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Singularities and Discontinuities of Feynman Amplitudes*

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The Landau singularities of the amplitude calculated from an arbitrary Feynman graph are considered. It is shown that the discontinuity across a branch cut starting from any Landau singularity is obtained by replacing Feynman propagators by delta functions for those lines which appear in the Landau diagram. The general formula is a simple generalization of the unitarity condition. The discontinuity is then considered as an analytic function of the momenta and masses; it is shown that its singularities are a subclass of the singularities of the original amplitude which corresponds to Landau diagrams with additional lines. The general results are illustrated by application to some single loop graphs. In particular, the general formula gives an immediate calculation of the Mandelstam spectral function for fourth-order scattering. Singularities not of the Landau type are discussed and illustrated by the third-order vertex part.

I. INTRODUCTION

KARPLUS, Sommerfield, and Wichman[1] and Landau[2] have emphasized the importance of examining the analyticity of the amplitudes corresponding to Feynman graphs, and have discussed some simple graphs in detail. Landau has also given a criterion for determining the position of certain singularities of the amplitude for an arbitrary graph. In this paper we shall derive a formula for the discontinuity across a cut starting from any one of Landau's branch points, and shall determine where this discontinuity is singular. The result is a very natural generalization of the well-known expression, given by the unitarity condition, for the discontinuity across a cut starting from any physical threshold. The general result is extremely useful for analyzing spectral representations. For example, it leads immediately to an explicit expression for the Mandelstam spectral function for the fourth-order scattering amplitude.[3]

Before proceeding with the calculation, let us recapitulate Landau's discussion. He considers the amplitude

$$F=\int B\prod(d^4k)A_1^{-1}\cdots A_N^{-1} \tag{1}$$

(where $A_i=M_i^2-q_i^2$ and B is an arbitrary polynomial) corresponding to a graph with N internal lines and n independent loops. In (1) and the following we adhere closely to Landau's notation. The q_i are linear combinations of the k_i and the external momenta p_i. On its principal branch F has no singularities for sufficiently small, real p_i^2; if the M_i^2 are positive, we may take the p_i^2 to be positive without passing a singularity, and begin the investigation with real p_{i4} and imaginary $\mathbf{p}_i$. We denote by z_a the independent invariants formed from the p_i.

Now introduce the Feynman parametrization

$$F=(N-1)!\int\prod(d\alpha)\prod(d^4k)BD^{-N}\delta(1-\tilde{\alpha}), \tag{2}$$

where $D=\sum_{i=1}^N\alpha_iA_i$ and $\tilde{\alpha}=\sum\alpha_i$. Let $\varphi=\max_k(D)$ (the maximization is carried out with real k_{i4} and imaginary $\mathbf{k}_i$). According to Landau, if $\min\varphi>0$, F is nonsingular, where the minimum is taken with respect to nonnegative α's satisfying $\tilde{\alpha}=1$. As the p_i^2 are increased, the first singularity of F occurs when $\min\varphi\to 0$. This, Landau shows, means that for each i

$$\alpha_iA_i=0, \tag{3}$$

and for each closed loop

$$\sum\alpha_iq_i=0, \tag{4}$$

where the sum is extended over all the lines in the loop; moreover, (4) must be satisfied with nonnegative α's. Landau pointed out that a singularity exists when (4) is satisfied with arbitrary α_i, but did not give an explicit proof of this; as this point is important to our subsequent discussion we show that this follows from an analytic continuation in the internal masses, and the continuity theorem for singularity surfaces.[4]

The following remarks are contained implicitly in Landau's paper.

Let D_m be obtained from D by setting the $\alpha_i=0$ for $i>m$, and let $\varphi_m=\max_k(D_m)$. If for some α_i, $\max_k(D_m)$ occurs for $q_i^2=M_i^2(i\leqslant m)$, then for any other nonnegative $\alpha_i(i\leqslant m)$, $\varphi_m\geqslant 0$. Now, we may choose the M_i^2 for $i>m$ so large that φ_m is the minimum of φ for nonnegative α's. For any $\alpha_i>0(i\leqslant m)$ and $p_i^2>0$ we determine q_i which satisfy (4) (this is just the maximization problem) and *define* for $i\leqslant m$ masses M_i by the equation $q_i^2=M_i^2$. Hence masses exist such that any

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[1] R. Karplus, C. M. Sommerfield, and E. H. Wichman, Phys. Rev. **111**, 1187 (1958); **114**, 375 (1959).
[2] L. D. Landau, Nuclear Phys. **13**, 181 (1959). *Note added in proof.* Results similar to Landau's were also obtained by J. C. Taylor [Phys. Rev. **117**, 261 (1960)], which the author received after submission of this paper.
[3] S. Mandelstam, Phys. Rev. **115**, 1742 (1959).
[4] H. Behnke and P. Thullen, *Theorie der Functionen Mehrerer Komplexer Veränderlichen* (Springer-Verlag, Berlin, 1934), p. 49.

