terms of the components as

$$\left(S^{O(6)}\right)_{\alpha\beta}^{\delta\gamma}(\theta) = b^{O(6)}(\theta)\delta_\alpha^\gamma\delta_\beta^\delta + c^{O(6)}(\theta)\delta_\alpha^\delta\delta_\beta^\gamma + d^{O(6)}(\theta)\mathbf{C}^{\delta\gamma}\mathbf{C}_{\alpha\beta} =$$



with the 'charge conjugation matrices'

$$\mathbf{C}_{\alpha\beta} = \delta_{\alpha\bar\beta} \text{ and } \mathbf{C}^{\alpha\beta} = \delta^{\alpha\bar\beta} \tag{7}$$

in the complex basis (see [11]). The three S-matrix eigenvalues are $S_\pm^{O(6)} = b^{O(6)} \pm c^{O(6)}$ and $S_0^{O(6)} = b^{O(6)} + c^{O(6)} + 6d^{O(6)}$ with

$$\left(S_0^{O(6)}, S_+^{O(6)}, S_-^{O(6)}\right) = \left(\frac{\theta + \mathrm{i}\pi}{\theta - \mathrm{i}\pi}, \frac{\theta - \frac{1}{2}\mathrm{i}\pi}{\theta + \frac{1}{2}\mathrm{i}\pi}, 1\right) S_-^{O(6)}. \tag{8}$$

Unitarity reads as

$$S_{0,+,-}^{O(6)}(-\theta) S_{0,+,-}^{O(6)}(\theta) = 1.$$

The highest weight amplitude is [21]

$$a^{O(6)}(\theta) = S_+^{O(6)}(\theta) = \frac{\Gamma\left(1 - \frac{1}{2\pi\mathrm{i}}\theta\right)\Gamma\left(\frac{1}{2} + \frac{1}{2\pi\mathrm{i}}\theta\right)}{\Gamma\left(1 + \frac{1}{2\pi\mathrm{i}}\theta\right)\Gamma\left(\frac{1}{2} - \frac{1}{2\pi\mathrm{i}}\theta\right)} \frac{\Gamma\left(\frac{3}{4} + \frac{1}{2\pi\mathrm{i}}\theta\right)\Gamma\left(\frac{1}{4} - \frac{1}{2\pi\mathrm{i}}\theta\right)}{\Gamma\left(\frac{3}{4} - \frac{1}{2\pi\mathrm{i}}\theta\right)\Gamma\left(\frac{1}{4} + \frac{1}{2\pi\mathrm{i}}\theta\right)}.$$

For later convenience we introduce again

$$\tilde S^{O(6)}(\theta) = S^{O(6)}(\theta)/a^{O(6)}(\theta) = \tilde b^{O(6)}(\theta)\mathbf{1} + \tilde c^{O(6)}(\theta)\mathbf{P} + \tilde d^{O(6)}(\theta)\mathbf{K} \tag{9}$$

with

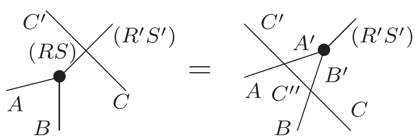
$$\tilde b^{O(6)}(\theta) = \frac{\theta}{\theta - \frac{1}{2}\mathrm{i}\pi}, \; \tilde c^{O(6)}(\theta) = \frac{-\frac{1}{2}\mathrm{i}\pi}{\theta - \frac{1}{2}\mathrm{i}\pi}, \; \tilde d^{O(6)}(\theta) = \frac{\theta}{\theta - \frac{1}{2}\mathrm{i}\pi}\frac{-\frac{1}{2}\mathrm{i}\pi}{\mathrm{i}\pi - \theta}.$$

Remark 1. Note, that the amplitudes $\tilde b$ and $\tilde c$ are the same for $SU(4)$ and $O(6)$.

2.3. Bound state S-matrix

The S-matrix eigenvalue $S_-^{SU(4)}(\theta)$ of (2) has a pole at $\theta = \frac{1}{2}\mathrm{i}\pi$ which means that two fundamental particles A and B form an anti-symmetric tensor bound state (AB). The S-matrix for the scattering of these bound states with fundamental particles is given by [14, 22]

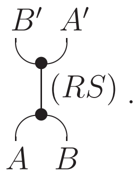
$$S_{(RS)C}^{C'(R'S')}(\theta_{(12)3})\Gamma_{AB}^{(RS)} = \Gamma_{A'B'}^{(R'S')} S_{AC''}^{C'A'}(\theta_{13}) S_{BC}^{C''B'}(\theta_{23})\Big|_{\theta_{12}=\frac{1}{2}i\pi}$$



(10)

where $\theta_{(12)} = \frac{1}{2}(\theta_1 + \theta_2)$ is the bound state rapidity and $\theta_{12}/i = \pi/2$ the bound state fusion angle. The bound state fusion intertwiner $\Gamma_{DE}^{(AB)}$ is defined by

$$i \operatorname*{Res}_{\theta=\frac{1}{2}i\pi} S_{AB}^{B'A'}(\theta) = \sum_{R<S} \Gamma_{(RS)}^{B'A'} \Gamma_{AB}^{(RS)} = \tag{11}$$



With a convenient choice of an undetermined phase factor one has

$$\Gamma_{AB}^{(RS)} = i\Gamma(\tfrac{3}{4})\,(2/\pi)^{1/4}\left(\delta_A^R \delta_B^S - \delta_A^S \delta_B^R\right). \tag{12}$$

Applying formula (10) twice we get the S-matrix for the scattering of two bound states. For example we obtain

$$b^{SU(4)} b^{SU(4)} b^{SU(4)} b^{SU(4)} + b^{SU(4)} c^{SU(4)} b^{SU(4)} b^{SU(4)} - b^{SU(4)} c^{SU(4)} b^{SU(4)} c^{SU(4)} = -b^{O(6)}(\theta)$$

where the arguments on the left hand side are to be taken as $\theta + \frac{1}{2}\mathrm{i}\pi, \theta, \theta, \theta - \frac{1}{2}\mathrm{i}\pi$. There are similar formulas for the other amplitudes. The result is the S-matrix for the $O(6)$ Gross–Neveu model up to a sign[3] (see [14]).

We have the map $M_{(RS)}^{\alpha}$ from the anti-symmetric tensor $SU(4)$ bound states to the $O(6)$ vector states (in the complex basis) (see [6, 11, 14])

$$\left.\begin{matrix} SU(4) \text{ bound states} \\ (RS) \in \{(12),(13),(14),(23),(24),(34)\} \end{matrix}\right\} \leftrightarrow \left\{\begin{matrix} O(6) \text{ vector states} \\ \alpha \in \{1,2,3,\bar{3},\bar{2},\bar{1}\} \end{matrix}\right., \tag{13}$$

which means that the no-zero matrix elements are

$$M_{(12)}^{1} = M_{(13)}^{2} = M_{(14)}^{3} = M_{(23)}^{\bar{3}} = M_{(24)}^{\bar{2}} = M_{(34)}^{\bar{1}} = 1.$$

3. Form factors

The matrix element of a local operator $\mathcal{O}(x)$ for a state of n particles of kind α_i with rapidities θ_i

$$\langle\, 0 \,|\, \mathcal{O}(x) \,|\, \theta_1, \ldots, \theta_n \,\rangle_{\underline{\alpha}}^{in} = \mathrm{e}^{-\mathrm{i}x(p_1+\cdots+p_n)} F_{\underline{\alpha}}^{\mathcal{O}}(\underline{\theta}),\ \theta_1 > \theta_2 > \cdots > \theta_n \tag{14}$$

defines the generalized form factor $F_{1\ldots n}^{\mathcal{O}}(\underline{\theta})$, which is a co-vector valued function with components $F_{\underline{\alpha}}^{\mathcal{O}}(\underline{\theta})$.

3.1. Form factor equations

The co-vector valued function $F_{1\ldots n}^{\mathcal{O}}(\underline{\theta})$ is meromorphic in all variables $\theta_1, \ldots, \theta_n$ and satisfies the following relations [23, 24]:

(i) The Watson's equations describe the symmetry property under the permutation of both, the variables θ_i, θ_j and the spaces $i, j = i + 1$ at the same time

$$F_{\ldots ij \ldots}^{\mathcal{O}}(\ldots, \theta_i, \theta_j, \ldots) = F_{\ldots ji \ldots}^{\mathcal{O}}(\ldots, \theta_j, \theta_i, \ldots)\, S_{ij}(\theta_{ij}) \tag{15}$$

for all possible arrangements of the θ's.

[3] This is because the fundamental Gross–Neveu particles are fermions (see [14]).

(ii) The crossing relation implies a periodicity property under the cyclic permutation of the rapidity variables and spaces

$$\begin{aligned}&{}^{\text{out},\bar 1}\langle\, p_1\,|\,\mathcal{O}(0)\,|\,p_2,\ldots,p_n\,\rangle^{\text{in,conn.}}_{2\ldots n}\\ &\qquad= F^{\mathcal{O}}_{1\ldots n}(\theta_1+\mathrm{i}\pi,\theta_2,\ldots,\theta_n)\sigma^{\mathcal{O}}_1\mathbf{C}^{\bar 1 1}=F^{\mathcal{O}}_{2\ldots n1}(\theta_2,\ldots,\theta_n,\theta_1-\mathrm{i}\pi)\mathbf{C}^{1\bar 1}\end{aligned} \tag{16}$$

where $\sigma^{\mathcal{O}}_\alpha$ takes into account the statistics of the particle α with respect to $\mathcal{O}$[4]. The charge conjugation matrix $\mathbf{C}^{\bar 1 1}$ will be discussed below.

(iii) There are poles determined by one-particle states in each sub-channel given by a subset of particles of the state in (14). In particular the function $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ has a pole at $\theta_{12}=\mathrm{i}\pi$ such that

$$\operatorname*{Res}_{\theta_{12}=\mathrm{i}\pi} F^{\mathcal{O}}_{1\ldots n}(\theta_1,\ldots,\theta_n)=2i\,\mathbf{C}_{12}\,F^{\mathcal{O}}_{3\ldots n}(\theta_3,\ldots,\theta_n)\left(\mathbf{1}-\sigma^{\mathcal{O}}_2 S_{2n}\ldots S_{23}\right). \tag{17}$$

(iv) If there are also bound states in the model the function $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ has additional poles. If for instance the particles 1 and 2 form a bound state (12), there is a pole at $\theta_{12}=\mathrm{i}\eta$ such that

$$\operatorname*{Res}_{\theta_{12}=\mathrm{i}\eta} F^{\mathcal{O}}_{12\ldots n}(\theta_1,\theta_2,\ldots,\theta_n)=F^{\mathcal{O}}_{(12)\ldots n}(\theta_{(12)},\ldots,\theta_n)\,\sqrt{2}\Gamma^{(12)}_{12} \tag{18}$$

where the bound state intertwiner $\Gamma^{(12)}_{12}$ is here given by (12) and the values of θ_1, θ_2, $\theta_{(12)}$ are given in general in [14, 22, 25].

