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The quantum inverse scattering method for Hubbard-like models

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Abstract

This work is concerned with various aspects of the formulation of the quantum inverse scattering method for the one-dimensional Hubbard model. We first establish the essential tools to solve the eigenvalue problem for the transfer matrix of the classical "covering" Hubbard model within the algebraic Bethe ansatz framework. The fundamental commutation rules exhibit a hidden 6-vertex symmetry which plays a crucial role in the whole algebraic construction. Next we apply this formalism to study the $SU(2)$ highest weights properties of the eigenvectors and the solution of a related coupled spin model with twisted boundary conditions. The machinery developed in this paper is applicable to many other models, and as an example we present the algebraic solution of the Bariev XY coupled model. © 1998 Elsevier Science B.V.

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1. Introduction

The discovery of the quantum version of the inverse scattering method in the late seventies was undoubtedly a remarkable contribution to the development of the field of exactly solvable models in $(1+1)$ dimensions [1]. This method provides a means for integrating models in two-dimensional classical statistical mechanics and $(1+1)$ quantum field theory, unifying major achievements such as the transfer matrix ideas, the Bethe ansatz and the Yang–Baxter equation. Nowadays detailed reviews on this subject are available in the literature, for instance see Refs. [2–5].

We shall start this paper illustrating the essential features of this method in the context of lattice models of statistical mechanics. For example, consider a vertex model

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on the square lattice and suppose that its row-to-row transfer matrix can be constructed from an elementary local vertex operator $\mathcal{L}_{\mathcal{A}i}(\lambda)$. This operator, known as the Lax operator, contains all information about the structure of the Boltzmann weights which are parametrized through the spectral parameter λ. The operator $\mathcal{L}_{\mathcal{A}i}(\lambda)$ is frequently viewed as a matrix on the auxiliary space $\mathcal{A}$, corresponding in the vertex model to the space of states of the horizontal degrees of freedom. Its matrix elements are operators on the Hilbert space $\prod_{i=1}^{L} \otimes V_i$, where V_i corresponds to the space of vertical degrees of freedom and i denotes the sites of a one-dimensional lattice of size L. In this paper we shall consider the situation in which the auxiliary space $\mathcal{A}$ and the quantum space V_i are equivalent. A sufficient condition for integrability of ultralocal models, i.e. those in which the matrices elements of the operator $\mathcal{L}_{\mathcal{A}i}(\lambda)$ commute for distinct values of index i, is the existence of an invertible matrix $R(\lambda,\mu)$ having the following property:

$$R(\lambda,\mu)\mathcal{L}_{\mathcal{A}i}(\lambda)\otimes\mathcal{L}_{\mathcal{A}i}(\mu)=\mathcal{L}_{\mathcal{A}i}(\mu)\otimes\mathcal{L}_{\mathcal{A}i}(\lambda)R(\lambda,\mu), \tag{1}$$

where the tensor product is taken only with respect to the auxiliary space $\mathcal{A}$. The matrix $R(\lambda,\mu)$ is defined on the tensor product $\mathcal{A}\otimes\mathcal{A}$ and its matrix elements are c-numbers. A ordered product of Lax operators gives rise to the monodromy operator $\mathcal{T}(\lambda)$

$$\mathcal{T}(\lambda)=\mathcal{L}_{\mathcal{A}L}(\lambda)\mathcal{L}_{\mathcal{A}L-1}(\lambda)\dots\mathcal{L}_{\mathcal{A}1}(\lambda). \tag{2}$$

It is possible to extend property (1) to the monodromy matrix, and such global intertwining relation reads

$$R(\lambda,\mu)\mathcal{T}(\lambda)\otimes\mathcal{T}(\mu)=\mathcal{T}(\mu)\otimes\mathcal{T}(\lambda)R(\lambda,\mu). \tag{3}$$

The transfer matrix of the vertex model, for periodic boundary conditions, can be written as the trace of the monodromy matrix on the auxiliary space $\mathcal{A}$

$$T(\lambda)=\mathrm{Tr}_{\mathcal{A}}\mathcal{T}(\lambda). \tag{4}$$

From the above definition and property (3) we can derive that the transfer matrix is the generating function of the conserved currents. Indeed, after multiplying this equation by the inverse of $R(\lambda,\mu)$, taking the trace on the tensor $\mathcal{A}\otimes\mathcal{A}$ space and using the trace cyclic property we find

$$\left[T(\lambda),T(\mu)\right]=0. \tag{5}$$

Consequently, the expansion of the transfer matrix in the spectral parameter yields an infinite number of conserved charges. We recall that local charges are in general obtained as logarithm derivatives of $T(\lambda)$ [6,7]. Furthermore, the compatibility condition of ordering three Lax operators $\mathcal{L}_{\mathcal{A}1}(\lambda_1)$, $\mathcal{L}_{\mathcal{A}2}(\lambda_2)$ and $\mathcal{L}_{\mathcal{A}3}(\lambda_3)$ through the intertwining relation (1) implies, under a plausible assumption, the famous Yang–Baxter equation

$$R_{23}(\lambda_1,\lambda_2)R_{12}(\lambda_1,\lambda_3)R_{23}(\lambda_2,\lambda_3)=R_{12}(\lambda_2,\lambda_3)R_{23}(\lambda_1,\lambda_3)R_{12}(\lambda_1,\lambda_2), \tag{6}$$

where $R_{ab}(\lambda,\mu)$ denotes the action of matrix $R(\lambda,\mu)$ on the spaces $V_a\otimes V_b$.

Eq. (3) is the starting point of solving two-dimensional classical statistical models by an exact operator formalism. This equation contains all possible commutation relation between the matrix elements of the monodromy operator $\mathcal{T}(\lambda)$. The diagonal terms of $\mathcal{T}(\lambda)$ define the transfer matrix eigenvalue problem and the off-diagonal ones play the role of creation and annihilation fields. The eigenvectors of the transfer matrix are constructed by applying the creation operators on a previously chosen reference state, providing us with an elegant formulation of the Bethe states. For this reason this framework is often denominated in the literature as the algebraic Bethe ansatz approach. This situation resembles much the matrix formulation of $(0+1)$ quantum mechanics. It is well known that the harmonic oscillator can either be solved by the Schrödinger formalism or by the Heisenberg algebra of creation and annihilation operators. The latter approach, however, is conceptually much simpler provided the relevant dynamical symmetry has been identified for a given quantum system. One successful example is the solution of the hydrogen atom through the $SO(4)$ algebra [8].

In this paper we are primarily interested in applying the quantum inverse scattering method for the one-dimensional Hubbard model. We recall that, after the Heisenberg model, the second one-dimensional lattice paradigm in the theory of magnetism solved by Bethe ansatz method was the Hubbard model. The solution was found by Lieb and Wu in 1968 [9] using the extension of the coordinate Bethe ansatz to the problem of fermions interacting via δ-functions [10]. Considering the success of the solution of the Heisenberg model by the inverse method [3], the next natural target for this program would then be the Hubbard model. However, it turns out that the solution of this problem followed a more arduous path than one could imagine from the very beginning. Indeed, nearly 18 years were to pass before it was found the classical statistical vertex model whose transfer matrix generates the conserved charges commuting with the Hubbard Hamiltonian. This remarkable step was done by Shastry [11–13] who also found the R-matrix solution and thus proved the integrability of the Hubbard model from the quantum inverse method point of view. Shastry himself attempted to complete the inverse scattering program, but he was only able to conjecture the eigenvalues of the transfer matrix guided by a phenomenological approach which goes by the name of analytical Bethe ansatz [13]. Subsequently Bariev presented a coordinate Bethe ansatz solution for the classical Shastry's model, however on the basis of the *diagonal-to-diagonal* transfer matrix method [14].

One of the main results of this paper is the solution of the one-dimensional Hubbard model by a first principle method, namely via the algebraic Bethe ansatz approach.[1] For this purpose we will use Shastry's R-matrix as well as the modifications introduced by Wadati and co-workers [15]. Apart from the fact that the solution of the one-dimensional Hubbard model by the algebraic Bethe ansatz framework remains an important unsolved theoretical challenge in the field of integrable models, there are also other motivations to pursue this program. Recent developments of new powerful methods to deal with finite temperature properties of integrable models (see for e.g. Refs. [17–19]) show

[1] A brief summary of some of our results has appeared in Ref. [16].

clearly that the central object to be diagonalized is the quantum transfer matrix rather the underlying one-dimensional Hamiltonian. The transfer matrix eigenvalues provide us with the spectrum of all conserved charges, a fact which could be helpful in the study of transport properties [20] and level statistics behaviour [21]. Lastly, there is a hope that this program is the first step towards the formulation of a general approach for computing lattice correlation functions [5].

We would like to remark that the ideas developed in this paper transcend the solution of the one-dimensional Hubbard model. In fact, the original basis of our approach might be traced back to the solution of the supersymmetric $spl(2|1)$ vertex model [22]. Very recently, we have shown that this method provide us with a unified way of solving a wider class of integrable models based on the braid monoid algebra [23]. Here we also will see that the lattice analog of the coupled XY Bariev chain [24] can be solved by this technique. The unusual feature of the Hubbard and Bariev models is that they both have a non-additive R-matrix solution.

We have organized this paper as follows. To make our presentation self-contained, in the next section we briefly review the basic properties of the embedding of the one-dimensional Hubbard model into a classical vertex model, originally due to Shastry [11–13]. In Section 3 we discuss the commutation rules coming from the Yang–Baxter algebra. In particular, a hidden symmetry of 6-vertex type, which is crucial for integrability, is found. We use these properties in Section 4 in order to construct the eigenvectors and the eigenvalues of the transfer matrix of the classical statistical model. Lieb's and Wu's results as well as the spectrum of higher conserved charges can be obtained from our expression for the transfer matrix eigenvalues. In Section 5 we present complementary results such as extra comments on systems with twisted boundary conditions and a discussion on the $SU(2)$ highest weights properties of the eigenvectors. Section 6 is dedicated to the solution of the classical analog of the coupled XY Bariev model. Our conclusions are presented in Section 7. Finally, four appendices summarize Boltzmann weights, extra commutation rules and technical details we omitted in the main text.

2. The classical covering Hubbard model

We begin this section reviewing the work of Shastry [11–13] on the identification of the classical statistical model whose row-to-row transfer matrix commutes with the one-dimensional Hubbard Hamiltonian. Originally, Shastry looked at this problem considering the coupled spin version of the Hubbard model, since in one-dimension fermions and spin-$\frac{1}{2}$ Pauli operators are related to each other via Jordan–Wigner transformation. In the context of statistical mechanics, however, the latter representation is sometimes more appealing. Here we will consider the coupled spin model introduced by Shastry with general twisted boundary conditions. Its Hamiltonian is

$$H=\sum_{i=1}^{L-1}\sigma_i^+\sigma_{i+1}^- +\sigma_i^-\sigma_{i+1}^+ +\tau_i^+\tau_{i+1}^- +\tau_i^-\tau_{i+1}^+ +\frac{U}{4}\sigma_i^z\tau_i^z$$
$$+e^{-i\phi_1}\sigma_L^+\sigma_1^- +e^{i\phi_1}\sigma_L^-\sigma_1^+ +e^{-i\phi_2}\tau_L^+\tau_1^- +e^{i\phi_2}\tau_L^-\tau_1^+ +\frac{U}{4}\sigma_L^z\tau_L^z \tag{7}$$

where $\{\sigma_i^\pm,\sigma_i^z\}$ and $\{\tau_i^\pm,\tau_i^z\}$ are two commuting sets of Pauli matrices acting on the site i of a lattice of size L. The second term in (7) stands for the boundary conditions $\sigma_{L+1}^\pm = e^{\pm i\phi_1}\sigma_1^\pm$, $\tau_{L+1}^\pm = e^{\pm i\phi_2}\tau_1^\pm$, $\sigma_{L+1}^z=\sigma_1^z$, and $\tau_{L+1}^z=\tau_1^z$ where ϕ_1 and ϕ_2 are arbitrary angles $0\leqslant\phi_1,\phi_2<2\pi$. The coupling constant U represents the Hubbard on-site Coulomb interaction.

In order to relate the coupled spin model to the Hubbard model we have to perform the following Jordan–Wigner transformation [11]:

$$c_{i\uparrow}=\prod_{k=1}^{i-1}\sigma_k^z\sigma_i^-,\qquad c_{i\downarrow}=\prod_{k=1}^{L}\sigma_k^z\prod_{k=1}^{i-1}\tau_k^z\tau_i^- \tag{8}$$

where $c_{i\sigma}$ are canonical Fermi operators of spins $\sigma=\uparrow,\downarrow$ on site i, with anti-commutation relations given by $\{c_{i\sigma}^\dagger,c_{j\sigma'}\}=\delta_{i,j}\delta_{\sigma,\sigma'}$. Defining the number operator $n_{i\sigma}=c_{i\sigma}^\dagger c_{i\sigma}$ for electrons with spin σ on site i and performing transformation (8) we find that

$$H=-\sum_{i=1}^{L-1}\sum_{\sigma=\uparrow,\downarrow}[c_{i\sigma}^\dagger c_{i+1\sigma}+c_{i+1\sigma}^\dagger c_{i\sigma}]+U\sum_{i=1}^{L-1}(n_{i\uparrow}-\tfrac12)(n_{i\downarrow}-\tfrac12)$$
$$-e^{-i\phi_\uparrow}c_{L\uparrow}^\dagger c_{1\uparrow}-e^{i\phi_\uparrow}c_{1\uparrow}^\dagger c_{L\uparrow}-e^{-i\phi_\downarrow}c_{L\downarrow}^\dagger c_{1\downarrow}-e^{i\phi_\downarrow}c_{1\uparrow}^\dagger c_{L\downarrow}$$
$$+U(n_{L\uparrow}-\tfrac12)(n_{L\downarrow}-\tfrac12), \tag{9}$$

where the angles $\phi_\uparrow$ and $\phi_\downarrow$ are given by

$$\phi_\uparrow=\phi_1+\pi+\pi N_\uparrow^h,\qquad \phi_\downarrow=\phi_2+\pi+\pi N_\downarrow^h \tag{10}$$

and N_σ^h is the number of holes (eigenvalues of the operator $\sum_{i=1}^{L}c_{i\sigma}c_{i\sigma}^\dagger$) of spin σ of a given sector of the Hubbard model. Therefore, the Hubbard model with periodic boundary conditions ($\phi_\uparrow=\phi_\downarrow$) is related to the coupled spin model with dynamically (sector dependent) twisted boundary conditions imposed. This was the reason why we started with a more general coupled spin model, since the two representations are fully equivalent only for free boundary conditions.

From the point of view of a vertex model, twisted boundary conditions correspond to the introduction of a seam of different Boltzmann weights along the infinite direction on the cylinder. In practice this is accomplished by multiplying one of the elementary vertex operator, $\mathcal{L}_{\mathcal{A}L}(\lambda)$ say, by a "gauge" matrix $G_{\mathcal{A}}$ (see Section 5). Such matrix is usually related to additional hidden invariances of the R-matrix [25]. Hence, although twisted boundary conditions may affect eigenvalues and Bethe ansatz equations in a significant way, the relevant features of the integrability still remain intact. Since this section is concerned with the latter point, we can assume periodic boundary conditions without losing generality. As Shastry [11–13] pointed out, the mapping of the Hubbard model (modulo the above subtlety) into a coupled spin system is quite illuminating in searching for a "covering" vertex model. It is known that the decoupled spin model ($U = 0$) can be derived in terms of a pair of uncoupled free-fermion 6-vertex models. This suggests that, for the interacting model, we have to look for a copy of two free-fermion 6-vertex models coupled in an appropriate way. Shastry [11–13] determined the nature of this coupling by demanding that it should reproduce the higher conserved charges [11] [2] when the corresponding transfer matrix $T(\lambda)$ was expanded in powers of the spectral parameter λ. The solution found by Shastry for the Lax operator is given by [12,13]

$$\mathcal{L}_{\mathcal{A}i}(\lambda) = \mathcal{L}^{\sigma}_{\mathcal{A}i}(\lambda)\mathcal{L}^{\tau}_{\mathcal{A}i}(\lambda)e^{h(\lambda)\sigma^z_{\mathcal{A}}\tau^z_{\mathcal{A}}\otimes I_i}. \tag{11}$$

The form of operators $\mathcal{L}^{\sigma}_{\mathcal{A}i}(\lambda)$ and $\mathcal{L}^{\tau}_{\mathcal{A}i}(\lambda)$ obey the 6-vertex structure

$$\mathcal{L}^{\sigma}_{\mathcal{A}i}(\lambda) = \frac{a(\lambda)+b(\lambda)}{2} + \frac{a(\lambda)-b(\lambda)}{2}\sigma^z_{\mathcal{A}}\sigma^z_i + (\sigma^+_{\mathcal{A}}\sigma^-_i + \sigma^-_{\mathcal{A}}\sigma^+_i) \tag{12}$$

and

$$\mathcal{L}^{\tau}_{\mathcal{A}i}(\lambda) = \frac{a(\lambda)+b(\lambda)}{2} + \frac{a(\lambda)-b(\lambda)}{2}\tau^z_{\mathcal{A}}\tau^z_i + (\tau^+_{\mathcal{A}}\tau^-_i + \tau^-_{\mathcal{A}}\tau^+_i), \tag{13}$$

where the weights $a(\lambda)$ and $b(\lambda)$ satisfy the free-fermion condition $a^2(\lambda)+b^2(\lambda) = 1$. Furthermore, the constraint $h(\lambda)$ is determined in terms of the weights and the coupling U by

$$\sinh[2h(\lambda)] = \frac{U}{2}a(\lambda)b(\lambda) \tag{14}$$

A second important result due to Shastry [12,13] was the solution of the Yang–Baxter algebra for the Lax operator (11), and thus determining the form of the R-matrix. The matrix $R(\lambda,\mu)$ is a 16×16 matrix whose non-null elements are given in terms of 10 distinct Boltzmann weights $\alpha_i(\lambda,\mu)$, $i = 1,\dots,10$. For practical calculations it is helpful to display its matrix form

$$R(\lambda,\mu) = \begin{pmatrix}
\alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha_5 & 0 & 0 & -\alpha_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha_5 & 0 & 0 & 0 & 0 & 0 & -\alpha_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_4 & 0 & 0 & \alpha_{10} & 0 & 0 & \alpha_{10} & 0 & 0 & -\alpha_7 & 0 & 0 & 0 \\
0 & \alpha_8 & 0 & 0 & \alpha_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_{10} & 0 & 0 & \alpha_3 & 0 & 0 & \alpha_6 & 0 & 0 & \alpha_{10} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_5 & 0 & 0 & 0 & 0 & 0 & \alpha_8 & 0 & 0 \\
0 & 0 & \alpha_8 & 0 & 0 & 0 & 0 & 0 & \alpha_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_{10} & 0 & 0 & \alpha_6 & 0 & 0 & \alpha_3 & 0 & 0 & \alpha_{10} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_5 & 0 & 0 & \alpha_8 & 0 \\
0 & 0 & 0 & -\alpha_7 & 0 & 0 & \alpha_{10} & 0 & 0 & \alpha_{10} & 0 & 0 & \alpha_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_9 & 0 & 0 & 0 & 0 & 0 & \alpha_5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_9 & 0 & 0 & \alpha_5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_2
\end{pmatrix}. \tag{15}$$

[2] For further discussion on Hubbard's conserved charges see Ref. [26].

where the expressions for the weights $\alpha_i(\lambda,\mu)$ in terms of the free-fermion weights $a(\lambda)$, $b(\lambda)$ and the constraint $h(\lambda)$ can be found in Appendix A. The striking feature of this solution is that R-matrix (15) is non-additive with respect the spectral parameters. In fact, after an unitary transformation, $R(\lambda,\mu)$ can be written in a more compact form [13] which shows that it depends on both the difference and the sum of the spectral parameters. As far we know, it is still an open question whether or not there exists an embedding for the Hubbard model having the standard difference property. As a final remark we mention that an analytical proof that $R(\lambda,\mu)$ indeed satisfies the Yang–Baxter equation (6) has been recently presented in Ref. [27].

We close this section presenting the graded Yang–Baxter formalism [28] for the Hubbard model. This interesting approach was pursued by Wadati and co-workers [15] and it has the advantage of making real distinction between bosonic and fermionic degrees of freedom. In the Hubbard model, the empty and doubly occupied sites play the role of bosonic states while the spin up and down states are the fermionic ones. This formalism is an elegant mathematical procedure[3] of avoiding the subtlety on boundary condition raised in the beginning of this section. In other words, the graded version of the inverse scattering method guarantees that the "non-local" anticommutation rules of fermionic degrees of freedom is satisfied for any lattice sites. In general, the basic changes we need to perform is to consider the analogs of the trace and the tensor product properties on the graded space. For example, the graded Yang–Baxter for the monodromy matrix now reads [28]

$$R_g(\lambda,\mu)\mathcal{T}(\lambda) \overset{s}{\otimes} \mathcal{T}(\mu) = \mathcal{T}(\mu) \overset{s}{\otimes} \mathcal{T}(\lambda) R_g(\lambda,\mu), \tag{16}$$

where the symbol $\overset{s}{\otimes}$ stands for the supertensor product

$$(A \overset{s}{\otimes} B)_{ab}^{cd} = (-1)^{p(b)[p(a)+p(c)]} A_{ac} B_{bd}.$$

The index $p(a)$ is the Grassmann parity of the ath degree of freedom, assuming values $p(a) = 0$ for bosonic specie and $p(a) = 1$ for fermionic ones. Other important change is on the transfer matrix definition, which is now given in terms of the supertrace of the monodromy matrix

$$T(\lambda) = \mathrm{Str}_{\mathcal{A}} \mathcal{T}(\lambda) = \sum_{a \in \mathcal{A}} (-1)^{p(a)} \mathcal{T}_{aa}(\lambda). \tag{17}$$

There is no extra effort to obtain the matrix $R_g(\lambda,\mu)$ from the original solution found by Shastry. One just have to perform a Jordan-Wigner transformation on the Lax operator (11), taking into account the gradation of the space of states [15]. It turns out that the graded R-matrix is related to Shastry's solution (15) by a unitary transformation, and its explicit form is given by [15]

[3] This scheme accommodates a particular class of models having "non-ultralocal" Yang–Baxter relations. For more general implications of non-ultralocality see the recent review [29].

$$R_g(\lambda,\mu)=\begin{pmatrix}
\alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha_5 & 0 & 0 & -i\alpha_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha_5 & 0 & 0 & 0 & 0 & 0 & -i\alpha_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_4 & 0 & 0 & -i\alpha_{10} & 0 & 0 & i\alpha_{10} & 0 & 0 & \alpha_7 & 0 & 0 & 0 \\
0 & -i\alpha_8 & 0 & 0 & \alpha_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & i\alpha_{10} & 0 & 0 & \alpha_3 & 0 & 0 & -\alpha_6 & 0 & 0 & -i\alpha_{10} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_5 & 0 & 0 & 0 & 0 & 0 & -i\alpha_8 & 0 & 0 \\
0 & 0 & -i\alpha_8 & 0 & 0 & 0 & 0 & 0 & \alpha_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -i\alpha_{10} & 0 & 0 & -\alpha_6 & 0 & 0 & \alpha_3 & 0 & 0 & i\alpha_{10} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_5 & 0 & 0 & -i\alpha_8 & 0 \\
0 & 0 & 0 & \alpha_7 & 0 & 0 & i\alpha_{10} & 0 & 0 & -i\alpha_{10} & 0 & 0 & \alpha_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -i\alpha_9 & 0 & 0 & 0 & 0 & 0 & \alpha_5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i\alpha_9 & 0 & 0 & \alpha_5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_2
\end{pmatrix} \tag{18}$$

where here we assumed that the first and the fourth degrees of freedom are bosonic ($p(1)=p(4)=0$) while the remaining ones are fermionic ($p(2)=p(3)=1$).

In the next sections we are going to use the graded formalism in order to find the appropriate commutation rules, the eigenvectors and the eigenvalues of the transfer matrix (17). Afterwards, we will get back to the standard quantum inverse formalism, but now with twisted boundary conditions.

3. The fundamental commutation rules

In addition to the Lax operator and the R-matrix the existence of a local reference state is another important object in the quantum inverse scattering program. This is a vector $|0\rangle_i$ such that the result of the action of the Lax operator on it is a matrix having a triangular form. We choose

$$|0\rangle_i=\begin{pmatrix}1\\0\end{pmatrix}_i\otimes\begin{pmatrix}1\\0\end{pmatrix}_i$$

as the standard spin up "ferromagnetic state", which in the fermionic language corresponds to the doubly occupied state. The action of the vertex operator in this state has the following property:

$$\mathcal{L}_{\mathcal{A}i}(\lambda)\,|0\rangle_i=\begin{pmatrix}
\omega_1(\lambda)\,|0\rangle_i & \ddagger & \ddagger & \ddagger \\
0 & \omega_2(\lambda)\,|0\rangle_i & 0 & \ddagger \\
0 & 0 & \omega_2(\lambda)\,|0\rangle_i & \ddagger \\
0 & 0 & 0 & \omega_3(\lambda)\,|0\rangle_i
\end{pmatrix}, \tag{19}$$

where the symbol $\ddagger$ represents arbitrary non-null values and the functions $\omega_1(\lambda)$, $\omega_2(\lambda)$ and $\omega_3(\lambda)$ are given by

$$\omega_1(\lambda)=[a(\lambda)]^2e^{h(\lambda)},\quad \omega_2(\lambda)=a(\lambda)b(\lambda)e^{-h(\lambda)},\quad \omega_3(\lambda)=[b(\lambda)]^2e^{h(\lambda)}. \tag{20}$$

The global reference state $|0\rangle$ is then defined by the tensor product $|0\rangle = \prod_{i=1}^{L} \otimes |0\rangle_i$. This state is an eigenstate of the transfer matrix since the triangular property is easily extended to the monodromy matrix. In order to construct other eigenstates it is necessary to seek for an appropriate representation of the monodromy matrix. By this we mean a structure which is able to distinguish creation and annihilation fields as well as possible hidden symmetries. The triangular property of the Lax operator suggests us the following form:

$$\mathcal{T}(\lambda) = \begin{pmatrix} B(\lambda) & \boldsymbol{B}(\lambda) & F(\lambda) \\ \boldsymbol{C}(\lambda) & \hat{A}(\lambda) & \boldsymbol{B}^*(\lambda) \\ C(\lambda) & \boldsymbol{C}^*(\lambda) & D(\lambda) \end{pmatrix}_{4\times4}, \tag{21}$$

where $\boldsymbol{B}(\lambda)$, $\boldsymbol{C}^*(\lambda)$ and $\boldsymbol{B}^*(\lambda)$, $\boldsymbol{C}(\lambda)$ are two component vectors with dimensions 1×2 and 2×1, respectively. The operator $\hat{A}(\lambda)$ is a 2×2 matrix and we shall denote its elements by $\hat{A}_{ab}(\lambda)$. The remaining operators $B(\lambda)$, $C(\lambda)$, $D(\lambda)$ and $F(\lambda)$ are scalars. In this paper we will use the symbol *ABCDF* to refer to the above way of representing the elements of the monodromy matrix. We recall that such ansatz is quite distinct from the traditional *ABCD* form proposed originally by Faddeev and co-workers [1–3].