"Landau diagram" corresponds to the first singularity. There are two cases to be considered, which can easily be distinguished upon inspection of the Landau diagram. If some of the masses obtained by the procedure described are constant, or satisfy a relation independent of the α's and the z's, then we do not in general have either a solution to (4) or a singularity. Otherwise, as the α's and z's are varied these masses take on all possible values, in which case it follows from the continuity theorem that for any internal masses there is always a singularity when Eqs. (3) and (4) are satisfied, although this singularity might not appear on the principal sheet of the Riemann surface.

In order to discuss the analytic continuations of F, we eliminate the delta function from (2), by replacing the α_i by $\lambda\alpha_i$, multiplying by a suitable entire function of λ (say $e^{-\lambda}$) and integrating over λ. This gives the equation

$$F=(N-1)!\int\prod(d\alpha)\prod(d^4k)BD^{-N}\tilde{\alpha}^{-1}\exp(-\tilde{\alpha}^{-1}). \quad (5)$$

In (5), the α_i vary independently over any suitable contours from 0 to ∞. We may use this equation to interpret Landau's conditions in the complex region. We use an idea introduced by Hadamard,[5] which has already been exploited in a similar problem by Eden.[6] If we first integrate over the $k_{i\mu}$, we obtain an integrand which is singular when φ vanishes, where in the general case φ is an extremum of D. The singularities of F occur when some of the α_i are fixed at the lower limit of integration, while the contours over which the remaining α_i are integrated are trapped between coalescing singularities. In other words, φ must have a double zero with respect to each of the free variables, which leads directly to Landau's conditions (3) and (4). It is also necessary that for z_a in the neighborhood of a singularity of F, the contours actually pass between these nearly coalescent zeros. We know that this occurs when we consider the first singularity; we obtain an illustration of the continuity theorem if we note that when the M_i are varied, if the z_a are simultaneously varied so as to keep the zeros in a nearly coalescent configuration, the contours must remain entrapped.

Since the integrand in (5) is always singular when $\tilde{\alpha}=0$, if D vanishes for $\tilde{\alpha}=0$ the condition of a double zero with respect to the free variables is relaxed. In this case we might have a singularity even if conditions (3) and (4) do not hold, although such a singularity could never appear on the principal sheet. We shall show, in Sec. III, that an "anomalous" singularity of this type actually occurs in the third-order vertex.

[5] J. Hadamard, Acta Math. **22**, 55 (1898).

[6] R. J. Eden, Proc. Roy. Soc. (London) **A210**, 388 (1952). *Note added in proof.* After submission of this paper, the author received two papers containing a similar discussion of the complex singularities: J. C. Polkinghorne and G. R. Screaton, Nuovo cimento **15**, 289 (1960); J. Tarski, J. Math. Phys. **1**, 154 (1960).

II. DISCONTINUITIES OF FEYNMAN AMPLITUDES

A. Calculation of the Discontinuity

We shall prove the following theorem: Let F denote the amplitude defined by Eq. (1), and let F_m denote the discontinuity of F across a branch cut starting from a singularity defined by Landau's conditions (3) and (4) in which $A_i=0$ for $i\leqslant m$; then

$$F_m=(2\pi i)^m \times\int\frac{B\prod(d^4k)\delta_p(q_1^2-M_1^2)\cdots\delta_p(q_m^2-M_m^2)}{A_{m+1}\cdots A_N}. \quad (6)$$

(The notation implies a particular ordering of the lines.) The subscript p on the delta functions means that only the contribution of the "proper" root of $q_i^2=M_i^2$ is to be taken. Equation (6) is a simple generalization of well-known results, and follows directly from the Hadamard-Eden analysis.