(v) Naturally, since we are dealing with relativistic quantum field theories we finally have

$$F^{\mathcal{O}}_{1\ldots n}(\theta_1+\mu,\ldots,\theta_n+\mu)=\mathrm{e}^{s\mu}\,F^{\mathcal{O}}_{1\ldots n}(\theta_1,\ldots,\theta_n) \tag{19}$$

if the local operator transforms under Lorentz transformations as $\mathcal{O}\to\mathrm{e}^{s\mu}\mathcal{O}$ where s is the 'spin' of $\mathcal{O}$.

For the $SU(4)$ S-matrix (4) the bound state pole is at $\theta=\frac{1}{2}\mathrm{i}\pi$, i.e. $\eta=\frac{1}{2}\pi$.

3.2. The general form factor formula

We write the general form factor $F^{\mathcal{O}}_{1\ldots n}(\underline{\theta})$ for n fundamental particles following [23] as

$$F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})=K^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})\prod_{1\leqslant i<j\leqslant n}F(\theta_{ij}) \tag{20}$$

where $F(\theta)$ is the minimal form factor function (see below). The K-function $K^{\mathcal{O}}_{1\ldots n}(\underline{\theta})$ is given in terms of a nested 'off-shell' Bethe ansatz (see e.g. [6, 10])[5]

$$\boxed{K^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})=\int_{\mathcal{C}^{(1)}_{\underline{\theta}}}\mathrm{d}z_1\cdots\int_{\mathcal{C}^{(m)}_{\underline{\theta}}}\mathrm{d}z_m\,h(\underline{\theta},\underline{z})\,p^{\mathcal{O}}(\underline{\theta},\underline{z})\,\tilde\Psi_{\underline{\alpha}}(\underline{\theta},\underline{z})} \tag{21}$$

written as a multiple contour integral. The scalar function $h(\underline{\theta},\underline{z})$ depends only on the S-matrix and not on the specific operator $\mathcal{O}(x)$

[4] The statistics factor $\sigma^{\mathcal{O}}_1$ is determined by the space-like commutation rule of the operator $\mathcal{O}$ and the field which creates the particle 1.

[5] A more general form of the nested Bethe ansatz where the p-function depends on all level z-variables is discussed in section 3.1 of [6].

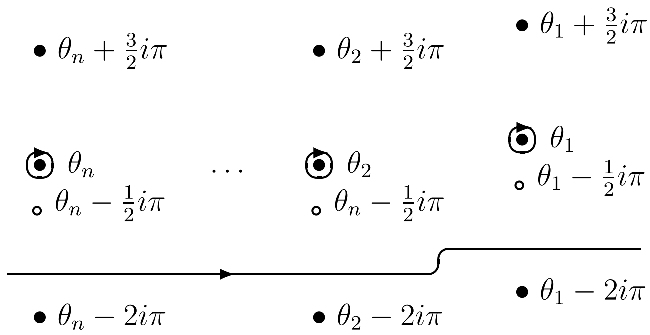


Figure 2. The integration contour for $SU(4)$.

$$h(\underline{\theta}, \underline{z}) = \prod_{i=1}^{n} \prod_{j=1}^{m} \tilde{\phi}(\theta_i - z_j) \prod_{1 \leqslant i < j \leqslant m} \tau(z_i - z_j) \tag{22}$$

$$\tau(z) = \frac{1}{\tilde{\phi}(-z)\tilde{\phi}(z)}. \tag{23}$$

The dependence on the specific operator $\mathcal{O}(x)$ is encoded in the scalar p-function[6] $p^{\mathcal{O}}(\underline{\theta}, \underline{z})$ which is in general a simple function of e^{θ_i} and e^{z_j}. The function $\tilde{\phi}(\theta)$ and the integration contours $\mathcal{C}_{\underline{\theta}}$ depend on the model and are given below.

3.3. *SU*(4) form factors

3.3.1. Minimal form factor. Let $S(\theta)$ be an S-matrix eigenvalue. The solution of Watson's and the crossing equations (i) and (ii) for two particles

$$\begin{aligned} F(\theta) &= S(\theta) F(-\theta) \\ F(\mathrm{i}\pi + \theta) &= F(\mathrm{i}\pi - \theta) \end{aligned} \tag{24}$$

with no poles in the physical strip $0 \leqslant \mathrm{Im}\theta \leqslant \pi$ and at most a simple zero at $\theta = 0$ is the minimal form factor [23]. For the construction of the 'off-shell' Bethe ansatz the minimal form factor for the highest weight eigenvalue of the $SU(4)$ S-matrix $a^{SU(4)}(\theta)$ of (3) is essential. The unique solution (up to a constant factor) is

$$F^{SU(4)}(\theta) = \frac{G\left(\frac{1}{2}\frac{\theta}{\mathrm{i}\pi}\right) G\left(1 - \frac{1}{2}\frac{\theta}{\mathrm{i}\pi}\right)}{G\left(\frac{3}{4} + \frac{1}{2}\frac{\theta}{\mathrm{i}\pi}\right) G\left(\frac{7}{4} - \frac{1}{2}\frac{\theta}{\mathrm{i}\pi}\right)} \tag{25}$$

where $G(z)$ is Barnes G-function, which satisfies (see e.g. [26])

$$G(1 + z) = \Gamma(z) G(z).$$

3.3.2. The n-particle form factor for SU(4). Is given by (21) and the function $\tilde{\phi}(\theta)$ in (22) and (23) is (see [6])

$$\tilde{\phi}(\theta) = \Gamma\left(\frac{3}{4} + \frac{1}{2\pi\mathrm{i}}\theta\right) \Gamma\left(-\frac{1}{2\pi\mathrm{i}}\theta\right). \tag{26}$$

The integration contour in (21) for $SU(4)$ is depicted in figure 2

[6] How the statistics factors enter the periodicity rules for the p-functions is presented in section 3.1 of [6].

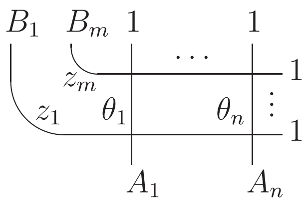
3.3.3. Nesting. The Bethe state in (21) for $SU(4)$ is written as

$$\tilde{\Psi}_{\underline{A}}(\underline{\theta},\underline{z}) = K^{(1)}_{\underline{B}}(\underline{z})\,\tilde{\Phi}^{\underline{B}}_{\underline{A}}(\underline{\theta},\underline{z}) \tag{27}$$

where $\underline{A} = (A_1,\ldots,A_n)$ with $1 \leqslant A_i \leqslant 4$ and summation over all $\underline{B} = (B_1,\ldots,B_m)$ with $2 \leqslant B_i \leqslant 4$ is assumed. The basic Bethe ansatz co-vectors (in the algebraic formulation) $\tilde{\Phi}^{\underline{B}}_{1\ldots n} \in \left(V^{1\ldots n}\right)^{\dagger}$ are defined as [6]

$$\tilde{\Phi}^{\underline{B}}_{1\ldots n}(\underline{\theta},\underline{z}) = \Omega_{1\ldots n}\tilde{C}^{B_m}_{1\ldots n}(\underline{\theta},z_m)\cdots\tilde{C}^{B_1}_{1\ldots n}(\underline{\theta},z_1)$$

$$\tilde{\Phi}^{\underline{B}}_{\underline{A}}(\underline{\theta},\underline{z}) =$$



$$, \quad \begin{array}{l} 2 \le B_i \le 4 \\ 1 \le A_i \le 4\,. \end{array} \tag{28}$$

The nested Bethe ansatz is obtained by writing for $K^{(1)}_{\underline{B}}(\underline{z})$ of (27) an ansatz as (21) and so on: for $K^{(1)}_{\underline{B}}(\underline{z}^{(1)})$ we have an $SU(3)$ and for $K^{(2)}_{\underline{C}}(\underline{z}^{(2)})$ an $SU(2)$ Bethe ansatz, which is well known. The number $m = n_1$ in (28) is the number of 'weight flip' operators. These numbers for the various levels of the nested Bethe ansatz satisfy [6]

$$(n - n_1, n_1 - n_2, n_2 - n_3, n_3) = w^{O} + L(1,1,1,1) \tag{29}$$

where w^{O} is the weight vector of the operator $\mathcal{O}$ and $L = 0, 1, 2, \ldots$.