In the *ABCDF* representation the eigenvalue problem for the graded transfer matrix becomes

$$\left[B(\lambda) - \sum_{a=1}^{2} A_{aa}(\lambda) + D(\lambda)\right] |\Phi\rangle = \Lambda(\lambda)\, |\Phi\rangle\,, \tag{22}$$

where $\Lambda(\lambda)$ and $|\Phi\rangle$ correspond to the eigenvalues and to the eigenvectors, respectively. As a consequence of the triangular property we can derive important relations for the monodromy matrix elements. For the diagonal part of $\mathcal{T}(\lambda)$ we have

$$\begin{aligned} &B(\lambda)\,|0\rangle = [\omega_1(\lambda)]^L\,|0\rangle\,, \qquad D(\lambda)\,|0\rangle = [\omega_3(\lambda)]^L\,|0\rangle\,, \\ &\hat{A}_{aa}(\lambda)\,|0\rangle = [\omega_2(\lambda)]^L\,|0\rangle \quad \text{for } a = 1,2. \end{aligned} \tag{23}$$

Also one expects that the operators $\boldsymbol{B}(\lambda)$, $\boldsymbol{B}^*(\lambda)$ and $F(\lambda)$ play the role of creation fields over the reference state $|0\rangle$. It also follows from the triangular property the annihilation properties

$$\begin{aligned} &\boldsymbol{C}(\lambda)\,|0\rangle = 0, \qquad \boldsymbol{C}^*(\lambda)\,|0\rangle = 0, \\ &C(\lambda)\,|0\rangle = 0, \qquad \hat{A}_{ab}(\lambda)\,|0\rangle = 0 \quad \text{for } a \neq b. \end{aligned} \tag{24}$$

To make further progress we have to recast the graded Yang–Baxter algebra in the form of commutation relations for the creation and annihilation fields. In general it is not known how and when such job can be performed for a particular representation, and one could surely say that the "artistic" part of the algebraic Bethe ansatz construction begins here. Within the *ABCDF* formalism, the solution of this problem turns out to be more complicated than a similar situation occurring for the 6-vertex model [1–3] and its multi-state generalizations [30,31]. The new feature present here is that we have a mixture of two classes of creation fields, the non-commutative vectors $\boldsymbol{B}(\lambda)$ or $\boldsymbol{B}^*(\lambda)$ and one commutative operator represented by $F(\lambda)$. We shall start our discussion by the commutation rule between the fields $\boldsymbol{B}(\lambda)$ and $\boldsymbol{B}(\mu)$. In this case the relation that comes out from the Yang-Baxter algebra is not the convenient one for further computations. It turns out to be necessary to perform a second step which consists in substituting the exchange relation for the scalar operators $B(\lambda)$ and $F(\mu)$ (see Eq. (38)) back on the original commutation rule we just derived for the fields $\boldsymbol{B}(\lambda)$ and $\boldsymbol{B}(\mu)$. The basic trick is to keep the diagonal operator $B(\lambda)$ always in the right-hand side position in the commutation rule [22]. After performing this two step procedure we are able to get the appropriate commutation rule, which is

$$\boldsymbol{B}(\lambda) \otimes \boldsymbol{B}(\mu) = \frac{\alpha_1(\lambda,\mu)}{\alpha_2(\lambda,\mu)}[\boldsymbol{B}(\mu) \otimes \boldsymbol{B}(\lambda)] \cdot \hat{r}(\lambda,\mu) - i\frac{\alpha_{10}(\lambda,\mu)}{\alpha_7(\lambda,\mu)}\{F(\lambda)\boldsymbol{B}(\mu) - F(\mu)\boldsymbol{B}(\lambda)\}\boldsymbol{\xi}, \tag{25}$$

where $\boldsymbol{\xi}$ is a 1×4 vector and $\hat{r}(\lambda,\mu)$ is an auxiliary 4×4 matrix given by

$$\boldsymbol{\xi} = (0 \quad 1 \quad -1 \quad 0), \quad \hat{r}(\lambda,\mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \bar{a}(\lambda,\mu) & \bar{b}(\lambda,\mu) & 0 \\ 0 & \bar{b}(\lambda,\mu) & \bar{a}(\lambda,\mu) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{26}$$

and the functions $\bar{a}(\lambda,\mu)$ and $\bar{b}(\lambda,\mu)$ are given in terms of the Boltzmann weights by

$$\bar{a}(\lambda,\mu) = \frac{\alpha_3(\lambda,\mu)\alpha_7(\lambda,\mu) + \alpha_{10}^2(\lambda,\mu)}{\alpha_1(\lambda,\mu)\alpha_7(\lambda,\mu)},$$
$$\bar{b}(\lambda,\mu) = -\frac{\alpha_6(\lambda,\mu)\alpha_7(\lambda,\mu) + \alpha_{10}^2(\lambda,\mu)}{\alpha_1(\lambda,\mu)\alpha_7(\lambda,\mu)}. \tag{27}$$

It turns out that the auxiliary matrix $\hat{r}(\lambda,\mu)$ is precisely the rational R-matrix of the isotropic 6-vertex model or the XXX spin chain. In order to see that, we first simplify a bit more the auxiliary weights $\bar{a}(\lambda,\mu)$ and $\bar{b}(\lambda,\mu)$ with the help of identities (A.10)-(A.12). We find that they satisfy the following relations:

$$\bar{a}(\lambda,\mu) = 1 - \bar{b}(\lambda,\mu), \qquad \bar{b}(\lambda,\mu) = \frac{\alpha_8(\lambda,\mu)\alpha_9(\lambda,\mu)}{\alpha_1(\lambda,\mu)\alpha_7(\lambda,\mu)}. \tag{28}$$

Next we simplify the ratios $\alpha_9(\lambda,\mu)/\alpha_1(\lambda,\mu)$ and $\alpha_8(\lambda,\mu)/\alpha_7(\lambda,\mu)$ in terms of the free-fermion Boltzmann weights and the constraint $h(\lambda)$ as much as possible. After some algebra we write these ratios as

$$\frac{\alpha_9(\lambda,\mu)}{\alpha_1(\lambda,\mu)} = \frac{a(\mu)b(\lambda)e^{2[h(\mu)-h(\lambda)]} - a(\lambda)b(\mu)}{b(\lambda)b(\mu) + a(\lambda)a(\mu)e^{2[h(\mu)-h(\lambda)]}}, \tag{29}$$

$$\frac{\alpha_8(\lambda,\mu)}{\alpha_7(\lambda,\mu)} = \frac{b(\lambda)b(\mu) + a(\lambda)a(\mu)e^{2[h(\mu)+h(\lambda)]}}{a(\mu)b(\lambda)e^{2[h(\mu)+h(\lambda)]} - a(\lambda)b(\mu)}. \tag{30}$$

Now if we take into account the identity

$$\frac{a(\lambda)}{b(\lambda)}e^{2h(\lambda)} - \frac{b(\lambda)}{a(\lambda)}e^{-2h(\lambda)} = \frac{a(\lambda)}{b(\lambda)}e^{-2h(\lambda)} - \frac{b(\lambda)}{a(\lambda)}e^{2h(\lambda)} + U \tag{31}$$

and perform the following reparametrization:

$$\tilde{\lambda} = \frac{a(\lambda)}{b(\lambda)}e^{2h(\lambda)} - \frac{b(\lambda)}{a(\lambda)}e^{-2h(\lambda)} - \frac{U}{2}, \tag{32}$$

we finally can rewrite the auxiliary weights as

$$\bar{a}(\tilde{\lambda}, \tilde{\mu}) = \frac{U}{\tilde{\mu} - \tilde{\lambda} + U}, \qquad \bar{b}(\tilde{\lambda}, \tilde{\mu}) = \frac{\tilde{\mu} - \tilde{\lambda}}{\tilde{\mu} - \tilde{\lambda} + U}. \tag{33}$$

Clearly, these are the non-trivial Boltzmann weights of the isotropic 6-vertex model. This is an important hidden symmetry, which is known to play a decisive role on the exact solution of the Hubbard model since the work of Lieb and Wu [9]. The derivation of this symmetry in the context of the quantum inverse scattering program is however a rather non-trivial result. One of the virtues of this result is that it becomes valid for the generator of the commuting conserved charges and not only for the Hubbard Hamiltonian. Moreover, we also recall that this symmetry is of relevance to the Yangian invariance of the Hubbard model which emerges in the thermodynamic limit [32,33].

To solve the eigenvalue problem (22) we still need the help of several other commutation relations. For instance, the commutation rules between the diagonal and creation operators play an important role in the eigenvalue construction. It turns out that in some cases we have to take into account similar trick discussed above. This is specially important for the field $\hat{A}(\lambda)$, where we have to use an auxiliary exchange relation between the operator $B(\mu)$ and $\boldsymbol{B}^*(\lambda)$, in order to obtain a more appropriate commutation rule with the creation operator $\boldsymbol{B}(\lambda)$. In general, the task is quite cumbersome and here we limit ourselves to list the final results. The commutation relations between the diagonal fields and the creation operator $\boldsymbol{B}(\lambda)$ are

$$\begin{aligned}\hat{A}(\lambda) \otimes \boldsymbol{B}(\mu) = & -i\frac{\alpha_1(\lambda,\mu)}{\alpha_9(\lambda,\mu)}[\boldsymbol{B}(\mu) \otimes \hat{A}(\lambda)] \cdot \hat{r}(\lambda,\mu) + i\frac{\alpha_5(\lambda,\mu)}{\alpha_9(\lambda,\mu)}\boldsymbol{B}(\lambda) \otimes \hat{A}(\mu) \\ & -i\frac{\alpha_{10}(\lambda,\mu)}{\alpha_7(\lambda,\mu)}\left[\boldsymbol{B}^*(\lambda)B(\mu) + i\frac{\alpha_5(\lambda,\mu)}{\alpha_9(\lambda,\mu)}F(\lambda)\boldsymbol{C}(\mu)\right. \\ & \left. -i\frac{\alpha_2(\lambda,\mu)}{\alpha_9(\lambda,\mu)}F(\mu)\boldsymbol{C}(\lambda)\right] \otimes \boldsymbol{\xi},\end{aligned} \tag{34}$$

$$B(\lambda)\boldsymbol{B}(\mu) = i\frac{\alpha_2(\mu,\lambda)}{\alpha_9(\lambda,\mu)}\boldsymbol{B}(\mu)B(\lambda) - i\frac{\alpha_5(\mu,\lambda)}{\alpha_9(\lambda,\mu)}\boldsymbol{B}(\lambda)B(\mu), \tag{35}$$

$$\begin{aligned}D(\lambda)\boldsymbol{B}(\mu) = & -i\frac{\alpha_8(\lambda,\mu)}{\alpha_7(\lambda,\mu)}\boldsymbol{B}(\mu)D(\lambda) + \frac{\alpha_5(\lambda,\mu)}{\alpha_7(\lambda,\mu)}F(u)\boldsymbol{C}^*(\lambda) \\ & -\frac{\alpha_4(\lambda,\mu)}{\alpha_7(\lambda,\mu)}F(\lambda)\boldsymbol{C}^*(\mu) - i\frac{\alpha_{10}(\lambda,\mu)}{\alpha_7(\lambda,\mu)}\boldsymbol{\xi} \cdot [\boldsymbol{B}^*(\lambda) \otimes \hat{A}(\mu)],\end{aligned} \tag{36}$$

while those for the scalar field $F(\lambda)$ are

$$\hat{A}_{ab}(\lambda)F(\mu)=\left[1+\frac{\alpha_5^2(\lambda,\mu)}{\alpha_9(\lambda,\mu)\alpha_8(\lambda,\mu)}\right]F(\mu)\hat{A}_{ab}(\lambda)$$
$$-\frac{\alpha_5^2(\lambda,\mu)}{\alpha_9(\lambda,\mu)\alpha_8(\lambda,\mu)}F(\lambda)\hat{A}_{ab}(\mu)$$
$$+i\frac{\alpha_5(\lambda,\mu)}{\alpha_9(\lambda,\mu)}[\boldsymbol{B}(\lambda)\otimes\boldsymbol{B}^*(\mu)]_{ba}+i\frac{\alpha_5(\lambda,\mu)}{\alpha_8(\lambda,\mu)}[\boldsymbol{B}^*(\lambda)\otimes\boldsymbol{B}(\mu)]_{ab}, \tag{37}$$

$$B(\lambda)F(\mu)=\frac{\alpha_2(\mu,\lambda)}{\alpha_7(\mu,\lambda)}F(\mu)B(\lambda)-\frac{\alpha_4(\mu,\lambda)}{\alpha_7(\mu,\lambda)}F(\lambda)B(\mu)$$
$$+i\frac{\alpha_{10}(\mu,\lambda)}{\alpha_7(\mu,\lambda)}\{\boldsymbol{B}(\lambda)\otimes\boldsymbol{B}(\mu)\}\cdot\boldsymbol{\xi}^t, \tag{38}$$

$$D(\lambda)F(\mu)=\frac{\alpha_2(\lambda,\mu)}{\alpha_7(\lambda,\mu)}F(\mu)D(\lambda)-\frac{\alpha_4(\lambda,\mu)}{\alpha_7(\lambda,\mu)}F(\lambda)D(\mu)$$
$$-i\frac{\alpha_{10}(\lambda,\mu)}{\alpha_7(\lambda,\mu)}\boldsymbol{\xi}\cdot\{\boldsymbol{B}^*(\lambda)\otimes\boldsymbol{B}^*(\mu)\}, \tag{39}$$

where $\boldsymbol{\xi}^t$ stands for the transpose of $\boldsymbol{\xi}$. Furthermore, the relations closing the commutation rules between the creation operators $\boldsymbol{B}(\lambda)$ and $F(\lambda)$ are

$$\left[F(\lambda),F(\mu)\right]=0, \tag{40}$$

$$F(\lambda)\boldsymbol{B}(\mu)=\frac{\alpha_5(\lambda,\mu)}{\alpha_2(\lambda,\mu)}F(\mu)\boldsymbol{B}(\lambda)-i\frac{\alpha_8(\lambda,\mu)}{\alpha_2(\lambda,\mu)}\boldsymbol{B}(\mu)F(\lambda), \tag{41}$$

$$\boldsymbol{B}(\lambda)F(\mu)=\frac{\alpha_5(\lambda,\mu)}{\alpha_2(\lambda,\mu)}\boldsymbol{B}(\mu)F(\lambda)-i\frac{\alpha_9(\lambda,\mu)}{\alpha_2(\lambda,\mu)}F(\mu)\boldsymbol{B}(\lambda). \tag{42}$$

Finally, it remains to consider the commutation rules for the creation field $\boldsymbol{B}^*(\lambda)$. To avoid overcrowding this section with more heavier formulae we have collected them in Appendix B. We see that they are quite similar to those we just derived for the field $\boldsymbol{B}(\lambda)$. In fact, it is possible to establish an equivalence between these two sets of commutation rules if we formally interpret the symbol $*$ as a mathematical operation acting on the elements of the monodromy matrix. For lack of a better name we call it "dual" transformation and we impose that it satisfies the following properties: $(O^*(\lambda))^*\equiv O(\lambda)$, $A^*(\lambda)\equiv -A^t(\lambda)$, $B^*(\lambda)\equiv D(\lambda)$, $F^*(\lambda)=F(\lambda)$ and $C^*(\lambda)=C(\lambda)$. Applying the "dual" transformation on the commutation rules of field $\boldsymbol{B}(\lambda)$ we obtain those for the field $\boldsymbol{B}^*(\lambda)$ with new Boltzmann weights $\alpha_j^*(\lambda,\mu;h)\equiv\alpha_j(\lambda,\mu;-h)$, where, for sake of clarity, we stressed the dependence on the constraint $h(\lambda)$. This means that the functional form of the weights remains unchanged but now we have to perform the transformation $h(\lambda)\rightarrow -h(\lambda)$ $(U\rightarrow -U)$. We recall that in this last step we used the following identities for the Boltzmann weights: $\alpha_j(\lambda,\mu;h)=\alpha_j(\mu,\lambda;-h)$, $j=1,\dots,7$, $\alpha_8(\lambda,\mu;h)=\alpha_9(\mu,\lambda;-h)$ and $\alpha_{10}(\lambda,\mu;h)=-\alpha_{10}(\mu,\lambda;-h)$. Therefore, we expect that the construction of the eigenvectors will be based either on the pair of fields $\boldsymbol{B}(\lambda)$ and $F(\lambda)$ or on the "dual" ones $\boldsymbol{B}^*(\lambda)$ and $F(\lambda)$ rather than on a general combination of the three creation fields. This redundancy is in accordance to what one would expect from the space of states of the Hubbard Hamiltonian, since at a given site we can either create a single electron (spin up or down) or a pair of electrons with opposite spins. We remark that such "duality" property is not particular to the Hubbard model but it is rather a general feature present in our framework [23].

At this point we have set up the basic tools to start the construction of the eigenvectors of the eigenvalue problem (22). In the next section we will show how this problem can be solved with the help of the commutations rules (25), (26); (34)–(42) and a few other relations presented in Appendix B.

4. The eigenvectors and the eigenvalue construction

The purpose of this section is to solve the eigenvalue problem for the graded transfer matrix. We shall begin by considering the construction of an ansatz for the corresponding eigenvectors. The multi-particle state are going to satisfy an important recurrence relation. We will see that the eigenvalue problem (22) has a nested structure, i.e. it will depend on the solution of an inhomogeneous auxiliary problem related to the 6-vertex hidden symmetry.

4.1. The eigenvalue problem

The eigenvectors of the transfer matrix are in principle built up in terms of a linear combination of products of the many creation fields acting on the reference state. These Bethe states are often thought as multi-particle states, characterized by a set of rapidities parametrizing the creation fields. Before embarking on the technicalities of the construction of an arbitrary n-particle state we first define it by the following scalar product

$$|\Phi_n(\lambda_1,\dots,\lambda_n)\rangle = \boldsymbol{\Phi}_n(\lambda_1,\dots,\lambda_n)\cdot\mathcal{F}\,|0\rangle\,, \tag{43}$$

where the mathematical structure of vector $\boldsymbol{\Phi}_n(\lambda_1,\dots,\lambda_n)$ will be described in terms of the creation fields. At this stage the components of vector $\mathcal{F}$ are simply thought as coefficients of an arbitrary linear combination which would be determined later on. This reflects the "spin" degrees of freedom of the space of states and we shall denote such coefficients by $\mathcal{F}^{a_n\dots a_1}$ where the index a_i run over two possible values $a_i = 1,2$.

Let us now turn our attention to the construction of vector $\boldsymbol{\Phi}_n(\lambda_1,\dots,\lambda_n)$. As mentioned at the end of the previous section, it is sufficient to look for combinations between the fields $\boldsymbol{B}(\lambda)$ and $F(\lambda)$. In general, there is no known recipe which is able to provide us with an educated ansatz for this vector and as it is customary we shall start the construction considering a few particle excitations over the reference state. A single particle excitation is made by creating a hole of spin up or down on the full band pseudovacuum $|0\rangle$. From the point of view of the inverse scattering method this excitation is represented by $\boldsymbol{\Phi}_1(\lambda_1) = \boldsymbol{B}(\lambda_1)$ and consequently the one-particle state is

$$|\Phi_1(\lambda_1)\rangle = \boldsymbol{B}(\lambda_1)\cdot\boldsymbol{\mathcal{F}}\,|0\rangle = B_a(\lambda_1)\mathcal{F}^a\,|0\rangle\,, \tag{44}$$

where from now on we assume sum over repeated index.

It is not difficult to solve the eigenvalue problem (22) for such one-particle state. If we use the commutation relations (34-36), and the pseudovacuum properties (23), (24) we find that the one-particle state satisfies the following relations:

$$\begin{aligned} B(\lambda)\,|\Phi_1(\lambda_1)\rangle &= i\frac{\alpha_2(\lambda_1,\lambda)}{\alpha_9(\lambda_1,\lambda)}[\omega_1(\lambda)]^L\,|\Phi_1(\lambda_1)\rangle \\ &\quad -i\frac{\alpha_5(\lambda_1,\lambda)}{\alpha_9(\lambda_1,\lambda)}[\omega_1(\lambda_1)]^L\boldsymbol{B}(\lambda)\cdot\boldsymbol{\mathcal{F}}\,|0\rangle\,, \end{aligned} \tag{45}$$

$$\begin{aligned} D(\lambda)\,|\Phi_1(\lambda_1)\rangle &= -i\frac{\alpha_8(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}[\omega_3(\lambda)]^L\,|\Phi_1(\lambda_1)\rangle \\ &\quad -i\frac{\alpha_{10}(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}[\omega_2(\lambda_1)]^L[\boldsymbol{\xi}\cdot(\boldsymbol{B}^*(\lambda)\otimes\hat{I})]\cdot\boldsymbol{\mathcal{F}}\,|0\rangle\,, \end{aligned} \tag{46}$$

$$\begin{aligned} \sum_{a=1}^{2} A_{aa}(\lambda)\,|\Phi_1(\lambda_1)\rangle &= -i\frac{\alpha_1(\lambda,\lambda_1)}{\alpha_9(\lambda,\lambda_1)}\hat{r}^{a_1b_1}_{c_1a_1}(\lambda,\lambda_1)[\omega_2(\lambda)]^L B_{c_1}(\lambda_1)\mathcal{F}^{b_1}\,|0\rangle \\ &\quad +i\frac{\alpha_5(\lambda,\lambda_1)}{\alpha_9(\lambda,\lambda_1)}[\omega_2(\lambda_1)]^L\boldsymbol{B}(\lambda)\cdot\boldsymbol{\mathcal{F}}\,|0\rangle \\ &\quad -i\frac{\alpha_{10}(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}[\omega_1(\lambda_1)]^L[\boldsymbol{\xi}.(\boldsymbol{B}^*(\lambda)\otimes\hat{I})]\cdot\boldsymbol{\mathcal{F}}\,|0\rangle\,, \end{aligned} \tag{47}$$

where $\hat{I}$ is the 2×2 identity matrix. The terms proportional to the eigenvector $|\Phi_1(\lambda_1)\rangle$ are denominated wanted terms because they contribute directly to the eigenvalue. The remaining ones are called unwanted terms and they can be eliminated by imposing further restriction on the rapidity λ_1. This constraint, known as the Bethe ansatz equation, is given by

$$\left[\frac{\omega_1(\lambda_1)}{\omega_2(\lambda_1)}\right]^L = 1. \tag{48}$$

It is now straightforward to go ahead and to determine the one-particle eigenvalue. However, it is convenient to start introducing suitable notation which can be extended to accommodate multi-particle states. With this in mind, we define the following auxiliary eigenvalue problem:

$$T^{(1)}(\lambda,\lambda_1)^{a_1}_{b_1}\mathcal{F}^{a_1} = \hat{r}^{\alpha a_1}_{b_1\alpha}(\lambda,\lambda_1)\mathcal{F}^{a_1} = \Lambda^{(1)}(\lambda,\lambda_1)\mathcal{F}^{b_1} \tag{49}$$

and we see that, in terms of Eq. (49), the one-particle eigenvalue can be expressed by

$$\begin{aligned} \Lambda(\lambda,\lambda_1) &= i\frac{\alpha_2(\lambda_1,\lambda)}{\alpha_9(\lambda_1,\lambda)}[\omega_1(\lambda)]^L - i\frac{\alpha_8(\lambda,\lambda_1)}{\alpha_7(\lambda,\lambda_1)}[\omega_3(\lambda)]^L \\ &\quad +i\frac{\alpha_1(\lambda,\lambda_1)}{\alpha_9(\lambda,\lambda_1)}\Lambda^{(1)}(\lambda,\lambda_1)[\omega_2(\lambda)]^L. \end{aligned} \tag{50}$$

Up to the level of the one-particle state there is no extra effort to solve the corresponding auxiliary problem. Considering the 6-vertex structure of matrix $\hat{r}(\lambda, \mu)$ it is easily seen that the solution is

$$\Lambda^{(1)}(\lambda, \lambda_1) = 1 + \bar{b}(\lambda, \lambda_1). \tag{51}$$

We next turn to the analysis of the two-particle state. We expect that such state will be a composition between two single hole excitations of arbitrary spins and a local hole pair with opposite spins. The former is made by tensoring two fields of $\boldsymbol{B}(\lambda)$ type while the latter should be represented by $F(\lambda)$. The vector $\boldsymbol{\xi}$ has also a physical meaning. It plays the role of an "exclusion" principle, forbidding two spin up or two spin down at the same site. Thus, an educate ansatz for the two-particle vector should be the linear combination

$$\boldsymbol{\Phi}_2(\lambda_1, \lambda_2) = \boldsymbol{B}(\lambda_1) \otimes \boldsymbol{B}(\lambda_2) + \boldsymbol{\xi} F(\lambda_1) B(\lambda_2) \hat{g}_0^{(2)}(\lambda_1, \lambda_2), \tag{52}$$

where $\hat{g}_0^{(2)}(\lambda_1, \lambda_2)$ is an arbitrary function to be determined. We found also convenient to add the diagonal field $B(\lambda_2)$ on the right-hand side of the two-particle vector ansatz. We see that when the ansatz (52) is projected out on the subspace of equal spins, no contribution coming from $F(\lambda)$ appears, which is in perfect accordance to what one would expect from the Pauli principle. In other words, using the definition (43) we have

$$|\Phi_2(\lambda_1, \lambda_2)\rangle = B_i(\lambda_1) B_j(\lambda_2) \mathcal{F}^{ji} |0\rangle + [\omega_1(\lambda_2)]^L F(\lambda_1) \hat{g}_0^{(2)}(\lambda_1, \lambda_2)(\mathcal{F}^{21} - \mathcal{F}^{12}). \tag{53}$$

In order to tackle the eigenvalue problem for the two-particle state, besides the commutation rules of the last section, we have to use extra relations between the fields $\boldsymbol{B}(\lambda)$, $\boldsymbol{B}^*(\lambda)$, $\boldsymbol{C}(\lambda)$ and $\boldsymbol{C}^*(\lambda)$. These relations have been summarized in the beginning of Appendix B. After turning the diagonal fields over the two-particle state, we find that there are two classes of unwanted terms. The first class we call "easy" unwanted terms because they are only produced by the same diagonal operator ($\hat{A}(\lambda)$ or $D(\lambda)$) and they can be eliminated by an appropriate choice of function $\hat{g}_0^{(2)}(\lambda_1, \lambda_2)$. There are three terms of this sort

$$F(\lambda) D(\lambda_1) B(\lambda_2), \qquad \boldsymbol{B}(\lambda) \cdot \boldsymbol{B}^*(\lambda_1) B(\lambda_2), \qquad \boldsymbol{\xi} \cdot [\boldsymbol{B}^*(\lambda) \otimes \boldsymbol{B}^*(\lambda_1)] B(\lambda_2) \tag{54}$$

and all of them are cancelled out provided we chose function $\hat{g}_0^{(2)}(\lambda_1, \lambda_2)$ as

$$\hat{g}_0^{(2)}(\lambda_1, \lambda_2) = i \frac{\alpha_{10}(\lambda_1, \lambda_2)}{\alpha_7(\lambda_1, \lambda_2)}. \tag{55}$$

Now, besides the wanted terms, we are only left with standard unwanted terms, i.e. those that require further restriction on the rapidities. We shall see below that these terms can be simplified in rather closed forms with the help of the two-particle auxiliary problem. Similar to the one-particle analysis, the auxiliary eigenvalue problem is figured out by looking at the wanted terms coming from the operator $\sum_{a=1}^{2} \hat{A}_{aa}(\lambda)$. Considering the commutation rule (34) we soon realize that the two-particle auxiliary problem is

$$T^{(1)}(\lambda, \{\lambda_l\})_{b_1 b_2}^{a_1 a_2} \mathcal{F}^{a_2 a_1} = \hat{r}_{b_1 d_1}^{c_1 a_1}(\lambda, \lambda_1) \hat{r}_{b_2 c_1}^{d_1 a_2}(\lambda, \lambda_2) \mathcal{F}^{a_2 a_1} = \Lambda^{(1)}(\lambda, \{\lambda_l\}) \mathcal{F}^{b_2 b_1}. \quad (56)$$