Consider the contracted Feynman graph obtained by fusing the vertices connected by the lines $i>m$. Let ν be the number of independent loops in this contracted graph. We can choose the k_j so that the $q_i (i\leqslant m)$ depend only on those k_j for which $j\leqslant\nu$. If the $m\times 4\nu$ matrix

$$J_{i,j\mu}=\partial q_i^2/\partial k_{j\mu}$$

is of rank m, we may choose as integration variables $\xi_i=q_i^2$ for $i\leqslant m$, and $4\nu-m$ additional variables. The q_i^2 are the squared distances between certain points in momentum space, and the ξ_i for $m<i\leqslant 4\nu$ may be interpreted as related angle variables. We shall discuss later the circumstance that $J_{i,j\mu}$ has a rank smaller than m for all $k_{j\mu}$. If the rank is too small only when the $k_{j\mu}$ satisfy particular relations, these exceptional points may in general be avoided by appropriate indentations of the $k_{j\mu}$ contours. We therefore obtain

$$F=\int_{a_1}^{b_1}dq_1^2\cdots\int_{a_m}^{b_m}dq_m^2 \times\int\frac{\prod_{m<i\leq 4\nu}(d\xi_i)\prod_{j>\nu}(d^4k_j)}{JA_1\cdots A_N}, \quad (7)$$

where

$$J=\det(\partial\xi_i/\partial k_{j\mu}).$$

The limits of integration (a_j,b_j) for the q_j^2 integration are the extrema of q_j^2 for fixed $q_i^2 (i<j)$. This leads to the equations (for each loop of the contracted graph)

$$\sum_{(i\leq j)}\beta_i q_{i\mu}=0, \quad (8)$$

where the β_i are Lagrange multipliers. From (8) for $j=m$ we see that Landau's conditions (3), (4) imply that when a singularity develops, the point where the $A_i=0$ for $i\leqslant m$ lies on the boundary of the region of integration. Equation (8) also shows that the rank of $J_{i,j\mu}$ is always too small on the boundary of the integration region, but this gives no difficulty. In certain cases each set of the q_i^2 corresponds to two points in momentum space; in these cases we interpret the q_m^2 integration as being taken over the closed contour which encloses the two points a_m and b_m where J is singular.[7]

For brevity we denote by z a point in the (many-sheeted) space of the invariants. Let z_0 denote any point on the singularity surface in question which does not also lie on some other singularity surface.

We first suppose that all the integrations in (7) have been performed, except that over q_1^2. Then we write

$$F=\int_{a_1}^{b_1} dq_1^2(M_1^2-q_1^2)^{-1}F_{(1)}(q_1^2). \tag{9}$$

Now, by hypothesis, (A) F is singular when $z\rightarrow z_0$, and (B) F would not be singular at z_0 if the factor $(M_1^2-q_1^2)^{-1}$ were absent or if the mass M_1 were changed. Therefore, the contour of the q_1^2 integration must pass between the pole $q_1^2=M_1^2$ and a singularity of $F_{(1)}(q_1^2)$ at $q_1^2=Q^2$, where $Q^2\rightarrow M_1^2$ when $z\rightarrow z_0$. We may replace this contour by one on the other side of the pole $q_1^2-M_1^2$ and a very small circle enclosing this pole, where the contour which avoids the pole gives a contribution to F which is regular in the neighborhood of z_0. The singular part of F is therefore

$$F_s=\pm 2\pi i F_{(1)}(M_1^2). \tag{10}$$

The argument given is not sufficient to determine the sign.

After applying the foregoing argument in succession to the variables $q_2^2\cdots q_{m-1}^2$, we obtain

$$F_s=\int_{a_m}^{b_m} dq_m^2(M_m^2-q_m^2)^{-1}F_{(m)}(q_m^2). \tag{11}$$

In (11), a_m and b_m are the limits calculated with $q_1^2=M_i^2$ for $i<m$. When $z\rightarrow z_0$, it follows from (8) that one of these limits coincides with the point $q_m^2=M_m^2$. It is obvious that the discontinuity across a branch cut starting from z_0 is $2\pi i F_{(m)}(M_m^2)$. When the q_m^2 integration is taken over a contour enclosing the points a_m and b_m, the two branches of F_s are determined by whether the pole $q_m^2=M_m^2$ lies inside this contour or not, so we obtain the same result.