3.4. $O(6)$ form factors

3.4.1. Minimal form factors. The solutions of Watson's and the crossing equations (i) and (ii) for two particles (24) with no poles in the physical strip $0 \leqslant \mathrm{Im}\theta \leqslant \pi$ and at most a simple zero at $\theta = 0$ are the minimal form factors [23]

$$\left(F_0^{O(6)}, F_+^{O(6)}, F_-^{O(6)}\right)^{\min} = \left(\frac{2\tanh\frac{1}{2}\left(\mathrm{i}\pi - \theta\right)}{\mathrm{i}\pi - \theta}, \frac{\Gamma\left(\frac{5}{4} - \frac{1}{2\pi\mathrm{i}}\theta\right)\Gamma\left(\frac{1}{4} + \frac{1}{2\pi\mathrm{i}}\theta\right)}{\Gamma^2\left(\frac{3}{4}\right)\cosh\frac{1}{2}\left(\mathrm{i}\pi - \theta\right)}, 1\right) F_-^{O(6)\,\min}. \tag{30}$$

They belong to the S-matrix eigenvalues $S_0^{O(6)}$ and $S_\pm^{O(6)}$ of (8). The full 2-particle form factors are

$$F^{O(6)}_{+,-,0}(\theta) = \frac{1}{\sinh\frac{1}{2}(\theta - \frac{1}{2}\mathrm{i}\pi)\sinh\frac{1}{2}(\theta + \frac{1}{2}\mathrm{i}\pi)} F^{O(6)\,\min}_{+,-,0}(\theta)\,. \tag{31}$$

They are non-minimal solutions of (24) having a pole at $\theta = \frac{1}{2}\mathrm{i}\pi$ (see (5.10) and (2.16) of [23]). For the construction of the 'off-shell' Bethe ansatz the minimal solution of the form factor equation (24) for the highest weight eigenvalue of the $O(N)$ S-matrix[7]

$$F^{O(6)}(\theta) = -a^{O(6)}(\theta)F^{O(6)}(-\theta) \tag{32}$$

[7] The minus sign in (32) is due to fermionic statistics of the fundamental particles (see also equation 4.12 of [27]).

is essential. The unique solution (up to a constant factor) is

$$
\begin{aligned}
F^{O(6)}(\theta) &= c\cosh\frac{1}{2}(\mathrm{i}\pi-\theta)\,F_{+}^{O(6)\,\min}(\theta)\\
&= \frac{G\left(\frac{1}{2}\frac{\theta}{\mathrm{i}\pi}\right)G\left(1-\frac{1}{2}\frac{\theta}{\mathrm{i}\pi}\right)}{G\left(\frac{1}{2}+\frac{1}{2}\frac{\theta}{\mathrm{i}\pi}\right)G\left(\frac{3}{2}-\frac{1}{2}\frac{\theta}{\mathrm{i}\pi}\right)}\frac{G\left(\frac{1}{4}+\frac{1}{2}\frac{\theta}{\mathrm{i}\pi}\right)G\left(\frac{5}{4}-\frac{1}{2}\frac{\theta}{\mathrm{i}\pi}\right)}{G\left(\frac{3}{4}+\frac{1}{2}\frac{\theta}{\mathrm{i}\pi}\right)G\left(\frac{7}{4}-\frac{1}{2}\frac{\theta}{\mathrm{i}\pi}\right)}.
\end{aligned} \tag{33}
$$

The function $\tilde{\phi}(\theta)$ in (22) is the same as (26) for $SU(4)$ and the integration contours in (21) can be found in [11].

3.5. Bound state form factors

The statistics factor of two fundamental particles in the chiral $SU(N)$ Gross–Neveu model [6, 19, 20] is $\sigma = \exp(2\pi \mathrm{i} s)$, where $s = \frac{1}{2}\left(1-\frac{1}{N}\right)$ is the spin. For $SU(4)$ this means that the spin is $s = \frac{3}{8}$, and the statistics factor is $\sigma = \exp\left(\frac{3}{4}\pi \mathrm{i}\right)$. In particular, the bound states of two fundamental $SU(4)$ particles are fermions because $\sigma^4 = -1$.

An $n' = n/2$-particle form factor for $O(6)$ is calculated from an n-particle one of $SU(4)$ using the bound state formula (iv) of (18)

$$
F_{\underline{\alpha}}^{O(6)}(\underline{\omega})\Gamma_{\underline{A}}^{\underline{\alpha}} = 2^{-n/4}\operatorname*{Res}_{\theta_{12}=\frac{1}{2}\mathrm{i}\pi}\dots\operatorname*{Res}_{\theta_{n-1n}=\frac{1}{2}\mathrm{i}\pi}F_{\underline{A}}^{SU(4)}(\underline{\theta}) \tag{34}
$$

where $\Gamma_{\underline{A}}^{\underline{\alpha}} = \Gamma_{A_1A_2}^{\alpha_1}\dots\Gamma_{A_{n-1}A_n}^{\alpha_{n'}}$ is the total intertwiner and $\omega_i = \frac{1}{2}(\theta_{2i-1}+\theta_{2i})$ are the bound state rapidities.

Lemma 2. *The bound state form factors defined by* (34) *satisfy the form factor equations* (*i*)—(*v*) *of* (15)–(18)*. The K-functions defined by* (20) *and* (34) *satisfy, in particular*

1.

$$
K_{\underline{\alpha}}^{O(6)}(\underline{\omega})\Gamma_{\underline{A}}^{\underline{\alpha}} = 2^{-n/4}\prod_{1\leqslant i<j\leqslant n'}\frac{1}{\tilde{\phi}(-\omega_{ij})\tilde{\phi}(-\omega_{ij}+\frac{1}{2}\mathrm{i}\pi)}\operatorname*{Res}_{\theta_{12}=\frac{1}{2}\mathrm{i}\pi}\dots\operatorname*{Res}_{\theta_{n-1n}=\frac{1}{2}\mathrm{i}\pi}K_{\underline{A}}^{SU(4)}(\underline{\theta}) \tag{35}
$$

2. *the form factor equation* (*iii*) *in the form* (*see* [11])

$$
\operatorname*{Res}_{\omega_{12}=\mathrm{i}\pi}K_{1\dots n'}^{O(6)}(\underline{\omega}) = \frac{2\mathrm{i}}{F^{O(6)}(\mathrm{i}\pi)}\mathbf{C}_{12}\prod_{i=3}^{n'}\tilde{\phi}(\omega_{i1}+\frac{1}{2}\mathrm{i}\pi)\tilde{\phi}(\omega_{i2})K_{3\dots n'}^{O(6)}(\underline{\breve{\omega}})\,(\mathbf{1}-S_{2n'}\dots S_{23}) \tag{36}
$$

with $\underline{\breve{\omega}} = \omega_3,\dots\omega_{n'}$.

Proof. In appendix E of [25] was proved that in general bound state form factors satisfy the form factor equations. We use the variables u, o with $\theta = \frac{1}{2}\mathrm{i}\pi u,\ \omega = \frac{1}{2}\mathrm{i}\pi o$.

1. Equation (34) implies for the K-functions (35) because from (25) and (33) we derive

$$
\frac{F^{SU(4)}\left(\omega+\frac{1}{2}\mathrm{i}\pi\right)\left(F^{SU(4)}(\omega)\right)^2F^{SU(4)}\left(\omega-\frac{1}{2}\mathrm{i}\pi\right)}{F^{O(6)}(\omega)} = \frac{1}{\tilde{\phi}(-\omega)\tilde{\phi}(-\omega+\frac{1}{2}\mathrm{i}\pi)}. \tag{37}
$$

2. This follows from the general proof of (iii) in [25] and (35). One can also prove it directly from (iii) for $F_{\underline{A}}^{SU(4)}(\underline{\theta})$, equation (35) and (up to a const.)

$$\frac{\prod\limits_{3\leqslant i<j\leqslant n'}\tilde{\phi}(-\omega_{ij})\tilde{\phi}(-\omega_{ij}+\frac{1}{2}\mathrm{i}\pi)}{\prod\limits_{1\leqslant i<j\leqslant n'}\tilde{\phi}(-\omega_{ij})\tilde{\phi}(-\omega_{ij}+\frac{1}{2}\mathrm{i}\pi)}\prod_{i=5}^{n}\prod_{j=2}^{4}\tilde{\phi}(\theta_{ij})=\prod_{i=3}^{n'}\tilde{\phi}(\omega_{i1}+\frac{1}{2}\mathrm{i}\pi)\tilde{\phi}(\omega_{i2})$$

for $\theta_{2i-1}=\omega_i+\frac{1}{4}\mathrm{i}\pi,\ \theta_{2i}=\omega_i-\frac{1}{4}\mathrm{i}\pi$ and $\omega_{12}=\mathrm{i}\pi$. ∎

This lemma implies the following

Corollary 3. *In* [14] *we demonstrated that the isomorphism $O(6)\simeq SU(4)$, leads to an equivalence between the $O(6)$ Gross–Neveu model and the $SU(4)$ chiral Gross–Neveu model for the S-matrices. The results of this section show, that this is also true for the form factors.*

4. Examples of operators

We use the results of [6] and [11].

4.1. The current $J^\mu(x)$

4.1.1. The $SU(4)$ form factor. The $SU(4)$ Noether current $J^\mu_{A\bar{B}}(x)$ transforms as the adjoint representation with weight vector $w^J=(2,1,1,0)$. Because the Bethe ansatz yields highest weight states we consider the highest weight component

$$J^\mu_{1\bar{4}}(x)=\epsilon^{\mu\nu}\partial_\nu J(x),$$

where the anti-particle $\bar{4}$ is defined by (5) and $J(x)$ is the pseudo potential with the p-function in (21) (see section 4.3 of [6])

$$p^J(\underline{\theta},\underline{\underline{z}})=e^{\frac{1}{2}\left(\sum\theta_i-\sum z_i^{(1)}-\sum z_i^{(3)}\right)}\Big/\sum \mathrm{e}^{\theta_i}. \tag{38}$$

The n-particle current form factor for $SU(4)$ is given by (20) and the nested 'off-shell' Bethe ansatz (21) with the p-function (38). The numbers of 'weight flip' operators in the various levels of the nested Bethe ansatz are given by (29) as $n=4+4L,\ n_1=2+3L,\ n_2=1+2L,\ n_3=L$. In particular we consider $L=0$, i.e. $n=4,\ n_1=2,\ n_2=1$ and $n_3=0$. The Bethe state in (21) is then

$$\begin{aligned}\Psi_{\underline{A}}(\underline{\theta},\underline{z})&=K^{(1)}_{\underline{B}}(\underline{z})\Phi^{\underline{B}}_{\underline{A}}(\underline{\theta},\underline{z})\\ K^{(1)}_{\underline{B}}(\underline{z})&=\int\mathrm{d}y\prod_{j=1}^{2}\tilde{\phi}(z_j-y)\Psi^{(1)}_{\underline{B}}(\underline{z},y)\\ \Psi^{(1)}_{\underline{B}}(\underline{z},y)&=\delta^2_{B_1}\delta^3_{B_2}\tilde{b}(z_1-y)\tilde{c}(z_2-y)+\delta^3_{B_1}\delta^2_{B_2}\tilde{c}(z_1-y)\end{aligned}\tag{39}$$

(see also figure 3). Below we use this formula to calculate the bound state form factor.

4.1.2. The $O(6)$ form factor. The $O(6)$ Noether current transforms as an antisymmetric $O(N)$ tensor with weights $w^J=(1,1,0)$. The bound state formula (34) applied to (20,21) with the p-function (38) yields the $O(6)$ current form factor for $n/2$ particles. In particular we consider the case $n=4$.