With the above information we move on simplifying as much as possible the action of the diagonal fields on the two-particle state. We keep in mind that we want to present the results in a way that would be amenable to multi-particle states generalization. After a cumbersome algebra we find that

$$\begin{aligned} B(\lambda) \left|\Phi_2(\lambda_1, \lambda_2)\right\rangle = {} & [\omega_1(\lambda)]^L \prod_{j=1}^{2} i \frac{\alpha_2(\lambda_j, \lambda)}{\alpha_9(\lambda_j, \lambda)} \left|\Phi_2(\lambda_1, \lambda_2)\right\rangle \\ & - \sum_{j=1}^{2} [\omega_1(\lambda_j)]^L \left|\Psi_1^{(1)}(\lambda, \lambda_j; \{\lambda_l\})\right\rangle \\ & + H_1(\lambda, \lambda_1, \lambda_2) [\omega_1(\lambda_1)\omega_1(\lambda_2)]^L \left|\Psi_0^{(3)}(\lambda, \lambda_j, \lambda_l; \{\lambda_k\})\right\rangle, \end{aligned} \quad (57)$$

$$\begin{aligned} D(\lambda) \left|\Phi_2(\lambda_1, \lambda_2)\right\rangle = {} & [\omega_3(\lambda)]^L \prod_{j=1}^{2} -i \frac{\alpha_8(\lambda, \lambda_j)}{\alpha_7(\lambda, \lambda_j)} \left|\Phi_2(\lambda_1, \lambda_2)\right\rangle \\ & - \sum_{j=1}^{2} [\omega_2(\lambda_j)]^L \Lambda^{(1)}(\lambda = \lambda_j, \{\lambda_l\}) \left|\Psi_1^{(2)}(\lambda, \lambda_j; \{\lambda_l\})\right\rangle \\ & + H_2(\lambda, \lambda_1, \lambda_2) [\omega_2(\lambda_1)\omega_2(\lambda_2)]^L \left|\Psi_0^{(3)}(\lambda, \lambda_j, \lambda_l; \{\lambda_k\})\right\rangle, \end{aligned} \quad (58)$$

$$\begin{aligned} \sum_{a=1}^{2} \hat{A}_{aa}(\lambda) \left|\Phi_2(\lambda_1, \lambda_2)\right\rangle = {} & [\omega_2(\lambda)]^L \prod_{j=1}^{2} -i \frac{\alpha_1(\lambda, \lambda_j)}{\alpha_9(\lambda, \lambda_j)} \Lambda^{(1)}(\lambda, \{\lambda_l\}) \left|\Phi_2(\lambda_1, \lambda_2)\right\rangle \\ & - \sum_{j=1}^{2} [\omega_2(\lambda_j)]^L \Lambda^{(1)}(\lambda = \lambda_j, \{\lambda_l\}) \left|\Psi_1^{(1)}(\lambda, \lambda_j; \{\lambda_l\})\right\rangle \\ & - \sum_{j=1}^{2} [\omega_1(\lambda_j)]^L \left|\Psi_1^{(2)}(\lambda, \lambda_j; \{\lambda_l\})\right\rangle \\ & + H_3(\lambda, \lambda_1, \lambda_2) [\omega_1(\lambda_1)\omega_2(\lambda_2)]^L \left|\Psi_0^{(3)}(\lambda, \lambda_j, \lambda_l; \{\lambda_k\})\right\rangle \\ & + H_4(\lambda, \lambda_1, \lambda_2) [\omega_1(\lambda_2)\omega_2(\lambda_1)]^L \left|\Psi_0^{(3)}(\lambda, \lambda_j, \lambda_l; \{\lambda_k\})\right\rangle. \end{aligned} \quad (59)$$

For the sake of clarity we have shortened the notation for the unwanted terms and represented them by the eigenfunctions $\left|\Psi_1^{(j)}(\lambda, \lambda_j; \{\lambda_k\})\right\rangle$ and $\left|\Psi_0^{(3)}(\lambda, \lambda_j; \{\lambda_k\})\right\rangle$. We see that there are three classes of unwanted terms and their explicit expressions in terms of the creation fields are

$$\left|\Psi_1^{(1)}(\lambda, \lambda_j; \{\lambda_l\})\right\rangle = i\frac{\alpha_5(\lambda_j, \lambda)}{\alpha_9(\lambda_j, \lambda)} \prod_{\substack{k=1 \\ k\neq j}}^{2} i\frac{\alpha_2(\lambda_k, \lambda_j)}{\alpha_9(\lambda_k, \lambda_j)} \times \boldsymbol{B}(\lambda) \otimes \boldsymbol{\Phi}_1(\lambda_k) \hat{O}_j^{(1)}(\lambda_j; \{\lambda_k\}) \cdot \boldsymbol{\mathcal{F}} |0\rangle , \quad (60)$$

$$\left|\Psi_1^{(2)}(\lambda, \lambda_j; \{\lambda_l\})\right\rangle = i\frac{\alpha_{10}(\lambda, \lambda_j)}{\alpha_7(\lambda, \lambda_j)} \prod_{\substack{k=1 \\ k\neq j}}^{2} i\frac{\alpha_2(\lambda_k, \lambda_j)}{\alpha_9(\lambda_k, \lambda_j)} \times [\boldsymbol{\xi} \cdot (\boldsymbol{B}^*(\lambda) \otimes \hat{I})] \otimes \boldsymbol{\Phi}_1(\lambda_k) \hat{O}_j^{(1)}(\lambda_j; \{\lambda_k\}) \cdot \boldsymbol{\mathcal{F}} |0\rangle , \quad (61)$$

$$\left|\Psi_0^{(3)}(\lambda, \lambda_j, \lambda_l; \{\lambda_k\})\right\rangle = F(\lambda) \boldsymbol{\xi} \cdot \boldsymbol{\mathcal{F}} |0\rangle , \quad (62)$$

where the operator $\hat{O}_j^{(1)}(\lambda_j; \{\lambda_k\})$ is a sort of "ordering" factor for the unwanted terms and it is given by the formula

$$\hat{O}_j^{(1)}(\lambda_j; \{\lambda_k\}) = \prod_{k=1}^{j-1} \frac{\alpha_1(\lambda_k, \lambda_j)}{\alpha_2(\lambda_k, \lambda_j)} \hat{r}_{k,k+1}(\lambda_k, \lambda_j). \quad (63)$$

Before proceeding with a discussion of the results, we should pause to comment on the "brute-force" analysis we performed so far for the two-particle state problem. Roughly speaking, one can estimate the wanted terms by keeping the first term of the right-hand side of the commutation rules (34)–(36) when we turn the diagonal fields over the creation operators $\boldsymbol{B}(\lambda_j)$. This procedure gives us the coefficients proportional to the first part of the eigenvector and to show that this is also true for the second part we need to use some identities between the Boltzmann weights. The situation for the unwanted terms is even worse due to the proliferation of many different terms, common in a Bethe ansatz "brute-force" analysis. The "ordering" factor just accounts for these many different contributions to the unwanted terms. Later on it will become clear that the origin of this factor is due to a permutation property satisfied by the two-particle eigenvector. In Appendix C we provide the details about the less straightforward simplifications carried out for the two-particle state, since some of them will be also useful to multi-particle states as well. Finally, within a "brute force" computation, we have found nine contributions to the third unwanted term which come from many different sources. It is possible to recast them in terms of four functions $H_i(x, y, z)$, $i = 1, \dots, 4$ whose expressions are

$$H_1(x, y, z) = i\frac{\alpha_2(y, x)\alpha_5(z, x)\alpha_{10}(y, x)}{\alpha_9(y, x)\alpha_9(z, x)\alpha_7(y, x)} - i\frac{\alpha_4(y, x)\alpha_{10}(y, z)}{\alpha_7(y, x)\alpha_7(y, z)},$$

$$H_2(x,y,z) = i\frac{\alpha_5(x,y)\alpha_{10}(x,z)}{\alpha_7(x,y)\alpha_7(x,z)} - i\frac{\alpha_4(x,y)\alpha_{10}(y,z)}{\alpha_7(x,y)\alpha_7(y,z)},$$

$$H_3(x,y,z) = i\frac{\alpha_{10}(x,y)\alpha_5(x,y)\alpha_5(y,z)}{\alpha_7(x,y)\alpha_9(x,y)\alpha_9(y,z)} - i\frac{\alpha_2(x,y)\alpha_5(x,z)\alpha_{10}(x,y)}{\alpha_9(x,y)\alpha_9(x,z)\alpha_7(x,y)},$$

$$H_4(x,y,z) = -i\frac{\alpha_{10}(x,y)\alpha_5(x,y)\alpha_5(y,z)}{\alpha_7(x,y)\alpha_9(x,y)\alpha_9(y,z)} + i\frac{\alpha_1(x,y)\alpha_{10}(x,z)\alpha_5(x,y)[1+\bar{a}(x,y)]}{\alpha_9(x,y)\alpha_7(x,z)\alpha_8(x,y)} - 2i\frac{\alpha_5^2(x,y)\alpha_{10}(y,z)}{\alpha_8(x,y)\alpha_9(x,y)\alpha_7(y,z)}. \tag{64}$$

Now we return to the discussion of the two-particle state results. For the first two classes of unwanted terms we only have two main contributions and from Eqs. (57)–(59) it is direct to see that they vanish provided that the rapidities satisfy the following Bethe ansatz equations:

$$\left[\frac{\omega_1(\lambda_i)}{\omega_2(\lambda_i)}\right]^L = \Lambda^{(1)}(\lambda=\lambda_i, \{\lambda_j\}), \quad i=1,2. \tag{65}$$

Furthermore, the above Bethe ansatz equations are also sufficient to cancel out altogether the four contributions proportional to the unwanted term $F(\lambda)\boldsymbol{\xi}\cdot\boldsymbol{\mathcal{F}}$. A simple way of seeing that is first to factorize a common factor $[\omega_2(\lambda_1)\omega_2(\lambda_2)]^L$ for all the four terms. This is done by substituting the values $[\omega_1(\lambda_1)]^L$ and $[\omega_1(\lambda_2)]^L$ given by the Bethe ansatz equations (65) and by using the following two-particle relations:

$$\Lambda^{(1)}(\lambda=\lambda_1,\{\lambda_l\})\boldsymbol{\xi}\cdot\boldsymbol{\mathcal{F}} = [\bar{b}(\lambda_1,\lambda_2) - \bar{a}(\lambda_1,\lambda_2)]\boldsymbol{\xi}\cdot\boldsymbol{\mathcal{F}},$$
$$\Lambda^{(1)}(\lambda=\lambda_2,\{\lambda_l\})\boldsymbol{\xi}\cdot\boldsymbol{\mathcal{F}} = [\bar{b}(\lambda_2,\lambda_1) - \bar{a}(\lambda_2,\lambda_1)]\boldsymbol{\xi}\cdot\boldsymbol{\mathcal{F}}. \tag{66}$$

After putting all these simplifications together, one is still left to verify that the identity

$$H_1(x,y,z) + H_2(x,y,z) = H_3(x,y,z)[\bar{b}(y,z) - \bar{a}(y,z)] + H_4(x,y,z)[\bar{b}(z,y) - \bar{a}(z,y)] \tag{67}$$

is satisfied. At this point we note that there is a way of rewriting the term $H_4(\lambda,\lambda_1,\lambda_2)$ in a more symmetrical form. This technical point is discussed in Appendix C and proved very useful in carrying out the cancellation mechanism for general multi-particle states.

Finally, from Eqs. (57)–(59) we can read directly the wanted terms, and the two-particle eigenvalue is

$$\Lambda(\lambda,\{\lambda_i\}) = [\omega_1(\lambda)]^L \prod_{i=1}^{2} i\frac{\alpha_2(\lambda_i,\lambda)}{\alpha_9(\lambda_i,\lambda)} + [\omega_3(\lambda)]^L \prod_{i=1}^{2} -i\frac{\alpha_8(\lambda,\lambda_i)}{\alpha_7(\lambda,\lambda_i)} - [\omega_2(\lambda)]^L \prod_{i=1}^{2} -i\frac{\alpha_1(\lambda,\lambda_i)}{\alpha_9(\lambda,\lambda_i)}\Lambda^{(1)}(\lambda,\{\lambda_j\}). \tag{68}$$

Now we reached a point which is typical of nested Bethe ansatz problems, i.e. the solution of the two-particle auxiliary problem is no longer trivial and it is necessary to implement a second Bethe ansatz. We will postpone this discussion until the next subsection in which we will present the solution of this problem for general multi-particle states. Although, for an integrable model, it is believed that the two-particle sector contains the essential features about the general structure of the eigenvalues and the Bethe ansatz equations, similar situation for the eigenvectors is still less clear. Before considering this problem, it is wise to look first for an alternative way of starting with a general ansatz, since a "brute force" analysis proved to be rather intricate even for the two-particle state. In fact, there is a symmetry which we have not yet explored. It consists of seeking for eigenvectors which are in some way related to each other via permutation of the rapidities. This idea goes along the lines the usual pseudomomenta symmetrization imposed to coordinate Bethe ansatz wave functions. For example, let us consider the two-particle vector in which the constraint $\hat{g}_0^{(2)}(\lambda_1, \lambda_2)$ has been fixed as in Eq. (55). Then, it is possible to verify that the exchange property

$$\boldsymbol{\Phi}_2(\lambda_1, \lambda_2) = \frac{\alpha_1(\lambda_1, \lambda_2)}{\alpha_2(\lambda_1, \lambda_2)} \boldsymbol{\Phi}_2(\lambda_2, \lambda_1) \cdot \hat{r}(\lambda_1, \lambda_2) \tag{69}$$

is satisfied. In order to show that, we used a remarkable relation between vector $\boldsymbol{\xi}$, the auxiliary matrix $\hat{r}(\lambda, \mu)$ and the Boltzmann weights given by

$$\boldsymbol{\xi} \cdot \hat{r}(\lambda, \mu) = \frac{\alpha_{10}(\lambda, \mu)\alpha_7(\mu, \lambda)\alpha_2(\lambda, \mu)}{\alpha_7(\lambda, \mu)\alpha_{10}(\mu, \lambda)\alpha_1(\lambda, \mu)} \boldsymbol{\xi}. \tag{70}$$

Alternatively, we can reverse the arguments demanding that the eigenvectors satisfy the exchange symmetry (69). This procedure gives us a restriction to function $\hat{g}_0^{(2)}(\lambda_1, \lambda_2)$ and it is an elegant way of fixing the linear combination from the very beginning. Now it is easy to understand the reason why an "ordering" factor had emerged in the "brute-force" analysis of the two-particle state. For example, the simplest way to generate the unwanted terms $\boldsymbol{B}(\lambda) \otimes \boldsymbol{\Phi}_1(\lambda_1)$ and $[\boldsymbol{\xi} \cdot (\boldsymbol{B}^*(\lambda) \otimes \hat{I})] \otimes \boldsymbol{\Phi}_1(\lambda_1)$ is by using the right-hand side of Eq. (69) instead of the left-hand side we used in the whole "brute force" analysis. In this way we obviously generate only one contribution to such unwanted terms which carries the "ordering" factor explicitly.

In principle, such symmetrization mechanism can be implemented to any multi-particle state, and as we shall see below, it indeed help us to handle the problem of constructing a general n-particle state ansatz. We will start our discussion considering the three-particle state. This state is expected to be a composition between the term representing the creation of three holes (arbitrary spins) on different sites and the three possible ways of combining pairs of holes with a single excitation. Within our algebraic framework the ansatz encoding these features is

$$\begin{aligned}\boldsymbol{\Phi}_3(\lambda_1, \lambda_2, \lambda_3) = {} & \boldsymbol{B}(\lambda_1) \otimes \boldsymbol{B}(\lambda_2) \otimes \boldsymbol{B}(\lambda_3) \\ & + [\boldsymbol{B}(\lambda_1) \otimes \boldsymbol{\xi} F(\lambda_2) B(\lambda_3)] \hat{g}_0^{(3)}(\lambda_1, \lambda_2, \lambda_3)\end{aligned}$$

$$+[\boldsymbol{\xi} \otimes F(\lambda_1)\boldsymbol{B}(\lambda_3)B(\lambda_2)]\hat{g}_1^{(3)}(\lambda_1, \lambda_2, \lambda_3)$$
$$+[\boldsymbol{\xi} \otimes F(\lambda_1)\boldsymbol{B}(\lambda_2)B(\lambda_3)]\hat{g}_2^{(3)}(\lambda_1, \lambda_2, \lambda_3), \tag{71}$$

where the coefficients $\hat{g}_j^{(3)}(\lambda_1, \lambda_2, \lambda_3)$ are going to be determined assuming a priori an exchange property (cf. Eq. (78)) for the $\lambda_1 \leftrightarrow \lambda_2$ and $\lambda_2 \leftrightarrow \lambda_3$ permutations. To see how this works in practice, let us first implement the permutation between the rapidities λ_2 and λ_3. To this end we use the commutation relation (25) to reorder these rapidities in the permuted three-particle vector $\boldsymbol{\Phi}_3(\lambda_1, \lambda_3, \lambda_2)$. This allows us to write the following relation:

$$\frac{\alpha_1(\lambda_2, \lambda_3)}{\alpha_2(\lambda_2, \lambda_3)}\boldsymbol{\Phi}_3(\lambda_1, \lambda_3, \lambda_2).\hat{r}_{23}(\lambda_2, \lambda_3)$$
$$= \boldsymbol{B}(\lambda_1) \otimes \boldsymbol{B}(\lambda_2) \otimes \boldsymbol{B}(\lambda_3) + i\frac{\alpha_{10}(\lambda_2, \lambda_3)}{\alpha_7(\lambda_2, \lambda_3)}[\boldsymbol{B}(\lambda_1) \otimes \boldsymbol{\xi}F(\lambda_2)B(\lambda_3)]$$
$$+[\boldsymbol{B}(\lambda_1) \otimes \boldsymbol{\xi}F(\lambda_3)B(\lambda_2)]\left[-i\frac{\alpha_{10}(\lambda_2, \lambda_3)}{\alpha_7(\lambda_2, \lambda_3)} + \frac{\alpha_1(\lambda_2, \lambda_3)}{\alpha_2(\lambda_2, \lambda_3)}\right.$$
$$\left.\times \hat{g}_0^{(3)}(\lambda_1, \lambda_3, \lambda_2).\hat{r}_{23}(\lambda_2, \lambda_3)\right]$$
$$+[\boldsymbol{\xi} \otimes F(\lambda_1)\boldsymbol{B}(\lambda_2)B(\lambda_3)]\frac{\alpha_1(\lambda_2, \lambda_3)}{\alpha_2(\lambda_2, \lambda_3)}\hat{g}_1^{(3)}(\lambda_1, \lambda_3, \lambda_2).\hat{r}_{23}(\lambda_2, \lambda_3)$$
$$+[\boldsymbol{\xi} \otimes F(\lambda_1)\boldsymbol{B}(\lambda_3)B(\lambda_2)]\frac{\alpha_1(\lambda_2, \lambda_3)}{\alpha_2(\lambda_2, \lambda_3)}\hat{g}_2^{(3)}(\lambda_1, \lambda_3, \lambda_2).\hat{r}_{23}(\lambda_2, \lambda_3). \tag{72}$$

Imposing the exchange property to the three-particle state, i.e. that the right-hand sides of Eqs. (71) and (72) are equal, we are able to derive constraints to functions $\hat{g}_j^{(3)}(\lambda_1, \lambda_2, \lambda_3)$. We find that it is sufficient to have

$$\hat{g}_0^{(3)}(\lambda_1, \lambda_2, \lambda_3) = i\frac{\alpha_{10}(\lambda_2, \lambda_3)}{\alpha_7(\lambda_2, \lambda_3)} \tag{73}$$

and

$$\hat{g}_2^{(3)}(\lambda_1, \lambda_2, \lambda_3) = \frac{\alpha_1(\lambda_2, \lambda_3)}{\alpha_2(\lambda_2, \lambda_3)}\hat{g}_1^{(3)}(\lambda_1, \lambda_3, \lambda_2) \cdot \hat{r}_{23}(\lambda_2, \lambda_3), \tag{74}$$

where we used the identities $\hat{r}_{23}(\lambda_2, \lambda_3) \cdot \hat{r}_{23}(\lambda_3, \lambda_2) = \hat{I}$ and $\alpha_1(\lambda, \mu) = \alpha_2(\mu, \lambda)$. We recall that relation (70) helps us to cancel out the third term of Eq. (72). Now it remains to determine function $\hat{g}_1^{(3)}(\lambda_1, \lambda_2, \lambda_3)$ and this can be done by using the permutation between the variables λ_1 and λ_2. The technical steps of this computation are more involving, since it is necessary to use other commutation rules and some identities between the Boltzmann weights. The details are presented in Appendix D and here we quote our result for the remaining functions

$$\hat{g}_1^{(3)}(\lambda_1, \lambda_2, \lambda_3) = i\frac{\alpha_{10}(\lambda_1, \lambda_2)}{\alpha_7(\lambda_1, \lambda_2)}i\frac{\alpha_2(\lambda_3, \lambda_2)}{\alpha_9(\lambda_3, \lambda_2)},$$
$$\hat{g}_2^{(3)}(\lambda_1, \lambda_2, \lambda_3) = i\frac{\alpha_{10}(\lambda_1, \lambda_3)}{\alpha_7(\lambda_1, \lambda_3)}i\frac{\alpha_1(\lambda_2, \lambda_3)}{\alpha_9(\lambda_2, \lambda_3)}\hat{r}_{23}(\lambda_2, \lambda_3). \tag{75}$$

To make sure we are on the right track, we have checked that the three-particle "easy" unwanted terms are automatically cancelled out provided we fix the constraints $\hat{g}_j^{(3)}(\lambda_1,\lambda_2,\lambda_3)$ as in Eqs. (73) and (75). We note that functions $\hat{g}_0^{(3)}(x,y,z)$ and $\hat{g}_0^{(2)}(y,z)$ are identical, and this allows us to rewrite the three-particle vector in terms of the following recurrence relation:

$$\begin{aligned}\boldsymbol{\Phi}_3(\lambda_1,\lambda_2,\lambda_3) = \boldsymbol{B}(\lambda_1)\otimes\boldsymbol{\Phi}_2(\lambda_2,\lambda_3)\\ +\sum_{j=2}^{3}\left[\boldsymbol{\xi}\otimes F(\lambda_1)\boldsymbol{\Phi}_1(\lambda_2,\dots,\lambda_{j-1},\lambda_{j+1},\dots,\lambda_3)B(\lambda_j)\right]\\ \times\hat{g}_{j-1}^{(3)}(\lambda_1,\lambda_2,\lambda_3).\end{aligned} \tag{76}$$

This expression is rather illuminating, because it suggests that we can write a general n-particle state in terms of the $(n-1)$-particle and $(n-2)$-particle states via a recurrence relation. From our expressions for the two-particle and the three-particle states it is not difficult to guess that the n-particle vector should be given by

$$\begin{aligned}\boldsymbol{\Phi}_n(\lambda_1,\dots,\lambda_n) = \boldsymbol{B}(\lambda_1)\otimes\boldsymbol{\Phi}_{n-1}(\lambda_2,\dots,\lambda_n)\\ +\sum_{j=2}^{n}\left[\boldsymbol{\xi}\otimes F(\lambda_1)\boldsymbol{\Phi}_{n-2}(\lambda_2,\dots,\lambda_{j-1},\lambda_{j+1},\dots,\lambda_n)B(\lambda_j)\right]\\ \times\hat{g}_{j-1}^{(n)}(\lambda_1,\dots,\lambda_n),\end{aligned} \tag{77}$$

where here we formally identified $\boldsymbol{\Phi}_0$ with the unity vector. Our next step is to implement the symmetrization scheme for such multi-particle state ansatz. The best way to proceed here is to use mathematical induction, i.e we assume that the $(n-2)$-particle and the $(n-1)$-particle states were already symmetrized to infer the constraints $\hat{g}_j^{(n)}(\lambda_1,\dots,\lambda_n)$ for the n-particle state. For this purpose we impose that any consecutive permutation between the rapidities λ_{j-1} and λ_j $(j=2,\dots,n)$ satisfies the following exchange property:

$$\begin{aligned}\boldsymbol{\Phi}_n(\lambda_1,\dots,\lambda_{j-1},\lambda_j,\dots,\lambda_n) = \frac{\alpha_1(\lambda_{j-1},\lambda_j)}{\alpha_2(\lambda_{j-1},\lambda_j)}\boldsymbol{\Phi}_n(\lambda_1,\dots,\lambda_j,\lambda_{j-1},\dots,\lambda_n)\\ \times\hat{r}_{j-1,j}(\lambda_{j-1},\lambda_j),\end{aligned} \tag{78}$$

where the indices under $\hat{r}_{j-1,j}(\lambda_{j-1},\lambda_j)$ emphasize the positions on the n-particle space $1\otimes\dots\otimes j-1\otimes j\dots\otimes n$ in which this matrix acts non-trivially.

Now starting with the latest permutation $j=n$ we go ahead comparing the terms proportional to

$$[\boldsymbol{\xi}\otimes F(\lambda_1)\boldsymbol{\Phi}_{n-2}(\lambda_2,\dots,\lambda_{j-2},\lambda_j,\dots,\lambda_n)B(\lambda_{j-1})]$$

and

$$[\boldsymbol{\xi}\otimes F(\lambda_1)\boldsymbol{\Phi}_{n-2}(\lambda_2,\dots,\lambda_{j-1},\lambda_{j+1},\dots,\lambda_n)B(\lambda_j)]$$

in both sides of the exchange relation (78). At each step, this yields a set of relations between the functions $\hat{g}_j^{(n)}(\lambda_1, \dots, \lambda_n)$ which are further simplified by using explicitly both the unitarity condition and the Yang-Baxter equation for the auxiliary r-matrix. Up to $j = 3$ we find that such functions satisfy the following recurrence relation:

$$\hat{g}_{j-1}^{(n)}(\lambda_1, \dots, \lambda_{j-1}, \lambda_j, \dots, \lambda_n) = \frac{\alpha_1(\lambda_{j-1}, \lambda_j)}{\alpha_2(\lambda_{j-1}, \lambda_j)} \hat{g}_{j-2}^{(n)}(\lambda_1, \dots, \lambda_j, \lambda_{j-1}, \dots, \lambda_n) \times \hat{r}_{j-1,j}(\lambda_{j-1}, \lambda_j). \tag{79}$$

Next we implement the symmetrization $\lambda_1 \leftrightarrow \lambda_2$ along the lines sketched in Appendix D for the three-particle state. In this case we have to eliminate the term proportional to $[\boldsymbol{\xi} \otimes F(\lambda_1)\boldsymbol{\Phi}_{n-2}(\lambda_3, \dots, \lambda_n)B(\lambda_2)]$ which only occurs in the left-hand side of the exchange relation (78). This condition helps us to determine the expression for the first constraint and we have

$$\hat{g}_1^{(n)}(\lambda_1, \dots, \lambda_n) = i\frac{\alpha_{10}(\lambda_1, \lambda_2)}{\alpha_7(\lambda_1, \lambda_2)} \prod_{k=3}^{n} i\frac{\alpha_2(\lambda_k, \lambda_2)}{\alpha_9(\lambda_k, \lambda_2)}. \tag{80}$$

Finally, the set of relations (79) and (80) are solved recursively and we find that the n-particle vector is

$$\boldsymbol{\Phi}_n(\lambda_1, \dots, \lambda_n) = \boldsymbol{B}(\lambda_1) \otimes \boldsymbol{\Phi}_{n-1}(\lambda_2, \dots, \lambda_n) + \sum_{j=2}^{n} i\frac{\alpha_{10}(\lambda_1, \lambda_j)}{\alpha_7(\lambda_1, \lambda_j)} \prod_{\substack{k=2 \\ k \neq j}}^{n} i\frac{\alpha_2(\lambda_k, \lambda_j)}{\alpha_9(\lambda_k, \lambda_j)}$$
$$\times \left[\boldsymbol{\xi} \otimes F(\lambda_1)\boldsymbol{\Phi}_{n-2}(\lambda_2, \dots, \lambda_{j-1}, \lambda_{j+1}, \dots, \lambda_n)B(\lambda_j)\right]$$
$$\times \prod_{k=2}^{j-1} \frac{\alpha_1(\lambda_k, \lambda_j)}{\alpha_2(\lambda_k, \lambda_j)} \hat{r}_{k,k+1}(\lambda_k, \lambda_j). \tag{81}$$

At this point it is fair to remark that the recursive way we found for the eigenvectors were inspired to some extent on an early work of Tarasov on the Izergin-Korepin model [34]. Our construction, however, has the important novelty of allowing a general "exclusion statistics" between the non-commutative and the commutative creation fields and therefore paving the way for further applications and extensions. Indeed, the non-trivial way that both the "exclusion" vector and the auxiliary matrix enters in the eigenvectors expression (81) makes our formula rather general, being able to accommodate the solution of a wider class of integrable models. This situation has to be contrasted to that of multi-state 6-vertex generalizations [30,31], in which the eigenvectors are easily given by tensoring the creation fields and there is no explicit dependence of the underlying algebra.