We now *define* the sign of F_m by analytic continuation from the case where the masses are such that the singularity in question is the first encountered as the z_a are continued through real values from the singularity-free region, and z is a real point just beyond this singularity. It was shown in the Introduction that it is possible to do this. We define the discontinuity $F_m(z)$ to be the difference between $F(z)$ as calculated by giving the masses small *negative* imaginary parts and that calculated with small *positive* imaginary parts; that is,

$$F_m(z)=F_{-i\epsilon}(z)-F_{+i\epsilon}(z). \tag{12}$$

Now consider the q_m^2 integration: Equation (12) implies that the discontinuity is given by a clockwise contour around the pole. But the same result must hold for all q_i^2. This proves Eq. (6) for the case that the rank of the matrix $\partial q_i^2/\partial k_{j\mu}$ is equal to m, except that in transforming back to the $k_{i\mu}$ we must be careful to keep only the contribution from the proper root of $q_i^2=M_i^2$.

There are two cases in which the rank of $J_{i,j\mu}$ is too small; either this happens only for z which satisfy some particular relation, which restricts these z to lie on some surface, or else it occurs identically, for all z. In the first case, (6) is valid for all nonexceptional z, but the discontinuity might be singular when $J_{i,j\mu}$ is singular. If the rank is always too small, as when $m>4\nu$, we consider the singularity obtained by eliminating a sufficient number of lines (say for $m'<i\leqslant m$) that the rank of the reduced matrix $\partial q_i^2/\partial k_{i\mu}$ is m'. The singularity of the larger matrices implies that the eliminated q_i^2 can be expressed in terms of the q_i^2 for $i\leqslant m'$. Hence when we evaluate the discontinuity $F_{m'}$ by Eq. (6), we find that $F_{m'}$ has not a branch point but a pole when one of the eliminated A_i vanishes. These exceptional cases will be illustrated in Sec. III.

B. Singularities of the Discontinuity Function

We may think of $F_m(z)$ as the difference between the values of $F(z)$ on two different sheets, so the singularity surfaces of $F_m(z)$ will be contained among those of $F(z)$. We discuss these singularities by introducing $N-m$ Feynman parameters $\alpha_i(i>m)$ and repeating Landau's calculation. When we integrate over the $k_{i\mu}$, we obtain a singularity for those values of the α_i for which

$$\varphi=\text{Extremum}_k(\textstyle\sum_{i>m}\alpha_i A_i)$$

vanishes. However, the variables $k_{i\mu}$ are not all independent, because they satisfy the constraints $A_i=0$ for $i\leqslant m$. These constraints are introduced into the extremization by using m Lagrange multipliers, which we also call $\alpha_i(i\leqslant m)$. This leads to the equation $\sum\alpha_i q_{i\mu}=0$, which is identical to (4). The integration over the Feynman parameters is singular when some of them are zero, and φ is a vanishing extremum with respect to the rest. This leads to Eq. (3) for $i>$m. We are not allowed to omit any of the conditions $A_i=0$ for $i\leqslant m$, so the singularities of $F(z)$ which are also singularities of $F_m(z)$ correspond to Landau diagrams in which lines have been added to the Landau diagram which defined the original singularity. The other singularities of F necessarily appear on both sheets and cancel when we calculate the difference. As we have pointed out before, there is also a possibility of non-Landauian singularities.

[7] For some graphs with more than one loop, several of the q_i^2 integrations need to be interpreted in this way.

Let us denote by $F_{m,m'-m}(z)$ the discontinuity of $F_m(z)$ across a branch cut starting from the branch point for which $A_i=0$ for $m<i\leqslant m'$. We calculate $F_{m,m'-m}$ by the same method used to calculate F_m; we use the q_i^2 as variables for $i\leqslant m'$. It is clear that all the steps in the proof (except for determination of the sign) are identical. Moreover, we find that

$$F_{m,m'-m}(z)\equiv F_{m'}(z). \qquad (13)$$

[We use Eq. (13) to define the sign of $F_{m,m'-m}$.] It may be noted that it can be proved independently (by extending the argument in the Introduction) that the singularity of F which corresponds to $A_i=0$ for $i\leqslant m'$ only appears on one of two adjacent sheets connected by the branch point corresponding to $A_i=0$ for $i\leqslant m$.