Proposition 4. *The bound state formula* (34) *for $n=4$ and* (20) *and* (21) *with the p-function* (38) *yield the two particle $O(6)$ form factor of the pseudo-potential $J^{\alpha\beta}(x)$ and the current $J^{\alpha\beta}_\mu(x)=\epsilon_{\mu\nu}\partial^\nu J^{\alpha\beta}(x)$*

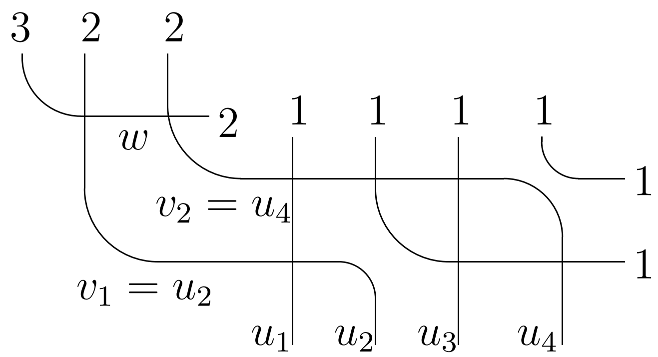


Figure 3. The Bethe state $\Psi_{\underline{A}}(\underline{u}, \underline{v})$ in (42) for $v_1 = u_2, v_2 = u_4$.

$$F^{O(6),J^{\alpha\beta}}_{\alpha_1\alpha_2}(\theta_1, \theta_2) = \mathrm{i}m\left(\delta^{\alpha}_{\alpha_1}\delta^{\beta}_{\alpha_2} - \delta^{\beta}_{\alpha_1}\delta^{\alpha}_{\alpha_2}\right)\frac{1}{\cosh\frac{1}{2}\theta_{12}}F^{O(6)}_{-}(\theta) \tag{40}$$

$$F^{O(6),J^{\alpha\beta}_{\mu}}_{\alpha_1\alpha_2}(\theta_1, \theta_2) = \mathrm{i}\left(\delta^{\alpha}_{\alpha_1}\delta^{\beta}_{\alpha_2} - \delta^{\beta}_{\alpha_1}\delta^{\alpha}_{\alpha_2}\right)\bar{v}(\theta_1)\gamma_\mu u(\theta_2)F^{O(6)}_{-}(\theta) \tag{41}$$

which agrees with the results of [11].

Proof. We have $n = 4$, $n_1 = 2$, $n_2 = 1$ and $n_3 = 0$. For convenience we use here the variables u, v, w with $\theta = \mathrm{i}\pi\frac{1}{2}u$, $z = \mathrm{i}\pi\frac{1}{2}v$, $y = \mathrm{i}\pi\frac{1}{2}w$ and calculate (always up to constants)

$$\begin{aligned}\operatorname*{Res}_{u_{12}=1}\operatorname*{Res}_{u_{34}=1} K^{SU(4),J}_{\underline{A}}(\underline{u}) &= \operatorname*{Res}_{u_{12}=1}\operatorname*{Res}_{u_{34}=1}\int_{\mathcal{C}_{\underline{u}}} \mathrm{d}\underline{v}\, h(\underline{u},\underline{v})\; p^J(\underline{u},\underline{v})\Psi_{\underline{A}}(\underline{u},\underline{v}) \\ &= \operatorname*{Res}_{u_{12}=1}\operatorname*{Res}_{u_{34}=1}\operatorname*{Res}_{v_1=u_2}\operatorname*{Res}_{v_2=u_4} h(\underline{u},\underline{v})\; p^J(\underline{u},\underline{v})\Psi_{\underline{A}}(\underline{u},\underline{v})\end{aligned} \tag{42}$$

because the residues are obtained by pinchings at $v_1 = u_2, v_2 = u_4$ which imply that the S-matrices $S(u_2 - v_1)$ and $S(u_4 - v_2)$ are replaced by the permutation operator (see figure 3). Using Yang-Baxter relations and the formula for the fusion intertwiner (11) we obtain

$$\operatorname*{Res}_{u_{12}=1}\operatorname*{Res}_{u_{34}=1} K^{SU(4),J}_{\underline{A}}(\underline{u}) = \tilde{\phi}(u_{14})\tilde{\phi}(u_{32})K^{(1)}_{\underline{B}}(u_{24})\tilde{b}(u_{14})\,p^J(\underline{u}, u_2, u_4)\left(\Gamma^{B_1 1}_{\alpha}\Gamma^{\alpha}_{A_1A_2}\Gamma^{B_2 1}_{\beta}\Gamma^{\beta}_{A_3A_4}\right).$$

With (39) we have (again up to constants)

$$\begin{aligned}K^{(1)}_{32}(\underline{v}) &= \int_{\mathcal{C}_{\underline{v}}} \mathrm{d}w\Gamma\left(-\frac{1}{4}(v_1 - w)\right)\Gamma\left(-\frac{1}{4} + \frac{1}{4}(v_1 - w)\right)\Gamma\left(-\frac{1}{4}(v_2 - w)\right)\Gamma\left(\frac{3}{4} + \frac{1}{4}(v_2 - w)\right) \\ &= \Gamma\left(\frac{3}{4} - \frac{1}{4}v_{12}\right)\Gamma\left(-\frac{1}{4} + \frac{1}{4}v_{12}\right)\end{aligned}$$

where $\tilde{\phi}(v_1 - w)\tilde{c}(v_1 - w) \propto \Gamma(-\frac{1}{4}(v_1 - w))\Gamma(-\frac{1}{4} + \frac{1}{4}(v_1 - w))$ and the Gauss formula

$${}_2F_1(a, b; c; 1) = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{\Gamma(a+n)}{\Gamma(a)}\frac{\Gamma(b+n)}{\Gamma(b)}\frac{\Gamma(c)}{\Gamma(c+n)} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{43}$$

have been used. Similarly, we calculate $K^{(1)}_{23}(\underline{v})$ and get $K^{(1)}_{23}(\underline{v}) = -K^{(1)}_{32}(\underline{v})$. Finally using (38)

$$\underset{u_{12}=1}{\text{Res}}\ \underset{u_{34}=1}{\text{Res}}\ K^{SU(4),J}_{\underline{A}}(\underline{u}) = K^J(o)\left(\Gamma^{(12)}_{A_1A_2}\Gamma^{(13)}_{A_3A_4} - \Gamma^{(13)}_{A_1A_2}\Gamma^{(12)}_{A_3A_4}\right)$$

$$K^J(o) = \tilde{\phi}(u_{14})\tilde{\phi}(u_{32})\Gamma\left(\frac{3}{4}-\frac{1}{4}u_{24}\right)\Gamma\left(-\frac{1}{4}+\frac{1}{4}u_{24}\right)\tilde{b}(u_{14})\,p^J\left(\underline{u},u_2,u_4\right)$$

$$= \frac{1}{\sin\frac{1}{2}\pi o}\left(\Gamma\left(\frac{3}{4}-\frac{1}{4}o\right)\Gamma\left(-\frac{1}{4}+\frac{1}{4}o\right)\right)^2$$

with $o = u_{(12)(34)} = u_{13} = u_{24} = u_{14} - \frac{1}{2} = u_{23} + \frac{1}{2}$. The result (40) follows then from (37), (30) and (31). ■

4.2. The iso-scalar operator $\mathcal{O}$

The $SU(4)$ n-particle form factor for the iso-scalar operator $\mathcal{O}(x)$ with weights $w^{\mathcal{O}} = (0,0,0,0)$ is given by (20) and the nested 'off-shell' Bethe ansatz (21). The numbers of 'weight flip' operators in the various levels of the nested Bethe ansatz are given by (29) as $n = 4+4L$, $n_1 = 3+3L$, $n_2 = 2+2L$, $n_3 = 1+L$. We propose for the iso-scalar operator $\mathcal{O}(x)$ the p-function

$$p^{\mathcal{O}}\left(\underline{\theta},\underline{\underline{z}}\right) = e^{\frac{1}{2}\sum\theta_i - \sum z_i^{(1)} + \sum z_i^{(3)}} - 1. \tag{44}$$

With this p-function in (21) the form factor equations (i)—(v) of (15)–(18) hold with statistics factor $\sigma_1^{\mathcal{O}} = -1$ and spin $s^{\mathcal{O}} = 0$. The bound state formula (34) applied to (20) and (21) with the p-function (44) yields the $O(6)$ form factor of the operator $\bar{\psi}\psi(x)$ for $n/2$ particles. In particular we consider the case $L = 0$, i.e $n = 4$, $n_1 = 3$, $n_2 = 2$ and $n_3 = 1$.

Proposition 5. *The bound state formula (34) applied to (20) and (21) with the p-function (44) yields the two particle $O(6)$ form factors of $\bar{\psi}\psi$*

$$F^{\bar{\psi}\psi}_{\alpha_1\alpha_2}(\underline{\theta}) = \langle\, 0\,|\,\bar{\psi}\psi(0)\,|\,p_1,p_2\,\rangle^{in}_{\alpha_1\alpha_2} = \mathbf{C}_{\alpha_1\alpha_2}\,\bar{v}(\theta_1)u(\theta_2)\,F_0(\theta_{12}) \tag{45}$$

which means for the energy momentum operator[8] $T^{\mu\nu}$

$$F^{T^{\mu\nu}}_{\alpha_1\alpha_2}(\underline{\theta}) = \langle\, 0\,|\,T^{\mu\nu}(0)\,|\,p_1,p_2\,\rangle^{in}_{\alpha_1\alpha_2} = \mathbf{C}_{\alpha_1\alpha_2}\bar{v}(\theta_1)\gamma^{\mu}u(\theta_2)\,\frac{1}{2}(p_1^{\nu}-p_2^{\nu})\,F_0(\theta_{12})$$

with $F_0(\theta)$ given by (30) and (31) which agrees with the results of [11].