Let us now return to the problem of finding the eigenvalues of the transfer matrix, keeping in mind the recurrence relation (81) for the eigenvectors. To gain some insight about this problem we first investigate how the wanted and unwanted terms are collected for the three-particle state. Besides the commutation rules for the diagonal fields, we also have to use our previous results for the two-particle state (cf. Eqs. (57)-(59)) wherever there is the need to carry the diagonal operators through the vector $\Phi_2(\lambda_2,\lambda_3)$. This recursive way not only helps us to better simplify the wanted terms but also makes it possible to gather the unwanted terms in rather closed forms. This analysis is presented in Appendix D since it still involves some extra technicalities. Having at hand the two-particle and the three-particle data we can move forward to the analysis of the four-particle state and so forth. In general, for $n \leqslant 3$, the knowledge of the $(n-1)$-particle and the $(n-2)$-particle results dictates the behaviour of the n-particle state. By using mathematical induction we are able to determine the general structure for the multi-particle states and the final results are

$$B(\lambda)\left|\Phi_n(\lambda_1,\dots,\lambda_n)\right\rangle = [\omega_1(\lambda)]^L \prod_{j=1}^{n} i\frac{\alpha_2(\lambda_j,\lambda)}{\alpha_9(\lambda_j,\lambda)} \left|\Phi_n(\lambda_1,\dots,\lambda_n)\right\rangle$$
$$-\sum_{j=1}^{n}[\omega_1(\lambda_j)]^L \left|\Psi^{(1)}_{n-1}(\lambda,\lambda_j;\{\lambda_l\})\right\rangle$$
$$+\sum_{j=2}^{n}\sum_{l=1}^{j-1} H_1(\lambda,\lambda_l,\lambda_j)[\omega_1(\lambda_l)\omega_1(\lambda_j)]^L \left|\Psi^{(3)}_{n-2}(\lambda,\lambda_j,\lambda_l;\{\lambda_k\})\right\rangle, \tag{82}$$

$$D(\lambda)\left|\Phi_n(\lambda_1,\dots,\lambda_n)\right\rangle = [\omega_3(\lambda)]^L \prod_{j=1}^{n} -i\frac{\alpha_8(\lambda,\lambda_j)}{\alpha_7(\lambda,\lambda_j)} \left|\Phi_n(\lambda_1,\dots,\lambda_n)\right\rangle$$
$$-\sum_{j=1}^{n}[\omega_2(\lambda_j)]^L \Lambda^{(1)}(\lambda=\lambda_j,\{\lambda_l\}) \left|\Psi^{(2)}_{n-1}(\lambda,\lambda_j;\{\lambda_l\})\right\rangle$$
$$+\sum_{j=2}^{n}\sum_{l=1}^{j-1} H_2(\lambda,\lambda_l,\lambda_j)[\omega_2(\lambda_1)\omega_2(\lambda_2)]^L \Lambda^{(1)}(\lambda=\lambda_j,\{\lambda_k\})$$
$$\times\Lambda^{(1)}(\lambda=\lambda_l,\{\lambda_k\}) \left|\Psi^{(3)}_{n-2}(\lambda,\lambda_j,\lambda_l;\{\lambda_k\})\right\rangle, \tag{83}$$

$$\sum_{a=1}^{2}\hat{A}_{aa}(\lambda)\left|\Phi_n(\lambda_1,\dots,\lambda_n)\right\rangle = [\omega_2(\lambda)]^L \prod_{j=1}^{n} -i\frac{\alpha_1(\lambda,\lambda_j)}{\alpha_9(\lambda,\lambda_j)}$$
$$\times\Lambda^{(1)}(\lambda,\{\lambda_l\})\left|\Phi_n(\lambda_1,\dots,\lambda_n)\right\rangle$$
$$-\sum_{j=1}^{n}[\omega_2(\lambda_j)]^L \Lambda^{(1)}(\lambda=\lambda_j,\{\lambda_l\}) \left|\Psi^{(1)}_{n-1}(\lambda,\lambda_j;\{\lambda_l\})\right\rangle$$
$$-\sum_{j=1}^{n}[\omega_1(\lambda_j)]^L \left|\Psi^{(2)}_{n-1}(\lambda,\lambda_j;\{\lambda_l\})\right\rangle$$
$$-\sum_{j=2}^{n}\sum_{l=1}^{j-1} H_3(\lambda,\lambda_l,\lambda_j)[\bar{a}(\lambda_l,\lambda_j)-\bar{b}(\lambda_l,\lambda_j)][\omega_1(\lambda_l)\omega_2(\lambda_j)]^L$$
$$\times\Lambda^{(1)}(\lambda=\lambda_j,\{\lambda_k\}) \left|\Psi^{(3)}_{n-2}(\lambda,\lambda_j,\lambda_l;\{\lambda_k\})\right\rangle$$

$$-\sum_{j=2}^{n}\sum_{l=1}^{j-1} H_3(\lambda,\lambda_j,\lambda_l)\frac{\alpha_1(\lambda_l,\lambda_j)}{\alpha_2(\lambda_l,\lambda_l)}[\omega_1(\lambda_j)\omega_2(\lambda_l)]^L$$
$$\times\Lambda^{(1)}(\lambda=\lambda_l,\{\lambda_k\})\left|\Psi_{n-2}^{(3)}(\lambda,\lambda_j,\lambda_l;\{\lambda_k\})\right\rangle. \tag{84}$$

Similarly to what happened to the two-particle and the three-particle cases we have three families of unwanted terms. As before they are written in terms of the creation operators and the general expressions are

$$\left|\Psi_{n-1}^{(1)}(\lambda,\lambda_j;\{\lambda_l\})\right\rangle = i\frac{\alpha_5(\lambda_j,\lambda)}{\alpha_9(\lambda_j,\lambda)}\prod_{\substack{k=1\\k\neq j}}^{n} i\frac{\alpha_2(\lambda_k,\lambda_j)}{\alpha_9(\lambda_k,\lambda_j)}$$
$$\times\boldsymbol{B}(\lambda)\otimes\boldsymbol{\Phi}_{n-1}(\lambda_1,\ldots,\check{\lambda}_j,\ldots,\lambda_n)$$
$$\times\hat{O}_j^{(1)}(\lambda_j;\{\lambda_k\})\cdot\mathcal{F}\left|0\right\rangle, \tag{85}$$

$$\left|\Psi_{n-1}^{(2)}(\lambda,\lambda_j;\{\lambda_l\})\right\rangle = i\frac{\alpha_{10}(\lambda,\lambda_j)}{\alpha_7(\lambda,\lambda_j)}\prod_{\substack{k=1\\k\neq j}}^{n} i\frac{\alpha_2(\lambda_k,\lambda_j)}{\alpha_9(\lambda_k,\lambda_j)}$$
$$\times[\boldsymbol{\xi}\cdot(\boldsymbol{B}^*(\lambda)\otimes\hat{I})]\otimes\boldsymbol{\Phi}_{n-1}(\lambda_1,\ldots,\check{\lambda}_j,\ldots,\lambda_n)$$
$$\times\hat{O}_j^{(1)}(\lambda_j;\{\lambda_k\})\cdot\mathcal{F}\left|0\right\rangle, \tag{86}$$

$$\left|\Psi_{n-2}^{(3)}(\lambda,\lambda_j,\lambda_l;\{\lambda_k\})\right\rangle = \prod_{\substack{k=1\\\neq j,l}}^{n} i\frac{\alpha_2(\lambda_k,\lambda_j)}{\alpha_9(\lambda_k,\lambda_j)}i\frac{\alpha_2(\lambda_k,\lambda_l)}{\alpha_9(\lambda_k,\lambda_l)}$$
$$\times F(\lambda)\boldsymbol{\xi}\otimes\boldsymbol{\Phi}_{n-2}(\lambda_1,\ldots,\check{\lambda}_l,\ldots,\check{\lambda}_j,\ldots,\lambda_n)$$
$$\times\hat{O}_{lj}^{(2)}(\lambda_l,\lambda_j;\{\lambda_k\})\cdot\mathcal{F}\left|0\right\rangle, \tag{87}$$

where the symbol $\check{\lambda}_j$ means that the rapidity λ_j is absent from the set $\{\lambda_1,\ldots,\lambda_n\}$. For $n\geqslant 3$ it was necessary to introduce a second "ordering" factor in order to better represent the third type of unwanted terms (cf. Appendix D). Its task is similar to that played by the first "ordering" factor with the difference that now two rapidities are reordered. In other words, this second "ordering" factor brings the rapidities λ_l and λ_j $(l<j)$ to the first two positions in the eigenvector formula (81), and a simple calculation shows that its expression is

$$\hat{O}_{lj}^{(2)}(\lambda_l,\lambda_j;\{\lambda_k\}) = \prod_{k=1}^{l-1}\frac{\alpha_1(\lambda_k,\lambda_j)}{\alpha_2(\lambda_k,\lambda_j)}\hat{r}_{k+1,k+2}(\lambda_k,\lambda_j)\prod_{k=l+1}^{j-1}\frac{\alpha_1(\lambda_k,\lambda_j)}{\alpha_2(\lambda_k,\lambda_j)}\hat{r}_{k,k+1}(\lambda_k,\lambda_j)$$
$$\times\prod_{k=1}^{l-1}\frac{\alpha_1(\lambda_k,\lambda_l)}{\alpha_2(\lambda_k,\lambda_l)}\hat{r}_{k,k+1}(\lambda_k,\lambda_l). \tag{88}$$

Before discussing the results, we should note that the above expressions for multi-particle states indeed reproduce our previous findings for the two-particle (after considering Appendix C) and the one-particle states. Now, from Eqs. (82)–(84), it is direct to read of the n-particle eigenvalue expression, namely

$$\Lambda(\lambda,\{\lambda_j\}) = [\omega_1(\lambda)]^L \prod_{j=1}^{n} i\frac{\alpha_2(\lambda_j,\lambda)}{\alpha_9(\lambda_j,\lambda)} + [\omega_3(\lambda)]^L \prod_{j=1}^{n} -i\frac{\alpha_8(\lambda,\lambda_j)}{\alpha_7(\lambda,\lambda_j)}$$

$$-[\omega_2(\lambda)]^L \prod_{j=1}^{n} -i\frac{\alpha_1(\lambda,\lambda_j)}{\alpha_9(\lambda,\lambda_j)} \Lambda^{(1)}(\lambda,\{\lambda_l\}). \tag{89}$$

Following the same arguments given for the two-particle state, and in particular the discussion presented at the end of Appendix C, we easily derive that the unwanted terms vanish provided the rapidities satisfy the following Bethe ansatz equations:

$$\left[\frac{\omega_1(\lambda_i)}{\omega_2(\lambda_i)}\right]^L = \Lambda^{(1)}(\lambda=\lambda_i,\{\lambda_j\}), \quad i=1,\dots,n. \tag{90}$$

Once again, the final results have been expressed in terms of the underlying auxiliary problem, which for a general multi-particle state is defined by

$$T^{(1)}(\lambda,\{\lambda_i\})_{a_1\dots a_n}^{b_1\dots b_n} \mathcal{F}^{b_n\dots b_1} = \Lambda^{(1)}(\lambda,\{\lambda_i\})\mathcal{F}^{a_n\dots a_1}, \tag{91}$$

where the inhomogeneous transfer matrix $T^{(1)}(\lambda,\{\lambda_i\})$ is

$$T^{(1)}(\lambda,\{\lambda_i\})_{b_1\dots b_n}^{a_1\dots a_n} = \hat{r}_{b_1 d_1}^{c_1 a_1}(\lambda,\lambda_1)\hat{r}_{b_2 c_2}^{d_1 a_2}(\lambda,\lambda_2)\dots\hat{r}_{b_n c_1}^{d_{n-1} a_n}(\lambda,\lambda_n). \tag{92}$$

As we have commented before these results are direct extensions of those obtained for the two-particle state. We see that the Bethe ansatz equations and the eigenvalues still depend on an additional auxiliary eigenvalue problem. In the language of condensed matter we would say that so far we managed to solve the "charge" degrees of freedom but still remains the diagonalization of the "spin" sector. As we shall see next the "spin" problem can also be solved in terms of the algebraic Bethe ansatz approach.

4.2. *The eigenvalues and the nested Bethe ansatz*

The purpose of this section will be the diagonalization of the auxiliary transfer matrix $T^{(1)}(\lambda,\{\lambda_j\})$. For this purpose we have to set up another Bethe ansatz which will result in "nested" Bethe ansatz equations for the rapidities we began with. This problem, however, is equivalent to the solution of the 6-vertex model in presence of inhomogeneities and it has been extensively discussed in the literature (see e.g. Refs. [5,30,31]). Therefore we will only sketch the main steps of the solution for sake of completeness. First we write the transfer matrix $T^{(1)}(\lambda,\{\lambda_j\})$ as the trace of the following monodromy matrix:

$$\mathcal{T}^{(1)}(\lambda,\{\lambda_j\}) = \mathcal{L}^{(1)}_{\mathcal{A}^{(1)}n}(\lambda,\lambda_n)\mathcal{L}^{(1)}_{\mathcal{A}^{(1)}n-1}(\lambda,\lambda_{n-1})\dots\mathcal{L}^{(1)}_{\mathcal{A}^{(1)}1}(\lambda,\lambda_1), \tag{93}$$

with $\mathcal{A}^{(1)}$ the two-dimensional "spin" auxiliary space. The Lax operator $\mathcal{L}^{(1)}_{\mathcal{A}^{(1)}j}(\lambda,\lambda_j)$ is related to the auxiliary matrix $\hat{r}(\lambda,\lambda_j)$ by a permutation on the $C^2 \times C^2$ space and its matrix elements are

$$\mathcal{L}^{(1)}_{\mathcal{A}^{(1)}j}(\lambda,\lambda_j)=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \bar{b}(\lambda,\lambda_j) & \bar{a}(\lambda,\lambda_j) & 0\\ 0 & \bar{a}(\lambda,\lambda_j) & \bar{b}(\lambda,\lambda_j) & 0\\ 0 & 0 & 0 & 1\end{pmatrix}. \tag{94}$$

We now go ahead applying the *ABCD* algebraic Bethe ansatz framework [1–3] for an inhomogeneous transfer matrix. Writing the monodromy matrix as

$$\mathcal{T}^{(1)}(\lambda,\{\lambda_j\})=\begin{pmatrix}A^{(1)}(\lambda,\{\lambda_j\}) & B^{(1)}(\lambda,\{\lambda_j\})\\ C^{(1)}(\lambda,\{\lambda_j\}) & D^{(1)}(\lambda,\{\lambda_j\})\end{pmatrix}, \tag{95}$$

and taking as the reference state the vector

$$|0^{(1)}\rangle=\prod_{j=1}^{n}\otimes\begin{pmatrix}1\\0\end{pmatrix}_j \tag{96}$$

we find the following relations:

$$A^{(1)}(\lambda,\{\lambda_j\})\,|0^{(1)}\rangle=|0^{(1)}\rangle,$$
$$D^{(1)}(\lambda,\{\lambda_j\})\,|0^{(1)}\rangle=\prod_{j=1}^{n}\bar{b}(\lambda,\lambda_j)\,|0^{(1)}\rangle,$$
$$C^{(1)}(\lambda,\{\lambda_j\})\,|0^{(1)}\rangle=0. \tag{97}$$

The field $B^{(1)}(\lambda,\{\lambda_j\})$ plays the role of a creation operator over the reference state. To get its commutation rules we solve the Yang–Baxter algebra for the monodromy matrix $\mathcal{T}^{(1)}(\lambda,\{\lambda_j\})$ using as intertwiner the auxiliary matrix (26). This yields the following relations:

$$\begin{aligned}A^{(1)}(\lambda,\{\lambda_j\})B^{(1)}(\mu,\{\lambda_j\})&=\frac{1}{\bar{b}(\mu,\lambda)}B^{(1)}(\mu,\{\lambda_j\})A^{(1)}(\lambda,\{\lambda_j\})\\&\quad-\frac{\bar{a}(\mu,\lambda)}{\bar{b}(\mu,\lambda)}B^{(1)}(\lambda,\{\lambda_j\})A^{(1)}(\mu,\{\lambda_j\}),\\ D^{(1)}(\lambda,\{\lambda_j\})B^{(1)}(\mu,\{\lambda_j\})&=\frac{1}{\bar{b}(\lambda,\mu)}B^{(1)}(\mu,\{\lambda_j\})D^{(1)}(\lambda,\{\lambda_j\})\\&\quad-\frac{\bar{a}(\lambda,\mu)}{\bar{b}(\lambda,\mu)}B^{(1)}(\lambda,\{\lambda_j\})D^{(1)}(\mu,\{\lambda_j\}),\\ \left[B^{(1)}(\mu,\{\lambda_j\}),B^{(1)}(\lambda,\{\lambda_j\})\right]&=0.\end{aligned} \tag{98}$$

Next we have to make an ansatz for the eigenstates of $T^{(1)}(\lambda,\{\lambda_j\})$. This is the "spin" part of the multi-particle states and it is given by $\prod_{l=1}^{m}B^{(1)}(\mu_l,\{\lambda_j\})\,|0^{(1)}\rangle$ whose components are precisely identified with the coefficients $\mathcal{F}^{a_n\ldots a_1}$. With the help of commutation rules (98) we are able to carry on the operators $A^{(1)}(\lambda,\{\lambda_j\})+D^{(1)}(\lambda,\{\lambda_j\})$ through all the creation fields $B^{(1)}(\mu_l,\{\lambda_j\})$ leading us to the following result for the auxiliary eigenvalue:

$$\Lambda^{(1)}(\lambda,\{\lambda_j\},\{\mu_l\}) = \prod_{l=1}^{m} \frac{1}{\bar{b}(\mu_l,\lambda)} + \prod_{j=1}^{n} \bar{b}(\lambda,\lambda_j) \prod_{l=1}^{m} \frac{1}{\bar{b}(\lambda,\mu_l)} \tag{99}$$

provided the numbers $\{\mu_l\}$ satisfy the additional restriction

$$\prod_{j=1}^{n} \bar{b}(\mu_l,\lambda_j) = -\prod_{k=1}^{m} \frac{\bar{b}(\mu_l,\mu_k)}{\bar{b}(\mu_k,\mu_l)}, \quad l = 1,\dots,m. \tag{100}$$

Finally, we use the auxiliary eigenvalue expression to rewrite our previous results for the eigenvalues and Bethe ansatz equations of the "covering" vertex model. Substituting the expression (99) in Eqs. (89,90) and using the second relation of Eq. (28) we obtain that the eigenvalue is

$$\begin{aligned}\Lambda(\lambda,\{\lambda_j\},\{\mu_l\}) &= [\omega_1(\lambda)]^L \prod_{j=1}^{n} i\frac{\alpha_2(\lambda_j,\lambda)}{\alpha_9(\lambda_j,\lambda)} + [\omega_3(\lambda)]^L \prod_{j=1}^{n} -i\frac{\alpha_8(\lambda,\lambda_j)}{\alpha_7(\lambda,\lambda_j)} \\ &-[\omega_2(\lambda)]^L \left\{ \prod_{j=1}^{n} -i\frac{\alpha_1(\lambda,\lambda_j)}{\alpha_9(\lambda,\lambda_j)} \prod_{l=1}^{m} \frac{1}{\bar{b}(\mu_l,\lambda)} + \prod_{j=1}^{n} -i\frac{\alpha_8(\lambda,\lambda_j)}{\alpha_7(\lambda,\lambda_j)} \prod_{l=1}^{m} \frac{1}{\bar{b}(\lambda,\mu_l)} \right\}\end{aligned} \tag{101}$$

while the Bethe ansatz equations for the rapidities $\{\lambda_j\}$ becomes

$$\left[\frac{\omega_1(\lambda_j)}{\omega_2(\lambda_j)}\right]^L = \prod_{l=1}^{m} \frac{1}{\bar{b}(\mu_l,\lambda_j)}. \tag{102}$$

Now we are almost ready to make a comparison with the Lieb's and Wu's results [9]. First we introduce a new set of variables $z_\pm(\lambda_j)$ defined by

$$z_-(\lambda_j) = \frac{a(\lambda_j)}{b(\lambda_j)} e^{2h(\lambda_j)}, \qquad z_+(\lambda_j) = \frac{b(\lambda_j)}{a(\lambda_j)} e^{2h(\lambda_j)}. \tag{103}$$

Considering this definition and taking into account the transformation (32) as well as the identities (29,30), we are able to rewrite the expression for the eigenvalue as

$$\begin{aligned}(-i)^n \Lambda(\lambda,\{z_\pm(\lambda_j)\},\{\tilde{\mu}_l\}) &= [\omega_1(\lambda)]^L \prod_{j=1}^{n} \frac{b(\lambda)}{a(\lambda)} \left[\frac{1+z_-(\lambda_j)/z_+(\lambda)}{1-z_-(\lambda_j)/z_-(\lambda)}\right] \\ &+[\omega_3(\lambda)]^L \prod_{j=1}^{n} \frac{b(\lambda)}{a(\lambda)} \left[\frac{1+z_-(\lambda_j)z_-(\lambda)}{1-z_-(\lambda_j)z_+(\lambda)}\right] \\ &-[\omega_2(\lambda)]^L \left\{ \prod_{j=1}^{n} \frac{b(\lambda)}{a(\lambda)} \left[\frac{1+z_-(\lambda_j)/z_+(\lambda)}{1-z_-(\lambda_j)/z_-(\lambda)}\right] \right. \\ &\times \prod_{l=1}^{m} \frac{z_-(\lambda) - 1/z_-(\lambda) - \tilde{\mu}_l + U/2}{z_-(\lambda) - 1/z_-(\lambda) - \tilde{\mu}_l - U/2}\end{aligned}$$

$$+\prod_{j=1}^{n}\frac{b(\lambda)}{a(\lambda)}\left[\frac{1+z_-(\lambda_j)z_-(\lambda)}{1-z_-(\lambda_j)z_+(\lambda)}\right]\prod_{l=1}^{n}\frac{1/z_+(\lambda)-z_+(\lambda)-\tilde{\mu}_l-U/2}{1/z_+(\lambda)-z_+(\lambda)-\tilde{\mu}_l+U/2}\Bigg\} \qquad (104)$$

and the nested Bethe ansatz equations are now given by

$$[z_-(\lambda_j)]^L=\prod_{l=1}^{m}\frac{z_-(\lambda_j)-1/z_-(\lambda_j)-\tilde{\mu}_l+U/2}{z_-(\lambda_j)-1/z_-(\lambda_j)-\tilde{\mu}_l-U/2},\quad j=1,\dots,n$$

$$\prod_{j=1}^{n}\frac{z_-(\lambda_j)-1/z_-(\lambda_j)-\tilde{\mu}_l-U/2}{z_-(\lambda_j)-1/z_-(\lambda_j)-\tilde{\mu}_l+U/2}=-\prod_{k=1}^{m}\frac{\tilde{\mu}_l-\tilde{\mu}_k+U}{\tilde{\mu}_l-\tilde{\mu}_k-U},\quad l=1,\dots,m \qquad (105)$$

From the above expressions we note that function $\Lambda(\lambda,\{z_\pm(\lambda_j)\},\{\tilde{\mu}_l\})$ is analytic in λ. This happens because the condition of having zero residues on both direct $z_-(\lambda_j)$ and "crossed" $z_+(\lambda_j)$ channels is clearly fulfilled by the nested Bethe ansatz equations. The next step is to expand the logarithm of the eigenvalue $\Lambda(\lambda,\{z_\pm(\lambda_j)\},\{\tilde{\mu}_l\})$ in powers of λ and up to second order in the expansion we find

$$\begin{aligned}\ln\left[\Lambda(\lambda,\{z_\pm(\lambda_j)\},\{\tilde{\mu}_l\})\right]&=\frac{i\pi}{2}n+\sum_{j=1}^{n}\ln[z_-(\lambda_j)]\\&+\lambda\left[\sum_{j=1}^{n}\left[z_-(\lambda_j)+1/z_-(\lambda_j)\right]+\frac{U}{4}(L-2n)\right]\\&+\lambda^2\left[\sum_{j=1}^{n}\left[z_-^2(\lambda_j)-1/z_-^2(\lambda_j)-U[z_-(\lambda_j)-1/z_-(\lambda_j)]\right]-2L\right]\\&+\mathcal{O}(\lambda^3).\end{aligned} \qquad (106)$$

The $\mathcal{O}(\lambda)$ term parametrizes the spectrum of the Hubbard Hamiltonian and to recover Lieb's and Wu's results we just have to reexpress the variable $z_-(\lambda_j)$ in terms of the hole momenta k_j by

$$z_-(\lambda_j)=e^{ik_j}. \qquad (107)$$

Considering this relation, the eigenenergies of the Hubbard model are

$$E_n(L)=\frac{U(L-2n)}{4}+\sum_{j=1}^{n}2\cos(k_j) \qquad (108)$$

and the momenta k_j satisfy the following Bethe ansatz equations:

$$e^{iLk_j}=\prod_{l=1}^{m}\frac{\sin(k_j)-\bar{\mu}_l-iU/4}{\sin(k_j)-\bar{\mu}_l+iU/4},\quad j=1,\dots,n$$

$$\prod_{j=1}^{n} \frac{\sin(k_j) - \tilde{\mu}_l + iU/4}{\sin(k_j) - \tilde{\mu}_l - iU/4} = -\prod_{k=1}^{m} \frac{\tilde{\mu}_l - \tilde{\mu}_k - iU/2}{\tilde{\mu}_l - \tilde{\mu}_k + iU/2}, \quad l = 1, \ldots, m \tag{109}$$

where we also used $\tilde{\mu}_l = 2i\bar{\mu}_l$ to bring our equations in the Lieb's and Wu's form. A careful reader might note that the above Bethe ansatz equations have an extra minus factor in front of the coupling U in comparison to the original ones. This is because we are using the language of holes instead of particles and this means that the integers n and m are the total number of holes and the number of holes with spin up, respectively. It is well known that via a particle–hole transformation the kinetic term of the Hamiltonian gets an extra minus sign, which changes the sign of factor U/t entering in the Bethe ansatz equations. Similar reasoning can be carried out for others conserved charges. For example, the first non-trivial current commuting with the Hamiltonian [12,13] is

$$\begin{aligned} J = \sum_{j=1}^{L} & c_{j\uparrow}^{\dagger} c_{j+2\uparrow} - c_{j+2\uparrow}^{\dagger} c_{j\uparrow} \\ & + U(c_{j\uparrow}^{\dagger} c_{j+1\uparrow} - c_{j+1\uparrow}^{\dagger} c_{j\uparrow})(n_{j+1\downarrow} + n_{j\downarrow} - 1) + [\uparrow \leftrightarrow \downarrow] \end{aligned} \tag{110}$$

and from Eq. (106) it follows that the spectrum (modulo a constant) of this charge is

$$E_n^J(L) = 2i \sum_{j=1}^{n} \left[\sin(2k_j) - U \sin(k_j) \right]. \tag{111}$$

We would like to close this section commenting on the construction of the eigenvectors in the terms of the "dual" field $\boldsymbol{B}^*(\lambda)$. The equivalence between the commutation rules for the fields $\boldsymbol{B}(\lambda)$ and $\boldsymbol{B}^*(\lambda)$ allow us to follow straightforwardly the whole construction of Section 4.1 and it is not difficult to derive formula for the "dual" eigenvectors $\boldsymbol{\Phi}^*{}_n(\lambda_1, \ldots, \lambda_n)$. Formally, we can apply the "dual" transformation in expression (81). This leads us to following "dual" recurrence relation:

$$\begin{aligned} \boldsymbol{\Phi}^*{}_n(\lambda_1, \ldots, \lambda_n) = {} & \boldsymbol{B}^*(\lambda_1) \otimes \boldsymbol{\Phi}^*{}_{n-1}(\lambda_2, \ldots, \lambda_n) + \sum_{j=2}^{n} i \frac{\alpha_{10}^*(\lambda_1, \lambda_j)}{\alpha_7^*(\lambda_1, \lambda_j)} \\ & \times \prod_{\substack{k=2 \\ k \neq j}}^{n} i \frac{\alpha_2^*(\lambda_k, \lambda_j)}{\alpha_9^*(\lambda_k, \lambda_j)} \prod_{k=2}^{j-1} \frac{\alpha_1^*(\lambda_k, \lambda_j)}{\alpha_2^*(\lambda_k, \lambda_j)} \hat{r}_{k,k+1}^*(\lambda_k, \lambda_j) \\ & \times \left[\boldsymbol{\xi} \otimes F(\lambda_1) \boldsymbol{\Phi}^*{}_{n-2}(\lambda_2, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_n) D(\lambda_j) \right]. \end{aligned} \tag{112}$$

We expect that the corresponding eigenvalues $\Lambda^*(\lambda, \{\lambda_j\}, \{\tilde{\mu}_l\})$ should also be related to $\Lambda(\lambda, \{\lambda_j\}, \{\tilde{\mu}_l\})$ in some way. This is indeed the case if we shift all the rapidities around the "crossing" point $\pi/2$, and the relation we found is

$$\Lambda^*(\pi/2 - \lambda, \{\pi/2 - \lambda_j\}, \{\pi/2 - \tilde{\mu}_l\}) = (-1)^n \Lambda(\lambda, \{\lambda_j\}, \{\tilde{\mu}_l\}). \tag{113}$$

With this we complete our analysis of the graded eigenvalue problem and in the next section we shall discuss some other complementary results which can be obtained within the $ABCDF$ formalism.