C. Unitarity Condition

Consider two graphs, each with m outgoing lines, and with r and s incoming lines, respectively. Let F and G denote the corresponding amplitudes. The unitarity of the S matrix implies that these two graphs give a contribution to the imaginary part of the T matrix (for r outgoing and s incoming particles) which is, apart from numerical factors and with neglect of the spins of the particles,

$$\mathcal{T}_{rs(m)}=\int d\tau_m F^*G, \qquad (14)$$

where $d\tau_m$ is the volume element in the phase space of m particles. Let $\mathbf{q}_i$ and W_i denote the momenta and energies of these m particles. As a consequence of momentum conservation, the $\mathbf{q}_i$ depend linearly on $m-1$ integration variables $\mathbf{k}_i$. With a covariant normalization of states, we have

$$d\tau_m=\frac{d^3k_1\cdots d^3k_{m-1}}{(2W_1)\cdots(2W_m)}\delta(\textstyle\sum W_i-E), \qquad (15)$$

where E is the total energy. We may introduce $m-1$ new integration variables k_{i4} and write (15) as follows:

$$d\tau_m=d^4k_1\cdots d^4k_{m-1}\delta_p(q_1^2-M_1^2)\cdots\delta_p(q_m^2-M_m^2). \qquad (16)$$

In (16) the q_{i4} are the same functions of the k_{i4} as the $\mathbf{q}_i$ are of the $\mathbf{k}_i$. The subscript p means that only the "proper" root of $q_i^2=M_i^2$, that for which q_{i4} is positive, is to be considered when the integrations are carried out.

Equation (14) is first obtained for real momenta. To continue it to the complex region we introduce the explicit forms of G and F, with the notation that q_i is the momentum of any internal line, and k_i is any integration variable. Then (14) becomes

$$\mathcal{T}_{rs(m)}=\int\frac{\prod(d^4k)B\delta_p(q_1^2-M_1^2)\cdots\delta_p(q_m^2-M_m^2)}{A_{m+1}\cdots A_N}, \qquad (17)$$

where $A_i=M_i^2+i\epsilon-q_i^2$ for lines belonging to the graph F, and $A_i=M_i^2-i\epsilon-q_i^2$ for lines belonging to the graph G.

Equation (17) is just a special case of the general discontinuity formula (6) for the graph obtained by joining the graphs F and G by the m common lines. In (17) the analytic continuations have been defined in a particular way (by the $\pm i\epsilon$ rule), while in (6) the masses may be considered to be arbitrary. The discussion in Sec. II.B of the location of the singularities of $F_m(z)$ applies without modification to $\mathcal{T}_{rs(m)}$.

The correspondence between the unitarity condition (17) and the general discontinuity formula (6) suggests that the general discontinuity may be looked on as a pseudounitarity condition. The particles, instead of being divided into the two groups of "initial" and "final" particles, may be divided into three or more groups.

III. ILLUSTRATIONS

In this section we illustrate the results derived in Sec. II by applying them to the three graphs shown in Fig. 1.

A. Fourth-Order Scattering

The singularities correspond to the vanishing of the following combinations of the A_i: (13), (24), (12), (23), (34), (41), (123), (134), (124), (234), and (1234). The ordinary threshold is the (13) singularity. The corresponding discontinuity is obtained by replacing A_1^{-1} and A_3^{-1} by $2\pi i\delta_p(A_1)$ and $2\pi i\delta_p(A_3)$. The discussion in Sec. II.B shows that this discontinuity has only the singularities (13), (123), (134), and (1234). The Mandelstam spectral function[3] is, apart from a factor of four, the discontinuity of this discontinuity function across the (1234) singularity, which is

$$F_4=\int d^4k\delta_p(q_1^2-M_1^2)\cdots\delta_p(q_4^2-M_4^2). \qquad (18)$$

Reverting to the variables used in the proof of (6),

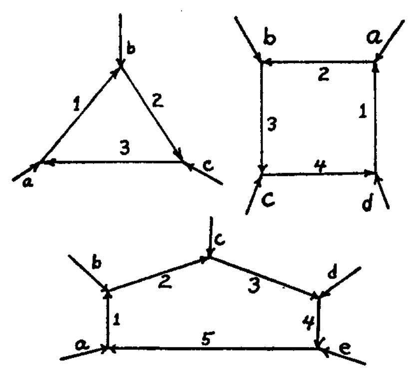


FIG. 1. Feynman graphs considered in Sec. III.

we have

$$F=\int J^{-1}dq_1^2\cdots dq_4^2\delta(q_1^2-M_1^2)\cdots\delta(q_4^2-M_4^2) \\ =J^{-1}, \tag{19}$$

where $J=\det\partial q_i^2/\partial k_\mu=2^4\det q_{i\mu}$ is evaluated for $q_i^2=M_i^2$. The result of Mandelstam[3] and Kibble[8] is obtained from (18) by noting that $[\det q_{i\mu}]^2=\det q_iq_j$.