Proof. The more general proof in appendix implies for $\nu = 1/2$

$$\underset{\theta_{12}=\frac{1}{2}\mathrm{i}\pi}{\text{Res}}\ \underset{\theta_{34}=\frac{1}{2}\mathrm{i}\pi}{\text{Res}}\ F^{SU(4),\mathcal{O}}_{1234}(\theta_1,\ldots,\theta_4) = \text{const.}\,\frac{\Gamma\left(\frac{3}{4}-\frac{1}{2}\frac{\omega}{\mathrm{i}\pi}\right)\Gamma\left(-\frac{1}{4}+\frac{1}{2}\frac{\omega}{\mathrm{i}\pi}\right)}{\Gamma\left(\frac{3}{2}-\frac{1}{2}\frac{\omega}{\mathrm{i}\pi}\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\frac{\omega}{\mathrm{i}\pi}\right)}F^{O(6)}(\omega)\,.$$

with $\omega = \theta_{(12)} - \theta_{(34)}$. Together with (30) and (31) the claim (45) follows. ■

4.3. The $O(6)$ Gross–Neveu field $\psi(x)$

4.3.1. The $SU(4)$ form factor. We follow [6] and define the $SU(4)$ operator $\mathcal{O}^{AB} = \left[\psi^A,\psi^B\right]$ where $\psi^A(x)$ is the fundamental field of the chiral $SU(4)$-Gross–Neveu model. It has the

[8] This follows from $\partial_\mu T^{\mu\nu} = 0$ and $T^{\mu}_{\mu} = m\bar{\psi}\psi$.

weight vector $w^{\mathcal{O}} = (1,1,0,0)$. We write the highest weight component $[\psi^1, \psi^2]$ as $\mathcal{O}$ and propose the p-function (see section 4.2 of [6])

$$p^{\mathcal{O}(\pm)}(\underline{\theta}, \underline{z}) = \left(p^{\psi(\pm)}(\underline{\theta}, \underline{z})\right)^2 = \mathrm{e}^{\pm\left(\sum_{i=1}^m z_i - \frac{3}{4}\sum_{i=1}^n \theta_i\right)} \tag{46}$$

belonging to the $\pm$ spinor components. The form factors are again given by (20) and (21). The numbers of 'weight flip' operators in the various levels of the nested Bethe ansatz are given by (29) as $n = 2 + 4L$, $n_1 = 1 + 3L$, $n_2 = 2L$, $n_3 = L$.

4.3.2. The $O(6)$ form factor. The fundamental $O(6)$ field ψ^α is fermionic and transforms as the vector representation with weight vector $w^\psi = (1,0,0)$ [11]. It is given in terms of $\mathcal{O}^{AB}$ by (12) and (13)

$$\psi^\alpha = M^\alpha_{(RS)} \Gamma^{(RS)}_{AB} \left[\psi^A, \psi^B\right].$$

The bound state formula (34) applied to (20) and (21) with the p-function (46) yields the $O(6)$ form factor for $n/2$ particles. In particular we consider the case $L = 0$, i.e. $n = 2,\ m = 1$

$$\begin{aligned}\operatorname*{Res}_{\theta_{12}=\mathrm{i}\pi 2/3} K^{SU(4),\mathcal{O}(\pm)}_{\underline{A}}(\underline{\theta}) &= \operatorname*{Res}_{\theta_{12}=\mathrm{i}\pi 2/3} \int_{\mathcal{C}_{\underline{\theta}}} \mathrm{d}z\, \tilde{\phi}(\theta_1 - z)\, \tilde{\phi}(\theta_2 - z)\, \mathrm{e}^{\pm\left(z - \frac{3}{4}(\theta_1+\theta_2)\right)}\, \tilde{\Psi}_{\underline{A}}(\underline{\theta}, z)\\ &= \tilde{\phi}(\theta_{12})\, \mathrm{e}^{\pm\left(\theta_2 - \frac{3}{4}(\theta_1+\theta_2)\right)} \operatorname*{Res}_{\theta_{12}=\mathrm{i}\pi 2/3} \tilde{S}^{21}_{A_1A_2}(\theta_{12})\end{aligned}$$

where pinching at $z = \theta_2$ was used. Therefore the $O(6)$ one particle form factor of the field is with $\theta = \frac{1}{2}(\theta_1 + \theta_2)$ (up to const.)

$$F_1^{O(6),\psi(\pm)}(\theta) = \mathrm{e}^{\mp\frac{1}{2}\theta} = u^{(\pm)}(\theta)$$

as expected.

5. $\mathbf{O(6) \simeq SU(4)}$ as a start of level iteration for $\mathbf{O(N)}$

5.1. The modified n*-particle K-function for* $O(6)$

The $O(N)$ Gross–Neveu form factors are given by the 'off-shell' nested Bethe ansatz [11]. Therefore we need the higher level $O(N-2k)$ Bethe ansatz for $k = 1, \dots, N/2 - 3$. The last one is of $O(6)$ type. For this discussion it is convenient to introduce the variables u, v with $\theta = \mathrm{i}\pi\nu_k u,\ z = \mathrm{i}\pi\nu_k v$ with $\nu_k = 2/(N - 2k - 2)$. For the $O(N-2k)$ S-matrix $S^{(k)}(u)$ we write

$$\begin{aligned}\tilde{S}^{(k)}(u) &= S^{(k)}/S^{(k)}_+ = \tilde{b}(u)\mathbf{1} + \tilde{c}(u)\mathbf{P} + \tilde{d}_k(u)\mathbf{K}\\ \tilde{b}(u) &= \frac{u}{u-1},\ \tilde{c}(u) = \frac{-1}{u-1},\ \tilde{d}_k(u) = \frac{u}{u-1}\frac{1}{u-1/\nu_k}.\end{aligned} \tag{47}$$

and define the higher level K-functions

$$\begin{aligned}K^{(k)}_{\underline{\alpha}}(\underline{u}) &= \tilde{N}^{(k)}_{m_k} \int_{\mathcal{C}^{(1)}_{\underline{u}}} \mathrm{d}v_1 \cdots \int_{\mathcal{C}^{(m_k)}_{\underline{u}}} \mathrm{d}v_{m_k}\, h(\underline{u}, \underline{v})\, p^{(k)}(\underline{u}, \underline{v})\ \tilde{\Psi}^{(k)}_{\underline{\alpha}}(\underline{u}, \underline{v})\\ \tilde{\Psi}^{(k)}_{\underline{\alpha}}(\underline{u}, \underline{v}) &= K^{(k+1)}_{\underline{\mathring{\beta}}}(\underline{v})\, \left(\tilde{\Phi}^{(k)}\right)^{\underline{\mathring{\beta}}}_{\underline{\alpha}}(\underline{u}, \underline{v})\end{aligned} \tag{48}$$

with $\underline{u} = u_1, \ldots, u_{n_k}$, $\underline{v} = v_1, \ldots, v_{m_k}$ and $m_k = n_{k+1}$. The basic Bethe ansatz co-vectors $\left(\tilde{\Phi}^{(k)}\right)^{\mathring{\underline{\beta}}}_{\underline{\alpha}}(\underline{u}, \underline{v})$ are defined analogously to (28). The function $h(\underline{u}, \underline{v})$ is given by (22) and (23) where $\tilde{\phi}(\theta)$ is replaced by

$$\tilde{\phi}_\nu(\theta) = \Gamma\left(1 - \frac{1}{2}\nu + \frac{1}{2\pi \mathrm{i}}\theta\right)\Gamma\left(-\frac{1}{2\pi \mathrm{i}}\theta\right), \ \nu = \nu_0 = 2/(N-2)$$

The higher level K-functions $K^{(k)}_{\underline{\alpha}}(\underline{u})$ for $k > 0$ satisfy the equations

(i)$^{(k)}$

$$K^{(k)}_{\ldots ij \ldots}(\ldots, u_i, u_j, \ldots) = K^{(k)}_{\ldots ji \ldots}(\ldots, u_j, u_i, \ldots)\,\tilde{S}^{(k)}_{ij}(u_{ij}) \tag{49}$$

(ii)$^{(k)}$

$$K^{(k)}_{1\ldots n_k}(u_1 + 2/\nu, u_2, \ldots, u_{n_k})\sigma_1^{\mathcal{O}}\mathbf{C}^{\bar{1}1} = K^{(k)}_{2\ldots n_k 1}(u_2, \ldots, u_{n_k}, u_1)\mathbf{C}^{1\bar{1}} \tag{50}$$

(iii)$^{(k)}$

$$\operatorname*{Res}_{u_{12}=1/\nu_k} K^{(k)}_{1\ldots n_k}(u_1, \ldots, u_{n_k}) = \prod_{i=3}^{n_k}\tilde{\phi}_\nu(u_{i1}+1)\tilde{\phi}_\nu(u_{i2})\mathbf{C}_{12}K^{(k)}_{3\ldots n_k}(u_3, \ldots, u_{n_k}). \tag{51}$$

The normal form factor equations (i)—(iii) for $O(N-2k)$ are similar to these higher level equations. There are, however, two differences:

1. The shift in (ii)$^{(k)}$ is the one of $O(N)$ but not that of $O(N-2k)$.
2. There is only one term on the right hand side in (iii)$^{(k)}$.

In particular for $k = N/2 - 3 = 1/\nu - 2$ we have $\nu_k = \frac{1}{2}$ and $K^{(k)}_{\underline{\alpha}}(\underline{u}) = K^{O(6),\nu}_{\underline{\alpha}}(\underline{u})$ is of $O(N-2k) = O(6)$ type, which means in particular that the S-matrix and the Bethe state are the ones of $O(6)$. We call $K^{O(6),\nu}_{\underline{\alpha}}$ a modified $\mathbf{O(6)}$ K-function.

5.2. The modified $\mathbf{n}$-particle K-function for $SU(4)$

Replacing in (21) and (26)

$$\tilde{\phi}(\theta) \to \tilde{\phi}_\nu(\theta) = \Gamma\left(1 - \frac{1}{2}\nu + \frac{1}{2\pi \mathrm{i}}\theta\right)\Gamma\left(-\frac{1}{2\pi \mathrm{i}}\theta\right)$$

we obtain the modified $\mathbf{n}$-particle K-function for $\mathbf{SU(4)}$ which satisfies the form factor equation (ii) (see (16)) not for the shift $\theta_1 \to \theta_1 + 2\pi i$ but for $\theta_1 \to \theta_1 + \mathrm{i}\pi/\nu$ and in (iii) (see (17)) the second term on the right hand side is missing. Again we use for convenience the variables u and v with $\theta = \mathrm{i}\pi\nu u$, $z = \mathrm{i}\pi\nu v$, then the K-function (the integration contour is shown in figure 4)

$$K^{SU(4),\nu}_{\underline{A}}(\underline{u}, \underline{\nu}) = \int_{\mathcal{C}_{\underline{u},\nu}} \mathrm{d}\underline{v} \prod_{i=1}^{n}\prod_{j=1}^{m}\tilde{\phi}_\nu(u_i - v_j)\prod_{i<j}\tau_\nu(v_{ij})\,p(\underline{u}, \underline{v})\tilde{\Psi}_{\underline{A}}(\underline{u}, \underline{v}) \tag{52}$$

satisfies for $\nu < \frac{1}{2}$ not the form factor equations (ii) and (iii) of (15)–(19) but the modified ones

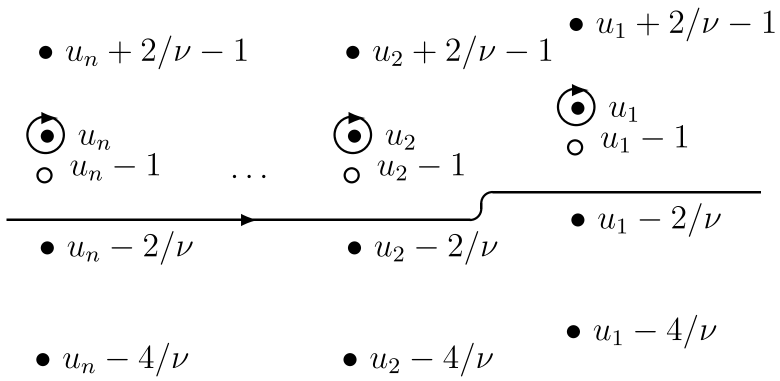


Figure 4. The integration contour $\mathcal{C}_{\underline{u},\nu}$ in (52). The bullets refer to poles of the integrand resulting from $\tilde{\phi}(u_i - v_j)$ and the small open circles refer to poles originating from $\tilde{b}(u_i - v_j)$ and $\tilde{c}(u_i - v_j)$.