5. Complementary results

In this section we shall first consider the solution of the coupled spin model with twisted boundary conditions. This allow us to illustrate the difference between the Hubbard and the coupled spin models from the viewpoint of their Bethe ansatz solution. Next we consider the well known $SU(2)$ symmetries of the Hubbard model [35–37]. We will show that the eigenvectors (81) are highest weights of both the $SU(2)$ Lie algebra of rotations and the η-paring $SU(2)$ symmetry. Thus we are able to recover the results by Essler, Korepin and Schoutens [36] from an algebraic point of view.

5.1. Twisted boundary conditions

We begin recalling that twisted boundary conditions are in general associated to certain gauge invariances of the Yang–Baxter algebra. The integrability condition (1) is still valid when $\mathcal{L}_{\mathcal{A}i}(\lambda) \to G_{\mathcal{A}}\mathcal{L}_{\mathcal{A}i}(\lambda)$ provided the gauge matrix $G_{\mathcal{A}}$ satisfies [25]

$$\left[R(\lambda,\mu), G_{\mathcal{A}} \otimes G_{\mathcal{A}}\right] = 0. \tag{114}$$

This means that a vertex model defined by the transfer matrix $T_G(\lambda) = \mathrm{Tr}_{\mathcal{A}}\mathcal{T}_G(\lambda)$ whose monodromy matrix is

$$\mathcal{T}_G(\lambda) = G_{\mathcal{A}}\mathcal{L}_{\mathcal{A}L}(\lambda)\mathcal{L}_{\mathcal{A}L-1}(\lambda)\dots\mathcal{L}_{\mathcal{A}1}(\lambda) \tag{115}$$

still remains integrable. One way of seeing the connection to twisted boundary conditions is, for example, to derive the quantum Hamiltonian H_G commuting with the transfer matrix $T_G(\lambda)$. The derivation of H_G is standard (see e.g. Ref. [38]) and it is given by

$$H_G = \sum_{i=1}^{L-1} h_{i,i+1} + G_L^{-1} h_{L,1} G_L, \tag{116}$$

where $h_{ij} = [P\mathcal{L}'(0)]_{ij}$. Here we assumed that the Lax operator is regular at $\lambda = 0$ and that $G_{\mathcal{A}}$ is invertible. The last term in the Hamiltonian (116) reflects the presence of non-trivial boundary conditions. In the context of the coupled spin model (7), it is straightforward to see that we get twisted boundary conditions by taking the following gauge:

$$G_{\mathcal{A}} = \begin{pmatrix} e^{-i\phi_1/2} & 0 \\ 0 & e^{i\phi_1/2} \end{pmatrix} \otimes \begin{pmatrix} e^{-i\phi_2/2} & 0 \\ 0 & e^{i\phi_2/2} \end{pmatrix}. \tag{117}$$

Clearly, such a gauge matrix fulfills the integrability condition (114). In order to diagonalize $T_G(\lambda)$ we only need to introduce a few modifications on the formalism developed in the previous sections. It is fundamental that this gauge does not spoil the triangular form of the monodromy $\mathcal{T}_G(\lambda)$ when it acts on the ferromagnetic reference $|0\rangle$. The diagonal operators of $\mathcal{T}_G(\lambda)$, however, pick up extra phase factors and now we have the following relations

$$B(\lambda)\ |0\rangle = e^{-i(\phi_1+\phi_2)/2}[\omega_1(\lambda)]^L\,|0\rangle\ D(\lambda)\ |0\rangle = e^{i(\phi_1+\phi_2)/2}[\omega_3(\lambda)]^L\,|0\rangle\,,$$
$$A_{11}(\lambda)\ |0\rangle = e^{-i(\phi_1-\phi_2)/2}[\omega_2(\lambda)]^L\,|0\rangle\ A_{22}(\lambda)\ |0\rangle = e^{i(\phi_1-\phi_2)/2}[\omega_2(\lambda)]^L\,|0\rangle\,. \tag{118}$$

The next step is to solve the commutation rules in the standard Yang–Baxter formalism, since we are considering the coupled spin model. These commutation rules have basically the same structure of those worked out for the graded case, apart from a few signs and imaginary factors. We have collected them in Appendix B and we note that the corresponding 6-vertex auxiliary matrix has now an extra sign in the amplitude $\bar{b}(\lambda,\mu)$. Therefore, the nested part always gets twisted, emphasizing the difference between the Hubbard and the coupled spin models for closed boundary conditions. Since now the basic ingredients have been set up we can follow closely the steps of Sections 3 and 4. Here we are interested in the eigenvalues of the twisted model and now we begin to summarize our final findings. Taking into account the relations (118) and the commutation rules (B.12)–(B.22) we derive that the eigenvalues of transfer matrix $T_G(\lambda)$ is

$$\begin{aligned}\Lambda_G(\lambda,\{\lambda_j\}) &= e^{-i(\phi_1+\phi_2)/2}[\omega_1(\lambda)]^L\prod_{j=1}^{n}-\frac{\alpha_2(\lambda_j,\lambda)}{\alpha_9(\lambda_j,\lambda)}\\ &\quad+e^{i(\phi_1+\phi_2)/2}[\omega_3(\lambda)]^L\prod_{j=1}^{n}-\frac{\alpha_8(\lambda,\lambda_j)}{\alpha_7(\lambda,\lambda_j)}\\ &\quad+\prod_{j=1}^{n}-\frac{\alpha_1(\lambda,\lambda_j)}{\alpha_9(\lambda,\lambda_j)}\Lambda_G^{(1)}(\lambda,\{\lambda_l\}),\end{aligned} \tag{119}$$

where the variables $\{\lambda_j\}$ satisfy the Bethe ansatz equations

$$\left[\frac{\omega_1(\lambda_j)}{\omega_2(\lambda_j)}\right]^L = -(-1)^n e^{i(\phi_1+\phi_2)/2}\Lambda_G^{(1)}(\lambda=\lambda_j,\{\lambda_l\}),\quad j=1,\dots,n. \tag{120}$$

It turns out that the auxiliary problem gets also an extra modification besides the sign on amplitude $\bar{b}(\lambda,\mu)$. The auxiliary problem absorbs the twisting on the diagonal fields $A_{11}(\lambda)$ and $A_{22}(\lambda)$ and now function $\Lambda_G^{(1)}(\lambda,\{\lambda_j\})$ is the eigenvalue of the following auxiliary transfer matrix:

$$\mathcal{T}_G^{(1)}(\lambda,\{\lambda_i\}) = \mathrm{Tr}_{\mathcal{A}}\left[G_{\mathcal{A}}^{(1)}\tilde{\mathcal{L}}_{\mathcal{A}n}^{(1)}(\lambda,\lambda_n)\tilde{\mathcal{L}}_{\mathcal{A}n-1}^{(1)}(\lambda,\lambda_{n-1})\dots\tilde{\mathcal{L}}_{\mathcal{A}1}^{(1)}(\lambda)\right], \tag{121}$$

where the Lax operator $\tilde{\mathcal{L}}_{\mathcal{A}j}^{(1)}(\lambda,\lambda_j)$ and the matrix $G_{\mathcal{A}}^{(1)}$ are given by

$$\tilde{\mathcal{L}}^{(1)}_{\mathcal{A}j}(\lambda,\lambda_j) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\bar{b}(\lambda,\lambda_j) & \bar{a}(\lambda,\lambda_j) & 0 \\ 0 & \bar{a}(\lambda,\lambda_j) & -\bar{b}(\lambda,\lambda_j) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$G^{(1)}_{\mathcal{A}} = \begin{pmatrix} e^{-i(\phi_1-\phi_2)/2} & 0 \\ 0 & e^{i(\phi_1-\phi_2)/2} \end{pmatrix}. \tag{122}$$

The solution of this auxiliary problem is once again standard. Following the lines of Section 4.2 we find that the auxiliary eigenvalue expression is

$$\Lambda^{(1)}\left(\lambda,\{\lambda_i\},\{\mu_j\}\right) = e^{-i(\phi_1-\phi_2)/2}\prod_{l=1}^{m}\frac{1}{-\bar{b}(\mu_l,\lambda)} + e^{i(\phi_1-\phi_2)/2}\prod_{j=1}^{n}-\bar{b}(\lambda,\lambda_j)\prod_{l=1}^{m}-\frac{1}{\bar{b}(\lambda,\mu_l)}. \tag{123}$$

Collecting these results altogether and substituting the variables $z_\pm(\lambda_j)$ and $\tilde{\mu}_l$ we find that the eigenvalues of $T_G(\lambda)$ can be written as

$$\begin{aligned}\Lambda_G(\lambda,\{z_\pm(\lambda_j)\},\{\tilde{\mu}_l\}) &= (-1)^n e^{-i(\phi_1+\phi_2)/2}[\omega_1(\lambda)]^L\prod_{j=1}^{n}\frac{b(\lambda)}{a(\lambda)}\frac{1+z_-(\lambda_j)/z_+(\lambda)}{1-z_-(\lambda_j)/z_-(\lambda)} \\ &+ e^{i(\phi_1+\phi_2)/2}[\omega_3(\lambda)]^L\prod_{j=1}^{n}\frac{b(\lambda)}{a(\lambda)}\frac{1+z_-(\lambda_j)z_-(\lambda)}{1-z_-(\lambda_j)z_+(\lambda)} \\ &+ (-)^m e^{-i(\phi_1-\phi_2)/2}[\omega_2(\lambda)]^L\left\{\prod_{j=1}^{n}\frac{b(\lambda)}{a(\lambda)}\frac{1+z_-(\lambda_j)/z_+(\lambda)}{1-z_-(\lambda_j)/z_-(\lambda)}\right. \\ &\times\prod_{l=1}^{m}\frac{z_-(\lambda)-1/z_-(\lambda)-\tilde{\mu}_l+U/2}{z_-(\lambda)-1/z_-(\lambda)-\tilde{\mu}_l-U/2} + (-1)^n e^{i(\phi_1-\phi_2)} \\ &\left.\times\prod_{j=1}^{n}\frac{b(\lambda)}{a(\lambda)}\frac{1+z_-(\lambda_j)z_-(\lambda)}{1-z_-(\lambda_j)z_+(\lambda)}\prod_{l=1}^{m}\frac{1/z_+(\lambda)-z_+(\lambda)-\tilde{\mu}_l-U/2}{1/z_+(\lambda)-z_+(\lambda)-\tilde{\mu}_l+U/2}\right\}\end{aligned} \tag{124}$$

while the nested Bethe ansatz equation are given by

$$(-1)^{m+n}e^{-i\phi_2}[z_-(\lambda_k)]^L = -\prod_{j=1}^{m}\frac{z_-(\lambda_k)-1/z_-(\lambda_k)-\tilde{\mu}_j+U/2}{z_-(\lambda_k)-1/z_-(\lambda_k)-\tilde{\mu}_j-U/2},\quad k=1,\dots,n.$$

$$\prod_{k=1}^{n}\frac{z_-(\lambda_k)-1/z_-(\lambda_k)-\tilde{\mu}_l-U/2}{z_-(\lambda_k)-1/z_-(\lambda_k)-\tilde{\mu}_l+U/2} = -(-1)^n e^{-i(\phi_1-\phi_2)}\prod_{j=1}^{m}\frac{\tilde{\mu}_l-\tilde{\mu}_j+U}{\tilde{\mu}_l-\tilde{\mu}_j-U},\quad l=1,\dots,m. \tag{125}$$

In order to get the results for the Hubbard model with twisted boundary conditions we substitute the angles (10) in the above expressions. We should also remember that we are using the language of holes and therefore the integers n and m are identified with the total number of holes N^h and the number of holes with spin up $N^h_\uparrow$, respectively. This cancels extra phase factors in the Bethe ansatz equations (125) and we recover the known set of non-linear equations parametrizing the spectrum of the twisted Hubbard model [39,40]. Let us close this discussion by mentioning a possible application of these twisted Bethe ansatz results. Consider the Hubbard model perturbed by a particle current term (see e.g. Ref. [41]) with periodic boundary conditions. This model is described by the Hamiltonian

$$H_c(U,\lambda_c) = H(U,\phi_\uparrow = 0, \phi_\downarrow = 0) - i\lambda_c \sum_{i=1}^{L} \sum_{\sigma=\pm} (c^\dagger_{i+1\sigma} c_{i\sigma} - c^\dagger_{i\sigma} c_{i+1\sigma}). \tag{126}$$

In the spin language, this perturbation is a Dzyaloshinsky–Moriya interaction in the azimuthal direction, playing the role of a "vertical" magnetic field. Similar to what happens in the spin case [42], the fermionic current perturbation can be gauged away by using the canonical transformation [40]

$$c_{k\sigma} \longrightarrow e^{i\frac{(2k-3)\phi}{2}} c_{k\sigma}, \qquad \tan(\phi) = \lambda_c \tag{127}$$

allowing us to derive the relation

$$H_c(U,\lambda_c) = \sqrt{1+\lambda_c^2}\, H\Big(\frac{U}{\sqrt{1+\lambda_c^2}}, \phi_\uparrow = \phi L, \phi_\downarrow = \phi L\Big). \tag{128}$$

Thus, the spectrum of $H_c(U,\lambda_c)$ is related to that of the Hubbard model with certain twisted boundary conditions and renormalized coupling. We recall that similar reasoning also works if we add a spin current term, but now $\phi_\uparrow = \phi L$ and $\phi_\downarrow = -\phi L$.

Before closing this section we would like to comment on possible extensions of the results we have obtained so far. First it is possible to diagonalize a two-parameter family of vertex models whose Lax operator is $\mathcal{L}^{(\theta_0)}(\lambda) = PR(\lambda,\theta_0)$ [13,27]. Its Bethe ansatz solution follows directly from the results of this section, since the main change is only concerned with the action of the fields on the reference state. It turns out that now the bare pseudomomenta (left-hand side of first Eq. (125)) depends on the variable θ_0 as $[\alpha_2(\lambda,\theta_0)/-\alpha_9(\lambda,\theta_0)]^L$. Also, the whole formalism can be extended to treat the Hubbard model in the presence of chemical potential [43]. Finally, for further results on twisted boundary conditions see for instance Ref. [44].

5.2. *SU(2) symmetries*

In this subsection we investigate the highest weights properties of the eigenvectors constructed in Section 4, with respect to the two $SU(2)$ symmetries of The Hubbard model [35]. Few years ago, Essler Korepin and Schoutens [36] have shown that certain "regular" states obtained from the coordinate Bethe ansatz wave function are highest weight states of both the $SU(2)$ algebra of rotations and η-pairing $SU(2)$ symmetry. The idea here is to explore the algebraic machinery we developed in the previous section to study this problem from an algebraic perspective, in close analogy with the discussion by Takhtajan and Faddeev [45] for the Heisenberg model. For this purpose we will use the results of Göhmann and Murakami [37] who recently showed that the graded monodromy matrix indeed commutes with these two $SU(2)$ Lie algebras. More precisely, following the notation of Ref. [37] we have

$$[\mathcal{T}(\lambda), S^{\alpha}]_Q = -[\mathcal{T}(\lambda), \sum\nolimits^{\alpha}]_A, \quad \alpha = +, -, z \tag{129}$$

and

$$[\mathcal{T}(\lambda), \eta^{\alpha}]_Q = -[\mathcal{T}(\lambda), \tilde{\sum}\nolimits^{\alpha}]_A, \quad \alpha = +, -, z, \tag{130}$$

where the subscripts Q and A emphasize in which space, quantum or auxiliary, the commutators are taken, respectively. The $SU(2)$ generators of rotations S^{α} and those of the η-pairing symmetry η^{α} are defined by [37]

$$S^{+} = -\sum_{j=1}^{L} c_{j\uparrow}^{\dagger} c_{j\downarrow}, \quad S^{-} = -\sum_{j=1}^{L} c_{j\downarrow}^{\dagger} c_{j\uparrow}, \quad S^{z} = \sum_{j=1}^{L} (n_{j\uparrow} - n_{j\downarrow}) \tag{131}$$

and

$$\eta^{+} = \sum_{j=1}^{L} (-1)^{j+1} c_{j\uparrow}^{\dagger} c_{j\downarrow}^{\dagger}, \quad \eta^{-} = \sum_{j=1}^{L} (-1)^{j+1} c_{j\downarrow} c_{j\uparrow}, \quad \eta^{z} = \sum_{j=1}^{L} (n_{j\uparrow} + n_{j\downarrow} - 1), \tag{132}$$

while the matrices $\sum^{\alpha}$ and $\tilde{\sum}^{\alpha}$ are [37]

$$\sum\nolimits^{+} = \sigma^{+} \otimes \sigma^{-}, \qquad \sum\nolimits^{-} = \sigma^{-} \otimes \sigma^{+}, \qquad \sum\nolimits^{z} = \tfrac{1}{2}(\sigma^{z} \otimes \hat{I} - \hat{I} \otimes \sigma^{z}), \tag{133}$$

$$\tilde{\sum}\nolimits^{+} = \sigma^{+} \otimes \sigma^{+}, \qquad \tilde{\sum}\nolimits^{-} = \sigma^{-} \otimes \sigma^{-}, \qquad \tilde{\sum}\nolimits^{z} = \tfrac{1}{2}(\sigma^{z} \otimes \hat{I} + \hat{I} \otimes \sigma^{z}). \tag{134}$$

Let us begin by considering the η-pairing symmetry. The identity (130) enables us to compute the commutators of the creation fields $\boldsymbol{B}(\lambda)$ and $F(\lambda)$ with the $SU(2)$ η-pairing generators. For the component η^{z} we find

$$[\eta^{z}, \boldsymbol{B}(\lambda)] = -\boldsymbol{B}(\lambda), \qquad [\eta^{z}, F(\lambda)] = -2F(\lambda) \tag{135}$$

while for η^{+} we have

$$[\eta^{+}, \boldsymbol{B}(\lambda)] = -\boldsymbol{C}^{*}(\lambda), \qquad [\eta^{+}, F(\lambda)] = B(\lambda) - D(\lambda). \tag{136}$$

We see that formula (135) corroborates the physical interpretation we have proposed for the creation fields $\boldsymbol{B}(\lambda)$ and $F(\lambda)$, i.e. that they create a single and a doubly occupied hole on the full band pseudovacuum. For example, from this equation it is straightforward to derive

$$\eta^z \left|\Phi_n(\lambda_1,\ldots,\lambda_n)\right\rangle = (L-n)\left|\Phi_n(\lambda_1,\ldots,\lambda_n)\right\rangle, \tag{137}$$

where we used the property $\eta^z\left|0\right\rangle = L\left|0\right\rangle$.

We note that the above result is valid for arbitrary values of the rapidities. However, this is no longer true when we consider the annihilation property of the raising operator η^+. In what follows we shall show that

$$\eta^+ \left|\Phi_n(\lambda_1,\ldots,\lambda_n)\right\rangle = 0 \tag{138}$$

provided the rapidities $\{\lambda_j\}$ satisfy the Bethe ansatz equations derived in Section 4.

To verify the above annihilation property it is instructive first to study the case of a few particles over the reference state and afterwards use mathematical induction for the general case. From Eq. (132) this is clearly correct for the reference state. For the one-particle state, by using the first commutator (136), it is easy to show that

$$\eta^+ \left|\Phi_1(\lambda_1)\right\rangle = \eta^+ \boldsymbol{B}(\lambda_1)\cdot\mathcal{F}\left|0\right\rangle = [\eta^+, \boldsymbol{B}(\lambda_1)]\cdot\mathcal{F}\left|0\right\rangle = -\boldsymbol{C}^*(\lambda_1)\cdot\mathcal{F}\left|0\right\rangle = 0. \tag{139}$$

The Bethe ansatz restrictions start to emerge in the two-particle state analysis. For this state the commutators (136) produce

$$\begin{aligned}\eta^+ \left|\Phi_2(\lambda_1,\lambda_2)\right\rangle =& \boldsymbol{B}(\lambda_1)\otimes\eta^+\boldsymbol{B}(\lambda_2)\cdot\mathcal{F}\left|0\right\rangle - \boldsymbol{C}^*(\lambda_1)\otimes\boldsymbol{B}(\lambda_2)\cdot\mathcal{F}\left|0\right\rangle \\ &+\frac{i\alpha_{10}(\lambda_1,\lambda_2)}{\alpha_7(\lambda_1,\lambda_2)}[B(\lambda_1)-D(\lambda_1)]B(\lambda_2)\boldsymbol{\xi}\cdot\mathcal{F}\left|0\right\rangle.\end{aligned} \tag{140}$$

The first term in the above equation vanishes by the same arguments used in the one-particle state analysis. To simplify the second term we use commutation rule (B.3) and finally the third term is easily estimated from the diagonal relation (23). Putting these simplifications together we find

$$\begin{aligned}\eta^+ \left|\Phi_2(\lambda_1,\lambda_2)\right\rangle =& \frac{i\alpha_{10}(\lambda_1,\lambda_2)}{\alpha_7(\lambda_1,\lambda_2)}\left[[w_1(\lambda_1)w_1(\lambda_2)]^L - [w_2(\lambda_1)w_2(\lambda_2)]^L\right]\boldsymbol{\xi}\cdot\mathcal{F}\left|0\right\rangle \\ =& \frac{i\alpha_{10}(\lambda_1,\lambda_2)}{\alpha_7(\lambda_1,\lambda_2)}\left[[w_1(\lambda_1)w_1(\lambda_2)]^L\right. \\ &\left.-[w_2(\lambda_1)w_2(\lambda_2)]^L\Lambda^{(1)}(\lambda=\lambda_1,\{\lambda_l\})\Lambda^{(1)}(\lambda=\lambda_2,\{\lambda_l\})\right]\boldsymbol{\xi}\cdot\mathcal{F}\left|0\right\rangle \\ =& 0,\end{aligned} \tag{141}$$

where in the second line we used the following two-particle identity:

$$\Lambda^{(1)}(\lambda=\lambda_1,\{\lambda_l\})\Lambda^{(1)}(\lambda=\lambda_2,\{\lambda_l\}) = 1.$$

Clearly, the term in brackets vanishes due to the Bethe ansatz equations (90).

Next we consider the three-particle state. We shall see that a general pattern in the analysis begins to emerge here. After using the commutator relations (136) we have

$$\eta^+ \left|\Phi_3(\lambda_1,\lambda_2,\lambda_3)\right\rangle = \boldsymbol{B}(\lambda_1)\otimes\eta^+\boldsymbol{\Phi}_2(\lambda_2,\lambda_3)\cdot\mathcal{F}\left|0\right\rangle - \boldsymbol{C}^*(\lambda_1)\otimes\boldsymbol{\Phi}_2(\lambda_2,\lambda_3)\cdot\mathcal{F}\left|0\right\rangle$$

$$+\boldsymbol{\xi}[B(\lambda_1) - D(\lambda_1)] \otimes \boldsymbol{B}(\lambda_3)B(\lambda_2)g_1^{(3)}(\lambda_1, \lambda_2, \lambda_3) \cdot \mathcal{F}|0\rangle$$
$$+\boldsymbol{\xi}[B(\lambda_1) - D(\lambda_1)] \otimes \boldsymbol{B}(\lambda_2)B(\lambda_3)g_2^{(3)}(\lambda_1, \lambda_2, \lambda_3) \cdot \mathcal{F}|0\rangle . \tag{142}$$

The first term is computable directly from the first line of Eq. (141), after making the replacements $\lambda_1 \longrightarrow \lambda_2$ and $\lambda_2 \longrightarrow \lambda_3$. The third and fourth terms are estimated with the help of commutations rules (35-36). The simplifications for the second term is more complicated since it involves the knowledge of an extra commutation rule, besides relation (B.3), between the fields $\boldsymbol{C}^*(\lambda)$ and $F(\mu)$. This relation is given by

$$C^*(\lambda)F(\mu) = -i\frac{\alpha_9(\lambda,\mu)}{\alpha_7(\lambda,\mu)}F(u)C^*(\lambda) + i\frac{\alpha_{10}(\lambda,\mu)}{\alpha_7(\lambda,\mu)}\boldsymbol{\xi}.[\hat{A}(\lambda) \otimes \boldsymbol{B}^*(\mu)]$$
$$-\frac{\alpha_4(\lambda,\mu)}{\alpha_7(\lambda,\mu)}\boldsymbol{B}(\lambda)D(\mu) + \frac{\alpha_5(\lambda,\mu)}{\alpha_7(\lambda,\mu)}\boldsymbol{B}(\mu)D(\lambda). \tag{143}$$

Collecting all the pieces together is remarkable to see that many terms have opposite signs and thus they are trivially cancelled out. However, there is a non-trivial simplification yet to be carried out. This is related to the terms proportional to $[\boldsymbol{\xi}.(\boldsymbol{B}^*(\lambda_1) \otimes \hat{I})]$ and they vanish thanks to the following identity:

$$\frac{\alpha_{10}(x,z)}{\alpha_7(x,z)}\frac{\alpha_9(x,y)}{\alpha_7(x,y)} + \frac{\alpha_5(y,z)}{\alpha_9(y,z)}\frac{\alpha_{10}(x,y)}{\alpha_7(x,y)} + \frac{\alpha_{10}(x,z)}{\alpha_7(x,z)}\frac{\alpha_2(z,y)}{\alpha_9(z,y)} = 0. \tag{144}$$

After these simplifications, the remaining terms are only proportional to $[\boldsymbol{\xi} \otimes \boldsymbol{B}(\lambda_j)]$ and they can be compactly written in the following way:

$$\eta^+ |\Phi_3(\lambda_1,\lambda_2,\lambda_3)\rangle = \sum_{j=2}^{3}\sum_{l=1}^{j-1}[\boldsymbol{\xi} \otimes \boldsymbol{\Phi}_1(\lambda_1,\dots,\check{\lambda}_l,\dots,\check{\lambda}_j,\dots,\lambda_3)]$$
$$\times\hat{Q}_{lj}^{(3)}(\lambda_l,\lambda_j;\{\lambda_k\}) \cdot \mathcal{F}|0\rangle . \tag{145}$$

The first term $\hat{Q}_{12}^{(3)}(\lambda_1,\lambda_2;\{\lambda_k\})$ is easily figured out because it has only two main contributions coming from the second and the third terms of Eq. (142). The other two are obtained from this term via consecutive permutation of rapidities through the exchange property (78). The expressions for these coefficients are

$$\hat{Q}_{lj}^{(3)}(\lambda_l,\lambda_j;\{\lambda_k\}) = \big[[w_1(\lambda_l)w_1(\lambda_j)]^L$$
$$-[w_2(\lambda_l)w_2(\lambda_j)]^L\Lambda^{(1)}(\lambda=\lambda_l,\{\lambda_k\})\Lambda^{(1)}(\lambda=\lambda_j,\{\lambda_k\})\big]$$
$$\times i\frac{\alpha_{10}(\lambda_l,\lambda_j)}{\alpha_7(\lambda_l,\lambda_j)}\prod_{\substack{k=1\\k\neq j,l}}^{3}\frac{\alpha_1(\lambda_l,\lambda_k)}{i\alpha_9(\lambda_l,\lambda_k)}\frac{\alpha_1(\lambda_j,\lambda_k)}{i\alpha_9(\lambda_j,\lambda_k)}\hat{O}_{lj}^{(2)}(\lambda_l,\lambda_j;\{\lambda_k\}) \tag{146}$$

and they vanish again as a consequence of the Bethe ansatz equations (90).