The reader will recognize $\det q_{i\mu}$ as the volume of the four-dimensional parallelepiped constructed with the q_i as edges. The vectors q_i have lengths M_i, and they have such directions that when drawn from a common vertex Q, their ends are vertices of the tetrahedron constructed from the external momenta (see Fig. 2). Complex vectors are to be used in drawing the figure, when necessary. This figure (a simplex) is one corner of the parallelepiped; its volume V is $1/4!$ times the volume of the parallelepiped. Hence $J=2^4 4!V$.

Landau's condition for the location of the (1234) singularity is that the point Q should lie in the hyperplane of the tetrahedron. In this case $V=0$. It should

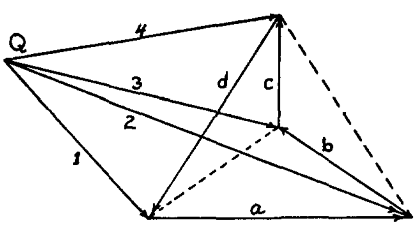


FIG. 2. The Mandelstam spectral function is the reciprocal of the volume of this figure.

be noted that the transformation from the k_μ to the q_i^2 is singular when the tetrahedron degenerates to a planar figure. But $4V$ is the product of the volume of the tetrahedron and the altitude of the point Q from the hyperplane of the tetrahedron, and when the volume of the tetrahedron vanishes, the altitude, for *fixed* lengths of the q_i, becomes infinite in such a way that V^{-1} is analytic.

B. Third-Order Vertex

The discontinuity across the (123) branch cut is

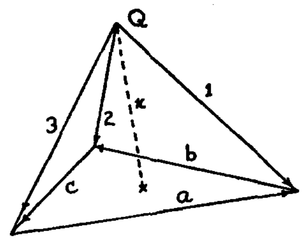
$$F_3=\frac{1}{2\pi}\int d^4k\delta_p(q_1^2-M_1^2)\delta_p(q_2^2-M_2^2)\delta_p(q_3^2-M_3^2). \tag{20}$$

Consider the point Q whose squared distances from the vertices of the triangle (p_a, p_b, p_c) are q_i^2 (see Fig. 3). The locus of Q in four-dimensional space is a circle whose radius κ is the altitude of Q from the plane of the triangle. Transforming to new variables, we have

$$d^4k=\kappa d\varphi d^3k=\kappa d\varphi\prod(dq_i^2)J_3^{-1}, \tag{21}$$

[8] T. W. B. Kibble, Phys. Rev. **117**, 1159 (1960).

FIG. 3. Geometrical construction associated with the third-order vertex.



where $J_3=8\det q_{ia}$ is a 3×3 determinant. Hence we obtain

$$F_3=\kappa J_3^{-1}. \tag{22}$$

Now $\det q_{ia}$ is 3! times the volume of the tetrahedron in Fig. 3, which in turn is $\frac{1}{3}\kappa\mathfrak{A}$, where $\mathfrak{A}$ is the area of the triangle. Therefore,

$$F_3=2^{-4}\mathfrak{A}^{-1} \\ =\tfrac{1}{4}\{p_a^4+p_b^4+p_c^4-2p_a^2p_b^2-2p_a^2p_c^2-2p_b^2p_c^2\}^{-\frac{1}{2}}. \tag{23}$$

We see that F_3, and therefore also F on at least one sheet, is singular when $\mathfrak{A}=0$. In this example, a singularity of the matrix $\partial q_i^2/\partial k_\mu$ actually is associated with a singularity of F. The singularity can be shown to correspond, in terms of the Feynman parametrization discussed in the Introduction, to the case $\alpha_1+\alpha_2+\alpha_3=0$.

C. Example of Redundant Lines

Consider the graph shown in Fig. 1 which has five lines in one loop. Landau's procedure shows there is a singularity when all five $A_i=0$, but this is not a branch point. The discontinuity across the (1234) branch cut is shown by the method of Sec. III.A to be

$$F_4=J^{-1}(q_5^2-M_5^2)^{-1}, \tag{24}$$

where J and q_5^2 are functions of the external momenta and of $M_1,\cdots,M_4$. When the external momenta are such that $q_5^2=M_5^2$, F_4 has a pole. Since F_4 is the difference between values of the amplitude F on two adjacent sheets, and since the (12345) singularity only appears on one of them, F also has a pole. The location of the pole corresponds to the possibility of drawing the Landau diagram with four-dimensional vectors; the nonexistence of a branch cut corresponds to the impossibility of buckling the diagram into an extra dimension.

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