(ii)$_\nu$

$$K^{SU(4),\nu}_{1\ldots n}(u_1 + 2/\nu, u_2, \ldots, u_n)\sigma_1^{\mathcal{O}}\mathbf{C}^{\bar{1}1} = K^{SU(4),\nu}_{2\ldots n1}(u_2, \ldots, u_n, u_1)\mathbf{C}^{1\bar{1}} \tag{53}$$

(iii)$_\nu$

$$\operatorname*{Res}_{u_{34}=1}\operatorname*{Res}_{u_{23}=1}\operatorname*{Res}_{u_{12}=1} K^{SU(4),\nu}_{1\ldots n}(\underline{u}) = \prod_{i=5}^{n}\prod_{j=2}^{4}\tilde{\phi}_\nu(u_{ij})\,\varepsilon_{1234}K^{SU(4),\nu}_{5\ldots n}(\underline{\check{u}}) \tag{54}$$

with $\underline{\check{u}} = u_5, \ldots, u_n$.

The proofs of these equations are quite analogous to the ones in [6] for the normal $SU(N)$ K-functions for $N = 4$.

5.3. $n' = n/2$ bound states of $SU(4)$ particles:

We apply the bound state formula (iv) to an n-particle modified K-function of $SU(4)$ and define correspondingly to (35) for $\nu = 2/(N-2)$ an $n' = n/2$-particle $O(6)$ K-function

$$K^{O(6),\nu}_{\underline{\alpha}}(\underline{o})\Gamma^{\underline{\alpha}}_{\underline{A}} = \prod_{1\leqslant i<j\leqslant n'}\frac{1}{\tilde{\phi}_\nu(-o_{ij})\tilde{\phi}_\nu(-o_{ij}+1)}\operatorname*{Res}_{u_{12}=1}\ldots\operatorname*{Res}_{u_{n-1n}=1}K^{SU(4),\nu}_{\underline{A}}(\underline{u}) \tag{55}$$

with $o_i = \frac{1}{2}(u_{2i-1} + u_{2i})$ and the intertwiner $\Gamma^{\underline{\alpha}}_{\underline{A}} = \Gamma^{\alpha_1}_{A_1A_2}\ldots\Gamma^{\alpha_{n'}}_{A_{n-1}A_n}$. Correspondingly to lemma 2 we prove

Lemma 6. *The K-function defined by (55) satisfies the modified form factor equations*

$(i)_\nu$

$$K^{O(6),\nu}_{\ldots ij\ldots}(\ldots o_i, o_j\ldots) = K^{O(6),\nu}_{\ldots ji\ldots}(\ldots o_j, o_i\ldots)\tilde{S}^{O(6)}(o_{ij})$$

$(ii)_\nu$

$$K^{O(6),\nu}_{12\ldots n'}(o_1 + 2/\nu, o_2, \ldots, o_{n'})\mathbf{C}^{\bar{1}1} = K^{O(6),\nu}_{2\ldots n'1}(o_2, \ldots, o_{n'}, o_1)\mathbf{C}^{1\bar{1}}$$

$(iii)_\nu$

$$\mathrm{Res}_{o_{12}=2} K^{O(6),\nu}_{1\dots n'}(\underline{o}) = \prod_{i=3}^{n'} \tilde{\phi}_\nu(o_{i1}+1)\tilde{\phi}_\nu(o_{i2})\mathbf{C}_{12} K^{O(6),\nu}_{3\dots n'}(\underline{\check{o}})$$

with $\underline{\check{o}} = o_3, \dots o_{n'}$.

Proof. We follow here the proof of proposition 7 in [25]. For $(i)_\nu$ and $(ii)_\nu$ the proofs are again obvious. To prove $(iii)_\nu$ one follows appendix E of [25] taking into account that also in (54) there is only one term on the right hand side. ∎

Corollary 7. *The K-function defined by* (55) *satisfies the higher level equations* $(i)^{(k)}$—$(iv)^{(k)}$ *or* (4.13)—(4.16) *of* [11] *for* $k = N/2 - 3$, *i.e.* $\nu_k = 1/2$. *Therefore it serves as a starting of the nesting for the construction of an* $O(N)$*-Gross–Neveu form factor for arbitrary even* $N > 6$.

To construct the form factors of the $O(N)$ Gross–Neveu model for the operators $\bar{\psi}\psi$, $J^{\alpha\beta}_\mu$ and ψ^α with weight vectors $w = (0, 0, \dots, 0)$, $(1, 0, \dots, 0)$ and $(1, 1, 0, \dots, 0)$, respectively, we need for the starting of the nested Bethe ansatz the modified $O(6)$ one for the iso-scalar with weight vectors $w = (0, 0, 0)$. Therefore we generalize the constructions of section 4.2 from $\nu = 1/2$ to general ν and prove

Lemma 8. *The bound state formula* (55) *applied to the modified* $SU(4)$ *K-function* (52) *with the p-function* (44)

$$p\left(\underline{u}, \underline{v}, \underline{w}, \underline{x}\right) = \mathrm{e}^{\mathrm{i}\pi\nu\left(\frac{1}{2}\sum_{i=1}^{4L} u_i - \sum_{i=1}^{3L} v_i + \sum_{i=1}^{L} x_i\right)} - 1 \tag{56}$$

for $L = 1, 2, \dots$ (*see* (29)) *yields the modified* $O(6)$ *K-function for the iso-scalar for* $n' = 2L$ *particles. This means that for* $L = 1$ *the the bound state formula* (55) *yields the modified* $O(6)$ *two-particle K-function*

$$K_{\alpha_1\alpha_2}(o_1, o_2) = \mathbf{C}_{\alpha_1\alpha_2} \frac{\Gamma\left(1 - \frac{1}{2}\nu - \frac{1}{2}\nu o_{12}\right)\Gamma\left(-\frac{1}{2}\nu + \frac{1}{2}\nu o_{12}\right)}{\Gamma\left(1 + \nu - \frac{1}{2}\nu o_{12}\right)\Gamma\left(\nu + \frac{1}{2}\nu o_{12}\right)}. \tag{57}$$

This is the higher level K-function needed as the starting for the nested $O(N)$ *Bethe ansatz* (*see* [11]).

The proof of this lemma can be found in appendix. It follows the main result of this article:

Corollary 9. *For all* $O(N)$ *Gross–Neveu form factors of operators* $\mathcal{O}(x)$ *with weights* $w^{\mathcal{O}} = (w_1, w_2, 0, \dots, 0, 0)$ *the start of the nesting is obtained by* (52) *with the p-function* (56) *and the bound state formula* (55).

6. Conclusions

The form factors for the $SU(N)$ chiral Gross–Neveu model were constructed in [6–8, 15, 16]. In [10] we used the isomorphism $O(4) \simeq SU(2) \times SU(2)$ as the starting point of the nesting procedure to construct the $O(N)$ σ-model form factors. Up to now we were not able to do the analog for the $O(N)$ Gross–Neveu model. However, the fundamental particles of the $O(6)$ Gross–Neveu model may by identified with the bound states of the $SU(4)$ chiral Gross–Neveu model [14]. Using this identification we showed in the present article how to use the $O(6)$ functions as the starting point of the nesting procedure to construct the $O(N)$ Gross–Neveu model form factors (for N even). In a forthcoming article we will consider the $O(4)$ Gross–Neveu model. Also the asymptotic behavior of the form factors and the short distance behavior of the correlation functions will be investigated.

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Appendix.

For simplicity the equations in the following are mostly written up to inessential constants.

A.1. Proof of lemma 8

Proof. We have $n = 4$, $n_1 = 3$, $n_2 = 2$ and $n_3 = 1$. For convenience we use again the variables u, v, w, x, o with $\theta = \mathrm{i}\pi\nu u$, $z^{(1)} = \mathrm{i}\pi\nu v$, $z^{(2)} = \mathrm{i}\pi\nu w$, $z^{(3)} = \mathrm{i}\pi\nu x$, $\omega = \mathrm{i}\pi\nu o$. We prove that (55) for $n = 4$ (with $o_1 = (u_1 + u_2)/2$, $o_2 = (u_3 + u_4)/2$)

$$K^{O(6),\nu}_{\underline{\alpha}}(\underline{o})\Gamma^{\underline{\alpha}}_{\underline{A}} = \frac{1}{\tilde{\phi}_\nu(-o_{12})\tilde{\phi}_\nu(-o_{12}+1)} \operatorname*{Res}_{u_{12}=1} \operatorname*{Res}_{u_{34}=1} K^{SU(4),\nu}_{\underline{A}}(\underline{u})$$

with the p-function (56) implies (57)

$$K^{O(6),\nu}_{\alpha_1\alpha_2\alpha}(o_1, o_2) = \mathbf{C}_{\alpha_1\alpha_2} \frac{\Gamma\left(1 - \frac{1}{2}\nu - \frac{1}{2}\nu o_{12}\right)\Gamma\left(-\frac{1}{2}\nu + \frac{1}{2}\nu o_{12}\right)}{\Gamma\left(1 + \nu - \frac{1}{2}\nu o_{12}\right)\Gamma\left(\nu + \frac{1}{2}\nu o_{12}\right)}.$$