Now using mathematical induction it is possible to write the action of the raising operator on a general n-particle state as

$$\eta^{+}\left|\Phi_{n}(\lambda_{1},\dots,\lambda_{n})\right\rangle=\sum_{j=2}^{n}\sum_{l=1}^{j-1}[\boldsymbol{\xi}\otimes\boldsymbol{\Phi}_{n-2}(\lambda_{1},\dots,\check{\lambda}_{l},\dots,\check{\lambda}_{j},\dots,\lambda_{n})]$$
$$\times\hat{Q}_{lj}^{(n)}(\lambda_{l},\lambda_{j};\{\lambda_{k}\})\cdot\boldsymbol{\mathcal{F}}\left|0\right\rangle. \quad (147)$$

As before, it is convenient first to compute the simplest coefficient $\hat{Q}_{12}^{(n)}(\lambda_1,\lambda_2;\{\lambda_k\})$ and then take advantage of the permutation property (78) to obtain the remaining ones. For this term we have just two contributions coming from

$$I:=\boldsymbol{\xi}\otimes B(\lambda_{1})\boldsymbol{\Phi}_{n-2}(\lambda_{3},\dots,\lambda_{n})B(\lambda_{2})\hat{g}_{1}^{(n)}(\lambda_{1},\dots,\lambda_{n})\cdot\boldsymbol{\mathcal{F}}\left|0\right\rangle \quad (148)$$

and

$$II:=-\boldsymbol{C}^{*}(\lambda_{1})\otimes\boldsymbol{\Phi}_{n-1}(\lambda_{2},\dots,\lambda_{n})]\cdot\boldsymbol{\mathcal{F}}\left|0\right\rangle. \quad (149)$$

We compute the first part by carrying the scalar operator $B(\lambda_1)$ through the vector $\boldsymbol{\Phi}_{n-2}(\lambda_3,\dots,\lambda_n)$ keeping only the "wanted terms" proportional to $B(\lambda_1)$. This is very similar to what we did in Appendix D and we find

$$I:=[w_{1}(\lambda_{1})w_{1}(\lambda_{2})]^{L}\frac{i\alpha_{10}(\lambda_{1},\lambda_{2})}{\alpha_{7}(\lambda_{1},\lambda_{2})}\prod_{\substack{k=1\\k\neq1,2}}^{n}\frac{i\alpha_{2}(\lambda_{k},\lambda_{1})}{\alpha_{9}(\lambda_{k},\lambda_{1})}\frac{i\alpha_{2}(\lambda_{k},\lambda_{2})}{\alpha_{9}(\lambda_{k},\lambda_{2})}$$
$$\times[\boldsymbol{\xi}\otimes\boldsymbol{\Phi}_{n-2}(\lambda_{3},\dots,\lambda_{n})]\cdot\boldsymbol{\mathcal{F}}\left|0\right\rangle. \quad (150)$$

The second part is more involving since we have to carry two operators of type $\hat{A}(\lambda)$ through vector $\boldsymbol{\Phi}_{n-2}(\lambda_3,\dots,\lambda_n)$. This means that we have to compute the expression

$$II:=-\frac{i\alpha_{10}(\lambda_{1},\lambda_{2})}{\alpha_{7}(\lambda_{1},\lambda_{2})}\xi_{\alpha\beta}\hat{A}_{\alpha b_{1}}(\lambda_{1})\hat{A}_{\beta b_{2}}(\lambda_{2})[\boldsymbol{\Phi}_{n-2}(\lambda_{3},\dots,\lambda_{n})]_{b_{3}\dots b_{n}}\mathcal{F}^{b_{n}\dots b_{1}}\left|0\right\rangle, \quad (151)$$

which after some algebra can be compacted back as

$$II:=-\frac{i\alpha_{10}(\lambda_{1},\lambda_{2})}{\alpha_{7}(\lambda_{1},\lambda_{2})}[w_{2}(\lambda_{1})w_{2}(\lambda_{1})]^{L}\prod_{\substack{k=1\\k\neq1,2}}^{n}\frac{\alpha_{1}(\lambda_{1},\lambda_{k})}{i\alpha_{9}(\lambda_{1},\lambda_{k})}\frac{\alpha_{1}(\lambda_{2},\lambda_{k})}{i\alpha_{9}(\lambda_{2},\lambda_{k})}$$
$$\times[\boldsymbol{\xi}\otimes\boldsymbol{\Phi}_{n-2}(\lambda_{3},\dots,\lambda_{n})]_{\alpha_{1}\dots\alpha_{n}}$$
$$\times[T^{(1)}(\lambda=\lambda_{1},\{\lambda_{l}\})T^{(1)}(\lambda=\lambda_{2},\{\lambda_{l}\})]_{\alpha_{1}\dots\alpha_{n}}^{b_{1}\dots b_{n}}\mathcal{F}^{b_{n}\dots b_{1}}\left|0\right\rangle. \quad (152)$$

Finally, putting together expressions (150) and (152) and also using the auxiliary eigenvalue definition (91) we find

$$\hat{Q}_{12}^{(n)}(\lambda_{1},\lambda_{2};\{\lambda_{k}\})=\big[[w_{1}(\lambda_{1})w_{1}(\lambda_{2})]^{L}$$
$$-[w_{2}(\lambda_{1})w_{2}(\lambda_{2})]^{L}\Lambda^{(1)}(\lambda=\lambda_{1},\{\lambda_{k}\})\Lambda^{(1)}(\lambda=\lambda_{2},\{\lambda_{k}\})\big]$$
$$\times\frac{i\alpha_{10}(\lambda_{1},\lambda_{2})}{\alpha_{7}(\lambda_{1},\lambda_{2})}\prod_{\substack{k=1\\k\neq1,2}}^{n}\frac{\alpha_{1}(\lambda_{1},\lambda_{k})}{i\alpha_{9}(\lambda_{1},\lambda_{k})}\frac{\alpha_{1}(\lambda_{2},\lambda_{k})}{i\alpha_{9}(\lambda_{2},\lambda_{k})}, \quad (153)$$

which once again vanishes due to the Bethe ansatz equations. All the other coefficients are obtained by permuting the rapidities and by taking into account the exchange property (78), and as a result they get an extra multiplicative "ordering" factor $\hat{O}^{(2)}_{lj}(\lambda_l, \lambda_j; \{\lambda_k\})$. Since the Bethe ansatz equations are invariant under indices relabeling, they vanish too. This completes the proof that the eigenvectors (81) are highest weight states of the η-pairing symmetry.

Next we turn to examine the highest weight property of the $SU(2)$ algebra of rotations. Now the commutators of the creation fields with the $SU(2)$ generators are obtained from Eq. (129). For the component S^z we find

$$[S^z, B_1(\lambda)] = B_1(\lambda), \qquad [S^z, B_2(\lambda)] = -B_2(\lambda), \qquad [S^z, F(\lambda)] = 0 \tag{154}$$

and for S^+ we have

$$[S^+, B_1(\lambda)] = 0, \qquad [S^+, B_2(\lambda)] = B_1(\lambda), \qquad [S^+, F(\lambda)] = 0. \tag{155}$$

First of all, it is not difficult to see that eigenvector (81) will be hardly annihilated by the raising operator S^+ unless further restriction are assumed. To illustrate this fact in a simple example let us consider the one-particle state. By using the commutators (155) we find

$$S^+ |\Phi_1(\lambda_1)\rangle = B_1(\lambda_1)\mathcal{F}^2 |0\rangle , \tag{156}$$

where we used that $S^+ |0\rangle = 0$. Therefore, to assure the highest weight property for the one-particle state we must set $\mathcal{F}^2 = 0$. This is an example of what was called "regular" Bethe states in Ref. [36], and in general these states are obtained by projecting out the negative sectors of the magnetization operator S^z. This latter condition is easily implemented for the eigenvector (81) if one uses the commutators (154).

To see how this works in practice let us consider the two-particle state. In this case it is obvious that we have to set $\mathcal{F}^{22} = 0$, and after that we find

$$\begin{aligned} S^+ |\Phi_2(\lambda_1, \lambda_2)\rangle_{\text{regular}} &= S^+ \Big[B_1(\lambda_1)B_1(\lambda_2)\mathcal{F}^{11} + B_1(\lambda_1)B_2(\lambda_2)\mathcal{F}^{21} \\ &\quad + B_2(\lambda_1)B_1(\lambda_2)\mathcal{F}^{12} + i\frac{\alpha_{10}(\lambda_1, \lambda_2)}{\alpha_7(\lambda_1, \lambda_2)} F(\lambda_1)B(\lambda_2)\boldsymbol{\xi} \cdot \boldsymbol{\mathcal{F}}\Big] |0\rangle \\ &= \Big[\sum_P \mathcal{F}^{12}\Big] B_1(\lambda_1)B_1(\lambda_2) |0\rangle , \end{aligned} \tag{157}$$

where the sum is over permutations on the indices of the coefficient $\mathcal{F}^{a_2 a_1}$. In this case it is straightforward to verify that this sum indeed vanishes by directly solving the auxiliary eigenvalue problem (56). The deeper reason behind this fact, however, is that the vanishing of such sum is precisely related to the highest weight property of the Bethe wave functions of the XXX Heisenberg model with *two* sites. We should recall here that the components of this wave function are identified with the coefficients $\mathcal{F}^{a_2 a_1}$. From this discussion, it becomes evident that the whole procedure can be applied to any multi-particle state. As an example, in Table 1 we summarize our findings up to the four-particle state

Table 1
The "regular" multi-particle states properties up to $n = 4$

n	$S^z \vert\Phi_n(\lambda_1,\dots,\lambda_n)\rangle_{\text{regular}}$	$S^+ \vert\Phi_n(\lambda_1,\dots,\lambda_n)\rangle_{\text{regular}} = 0$
2	2	none
2	0	$\sum_P \mathcal{F}^{12} = 0$
3	3	none
3	1	$\sum_P \mathcal{F}^{112} = 0$
4	4	none
4	2	$\sum_P \mathcal{F}^{1112} = 0$
4	0	$\sum_P \mathcal{F}^{112\bar{2}} = \sum_P \mathcal{F}^{11\bar{2}2} = \sum_P \mathcal{F}^{1\bar{2}12} = \sum_P \mathcal{F}^{\bar{2}112} = 0$

The columns of Table 1 refer to the particle number, magnetization values and the sufficient vanishing condition for S^+ annihilate the "regular" part of eigenvector (81), respectively. In the sum the symbol $\bar{a}$ means that the ath element is maintain fixed under permutations. The generalization to multi-particle state is done by induction and the sufficient vanishing conditions are made of the many possible permutation over the coefficients $\mathcal{F}^{a_n \dots a_1}$ having positive magnetization. As before, these conditions are fulfilled as a consequence of the highest weight property of the Bethe states of the XXX Heisenberg spin chain in a lattice with size n. Since this latter point has been well explained by Essler, Korepin and Schoutens [36], there is no need to proceed with details, and thus we conclude our proof that $S^+ |\Phi_n(\lambda_1,\dots,\lambda_n)\rangle_{\text{regular}} = 0$ here.

Finally, we remark that similar properties can be also verified for the "dual" eigenvector. The only difference is that now the "regular" states are defined by projecting out the *positive* sector of the magnetization. At this level, the eigenvector and its "dual" becomes complementary eigenstates.

6. The *ABCDF* framework for the Bariev model

The purpose of this section is to illustrate that the *ABCDF* framework developed in the previous sections is by no means only applicable to the Hubbard model. In order to show that, we consider a second interesting model of interacting XY chains whose corresponding R-matrix also does not have the difference property. The model was originally formulated by Bariev [24] and its one-dimensional Hamiltonian is

$$H = \sum_{i=1}^{L} (\sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+)(1 + V\tau_{i+1}^z) + (\tau_i^+ \tau_{i+1}^- + \tau_i^- \tau_{i+1}^+)(1 + V\sigma_i^z), \quad (158)$$

where V is a coupling constant. In the language of fermions V plays the role of a bond-charge interaction and Hamiltonian (158) resembles the model of hole superconductivity proposed by Hirsch [46].

In the context of the quantum inverse scattering method this model has recently been investigated by Zhou [47] and Shiroishi and Wadati [50] who found two distinct covering vertex models for the Bariev Hamiltonian. In this section we apply the *ABCDF* formalism for the former solution.[4] In this case, the proposed Lax operator was [47]

$$\mathcal{L}^{(B)}_{\mathcal{A}j}(\lambda) = \mathcal{L}^{(1)}_{\mathcal{A}j}(\lambda)\mathcal{L}^{(2)}_{\mathcal{A}j}(\lambda), \tag{159}$$

where

$$\begin{aligned}\mathcal{L}^{(1)}_{\mathcal{A}j}(\lambda) &= \tfrac{1}{2}(1+\sigma^z_j\sigma^z_{\mathcal{A}}) + \frac{\lambda}{2}(1-\sigma^z_j\sigma^z_{\mathcal{A}})\exp(\beta\tau^+_{\mathcal{A}}\tau^-_{\mathcal{A}}) \\ &\quad + (\sigma^+_j\sigma^-_{\mathcal{A}} + \sigma^-_j\sigma^+_{\mathcal{A}})\sqrt{1+\lambda^2\exp(2\beta\tau^+_{\mathcal{A}}\tau^-_{\mathcal{A}})}\end{aligned} \tag{160}$$

and

$$\begin{aligned}\mathcal{L}^{(2)}_{\mathcal{A}j}(\lambda) &= \tfrac{1}{2}(1+\tau^z_j\tau^z_{\mathcal{A}}) + \frac{\lambda}{2}(1-\tau^z_j\tau^z_{\mathcal{A}})\exp(\beta\sigma^+_{\mathcal{A}}\sigma^-_{\mathcal{A}}) \\ &\quad + (\tau^+_j\tau^-_{\mathcal{A}} + \tau^-_j\tau^+_{\mathcal{A}})\sqrt{1+\lambda^2\exp(2\beta\sigma^+_{\mathcal{A}}\sigma^-_{\mathcal{A}})}.\end{aligned} \tag{161}$$

The relation between the parameter β and the coupling constant V is determined by computing the expression $P(d/d\lambda)\mathcal{L}^{(B)}(\lambda)$ on $\lambda = 0$. After performing the rescaling $\lambda \to \lambda e^{-\beta/2}/\cosh(\beta/2)$ we found

$$\bar{h} = e^{\beta} = \frac{1+V}{1-V}. \tag{162}$$

The R-matrix solving the Yang–Baxter algebra for this choice of Lax operator was also found by Zhou. Its explicit 16×16 form is [47]

$$R(\lambda,\mu) = \begin{pmatrix}
\rho_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_2 & 0 & 0 & \rho_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & \rho_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \rho_4 & 0 & 0 & \rho_5 & 0 & 0 & \rho_6 & 0 & 0 & \rho_9 & 0 & 0 & 0 \\
0 & \rho_3 & 0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \rho_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \rho_{12} & 0 & 0 & \rho_7 & 0 & 0 & \rho_{15} & 0 & 0 & \rho_5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_8 & 0 & 0 & 0 & 0 & 0 & \rho_{11} & 0 & 0 \\
0 & 0 & \rho_3 & 0 & 0 & 0 & 0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \rho_{13} & 0 & 0 & \rho_{15} & 0 & 0 & \rho_{10} & 0 & 0 & \rho_6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_8 & 0 & 0 & \rho_{11} & 0 \\
0 & 0 & 0 & \rho_{14} & 0 & 0 & \rho_{12} & 0 & 0 & \rho_{13} & 0 & 0 & \rho_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_{11} & 0 & 0 & 0 & 0 & 0 & \rho_8 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_{11} & 0 & 0 & \rho_8 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_1
\end{pmatrix} \tag{163}$$

where the fifteen non-null Boltzmann weights $\rho_j(\lambda,\mu)$, $j = 1,\dots,15$ can be found in Ref. [47]. We remark that we have verified that this R-matrix indeed satisfies the Yang–Baxter equation (6).

We note that the structure of such R-matrix is very similar to that found for the Hubbard model and consequently one could easily guess that the *ABCDF* formalism should work for this embedding as well. It is not difficult to adapt the main steps of Section 3 to obtain the commutation rules for such classical vertex analog of the Bariev

[4] Part of our results were first announced in Ref. [48]. See also Ref. [51].

model and therefore we omit further details. The interesting feature here is the structure which comes up for both the "exclusion" vector and the auxiliary r-matrix. We found that they are given by

$$\xi^{(B)} = (0 \quad 1 \quad 1/\bar{h} \quad 0), \qquad \hat{r}^{(B)}(\lambda,\mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a^{(B)}(\lambda,\mu) & b^{(B)}(\lambda,\mu) & 0 \\ 0 & b^{(B)}(\lambda,\mu) & a_1^{(B)}(\lambda,\mu) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{164}$$

where the weights $a^{(B)}(\lambda,\mu)$ and $b^{(B)}(\lambda,\mu)$ are

$$a^{(B)}(\lambda,\mu) = \frac{\lambda(1-\bar{h}^2)}{\lambda-\bar{h}^2\mu}, \qquad a_1^{(B)}(\lambda,\mu) = \frac{\mu(1-\bar{h}^2)}{\lambda-\bar{h}^2\mu},$$
$$b^{(B)}(\lambda,\mu) = -\frac{\bar{h}(\lambda-\mu)}{\lambda-\bar{h}^2\mu}. \tag{165}$$

From Eq. (165), it is easily recognizable that the auxiliary r-matrix has the structure of an asymmetrical and anisotropic 6-vertex model because the parametrization leading to the difference property for $\hat{r}^{(B)}(\lambda,\mu)$ is now standard, namely $\lambda = \exp(ik)$. In this case the hidden symmetry is of Hecke type because such auxiliary r-matrix can be produced as a result of Baxterization of the Hecke algebra (see e.g. Ref. [52]). We recall here that this latter symmetry was first noted by Hikami and Murakami by exploiting the continuum limit of the Bariev Hamiltonian [49]. Interesting enough, we note that the "exclusion" statistics for "spins" degrees of freedom seems to be of anyonic type with a phase β which depends on the strength of the coupling constant V (see Eq. (162)). It remains to be seen if this feature will also be manifested in physical quantities computable by Bethe ansatz methods such as in the low temperature behaviour of the free energy (conformal limit) and in the scattering of the elementary excitations.

Let us now discuss the construction of the eigenvalues and the eigenvectors for this classical analog of Bariev model. It turns out that such formulation goes fairly parallel to the one already presented in Section 4 and in Appendix D. For this reason we shall avoid unnecessary repetition, and from now on we concentrate our attention only to the basic points. We start directly with the two-particle state analysis since it has already proved to contain sufficient information about the main steps entering in the relevant computations. Afterwards, generalization to multi-particle states is made following similar discussion presented in Appendix D. Our previous experience with the Hubbard model suggests us to begin with a symmetrized two-particle vector. As before, the main trick is to look at the commutation rule between the two creation fields of type $\boldsymbol{B}(\lambda)$, providing us the following ansatz:

$$\boldsymbol{\Phi}_2^{(B)}(\lambda_1,\lambda_2) = \boldsymbol{B}(\lambda_1)\otimes\boldsymbol{B}(\lambda_2) - \frac{\rho_5(\lambda_1,\lambda_2)}{\rho_9(\lambda_1,\lambda_2)}\xi^{(B)}F(\lambda_1)B(\lambda_2), \tag{166}$$

which is indeed the case thanks to the following identity:

$$\xi^{(B)} \cdot \hat{r}^{(B)}(\lambda, \mu) = \frac{\rho_{12}(\lambda, \mu)\rho_9(\mu, \lambda)}{\rho_9(\lambda, \mu)\rho_5(\mu, \lambda)} \xi^{(B)}. \tag{167}$$

We proceed by computing the action of the diagonal fields on the two-particle state ansatz. Here we shall make full use of the permutation property of the eigenvector, especially the simplifications mentioned at the end of Appendix C, and our final results are

$$B(\lambda) \left|\Phi_2^{(B)}(\lambda_1, \lambda_2)\right\rangle = \prod_{j=1}^{2} \frac{\rho_1(\lambda_j, \lambda)}{\rho_3(\lambda_j, \lambda)} \left|\Phi_2^{(B)}(\lambda_1, \lambda_2)\right\rangle$$

$$- \sum_{j=1}^{2} \left|\Omega_1^{(1)}(\lambda, \lambda_j; \{\lambda_k\})\right\rangle + H_1^{(B)}(\lambda, \lambda_1, \lambda_2) \left|\Omega_0^{(3)}(\lambda, \lambda_j, \lambda_l; \{\lambda_k\})\right\rangle, \tag{168}$$

$$D(\lambda) \left|\Phi_2^{(B)}(\lambda_1, \lambda_2)\right\rangle = [\lambda^2]^L \prod_{j=1}^{2} \frac{\rho_{11}(\lambda, \lambda_j)}{\rho_9(\lambda, \lambda_j)} \left|\Phi_2^{(B)}(\lambda_1, \lambda_2)\right\rangle$$

$$- \sum_{j=1}^{2} [\bar{h}\lambda_j]^L \Lambda_{(B)}^{(1)}(\lambda = \lambda_j, \{\lambda_l\}) \left|\Omega_1^{(2)}(\lambda, \lambda_j; \{\lambda_k\})\right\rangle$$

$$+ H_2^{(B)}(\lambda, \lambda_1, \lambda_2) [\bar{h}^2 \lambda_1 \lambda_2]^L \left|\Omega_0^{(3)}(\lambda, \lambda_j, \lambda_l; \{\lambda_k\})\right\rangle, \tag{169}$$

$$\sum_{a=1}^{2} \hat{A}_{aa}(\lambda) \left|\Phi_2^{(B)}(\lambda_1, \lambda_2)\right\rangle = [\bar{h}\lambda]^L \prod_{j=1}^{2} \frac{\rho_1(\lambda, \lambda_j)}{\rho_3(\lambda, \lambda_j)} \Lambda_{(B)}^{(1)}(\lambda, \{\lambda_l\}) \left|\Phi_2^{(B)}(\lambda_1, \lambda_2)\right\rangle$$

$$- \sum_{j=1}^{2} [\bar{h}(\lambda_j)]^L \Lambda_{(B)}^{(1)}(\lambda = \lambda_j, \{\lambda_k\}) \left|\Omega_1^{(1)}(\lambda, \lambda_j; \{\lambda_k\})\right\rangle$$

$$- \sum_{j=1}^{2} \left|\Omega_1^{(2)}(\lambda, \lambda_j; \{\lambda_k\})\right\rangle$$

$$+ H_3^{(B)}(\lambda, \lambda_1, \lambda_2) [\bar{h}\lambda_2]^L \Lambda_{(B)}^{(1)}(\lambda = \lambda_2), \{\lambda_k\}) \left|\Omega_0^{(3)}(\lambda, \lambda_j, \lambda_l; \{\lambda_k\})\right\rangle$$

$$+ H_4^{(B)}(\lambda, \lambda_1, \lambda_2) [\bar{h}\lambda_1]^L \Lambda_{(B)}^{(1)}(\lambda = \lambda_1), \{\lambda_k\}) \left|\Omega_0^{(3)}(\lambda, \lambda_j, \lambda_l; \{\lambda_k\})\right\rangle, \tag{170}$$

where we used the relations $B(\lambda)|0\rangle = |0\rangle$, $A_{aa}(\lambda) = [\bar{h}\lambda]^L |0\rangle$ and $D(\lambda)|0\rangle = [\lambda^2]^L |0\rangle$ which are determined by acting the Lax operator on the ferromagnetic pseudovacuum. As before, $\Lambda_{(B)}^{(1)}(\lambda, \{\lambda_l\})$ is the eigenvalue of the auxiliary problem (56) whose r-matrix is now $\hat{r}^{(B)}(\lambda, \mu)$. Furthermore, the expressions for the unwanted terms are

$$\left|\Omega_1^{(1)}(\lambda,\lambda_j;\{\lambda_l\})\right\rangle = \frac{\rho_2(\lambda_j,\lambda)}{\rho_3(\lambda_j,\lambda)}\prod_{\substack{k=1\\k\neq j}}^{2}\frac{\rho_1(\lambda_k,\lambda_j)}{\rho_3(\lambda_k,\lambda_j)}\left[\boldsymbol{B}(\lambda)\otimes\boldsymbol{B}(\lambda_k)\right]$$

$$\times\prod_{k=1}^{j-1}\hat{r}^{(B)}_{k,k+1}(\lambda_k,\lambda_j)\cdot\mathcal{F}\left|0\right\rangle, \tag{171}$$

$$\left|\Omega_1^{(2)}(\lambda,\lambda_j;\{\lambda_l\})\right\rangle = \frac{\rho_5(\lambda,\lambda_j)}{\rho_9(\lambda,\lambda_j)}\prod_{\substack{k=1\\k\neq j}}^{2}\frac{\rho_1(\lambda_j,\lambda_k)}{\rho_3(\lambda_j,\lambda_k)}\left[\boldsymbol{\xi}^{(B)}.(\boldsymbol{B}^*(\lambda)\otimes\hat{I})\right]\otimes\boldsymbol{B}(\lambda_k)$$

$$\times\prod_{k=1}^{j-1}\hat{r}^{(B)}_{k,k+1}(\lambda_k,\lambda_j)\cdot\mathcal{F}\left|0\right\rangle, \tag{172}$$

$$\left|\Omega_0^{(3)}(\lambda,\lambda_j,\lambda_l;\{\lambda_k\})\right\rangle = F(\lambda)\boldsymbol{\xi}^{(B)}\cdot\mathcal{F}\left|0\right\rangle. \tag{173}$$

Finally, the functions $H_l^{(B)}(x,y,z)$, $l=1,\dots,4$ are given by

$$H_1^{(B)}(x,y,z) = \frac{\rho_5(y,z)\rho_4(y,x)}{\rho_9(y,z)\rho_9(y,x)} + \frac{\rho_1(y,x)\rho_2(z,x)\rho_{12}(y,x)}{\rho_3(y,x)\rho_3(z,x)\rho_9(y,x)}, \tag{174}$$

$$H_2^{(B)}(x,y,z) = \frac{\rho_4(x,y)\rho_5(y,z)}{\rho_9(x,y)\rho_9(y,z)} - \frac{\rho_5(x,z)\rho_2(x,y)}{\rho_9(x,z)\rho_9(x,y)}, \tag{175}$$

$$H_3^{(B)}(x,y,z) = \left[\frac{\rho_1(x,y)\rho_2(x,z)\rho_5(x,y)}{\rho_3(x,y)\rho_3(x,z)\rho_9(x,y)} - \frac{\rho_5(x,y)\rho_2(x,y)\rho_2(y,z)}{\rho_9(x,y)\rho_3(x,y)\rho_3(y,z)}\right]$$

$$\times\left[a^{(B)}(y,z) + \frac{b^{(B)}(y,z)}{\bar{h}}\right], \tag{176}$$

$$H_4^{(B)}(x,y,z) = \frac{\rho_1(x,z)\rho_2(x,y)\rho_5(x,z)}{\rho_3(x,z)\rho_3(x,y)\rho_9(x,z)} - \frac{\rho_5(x,z)\rho_2(x,z)\rho_2(z,y)}{\rho_9(x,z)\rho_3(x,z)\rho_3(z,y)}. \tag{177}$$

In order to cancel out the unwanted terms it is sufficient to impose the following Bethe ansatz restriction to the rapidities:

$$[\lambda_i\bar{h}]^{-L} = -\Lambda_B^{(1)}(\lambda=\lambda_i,\{\lambda_j\}), \quad i=1,2. \tag{178}$$

since this condition eliminates automatically the first two kind of unwanted terms. Moreover, this helps us to gather the four unwanted terms proportional to $F(\lambda)\boldsymbol{\xi}^{(B)}\cdot\mathcal{F}$ which are finally vanished due to the identity

$$H_1^{(B)}(x,y,z) + H_2^{(B)}(x,y,z) = H_3^{(B)}(x,y,z) + H_4^{(B)}(x,y,z). \tag{179}$$

To obtain the two-particle eigenvalue we collect the wanted terms and by using the expression for the Boltzmann weights [47] we find

$$\Lambda^{(B)}(\lambda,\{\lambda_j\})=\prod_{j=1}^{2}\frac{\bar{h}^{-1}+\bar{h}\lambda_j\lambda}{\lambda_j-\lambda}+\lambda^{2L}\prod_{j=1}^{2}\frac{1+\bar{h}^2\lambda_j\lambda}{\lambda-\bar{h}^2\lambda_j}$$
$$+[\bar{h}\lambda]^L\prod_{j=1}^{2}\frac{\bar{h}^{-1}+\bar{h}\lambda_j\lambda}{\lambda-\lambda_j}\Lambda_B^{(1)}(\lambda,\{\lambda_j\}). \tag{180}$$

The generalization of these results for multi-particle states goes much along the lines discussed in Appendix D. We start constructing a symmetrized n-particle vector state which satisfies

$$\boldsymbol{\Phi}^{(B)}(\lambda_1,\dots,\lambda_j,\lambda_{j+1},\dots,\lambda_n)=\boldsymbol{\Phi}^{(B)}(\lambda_1,\dots,\lambda_{j+1},\lambda_j,\dots,\lambda_n).\hat{r}^{(B)}(\lambda_j,\lambda_{j+1}) \tag{181}$$

and after solving these constraints we have

$$\boldsymbol{\Phi}_n^{(B)}(\lambda_1,\dots,\lambda_n)=\boldsymbol{B}(\lambda_1)\otimes\boldsymbol{\Phi}_{n-1}^{(B)}(\lambda_2,\dots,\lambda_n)-\sum_{j=2}^{n}\frac{\rho_5(\lambda_1,\lambda_j)}{\rho_9(\lambda_1,\lambda_j)}\prod_{\substack{k=2\\k\neq j}}^{n}\frac{\rho_1(\lambda_k,\lambda_j)}{\rho_9(\lambda_k,\lambda_j)}$$
$$\times\left[\boldsymbol{\xi}\otimes F(\lambda_1)\boldsymbol{\Phi}_{n-2}^{(B)}(\lambda_2,\dots,\lambda_{j-1},\lambda_{j+1},\dots,\lambda_n)B(\lambda_j)\right]\prod_{k=2}^{j-1}\hat{r}_{k,k+1}^{(B)}(\lambda_k,\lambda_j). \tag{182}$$