We calculate the residues (first for $p = 1$) of the component with $\underline{A} = (1, 2, 3, 4)$ (using pinching at $v_1 = u_2 \to u_1 - 1$, $v_2 = u_4 \to u_3 - 1$)

$$\begin{aligned} X &= \operatorname*{Res}_{u_{12}=1} \operatorname*{Res}_{u_{34}=1} K^{SU(4)}_{1234}(\underline{u}) = \int_{\mathcal{C}_{\underline{u}}} \mathrm{d}v_3 \operatorname*{Res}_{u_{12}=1} \operatorname*{Res}_{u_{34}=1} \operatorname*{Res}_{v_1=u_2} \operatorname*{Res}_{v_2=u_4} h\left(\underline{u}, \underline{v}\right) \Psi_{1234}(\underline{u}, \underline{v}) \\ &= \int_{\mathcal{C}_{\underline{u}}} \mathrm{d}v_3 \operatorname*{Res}_{u_{12}=1} \operatorname*{Res}_{u_{34}=1} \operatorname*{Res}_{v_1=u_2} \operatorname*{Res}_{v_2=u_4} h\left(\underline{u}, \underline{v}\right) K^{(1)}_{\underline{B}}(\underline{v}) \Phi^{\underline{B}}_{1234}(\underline{u}, \underline{v}) \\ &= \left[\tilde{b}\left(u_{14}\right) \int_{\mathcal{C}_{\underline{u}}} \mathrm{d}v_3 h_r\left(\underline{u}, \underline{v}\right) \left(K^{(1)}_{234}(\underline{v}) - K^{(1)}_{243}(\underline{v})\right) \tilde{b}\left(u_1 - v_3\right) \tilde{c}\left(u_3 - v_3\right)\right]_{\substack{v_1=u_2, v_2=u_4 \\ u_1=u_2+1, u_3=u_4+1}} \end{aligned} \tag{A.1}$$

with $h_r\left(\underline{u}, \underline{v}\right) = \operatorname*{Res}_{v_1=u_2} \operatorname*{Res}_{v_2=u_4} h\left(\underline{u}, \underline{v}\right)$. It was used that for $v_1 = u_2$ and $v_2 = u_4$ (see figure A1)

$$\operatorname*{Res}_{u_{12}=1} \operatorname*{Res}_{u_{34}=1} \Phi^{\underline{B}}_{1234}(\underline{u}, u_2, u_4) = \delta^{B_1}_2 \left(\delta^{B_2}_3 \delta^{B_3}_4 - \delta^{B_2}_4 \delta^{B_3}_3\right) \tilde{b}\left(u_1 - u_4\right) \tilde{b}\left(u_1 - v_3\right) \tilde{c}\left(u_3 - v_3\right)$$

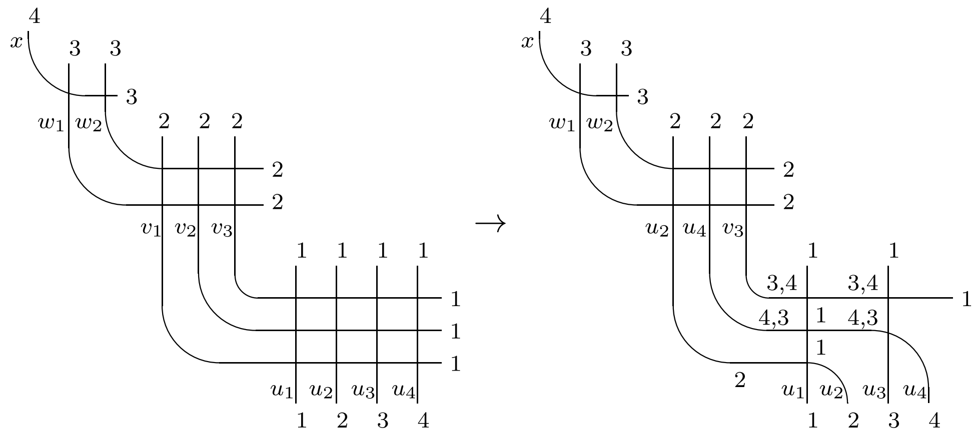

Figure A1. The Bethe state $\Psi_{\underline{A}}(\underline{u},\underline{v})$ in (21) for an iso-scalar operator where $\underline{A}=(1,2,3,4)$ and $v_1\to u_2, v_2\to u_4$.

and therefore

$$\begin{aligned}&K^{(1)}_{\underline{B}}(u_2,u_4,v_3)\operatorname*{Res}_{u_{12}=1}\operatorname*{Res}_{u_{34}=1}\Phi^{\underline{B}}_{1234}(\underline{u},u_2,u_4,v_3)\\&=\left(K^{(1)}_{234}(u_2,u_4,v_3)-K^{(1)}_{243}(u_2,u_4,v_3)\right)\tilde{b}\,(u_1-u_4)\,\tilde{b}\,(u_1-v_3)\,\tilde{c}\,(u_3-v_3)\,,\end{aligned}$$

further with $o=o_{12}=u_{24}=v_{12}$

$$\begin{aligned}X(o)&=\operatorname*{Res}_{u_{12}=1}\operatorname*{Res}_{u_{34}=1}K^{SU(4)}_{1234}(\underline{u})\\&=\int_{\mathcal{C}_{\underline{u}}}\mathrm{d}v_3\prod_{i=1}^{4}\prod_{\substack{j=1\\ i,j\neq 2,1;4,2}}^{2}\tilde{\phi}_\nu(u_i-v_j)\tau(v_{12})\prod_{i=1}^{4}\tilde{\phi}_\nu(u_i-v_3)\tau(v_{13})\tau(v_{23})\\&\quad\times\left(K^{(1)}_{234}(u_2,u_4,v_3)-K^{(1)}_{243}(u_2,u_4,v_3)\right)\tilde{b}\,(u_1-u_4)\,\tilde{b}\,(u_1-v_3)\,\tilde{c}\,(u_3-v_3)\end{aligned}$$

with $v_1\to u_2,\ v_2\to u_4,\ u_1\to u_2+1,\ u_3\to u_4+1$. We get X as

$$\begin{aligned}X(o)&=\frac{\Gamma\left(1-\frac{1}{2}\nu\,(1+o)\right)\Gamma\left(\frac{1}{2}\nu\,(o-1)\right)}{\sin\frac{1}{2}\pi\nu o}Y(o)\\Y(o)&=\int_{\mathcal{C}_{\underline{u}}}\mathrm{d}v_3\tilde{c}\,(-u_4+v_3)\left(K^{(1)}_{234}(u_2,u_4,v_3)-K^{(1)}_{243}(u_2,u_4,v_3)\right)\end{aligned}$$

where it was used that for $v_1=u_2,\ v_2=u_4,\ u_1=u_2+1,\ u_3=u_4+1$

$$\begin{aligned}&\tilde{b}\,(u_1-u_4)\,\frac{\tilde{\phi}_\nu(u_1-v_1)\tilde{\phi}_\nu(u_1-v_2)\tilde{\phi}_\nu(u_2-v_2)\tilde{\phi}_\nu(u_3-v_1)\tilde{\phi}_\nu(u_3-v_2)\tilde{\phi}_\nu(u_4-v_1)}{\tilde{\phi}_\nu(v_1-v_2)\tilde{\phi}_\nu(-v_1+v_2)}\\&=\frac{1}{\sin\frac{1}{2}\nu\pi\,(u_4-u_2)}\Gamma\left(1+\frac{1}{2}\nu\,(-u_2-1+u_4)\right)\Gamma\left(-\frac{1}{2}\nu\,(u_4+1-u_2)\right)\end{aligned}$$

and

$$\frac{\tilde{\phi}_\nu(u_1-v_3)\tilde{\phi}_\nu(u_2-v_3)\tilde{\phi}_\nu(u_3-v_3)\tilde{\phi}_\nu(u_4-v_3)}{\tilde{\phi}_\nu(v_1-v_3)\tilde{\phi}_\nu(-v_1+v_3)\tilde{\phi}_\nu(v_2-v_3)\tilde{\phi}_\nu(-v_2+v_3)}\frac{\tilde{b}\,(u_1-v_3)\,\tilde{c}\,(u_3-v_3)}{\tilde{c}\,(-u_4+v_3)}=-1.$$

Therefore we have

$$K_{\underline{\alpha}}^{O(6),\nu}(\underline{o})=\mathbf{C}_{\alpha_1\alpha_2}\frac{\Gamma\left(1-\frac{1}{2}\nu\,(1+o)\right)\Gamma\left(\frac{1}{2}\nu\,(o-1)\right)}{\tilde{\phi}_\nu(-o)\tilde{\phi}_\nu(-o+1)\sin\frac{1}{2}\pi\nu o}Y(o)=\mathbf{C}_{\alpha_1\alpha_2}Y(o).$$

Next we calculate

$$K_{\underline{B}}^{(1)}(\underline{v})=\int_{\mathcal{C}_{\underline{v}}}\mathrm{d}\underline{w}h\,(\underline{v},\underline{w})\,K_{\underline{C}}^{(2)}(\underline{w})\Phi^{(1)}{}_{\underline{B}}^{\underline{C}}(\underline{v},\underline{w})$$

with (see figure A1)

$$\Phi^{(1)}{}_{2B_2B_3}^{\;C_1C_2}(\underline{v},\underline{w})=\delta_{B_2}^{C_1}\delta_{B_3}^{C_2}\Phi_1+\delta_{B_2}^{C_2}\delta_{B_3}^{C_1}\Phi_2$$

$$\begin{aligned}\Phi_2&=\tilde{b}\,(v_1-w_1)\,\tilde{b}\,(v_1-w_2)\,\tilde{b}\,(v_2-w_1)\,\tilde{c}\,(v_2-w_2)\,\tilde{c}\,(v_3-w_1)\\ \Phi_1&=\tilde{b}\,(v_1-w_1)\,\tilde{b}\,(v_1-w_2)\,\tilde{c}\,(v_2-w_1)\\ &\quad\times\left(\tilde{b}\,(v_2-w_2)\,\tilde{b}\,(v_3-w_1)\,\tilde{c}\,(v_3-w_2)+\tilde{c}\,(v_2-w_2)\,\tilde{c}\,(v_3-w_1)\right)\end{aligned}$$

and

$$\begin{aligned}K_{\underline{C}}^{(2)}(\underline{w})&=\int_{\mathcal{C}_{\underline{w}}}\mathrm{d}x\tilde{\phi}_\nu(w_1-x)\tilde{\phi}_\nu(w_2-x)\left(\delta_{\underline{C}}^{34}\tilde{b}(w_1-x)\tilde{c}(w_2-x)+\delta_{\underline{C}}^{43}\tilde{c}(w_1-x)\right)\\ &=\left(\delta_{\underline{C}}^{34}-\delta_{\underline{C}}^{43}\right)\Gamma\left(-\frac{1}{2}\nu+\frac{1}{2}\nu w_{12}\right)\Gamma\left(1-\frac{1}{2}\nu-\frac{1}{2}\nu w_{12}\right)\end{aligned}\tag{A.2}$$

which follows from

$$\frac{1}{2\pi\mathrm{i}}\left(\int_{\mathcal{C}_a}+\int_{\mathcal{C}_b}\right)\mathrm{d}z\Gamma(a-z)\Gamma(b-z)\Gamma\,(c+z)\,\Gamma\,(d+z)=-\frac{\Gamma\,(c+a)\,\Gamma\,(d+a)\,\Gamma\,(c+b)\,\Gamma\,(d+b)}{\Gamma\,(c+d+a+b)}.$$