From the two-particle analysis it is not difficult to see what should be the expressions for the multi-particle eigenvalues and Bethe ansatz equations. For example, the auxiliary eigenvalue expression is the same as given in Eq. (99), replacing $\bar{b}(\lambda,\mu)$ by $b^{(B)}(\lambda,\mu)$. To make a comparison with the previous Bethe ansatz results derived by Bariev [24] it is convenient to redefine the spectral parameter λ, the rapidities $\{\lambda_i\}$ and the nesting variables $\{\mu_j\}$ [48]. Here we set

$$\lambda=e^{ik},\qquad \bar{h}\lambda_j=e^{ik_j},\qquad \mu_j=e^{i\Lambda_j}. \tag{183}$$

In terms of these new rapidities, our final results for the eigenvalues are

$$\Lambda(k,\{k_i\},\{\Lambda_j\})=\prod_{i=1}^{n}\frac{\cos(k/2+k_i/2-i\beta/2)}{i\sin(k_i/2-k/2+i\beta/2)}$$
$$+\exp(i2Lk)\prod_{i=1}^{n}\frac{\cos(k_i/2+k/2-i\beta/2)}{i\sin(k/2-k_i/2+i\beta/2)}$$
$$+\exp[i(k-i\beta)L]\left\{\prod_{i=1}^{n}\frac{i\cos(k/2+k_i/2-i\beta/2)}{\sin(k_i/2-k/2+i\beta/2)}\prod_{j=1}^{m}-\frac{\sin(\Lambda_j/2-k/2+i\beta)}{\sin(\Lambda_j/2-k/2)}\right.$$
$$\left.+\prod_{i=1}^{n}\frac{i\cos(k/2+k_i/2-i\beta/2)}{\sin(k/2-k_i/2+i\beta/2)}\prod_{j=1}^{m}-\frac{\sin(k/2-\Lambda_j/2+i\beta)}{\sin(k/2-\Lambda_j/2)}\right\}, \tag{184}$$

while the nested Bethe ansatz equations for the rapidities $\{k_i\}$ and $\{\Lambda_j\}$ are

$$\exp(ik_iL) = -(-1)^{n-m}\prod_{j=1}^{m}\frac{\sin(k_i/2-\Lambda_j/2+i\beta/2)}{\sin(k_i/2-\Lambda_j/2-i\beta/2)}, \quad i=1,\dots,n$$

$$(-1)^n\prod_{i=1}^{n}\frac{\sin(\Lambda_j/2-k_i/2-i\beta/2)}{\sin(\Lambda_j/2-k_i/2+i\beta/2)} = -\prod_{k=1}^{m}\frac{\sin(\Lambda_j/2-\Lambda_k/2-i\beta)}{\sin(\Lambda_j/2-\Lambda_k/2+i\beta)}, \quad j=1,\dots,m. \tag{185}$$

Finally, to obtain the eigenspectrum of the Hamiltonian (158) we expand the transfer matrix eigenvalues in power of the spectral parameter. Up to second order we have

$$\ln[\Lambda(\lambda,\{\lambda_j\},\{\mu_j\}] = \sum_i^n \frac{1}{\bar{h}\lambda_i} + \bar{h}\lambda\sum_i^n\left(\bar{h}\lambda_i + \frac{1}{\bar{h}\lambda_i}\right) + \frac{\bar{h}^2\lambda^2}{2!}\sum_i^n\left[\left(\frac{1}{\bar{h}\lambda_i}\right)^2 - (\bar{h}\lambda_i)^2\right] + \mathcal{O}(\lambda^3). \tag{186}$$

Considering the $\mathcal{O}(\lambda)$ term of the above equation and remembering to perform the rescaling $\lambda \to \lambda e^{-\beta/2}/\cosh(\beta/2)$ we conclude that the eigenenergies of the Hamiltonian (158) are

$$E_n = 2(1+V)\sum_{i=1}^{n}\cos(k_i) \tag{187}$$

We conclude remarking that this model can also be solved with twisted boundary conditions following precisely the same steps presented in Section 5.1.

7. Conclusions

The main purpose of this paper was to apply the quantum inverse scattering program for the one-dimensional Hubbard model. We succeeded in developing a framework which allowed us to present an algebraic formulation for the Bethe states of the transfer matrix of the classical "covering" Hubbard model proposed earlier by Shastry [12,13]. A hidden 6-vertex symmetry has been revealed, and it played a fundamental role in the solution of the transfer matrix eigenvalue problem. We have found the eigenvalues of the transfer matrix and showed that its eigenstates are highest weights states of both the rotational and the η-paring $SU(2)$ symmetries. This latter result corroborates the original proof given by Essler, Korepin and Schoutens [36] in terms of coordinate wave functions. We have also discussed the algebraic solution of models with twisted boundary conditions and applied the results to the Hubbard model perturbed by charge and spin currents.

The framework developed in this paper, the *ABCDF* formalism, is indeed suitable to solve a broad class of integrable systems. As an example, we solved, in Section 6, the classical analog of the Bariev model by this method. There are also other models that fit in the *ABCDF* framework, such as the trigonometric vertex models based on the B_n, C_n, D_n, A^2_{2n} and A^2_{2n-1} algebras as well as certain related supersymmetric models [23]. Interesting enough, the former models almost exhaust the Jimbo's and Bazhanov's list of $U_q(G)$ R-matrices [53], and only the D^2_{n+1} model appears to be not solvable within our framework. Anyhow, these examples suggest us that the *ABCDF* formalism is capable of solving integrable models having one less trivial conserved quantum number when compared to the A_n multi-state 6-vertex models with an equivalent Hilbert space.

Finally, the possibility of bringing a variety of models under one unifying approach not only highlight the qualities of the quantum inverse scattering program but also allows us to better understand the relevant properties entering their Bethe ansatz solution. This also motives us to look for further extensions which could shape our knowledge towards a possible classification of integrable models from an algebraic point of view. An interesting example seems to be the D^2_{n+1} vertex model, which we plan to investigate in a future work.

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Appendix A. Boltzmann weights of the Shastry model

We start this appendix by presenting the ten non-null Boltzmann weights of Shastry's R-matrix (15). They are given by

$$\alpha_1(\lambda,\mu) = \left\{e^{[h(\mu)-h(\lambda)]}a(\lambda)a(\mu) + e^{-[h(\mu)-h(\lambda)]}b(\lambda)b(\mu)\right\}\alpha_5(\lambda,\mu), \quad (A.1)$$

$$\alpha_2(\lambda,\mu) = \left\{e^{-[h(\mu)-h(\lambda)]}a(\lambda)a(\mu) + e^{[h(\mu)-h(\lambda)]}b(\lambda)b(\mu)\right\}\alpha_5(\lambda,\mu), \quad (A.2)$$

$$\alpha_3(\lambda,\mu) = \frac{e^{[h(\mu)+h(\lambda)]}a(\lambda)b(\mu) + e^{-[h(\mu)+h(\lambda)]}b(\lambda)a(\mu)}{a(\lambda)b(\lambda)+a(\mu)b(\mu)} \times \left\{\frac{\cosh[h(\mu)-h(\lambda)]}{\cosh[h(\mu)+h(\lambda)]}\right\}\alpha_5(\lambda,\mu), \quad (A.3)$$

$$\alpha_4(\lambda,\mu) = \frac{e^{-[h(\mu)+h(\lambda)]}a(\lambda)b(\mu) + e^{[h(\mu)+h(\lambda)]}b(\lambda)a(\mu)}{a(\lambda)b(\lambda)+a(\mu)b(\mu)} \times \left\{\frac{\cosh(h(\mu)-h(\lambda))}{\cosh(h(\mu)+h(\lambda))}\right\}\alpha_5(\lambda,\mu), \quad (A.4)$$

$$\alpha_6(\lambda,\mu)=\left\{\frac{e^{[h(\mu)+h(\lambda)]}a(\lambda)b(\mu)-e^{-[h(\mu)+h(\lambda)]}b(\lambda)a(\mu)}{a(\lambda)b(\lambda)+a(\mu)b(\mu)}\right\}$$
$$\times[b^2(\mu)-b^2(\lambda)]\frac{\cosh[h(\mu)-h(\lambda)]}{\cosh[h(\mu)+h(\lambda)]}\alpha_5(\lambda,\mu), \quad (A.5)$$

$$\alpha_7(\lambda,\mu)=\left\{\frac{-e^{-[h(\mu)+h(\lambda)]}a(\lambda)b(\mu)+e^{[h(\mu)+h(\lambda)]}b(\lambda)a(\mu)}{a(\lambda)b(\lambda)+a(\mu)b(\mu)}\right\}$$
$$\times[b^2(\mu)-b^2(\lambda)]\frac{\cosh[h(\mu)-h(\lambda)]}{\cosh[h(\mu)+h(\lambda)]}\alpha_5(\lambda,\mu), \quad (A.6)$$

$$\alpha_8(\lambda,\mu)=\left\{e^{[h(\mu)-h(\lambda)]}a(\lambda)b(\mu)-e^{-[h(\mu)-h(\lambda)]}b(\lambda)a(\mu)\right\}\alpha_5(\lambda,\mu), \quad (A.7)$$

$$\alpha_9(\lambda,\mu)=\left\{-e^{-[h(\mu)-h(\lambda)]}a(\lambda)b(\mu)+e^{[h(\mu)-h(\lambda)]}b(\lambda)a(\mu)\right\}\alpha_5(\lambda,\mu), \quad (A.8)$$

$$\alpha_{10}(\lambda,\mu)=\frac{b^2(\mu)-b^2(\lambda)}{a(\lambda)b(\lambda)+a(\mu)b(\mu)}\left\{\frac{\cosh[h(\mu)-h(\lambda)]}{\cosh[h(\mu)+h(\lambda)]}\right\}\alpha_5(\lambda,\mu), \quad (A.9)$$

where the weight $\alpha_5(\lambda,\mu)$ has been used as a normalization. We recall that functions $a(\lambda)$ and $b(\lambda)$ satisfy the free-fermion condition $a^2(\lambda)+b^2(\lambda)=1$, and in this paper we shall use the parametrization is $a(\lambda)=\cos(\lambda)$ and $b(\lambda)=\sin(\lambda)$. There are certain useful identities satisfied by these weights we have used to simplify commutation rules and the multi-particle problem. These relations are given by [15]

$$\alpha_3(\lambda,\mu)=\alpha_1(\lambda,\mu)+\alpha_6(\lambda,\mu), \qquad \alpha_4(\lambda,\mu)+\alpha_7(\lambda,\mu)=\alpha_2(\lambda,\mu), \quad (A.10)$$

$$\alpha_2(\lambda,\mu)\alpha_1(\lambda,\mu)-\alpha_9(\lambda,\mu)\alpha_8(\lambda,\mu)=\alpha_4(\lambda,\mu)\alpha_3(\lambda,\mu)-\alpha_{10}^2(\lambda,\mu)$$
$$=\alpha_5^2(\lambda,\mu), \quad (A.11)$$

$$\alpha_2(\lambda,\mu)\alpha_3(\lambda,\mu)+\alpha_4(\lambda,\mu)\alpha_1(\lambda,\mu)=2\alpha_5^2(\lambda,\mu). \quad (A.12)$$

Appendix B. Extra commutation rules

This appendix is devoted to complement the commutation relations presented in the main text. For instance, there are some additional commutation rules which are important for the complete solution of the two-particle state problem. These are relations between the fields $\boldsymbol{B}(\lambda)$, $\boldsymbol{B}^*(\lambda)$, $\boldsymbol{C}(\lambda)$ and $\boldsymbol{C}^*(\lambda)$ given by

$$C_a(\lambda)B_b(\mu)=-\frac{\alpha_8(\lambda,\mu)}{\alpha_9(\lambda,\mu)}B_b(\mu)C_a(\lambda)$$
$$+i\frac{\alpha_5(\lambda,\mu)}{\alpha_9(\lambda,\mu)}[B(\mu)A_{ab}(\lambda)-B(\lambda)A_{ab}(\mu)], \quad (B.1)$$

$$B_a^*(\lambda)B_b(\mu)=-\frac{\alpha_8(\lambda,\mu)}{\alpha_9(\lambda,\mu)}B_b(\mu)B_a^*(\lambda)$$
$$+i\frac{\alpha_5(\lambda,\mu)}{\alpha_9(\lambda,\mu)}[F(\mu)A_{ab}(\lambda)-F(\lambda)A_{ab}(\mu)], \quad (B.2)$$

$$C_a^*(\lambda)B_b(\mu) = \frac{\alpha_3(\lambda,\mu)}{\alpha_7(\lambda,\mu)}B_a(\mu)C_b^*(\lambda) - \frac{\alpha_4(\lambda,\mu)}{\alpha_7(\lambda,\mu)}B_a(\lambda)C_b^*(\mu)$$
$$-\frac{\alpha_6(\lambda,\mu)}{\alpha_7(\lambda,\mu)}B_b(\mu)C_a^*(\lambda) + i\frac{\alpha_{10}(\lambda,\mu)}{\alpha_7(\lambda,\mu)}\xi_{lm}A_{la}(\lambda)A_{mb}(\mu)$$
$$+i\frac{\alpha_{10}(\lambda,\mu)}{\alpha_7(\lambda,\mu)}\xi_{ab}[F(\mu)C(\lambda) - B(\mu)D(\lambda)]. \tag{B.3}$$

In particular, the commutation rule (B.3) is of considerable importance in the proof that the eigenvectors constructed in Section 4 are highest weights states of the $SU(2)$ η-pairing symmetry (see Section 5.2). In order to understand the role of the creation field $\boldsymbol{B}^*(\lambda)$ it is indispensable to derive its commutations relations with the other relevant fields. Between $\boldsymbol{B}^*(\lambda)$ and the diagonal operators we have

$$\hat{A}(\lambda)\otimes \boldsymbol{B}^*(\mu) = -i\frac{\alpha_1(\mu,\lambda)}{\alpha_8(\mu,\lambda)}\hat{r}(\mu,\lambda)\cdot[\boldsymbol{B}^*(\mu)\otimes\hat{A}(\lambda)] + i\frac{\alpha_5(\mu,\lambda)}{\alpha_8(\mu,\lambda)}\boldsymbol{B}^*(\lambda)\otimes\hat{A}(\mu)$$
$$-i\frac{\alpha_{10}(\mu,\lambda)}{\alpha_7(\mu,\lambda)}\boldsymbol{\xi}^t\otimes\left[\boldsymbol{B}(\lambda)D(\mu) + i\frac{\alpha_5(\mu,\lambda)}{\alpha_8(\mu,\lambda)}F(\lambda)\boldsymbol{C}^*(\mu)\right.$$
$$\left.-i\frac{\alpha_2(\mu,\lambda)}{\alpha_8(\mu,\lambda)}F(\mu)\boldsymbol{C}^*(\lambda)\right], \tag{B.4}$$

$$D(\lambda)\boldsymbol{B}^*(\mu) = i\frac{\alpha_2(\lambda,\mu)}{\alpha_8(\lambda,\mu)}\boldsymbol{B}^*(\mu)D(\lambda) - i\frac{\alpha_5(\lambda,\mu)}{\alpha_8(\lambda,\mu)}\boldsymbol{B}^*(\lambda)D(\mu), \tag{B.5}$$

$$B(\lambda)\boldsymbol{B}^*(\mu) = -i\frac{\alpha_9(\mu,\lambda)}{\alpha_7(\mu,\lambda)}\boldsymbol{B}^*(\mu)B(\lambda) + \frac{\alpha_5(\mu,\lambda)}{\alpha_7(\mu,\lambda)}F(u)\boldsymbol{C}(\lambda)$$
$$-\frac{\alpha_4(\mu,\lambda)}{\alpha_7(\mu,\lambda)}F(\lambda)\boldsymbol{C}(\mu) - i\frac{\alpha_{10}(\mu,\lambda)}{\alpha_7(\mu,\lambda)}[\boldsymbol{B}(\lambda)\otimes\hat{A}(\mu)]\cdot\boldsymbol{\xi}^t, \tag{B.6}$$

while with itself and with the scalar operator $F(\lambda)$ we have

$$\boldsymbol{B}^*(\lambda)\otimes\boldsymbol{B}^*(\mu) = \frac{\alpha_1(\mu,\lambda)}{\alpha_2(\mu,\lambda)}\hat{r}(\mu,\lambda)\cdot[\boldsymbol{B}^*(\mu)\otimes\boldsymbol{B}^*(\lambda)]$$
$$+i\frac{\alpha_{10}(\mu,\lambda)}{\alpha_7(\mu,\lambda)}\{F(\lambda)D(\mu) - F(\mu)D(\lambda)\}\boldsymbol{\xi}^t, \tag{B.7}$$

$$F(\lambda)\boldsymbol{B}^*(\mu) = \frac{\alpha_5(\mu,\lambda)}{\alpha_2(\mu,\lambda)}F(\mu)\boldsymbol{B}^*(\lambda) - i\frac{\alpha_9(\mu,\lambda)}{\alpha_2(\mu,\lambda)}\boldsymbol{B}^*(\mu)F(\lambda), \tag{B.8}$$

$$\boldsymbol{B}^*(\lambda)F(\mu) = \frac{\alpha_5(\mu,\lambda)}{\alpha_2(\mu,\lambda)}\boldsymbol{B}^*(\mu)F(\lambda) - i\frac{\alpha_8(\mu,\lambda)}{\alpha_2(\mu,\lambda)}F(\mu)\boldsymbol{B}^*(\lambda). \tag{B.9}$$

Lastly, the commutation rules with the annihilation fields $\boldsymbol{C}(\lambda)$ and $\boldsymbol{C}^*(\lambda)$ are

$$C_a^*(\lambda)B_b^*(\mu) = -\frac{\alpha_9(\mu,\lambda)}{\alpha_8(\mu,\lambda)}B_b^*(\mu)C_a^*(\lambda)$$
$$-i\frac{\alpha_5(\mu,\lambda)}{\alpha_8(\mu,\lambda)}[D(\mu)A_{ba}(\lambda) - D(\lambda)A_{ba}(\mu)], \tag{B.10}$$

$$C_a(\lambda)B^*_b(\mu)=\frac{\alpha_3(\mu,\lambda)}{\alpha_7(\mu,\lambda)}B^*_a(\mu)C_b(\lambda)-\frac{\alpha_4(\mu,\lambda)}{\alpha_7(\mu,\lambda)}B^*_a(\lambda)C_b(\mu)$$
$$-\frac{\alpha_6(\mu,\lambda)}{\alpha_7(\mu,\lambda)}B^*_b(\mu)C_a(\lambda)-i\frac{\alpha_{10}(\mu,\lambda)}{\alpha_7(\mu,\lambda)}\xi_{lm}A_{al}(\lambda)A_{bm}(\mu)$$
$$-i\frac{\alpha_{10}(\mu,\lambda)}{\alpha_7(\mu,\lambda)}\xi_{ab}[F(\mu)C(\lambda)-D(\mu)B(\lambda)]. \tag{B.11}$$

The best way of seeing that the latter commutations relations are connected to those for the field $\boldsymbol{B}(\lambda)$ is to read the equations in terms of their components. For instance, we note that commutation rule (B.2) is self-dual under the "dual" transformation described in Section 3. Several other relations have similar property as well.

We close this appendix by presenting the expressions for the fundamental commutation rules when we solve the standard Yang–Baxter algebra (3). These relations lack the presence of the imaginary factors "i" and certain extra signs when compared to their graded counterparts. Below we list the most important relations for the creation fields $\boldsymbol{B}(\lambda)$ and $F(\lambda)$

$$\hat{A}(\lambda)\otimes\boldsymbol{B}(\mu)=-\frac{\alpha_1(\lambda,\mu)}{\alpha_9(\lambda,\mu)}[\boldsymbol{B}(\mu)\otimes\hat{A}(\lambda)]\cdot\hat{r}_{TW}(\lambda,\mu)+\frac{\alpha_5(\lambda,\mu)}{\alpha_9(\lambda,\mu)}\boldsymbol{B}(\lambda)\otimes\hat{A}(\mu)$$
$$-\frac{\alpha_{10}(\lambda,\mu)}{\alpha_7(\lambda,\mu)}[\boldsymbol{B}^*(\lambda)B(\mu)-\frac{\alpha_5(\lambda,\mu)}{\alpha_9(\lambda,\mu)}F(\lambda)\boldsymbol{C}(\mu)$$
$$+\frac{\alpha_2(\lambda,\mu)}{\alpha_9(\lambda,\mu)}F(\mu)\boldsymbol{C}(\lambda)]\otimes\boldsymbol{\xi}_{TW}, \tag{B.12}$$

$$B(\lambda)\boldsymbol{B}(\mu)=-\frac{\alpha_2(\mu,\lambda)}{\alpha_9(\mu,\lambda)}\boldsymbol{B}(\mu)B(\lambda)+\frac{\alpha_5(\mu,\lambda)}{\alpha_9(\mu,\lambda)}\boldsymbol{B}(\lambda)B(\mu), \tag{B.13}$$

$$D(\lambda)\boldsymbol{B}(\mu)=-\frac{\alpha_8(\lambda,\mu)}{\alpha_7(\lambda,\mu)}\boldsymbol{B}(\mu)D(\lambda)-\frac{\alpha_5(\lambda,\mu)}{\alpha_7(\lambda,\mu)}F(u)\boldsymbol{C}^*(\lambda)$$
$$+\frac{\alpha_4(\lambda,\mu)}{\alpha_7(\lambda,\mu)}F(\lambda)\boldsymbol{C}^*(\mu)+\frac{\alpha_{10}(\lambda,\mu)}{\alpha_7(\lambda,\mu)}\boldsymbol{\xi}_{TW}\cdot[\boldsymbol{B}^*(\lambda)\otimes\hat{A}(\mu)], \tag{B.14}$$

$$\hat{A}_{ab}(\lambda)F(\mu)=\left[1+\frac{\alpha_5^2(\lambda,\mu)}{\alpha_9(\lambda,\mu)\alpha_8(\lambda,\mu)}\right]F(\mu)\hat{A}_{ab}(\lambda)$$
$$-\frac{\alpha_5^2(\lambda,\mu)}{\alpha_9(\lambda,\mu)\alpha_8(\lambda,\mu)}F(\lambda)\hat{A}_{ab}(\mu)+\frac{\alpha_5(\lambda,\mu)}{\alpha_9(\lambda,\mu)}[\boldsymbol{B}(\lambda)\otimes\boldsymbol{B}^*(\mu)]_{ba}$$
$$+\frac{\alpha_5(\lambda,\mu)}{\alpha_8(\lambda,\mu)}[\boldsymbol{B}^*(\lambda)\otimes\boldsymbol{B}(\mu)]_{ab}, \tag{B.15}$$

$$B(\lambda)F(\mu)=-\frac{\alpha_2(\mu,\lambda)}{\alpha_7(\mu,\lambda)}F(\mu)B(\lambda)+\frac{\alpha_4(\mu,\lambda)}{\alpha_7(\mu,\lambda)}F(\lambda)B(\mu)$$
$$+\frac{\alpha_{10}(\mu,\lambda)}{\alpha_7(\mu,\lambda)}\{\boldsymbol{B}(\lambda)\otimes\boldsymbol{B}(\mu)\}\cdot\boldsymbol{\xi}^t_{TW}, \tag{B.16}$$

$$D(\lambda)F(\mu) = -\frac{\alpha_2(\lambda,\mu)}{\alpha_7(\lambda,\mu)}F(\mu)D(\lambda) + \frac{\alpha_4(\lambda,\mu)}{\alpha_7(\lambda,\mu)}F(\lambda)D(\mu) + \frac{\alpha_{10}(\lambda,\mu)}{\alpha_7(\lambda,\mu)}\boldsymbol{\xi}_{TW}\cdot\{\boldsymbol{B}^*(\lambda)\otimes\boldsymbol{B}^*(\mu)\}, \tag{B.17}$$

where $\boldsymbol{\xi}_{TW}$ and $\hat{r}(\lambda,\mu)_{TW}$ are given by

$$\boldsymbol{\xi}_{TW} = \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix}, \quad \hat{r}_{TW}(\lambda,\mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \bar{a}(\lambda,\mu) & -\bar{b}(\lambda,\mu) & 0 \\ 0 & -\bar{b}(\lambda,\mu) & \bar{a}(\lambda,\mu) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{B.18}$$

Furthermore, the relations closing the commutation rules between the creation operators $\boldsymbol{B}(\lambda)$ and $F(\lambda)$ are

$$\boldsymbol{B}(\lambda)\otimes\boldsymbol{B}(\mu) = \frac{\alpha_1(\lambda,\mu)}{\alpha_2(\lambda,\mu)}[\boldsymbol{B}(\mu)\otimes\boldsymbol{B}(\lambda)].\hat{r}_{TW}(\lambda,\mu) - \frac{\alpha_{10}(\lambda,\mu)}{\alpha_7(\lambda,\mu)}\{F(\lambda)B(\mu) - F(\mu)B(\lambda)\}\boldsymbol{\xi}_{TW}, \tag{B.19}$$

$$[F(\lambda), F(\mu)] = 0, \tag{B.20}$$

$$F(\lambda)\boldsymbol{B}(\mu) = \frac{\alpha_5(\lambda,\mu)}{\alpha_2(\lambda,\mu)}F(\mu)\boldsymbol{B}(\lambda) + \frac{\alpha_8(\lambda,\mu)}{\alpha_2(\lambda,\mu)}\boldsymbol{B}(\mu)F(\lambda), \tag{B.21}$$

$$\boldsymbol{B}(\lambda)F(\mu) = \frac{\alpha_5(\lambda,\mu)}{\alpha_2(\lambda,\mu)}\boldsymbol{B}(\mu)F(\lambda) - \frac{\alpha_9(\lambda,\mu)}{\alpha_2(\lambda,\mu)}F(\mu)\boldsymbol{B}(\lambda). \tag{B.22}$$

Appendix C. The two-particle state

In this appendix we provide details about the technical points entering the analysis of the two-particle eigenvalue problem. We begin the discussion by first considering the wanted terms. We recall that the amplitudes proportional to the first part of the two-particle eigenstate are easily estimated as a product of the first right-hand side terms of the commutation rules (34-36). For the second part, however, there are more contributions since the action of diagonal operators on the first part $\boldsymbol{B}(\lambda_1)\otimes\boldsymbol{B}(\lambda_2)\cdot\mathcal{F}|0\rangle$ produce at least one extra term proportional to the second part $F(\lambda_1)\boldsymbol{\xi}\cdot\mathcal{F}|0\rangle$ as well. It turns out, however, that these contributions miraculously factorize in the same product forms we have obtained for the first part of the eigenstate. This happens thanks to remarkable identities between the Boltzmann weights we begin listing below. For the field $B(\lambda)$ there are two contributions and they factorize as

$$\frac{\alpha_2(y,x)}{\alpha_7(y,x)} - \frac{\alpha_2(y,x)}{\alpha_9(y,x)}\frac{\alpha_5(z,x)}{\alpha_9(z,x)}\frac{\alpha_{10}(y,x)}{\alpha_7(y,x)}\frac{\alpha_7(y,z)}{\alpha_{10}(y,z)} = -\frac{\alpha_2(y,x)}{\alpha_9(y,x)}\frac{\alpha_2(z,x)}{\alpha_9(z,x)}. \tag{C.1}$$

Analogously, for the field $D(\lambda)$ we have

$$\frac{\alpha_2(x,y)}{\alpha_7(x,y)} - \frac{\alpha_5(x,y)}{\alpha_7(x,y)}\frac{\alpha_{10}(x,z)}{\alpha_7(x,z)}\frac{\alpha_7(y,z)}{\alpha_{10}(y,z)} = -\frac{\alpha_8(x,y)}{\alpha_7(x,y)}\frac{\alpha_8(x,z)}{\alpha_7(x,z)}. \tag{C.2}$$

For the diagonal field $\sum_{a=1}^{2} \hat{A}_{aa}(\lambda)$ we have three contributions, where two of them are generated by the first part of the eigenstate. The identity that brings these terms together and also gives rise to the auxiliary eigenvalue function is

$$2\left[1+\frac{\alpha_5^2(x,y)}{\alpha_8(x,y)\alpha_9(x,y)}\right]-\frac{\alpha_1(x,y)}{\alpha_9(x,y)}\frac{\alpha_{10}(x,z)}{\alpha_7(x,z)}\frac{\alpha_5(x,y)}{\alpha_8(x,y)}\frac{\alpha_7(y,z)}{\alpha_{10}(y,z)}\left[1+\bar{a}(x,y)\right]$$
$$+\frac{\alpha_{10}(x,y)}{\alpha_7(x,y)}\frac{\alpha_2(x,y)}{\alpha_9(x,y)}\frac{\alpha_5(x,z)}{\alpha_9(x,z)}\frac{\alpha_7(y,z)}{\alpha_{10}(y,z)}$$
$$=-\frac{\alpha_1(x,y)}{\alpha_9(x,y)}\frac{\alpha_1(x,z)}{\alpha_9(x,z)}\left[\bar{b}(x,y)+\bar{b}(x,z)-\bar{a}(x,y)\bar{a}(x,z)\right]. \tag{C.3}$$

Next we turn to the analysis of the unwanted terms proportional to $\boldsymbol{B}(\lambda)\otimes\boldsymbol{B}(\lambda_j)$ and $[\boldsymbol{\xi}\otimes(\boldsymbol{B}^*(\lambda)\otimes\hat{I})]\otimes\boldsymbol{B}(\lambda_j)$. The terms with $\lambda_j=\lambda_2$ are straightforwardly read from the commutation rules (34)–(36) because only single contributions occur for each diagonal field. However, for $\lambda_j=\lambda_1$, the situation is more complicated because it involves many different contributions whose origin is due to the fact that the rapidity λ_1 is wrongly ordered when compared with λ_2. Nevertheless, one expects that there should be a better way of recasting these terms since the Bethe ansatz equations are usually independent of indices relabeling. Indeed, it turns out that these many contributions can be compactly written by introducing the "ordering" factor $O_j^{(1)}(\lambda,\lambda_j;\{\lambda_k\})$. As before, in order to factorize these contributions to a single term, we had to use extra identities between the Boltzmann weights. For example, for the field $B(\lambda)$ they are

$$\frac{\alpha_1(y,x)}{\alpha_9(y,x)}\frac{\alpha_5(z,x)}{\alpha_9(z,x)}\bar{a}(y,x)-\frac{\alpha_{10}(y,x)}{\alpha_7(y,x)}\frac{\alpha_{10}(y,z)}{\alpha_7(y,z)}-\frac{\alpha_5(y,x)}{\alpha_9(y,x)}\frac{\alpha_5(z,y)}{\alpha_9(z,y)} \tag{C.4}$$
$$=\frac{\alpha_5(z,x)}{\alpha_9(z,x)}\frac{\alpha_1(y,z)}{\alpha_9(y,z)}\bar{a}(y,z) \tag{C.5}$$

and

$$\frac{\alpha_{10}(y,x)}{\alpha_7(y,x)}\frac{\alpha_{10}(y,z)}{\alpha_7(y,z)}+\frac{\alpha_1(y,x)}{\alpha_9(y,x)}\frac{\alpha_5(z,x)}{\alpha_9(z,x)}\bar{b}(y,x)=\frac{\alpha_5(z,x)}{\alpha_9(z,x)}\frac{\alpha_1(y,z)}{\alpha_9(y,z)}\bar{b}(y,z), \tag{C.6}$$

where the left-hand side of the above equations represents the contributions coming from the "brute force" calculations while the right-hand side exhibits the "ordering" factor explicitly.

Similar simplifications can be carried out for the fields $\sum_{a=1}^{2}\hat{A}_{aa}(\lambda)$ and $D(\lambda)$, but we skip further details since there is a much simpler way to understand the origin of such "ordering" factor. As it has been explained in Section 4, this factor can be easily derived with the help of the exchange property (69). Anyhow, the coincidence between the "brute-force" computations and the symmetrization results gives us confidence to go ahead using the symmetrization procedure for multi-particle states.

Finally, we show how the third type of unwanted terms generated by the diagonal field $\sum_{a=1}^{2}\hat{A}_{aa}(\lambda)$ can be further simplified. First it is convenient to rewrite the term proportional to $[\omega_1(\lambda_1)\omega_2(\lambda_2)]^L F(\lambda)\boldsymbol{\xi}\cdot\boldsymbol{\mathcal{F}}$ in a way that the auxiliary eigenvalue function appears explicitly. For this purpose we use the second identity (66), and rewrite the contribution to the above mentioned unwanted term as

$$[w_1(\lambda_1)w_2(\lambda_2)]^L \Lambda^{(1)}(\lambda=\lambda_2,\{\lambda_l\}) H_3(\lambda,\lambda_1,\lambda_2) \times[\bar{b}(\lambda_1,\lambda_2)-\bar{a}(\lambda_1,\lambda_2)] F(\lambda)\boldsymbol{\xi}\cdot\boldsymbol{\mathcal{F}}. \tag{C.7}$$

Next we take advantage of the symmetrization property of the two-particle eigenstate and evaluate the contribution proportional to $[\omega_1(\lambda_2)\omega_2(\lambda_1)]^L F(\lambda)\boldsymbol{\xi}\cdot\boldsymbol{\mathcal{F}}$ as follows. The idea is to begin with the right-hand side of Eq. (69), which remarkably gives us precisely the extra r-matrix necessary to produce the auxiliary eigenvalue at $\lambda=\lambda_1$. Obviously, the amplitude contributing to this term is proportional to function obviously $H_3(\lambda,\lambda_2,\lambda_1)$ multiplied by the extra factor $\alpha_1(\lambda_1,\lambda_2)/\alpha_2(\lambda_1,\lambda_2)$ coming from the exchange relation. Putting these information together we are able to rewrite the second contribution as

$$-[w_1(\lambda_2)w_2(\lambda_1)]^L \Lambda^{(1)}(\lambda=\lambda_1,\{\lambda_l\}) H_3(\lambda,\lambda_2,\lambda_1)\frac{\alpha_1(\lambda_1,\lambda_2)}{\alpha_2(\lambda_1,\lambda_2)} F(\lambda)\boldsymbol{\xi}\cdot\boldsymbol{\mathcal{F}}. \tag{C.8}$$

These manipulations make the cancellation of the third type of unwanted terms more transparent, since it allows us to use the Bethe ansatz equations in a more direct way. Indeed, using the Bethe ansatz equations (65) in the terms (C.6) and (C.7) and adding them to those coming from the fields $B(\lambda)$ and $D(\lambda)$, we find that the unwanted terms proportional to $F(\lambda)\boldsymbol{\xi}\cdot\boldsymbol{\mathcal{F}}$ are cancelled out thanks to the following identity:

$$H_1(x,y,z)+H_2(x,y,z)=H_3(x,y,z)[\bar{b}(y,z)-\bar{a}(y,z)]-H_3(x,z,y)\frac{\alpha_1(y,z)}{\alpha_2(y,z)}. \tag{C.9}$$

This gives us another opportunity to verify the symmetrization scheme. Comparing (C.8) and (67) we conclude that the identity

$$H_3(x,z,y)\frac{\alpha_1(y,z)}{\alpha_2(y,z)}=H_4(x,y,z)[\bar{a}(z,y)-\bar{b}(z,y)] \tag{C.10}$$

is indeed satisfied.

We remark that the above technicalities are of enormous help when we consider generalization to multi-particle states. In Appendix D we shall discuss this fact for the three-particle state.

Appendix D. The three-particle state

We shall start this appendix showing how the permutation symmetry $\lambda_1\leftrightarrow\lambda_2$ is implemented for the three-particle state. As before, our strategy consists in reordering the rapidities λ_1 and λ_2 with the help of the commutation rule (25). This allows us to write the ansatz (76) as

$$\begin{aligned}\boldsymbol{\Phi}_3(\lambda_1,\lambda_2,\lambda_3) = & \frac{\alpha_1(\lambda_1,\lambda_2)}{\alpha_2(\lambda_1,\lambda_2)}\boldsymbol{B}(\lambda_2)\otimes\boldsymbol{B}(\lambda_1)\otimes\boldsymbol{B}(\lambda_3)\cdot\hat{r}_{12}(\lambda_1,\lambda_2) \\ & -i\frac{\alpha_{10}(\lambda_1,\lambda_2)}{\alpha_7(\lambda_1,\lambda_2)}[F(\lambda_1)B(\lambda_2)\boldsymbol{\xi}\otimes\boldsymbol{B}(\lambda_3)] \\ & i\frac{\alpha_{10}(\lambda_1,\lambda_2)}{\alpha_7(\lambda_1,\lambda_2)}[F(\lambda_2)B(\lambda_1)\boldsymbol{\xi}\otimes\boldsymbol{B}(\lambda_3)] \\ & +i\frac{\alpha_{10}(\lambda_2,\lambda_3)}{\alpha_7(\lambda_2,\lambda_3)}[\boldsymbol{B}(\lambda_1)\otimes\boldsymbol{\xi}F(\lambda_2)B(\lambda_3)] \\ & +[\boldsymbol{\xi}\otimes F(\lambda_1)\boldsymbol{B}(\lambda_3)B(\lambda_2)]\hat{g}_1^{(3)}(\lambda_1,\lambda_2,\lambda_3) \\ & +[\boldsymbol{\xi}\otimes F(\lambda_1)\boldsymbol{B}(\lambda_2)B(\lambda_3)]\hat{g}_2^{(3)}(\lambda_1,\lambda_2,\lambda_3). \end{aligned} \tag{D.1}$$

Next we use the commutation rule (35) to simplify the second and the third parts of the above equation, carrying the scalar field $B(\lambda_j)$ ($j = 1,2$) through the creation operator $\boldsymbol{B}(\lambda_3)$. This procedure not only helps us to eliminate the fifth term of equation (D.1) but also prompts the appearance of a desirable term proportional to $[\boldsymbol{\xi}\otimes F(\lambda_2)\boldsymbol{B}(\lambda_3)B(\lambda_1)]$. Now, imposing the exchange property (78) for the rapidities λ_1 and λ_2 we find the following necessary condition:

$$\hat{g}_1^{(3)}(\lambda_2,\lambda_1,\lambda_3)\frac{\alpha_7(\lambda_2,\lambda_1)}{\alpha_{10}(\lambda_2,\lambda_1)} = -\frac{\alpha_2(\lambda_3,\lambda_1)}{\alpha_9(\lambda_3,\lambda_1)}. \tag{D.2}$$

This relation together with the previous restrictions found in Section 4, cf. Eqs. (73), (74), are able to determine unambiguously the constraints for the three-particle state. The next step is to show the consistency of the whole procedure, i.e. that the equality between the remaining terms are indeed satisfied. By using the commutation rules (41), (42) we derive two consistency conditions, given by

$$\begin{aligned} & [\boldsymbol{B}(\lambda_2)\otimes\boldsymbol{\xi}]F(\lambda_1)B(\lambda_3) \\ & \quad\times\left[\frac{\alpha_{10}(\lambda_2,\lambda_3)}{\alpha_7(\lambda_2,\lambda_3)}\frac{\alpha_5(\lambda_1,\lambda_2)}{\alpha_2(\lambda_1,\lambda_2)} - \frac{\alpha_1(\lambda_1,\lambda_2)}{\alpha_2(\lambda_1,\lambda_2)}\frac{\alpha_{10}(\lambda_1,\lambda_3)}{\alpha_7(\lambda_1,\lambda_3)}\hat{r}_{12}(\lambda_1,\lambda_2)\right] \\ & = [\boldsymbol{\xi}\otimes\boldsymbol{B}(\lambda_2)]F(\lambda_1)B(\lambda_3) \\ & \quad\times\left[-\frac{\alpha_{10}(\lambda_1,\lambda_2)}{\alpha_7(\lambda_1,\lambda_2)}\frac{\alpha_5(\lambda_3,\lambda_2)}{\alpha_9(\lambda_3,\lambda_2)}\frac{\alpha_8(\lambda_1,\lambda_2)}{\alpha_2(\lambda_1,\lambda_2)} + \frac{\alpha_8(\lambda_1,\lambda_2)}{\alpha_2(\lambda_1,\lambda_2)}\hat{g}_2^{(3)}(\lambda_1,\lambda_2,\lambda_3)\right] \end{aligned} \tag{D.3}$$

and

$$\begin{aligned} & F(\lambda_2)[\boldsymbol{\xi}\otimes\boldsymbol{B}(\lambda_1)]B(\lambda_3)\left[\frac{\alpha_{10}(\lambda_1,\lambda_2)}{\alpha_7(\lambda_1,\lambda_2)}\frac{\alpha_5(\lambda_3,\lambda_1)}{\alpha_9(\lambda_3,\lambda_1)} + \frac{\alpha_5(\lambda_1,\lambda_2)}{\alpha_2(\lambda_1,\lambda_2)}\hat{g}_2^{(3)}(\lambda_1,\lambda_2,\lambda_3)\right. \\ & \quad -\frac{\alpha_{10}(\lambda_1,\lambda_2)}{\alpha_7(\lambda_1,\lambda_2)}\frac{\alpha_5(\lambda_3,\lambda_2)}{\alpha_9(\lambda_3,\lambda_2)}\frac{\alpha_5(\lambda_1,\lambda_2)}{\alpha_2(\lambda_1,\lambda_2)} \\ & \quad\left.-\frac{\alpha_1(\lambda_1,\lambda_2)}{\alpha_2(\lambda_1,\lambda_2)}\hat{g}_2^{(3)}(\lambda_2,\lambda_1,\lambda_3).\hat{r}_{12}(\lambda_1,\lambda_2)\right] \\ & = -F(\lambda_2)[\boldsymbol{B}(\lambda_1)\otimes\boldsymbol{\xi}]B(\lambda_3)\frac{\alpha_{10}(\lambda_2,\lambda_3)}{\alpha_7(\lambda_2,\lambda_3)}\frac{\alpha_9(\lambda_1,\lambda_2)}{\alpha_2(\lambda_1,\lambda_2)}. \end{aligned} \tag{D.4}$$

In order to disentangle the above expressions we need the help of certain useful identities between the "exclusion" vector and the auxiliary r-matrix. More precisely, they are

$$[\xi \otimes B(y)]\hat{r}_{12}(\lambda,\mu) = [\bar{a}(\lambda,\mu) - \bar{b}(\lambda,\mu)][\xi \otimes B(y)], \tag{D.5}$$

$$[\xi \otimes B(y)]\hat{r}_{23}(\lambda,\mu) = [\xi \otimes B(y)] + \bar{b}(\lambda,\mu)[B(y) \otimes \xi], \tag{D.6}$$

$$[B(y) \otimes \xi]\hat{r}_{23}(\lambda,\mu) = [\bar{a}(\lambda,\mu) - \bar{b}(\lambda,\mu)][B(y) \otimes \xi], \tag{D.7}$$

$$[B(y) \otimes \xi]\hat{r}_{12}(\lambda,\mu) = [B(y) \otimes \xi] + \bar{b}(\lambda,\mu)[\xi \otimes B(y)]. \tag{D.8}$$

Inserting the identities (D.5)–(D.8) into Eqs. (D.3), (D.4) we end up with four identities among the Boltzmann weights which have been verified by using Mathematica™. With this we complete the symmetrization analysis for the three-particle state.

We now turn to the analysis of the eigenvalue problem for the three-particle state. Let us begin by investigating the action of the scalar field $B(\lambda)$ on the state (76). The first step consists to carry the field $B(\lambda)$ through the creation fields $\boldsymbol{B}(\lambda_1)$ and $F(\lambda_1)$ by using the commutation rules (35) and (41), (42). Afterwards, we use directly the known results for the two-particle state, cf. (57), in order to turn one more time the scalar fields $B(\lambda)$ and $B(\lambda_1)$ over the two-particle state $|\Phi_2(\lambda_2,\lambda_3)\rangle$. As a third step, we need to reorder creation fields such as $\boldsymbol{B}(\lambda_1)$ and $\boldsymbol{B}(\lambda)$ with the help of commutation rule (25) as well as keep on carrying the scalar field $B(\lambda)$ until it reaches the vacuum. After this long but straightforward computations we find the following result:

$$\begin{aligned}
B(\lambda)|\Phi_3(\lambda_1,\lambda_2,\lambda_3)\rangle &= [w_1(\lambda)]^L \prod_{j=1}^{3} i\frac{\alpha_2(\lambda_j,\lambda)}{\alpha_1(\lambda_j,\lambda)} \boldsymbol{B}(\lambda_1) \otimes \boldsymbol{\Phi}_2(\lambda_2,\lambda_3)\cdot \mathcal{F}|0\rangle \\
&+ [w_1(\lambda)]^L i\frac{\alpha_2(\lambda_3,\lambda)}{\alpha_9(\lambda_3,\lambda)} L[\lambda,\lambda_1,\lambda_2]\xi \otimes F(\lambda_1)\boldsymbol{B}(\lambda_3)B(\lambda_2)\hat{g}_1^{(3)}(\lambda_1,\lambda_2,\lambda_3)\cdot\mathcal{F}|0\rangle \\
&+ [w_1(\lambda)]^L i\frac{\alpha_2(\lambda_2,\lambda)}{\alpha_9(\lambda_2,\lambda)} L[\lambda,\lambda_1,\lambda_3]\xi \otimes F(\lambda_1)\boldsymbol{B}(\lambda_2)B(\lambda_3)\hat{g}_2^{(3)}(\lambda_1,\lambda_2,\lambda_3)\cdot\mathcal{F}|0\rangle \\
&+ \text{unwanted terms}.
\end{aligned} \tag{D.9}$$

where function $L[x,y,z]$ is precisely the left-hand side of identity (C.1) we have worked out for the two-particle state. This allows us to factorize the amplitudes for the second and the third terms of the above equation, and as result all the three wanted terms have a common amplitude as it should be. To what concerns the unwanted terms, our computation shows that they can be gathered in two basic families. More specifically, they are proportional to

$$[w_1(\lambda_j)]^L \boldsymbol{B}(\lambda) \otimes \boldsymbol{\Phi}_2(\lambda_l,\lambda_k) \tag{D.10}$$

and

$$[w_1(\lambda_j)w_1(\lambda_l)]^L \xi \otimes F(\lambda)\boldsymbol{\Phi}_1(\lambda_k). \tag{D.11}$$

The first term in the family (D.10), say $\lambda_j = \lambda_1$, $\lambda_l = \lambda_2$ and $\lambda_k = \lambda_3$, is originated from the first part of the three-particle state when we turn the scalar field $B(\lambda)$ through $\boldsymbol{B}(\lambda_1)$. Keeping the second term of the commutation rule (35), and by using the two-particle results (57) to carry $B(\lambda_1)$ through $|\Phi_2(\lambda_2, \lambda_3)\rangle$, we find that its amplitude is

$$-i\frac{\alpha_5(\lambda_1, \lambda)}{\alpha_9(\lambda_1, \lambda)} \prod_{k=2}^{3} i\frac{\alpha_2(\lambda_k, \lambda_1)}{\alpha_9(\lambda_k, \lambda_1)}. \tag{D.12}$$

We estimate the amplitudes of the remaining terms in the family (D.10) by taking into account the exchange property (78), in much the same way we did for the two-particle state. This means that the amplitudes are going to be multiplied by the first "ordering" factors $\hat{O}_j^{(1)}(\lambda_j; \{\lambda_k\})$, and three possible unwanted terms $j = 1,2,3$ can be compactly written as

$$-[w_1(\lambda_j)]^L i\frac{\alpha_5(\lambda_j, \lambda)}{\alpha_9(\lambda_j, \lambda)} \prod_{\substack{k=1 \\ i \neq j}}^{3} i\frac{\alpha_2(\lambda_k, \lambda_j)}{\alpha_9(\lambda_k, \lambda_j)} \boldsymbol{B}(\lambda) \otimes \boldsymbol{\Phi}_2(\lambda_1, \ldots, \check{\lambda}_j, \ldots, \lambda_3)$$
$$\times \hat{O}_j^{(1)}(\lambda_j; \{\lambda_k\}) \cdot \mathcal{F}|0\rangle. \tag{D.13}$$

The contributions to the second family of unwanted terms come from all the pieces composing the three-particle state. It turns out that for $k = 2,3$ their amplitudes can be computed in a very similar way we did for the second and third parts of the wanted terms, respectively. The main difference is that now we have to keep track of terms proportional to $F(\lambda)$ rather than $F(\lambda_1)$. We find that the amplitudes for these unwanted terms are

$$[w_1(\lambda_1) w_1(\lambda_j)]^L H_1(\lambda, \lambda_1, \lambda_j) \prod_{\substack{k=1 \\ k \neq 1,j}}^{3} i\frac{\alpha_2(\lambda_k, \lambda_1)}{\alpha_9(\lambda_k, \lambda_1)} i\frac{\alpha_2(\lambda_k, \lambda_j)}{\alpha_9(\lambda_k, \lambda_j)}$$
$$\times [F(\lambda)\boldsymbol{\xi} \otimes \boldsymbol{\Phi}_1(\lambda_2, \ldots, \check{\lambda}_j, \ldots, \lambda_3)] \hat{O}_{1j}^{(2)}(\lambda_1, \lambda_j; \{\lambda_k\}) \cdot \mathcal{F}|0\rangle, \tag{D.14}$$

where $\hat{O}_{1j}^{(2)}(\lambda_1, \lambda_j; \{\lambda_k\}), j = 1,2$, is the second type of "ordering" factor which has been already defined in the main text, see Eq. (88). It should be emphasized that we have derived the above factor from a "brute force " analysis, and similar to what happened to the two-particle state, this gives us the clue to proceed in order to better estimate the remaining unwanted terms appearing in this family. We easily recognize that this factor is related to the operation of bringing two rapidities in the first two positions of the eigenvector. Keeping this in mind, we see that all the contributions to the second family of unwanted terms can be written by

$$[w_1(\lambda_l) w_1(\lambda_j)]^L H_1(\lambda, \lambda_l, \lambda_j) \prod_{\substack{k=1 \\ k \neq l,j}}^{3} i\frac{\alpha_2(\lambda_k, \lambda_l)}{\alpha_9(\lambda_k, \lambda_l)} i\frac{\alpha_2(\lambda_k, \lambda_j)}{\alpha_9(\lambda_k, \lambda_j)}$$
$$\times [F(\lambda)\boldsymbol{\xi} \otimes \boldsymbol{\Phi}_1(\lambda_1, \ldots, \check{\lambda}_l, \ldots, \check{\lambda}_j, \ldots, \lambda_3)] \hat{O}_{lj}^{(2)}(\lambda_l, \lambda_j; \{\lambda_k\}) \cdot \mathcal{F}|0\rangle. \tag{D.15}$$

Collecting the expressions (D.9), (D.13) and (D.15) we find that the action of the scalar field $B(\lambda)$ on the three-particle state is described by the formula (82) with $n=3$. Similar reasoning can be repeated for the fields $D(\lambda)$ and $\sum_{a=1}^{2}\hat{A}_{aa}(\lambda)$, and only when we are estimating the third type of unwanted terms new technicalities emerge. In what follows we present the details of these computations in the simplest case, i.e. the situation where no "ordering" factors are needed. Generalization for the remaining terms is along the lines of formula (D.15). For the field $D(\lambda)$ we find that such amplitude is

$$H_2(\lambda,\lambda_1,\lambda_2)F(\lambda)\xi_{bc}\hat{A}_{bb_1}(\lambda_1)\hat{A}_{cb_2}(\lambda_2)B_{b_3}(\lambda_3)\mathcal{F}^{b_3b_2b_1}|0\rangle . \tag{D.16}$$

Now, carrying the operators $\hat{A}_{bb_1}(\lambda_1)$ and $\hat{A}_{cb_2}(\lambda_2)$ through $B_{b_3}(\lambda_3)$ with the help of commutation rule (34) we find

$$[w_2(\lambda_1)w_2(\lambda_2)]^L H_2(\lambda,\lambda_1,\lambda_2)\prod_{k=1}^{2}\frac{\alpha_1(\lambda_k,\lambda_3)}{i\alpha_9(\lambda_k,\lambda_3)}$$
$$\times\xi_{bc}F(\lambda)B_\gamma(\lambda_3)\hat{r}_{\alpha c}^{b_2b_3}(\lambda_2,\lambda_3)\hat{r}_{\gamma b}^{b_1\alpha}(\lambda_1,\lambda_3)\mathcal{F}^{b_3b_2b_1}|0\rangle , \tag{D.17}$$

which is further simplified by using the following identity:

$$T^{(1)}(\lambda=\lambda_2,\{\lambda_l\})_{c_1c_2c_3}^{b_1b_2b_3}T^{(1)}(\lambda=\lambda_1,\{\lambda_l\})_{bc\gamma}^{c_1c_2c_3}=\hat{r}_{\alpha c}^{b_2b_3}(\lambda_2,\lambda_3)\hat{r}_{\gamma b}^{b_1\alpha}(\lambda_1,\lambda_3). \tag{D.18}$$

Inserting (D.18) into (D.17) we finally obtain

$$[w_2(\lambda_1)w_2(\lambda_2)]^L H_2(\lambda,\lambda_1,\lambda_2)\Lambda^{(1)}(\lambda=\lambda_1,\{\lambda_l\})\Lambda^{(1)}(\lambda=\lambda_2,\{\lambda_l\})$$
$$\times\prod_{k=1}^{2}\frac{\alpha_1(\lambda_k,\lambda_3)}{i\alpha_9(\lambda_k,\lambda_3)}F(\lambda)\boldsymbol{\xi}\otimes\boldsymbol{B}(\lambda_3). \tag{D.19}$$

For the field $\sum_{a=1}^{2}\hat{A}_{aa}(\lambda)$ we find that one of the contributions is

$$H_3(\lambda,\lambda_1,\lambda_2)\xi_{ab_1}F(\lambda)B(\lambda_1)\hat{A}_{ab_2}(\lambda_2)B_{b_3}(\lambda_3)\mathcal{F}^{b_3b_2b_1}|0\rangle \tag{D.20}$$

and when we carry $B(\lambda_1)$ and $\hat{A}_{ab_2}(\lambda_1)$ through $B_{b_3}(\lambda_3)$ we have

$$[w_1(\lambda_1)w_2(\lambda_2)]^L H_3(\lambda,\lambda_1,\lambda_2)\prod_{k=1}^{2}\frac{\alpha_1(\lambda_k,\lambda_3)}{i\alpha_9(\lambda_k,\lambda_3)}$$
$$\times\xi_{ab_1}F(\lambda)\hat{r}_{da}^{b_2b_3}(\lambda_2,\lambda_3)B_d(\lambda_3)\mathcal{F}^{b_3b_2b_1}|0\rangle . \tag{D.21}$$

Next, using the following identity,

$$\xi_{ab_1}\hat{r}_{da}^{b_2b_3}(\lambda_2,\lambda_3)B_d(\lambda_3)\mathcal{F}^{b_3b_2b_1}$$
$$=\xi_{\gamma\delta}\hat{r}_{\gamma\delta}^{\alpha\beta}(\lambda_1,\lambda_2)T^{(1)}(\lambda=\lambda_2,\{\lambda_l\})_{\alpha\beta d}^{b_1b_2b_3}B_d(\lambda_3)\mathcal{F}^{b_3b_2b_1}, \tag{D.22}$$

we finally find

$$[w_1(\lambda_1)w_2(\lambda_2)]^L H_3(\lambda,\lambda_1,\lambda_2)[\bar{a}(\lambda_1,\lambda_2)-\bar{b}(\lambda_1,\lambda_2)]\Lambda^{(1)}(\lambda=\lambda_2,\{\lambda_l\})$$
$$\times\prod_{k=1}^{2}\frac{\alpha_1(\lambda_k,\lambda_3)}{i\alpha_9(\lambda_k,\lambda_3)}F(\lambda)\boldsymbol{\xi}\otimes\boldsymbol{B}(\lambda_3). \tag{D.23}$$

Finally, the second contribution coming from the field $\sum_{a=1}^{2}\hat{A}_{aa}(\lambda)$ is estimated by using the same trick explained in the previous appendix for the two-particle state. We further remark that the technical points explained in Appendices C and D are valid for many other models such as the Bariev *XY* chain and those solved in Ref. [23].

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