Therefore

$$\begin{aligned}K_{2B_2B_3}^{(1)}(\underline{v})=\int_{\mathcal{C}_{\underline{v}}}\mathrm{d}\underline{w}h\,(\underline{v},\underline{w})\,\Gamma\left(-\frac{1}{2}\nu+\frac{1}{2}\nu w_{12}\right)\Gamma\left(1-\frac{1}{2}\nu-\frac{1}{2}\nu w_{12}\right)\\ \times\left(\delta_{B_2}^{3}\delta_{B_3}^{4}-\delta_{B_2}^{4}\delta_{B_3}^{3}\right)(\Phi_1-\Phi_2)\end{aligned}$$

because $\left(\delta_{\underline{C}}^{34}-\delta_{\underline{C}}^{43}\right)\left(\delta_{B_2}^{C_1}\delta_{B_3}^{C_2}\Phi_1+\delta_{B_2}^{C_2}\delta_{B_3}^{C_1}\Phi_2\right)=\left(\delta_{B_2}^{3}\delta_{B_3}^{4}-\delta_{B_2}^{4}\delta_{B_3}^{3}\right)(\Phi_1-\Phi_2)$ and

$$\Phi_1 - \Phi_2 = \frac{\tilde{b}\,(v_1 - w_1)\,\tilde{b}\,(v_1 - w_2)\,\tilde{c}\,(v_2 - w_1)\,\tilde{c}\,(v_2 - w_2)}{\tilde{c}\,(w_1 - w_2)} \frac{\tilde{c}\,(v_3 - w_1)\,\tilde{c}\,(v_3 - w_2)}{\tilde{c}\,(v_3 - v_2)}.$$

Finally exchanging the integrations

$$\begin{aligned} Y(o) &= \int_{\mathcal{C}_{\underline{u}}} \mathrm{d}v_3 \tilde{c}\,(-u_4 + v_3) \left(K^{(1)}_{234}(u_2, u_4, v_3) - K^{(1)}_{243}(u_2, u_4, v_3) \right) \\ &= \int_{\mathcal{C}_{\underline{v}}} \mathrm{d}\underline{w} \tau_\nu(w_{12}) \Gamma\left(-\frac{1}{2}\nu + \frac{1}{2}\nu w\right) \Gamma\left(-\frac{1}{2}\nu w\right) \\ &\quad \times \left[\prod_{i=1}^{2} \prod_{j=1}^{2} \tilde{\phi}_\nu(v_i - w_j) \frac{\tilde{b}\,(v_1 - w_1)\,\tilde{b}\,(v_1 - w_2)\,\tilde{c}\,(v_2 - w_1)\,\tilde{c}\,(v_2 - w_2)}{\tilde{c}\,(w_1 - w_2)} \right]_{v_1 = u_2, v_2 = u_4} \\ &\quad \times \int_{\mathcal{C}_{\underline{w}}} \mathrm{d}v_3 \tilde{c}\,(-u_4 + v_3)\, \tilde{\phi}_\nu(v_3 - w_1) \tilde{\phi}_\nu(v_3 - w_2) \frac{\tilde{c}\,(v_3 - w_1)\,\tilde{c}\,(v_3 - w_2)}{\tilde{c}\,(v_3 - u_4)} \end{aligned}$$

the v_3-integration can be done as above in (A.2)

$$\int_{\mathcal{C}_{\underline{w}}} \mathrm{d}v_3 \tilde{\phi}_\nu(v_3 - w_1) \tilde{\phi}_\nu(v_3 - w_2) \tilde{c}\,(v_3 - w_1)\,\tilde{c}\,(v_3 - w_2) = \Gamma(-\frac{1}{2}\nu + \frac{1}{2}\nu w_{12}) \Gamma(-\frac{1}{2}\nu - \frac{1}{2}\nu w_{12}) \tag{A.3}$$

and therefore (for $v_1 = u_2, v_2 = u_4,\ o = u_2 - u_4$)

$$\begin{aligned} Y(o) &= \int_{\mathcal{C}_{\underline{v}}} \mathrm{d}\underline{w} \left(\prod_{i=1}^{2} \prod_{j=1}^{2} \tilde{\phi}_\nu(v_i - w_j) \right) \\ &\quad \times \tilde{b}\,(v_1 - w_1)\,\tilde{b}\,(v_1 - w_2)\,\tilde{c}\,(v_2 - w_1)\,\tilde{c}\,(v_2 - w_2)\,\Psi\,(w_1 - w_2) \end{aligned}$$

with

$$\begin{aligned} \Psi(w) &= \frac{\Gamma\left(-\frac{1}{2}\nu + \frac{1}{2}\nu w\right) \Gamma\left(1 - \frac{1}{2}\nu - \frac{1}{2}\nu w\right) \Gamma\left(-\frac{1}{2}\nu + \frac{1}{2}\nu w\right) \Gamma\left(-\frac{1}{2}\nu - \frac{1}{2}\nu w\right)}{\tilde{c}(w) \tilde{\phi}_\nu(w) \tilde{\phi}_\nu(-w)} \\ &= \frac{1}{\pi} w \left(\sin \frac{1}{2} \pi \nu w \right) \Gamma\left(-\frac{1}{2}\nu + \frac{1}{2}\nu w\right) \Gamma\left(-\frac{1}{2}\nu - \frac{1}{2}\nu w\right). \end{aligned}$$

In (C.10) of [11] was shown that

$$\int_{\mathcal{C}_{\underline{v}}} \mathrm{d}\underline{w} \prod_{j=1}^{2} \left(\tilde{\phi}_\nu(v_1 - w_j) \tilde{\phi}_\nu(v_2 - w_j) \tilde{c}\,(v_2 - w_j) \right) \varphi\,(w_{12}, k) = K(v_{12}, k) \tag{A.4}$$

with

$$\begin{aligned} \varphi\,(w, k) &= \frac{(1 - w)\, K(w, k+1)}{\tilde{\phi}_\nu(w) \tilde{\phi}_\nu(-w)\,(w + 1/\nu - k - 1)} \\ K(u, k) &= \frac{\Gamma\left(1 - \frac{1}{2}\nu - \frac{1}{2}\nu u\right) \Gamma\left(-\frac{1}{2}\nu + \frac{1}{2}\nu u\right)}{\Gamma\left(\frac{3}{2} - \frac{1}{2}\nu k - \frac{1}{2}\nu u\right) \Gamma\left(\frac{1}{2} - \frac{1}{2}\nu k + \frac{1}{2}\nu u\right)}. \end{aligned}$$

Note that for $k = 1/\nu - 2$

$$\Psi(w) = \frac{1}{\sin\frac{1}{2}\pi\nu\,(w-1)\sin\frac{1}{2}\pi\nu\,(w+1)}\varphi\,(w, 1/\nu - 2)\,.$$

Similarly to (A.4) we have here[9]

$$\begin{aligned} Y(o) &= \int_{\mathcal{C}_{\underline{v}}} \mathrm{d}\underline{w} \prod_{j=1}^{2} \left(\tilde{\phi}_\nu(v_1 - w_j)\tilde{b}\,(v_1 - w_j)\,\tilde{\phi}_\nu(v_2 - w_j)\tilde{c}\,(v_2 - w_j)\right)\Psi\,(w_{12}) \\ &= \frac{K(o, k = 1/\nu - 2)}{\sin\frac{1}{2}\pi\nu\,(o-1)\sin\frac{1}{2}\pi\nu\,(o+1)} = 2\frac{K(o, k = 1/\nu - 2)}{\cos\pi\nu - \cos\pi\nu o} \end{aligned} \tag{A.5}$$

with $o = o_{12} = u_{24} = v_{12}$. The arguments are as follows: The function $Y(o)$ satisfies the equations (24) with the S-matrix eigenvalue $\tilde{S}_0^{O(6)}$ of (8). Therefore the minimal solution is $Y^{\min}(o) = K(o, 1/\nu - 2)\sin\frac{1}{2}\pi\nu\,(o-1)\sin\frac{1}{2}\pi\nu\,(o+1)$. Pinching at $w_1 \to v_1 - 2/\nu$, $w_2 \to v_2$ and produces a double pole at $o = 1$, wich implies (A.5).

Now we consider the p-function (56) in (A.1), then (up to a constant)

$$Y_p(o) = K(o, k = 1/\nu - 2).$$

This result is obtained by applying to the equations which correspond to (A.2) and (A.3) the formula

$$\begin{aligned} &\frac{1}{2\pi\mathrm{i}}\left(\int_{\mathcal{C}_a} + \int_{\mathcal{C}_b}\right)\mathrm{d}z\Gamma(a-z)\Gamma(b-z)\Gamma\,(c+z)\,\Gamma\,(d+z)\,f(z) \\ &\quad = \Gamma\,(1-c-d-a-b)\,\Gamma\,(c+a)\,\Gamma\,(d+a)\,\Gamma\,(c+b)\,\Gamma\,(d+b) \\ &\qquad \times\left(f(a)\frac{\sin\pi\,(c+b)\sin\pi\,(d+b)}{\pi\sin\pi\,(a-b)} - f(b)\frac{\sin\pi\,(c+a)\sin\pi\,(d+a)}{\pi\sin\pi\,(a-b)}\right) \end{aligned}$$

where $\mathcal{C}_a$ encloses the poles of $\Gamma(a-z)$ and $f(z+1) = f(z)$ holds. ■

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