S Matrix for the One-Dimensional N-Body Problem with Repulsive or Attractive $\delta$-Function Interaction

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For $N$ particles with equal mass, interacting with repulsive or attractive $\delta$-function interaction of the same strength, the $S$ matrix is explicitly given and shown to be symmetrical and unitary. The incoming and outgoing states may consist of bound compounds as well as single particles. The momenta of the particles and compounds are not changed in the scattering, but particles are exchanged, such as $ABC+DE \rightarrow ADC+BE$. Only distinguishable particles are considered.

1. INTRODUCTION

For the one-dimensional $N$-body problem

$$H = -\sum_{i=1}^{N} \frac{\partial^{2}}{\partial x_{i}^{2}} + 2e \sum_{i<j} \delta(x_{i} - x_{j}),$$

(1)

with positive or negative $e$, the $S$ matrix was discussed by McGuire and by Zinn-Justine and Brezin. (Note added in proof. K. Hepp kindly informed the author that F. A. Berezin and V. N. Sushko, Zh. Eksp. Teor. Fiz. 48, 1293 (1965) [English transl.: Soviet Phys.—JETP 21, 865 (1965)] have also discussed this problem.) We give in this paper a complete explicit expression for $S$. Only distinguishable particles are considered.

2. METHOD

The method used follows that of Sec. 1 of a recent paper. We observe that all formulas there are also applicable to the case $e<0$.

If boundary conditions are not imposed, it is clear that all solutions of the Schrödinger equation are superpositions of solutions of the type (Y2). In other words, Bethe’s hypothesis is proved in such a case.

3. INCOMING AND OUTGOING STATES

To construct scattering states, we need real values of the $p$'s. Let us choose them so that

$$p_1 < p_2 < \cdots < p_N.$$  

(2)

A term in (Y2) that has $P=\text{identity permutation}=I$, then, represents an outgoing wave. [A wave packet constructed out of such a term would have the leftmost particle (at $X_{p1}$) travel with velocity $2p_1$; the second leftmost particle (at $X_{p2}$) travel with velocity $2p_2$, etc. Thus the wave packet in future movement develops no collisions, meaning it is an outgoing wave packet.] A term in (Y2) that has $P=[N, N-1, \cdots, 1]=I'$, i.e., the “reversed” permutation, represents an incoming wave.

Now each permutation $Q$ represents a definite ordering of the coordinates and represents a scattering channel. A scattering state $Q_1 \rightarrow Q_0$ is obtained if there are only incoming waves in channel $Q_1$:

$$[Q_1 I'] = 1,$$
$$[Q_1 I'] = 0 \quad \text{for} \quad Q \neq Q_1.$$  

(3)

In other words,

$$\langle Q_1 | \xi_{\ell} \rangle = 1,$$
$$\langle Q | \xi_{\ell} \rangle = 0 \quad \text{for} \quad Q \neq Q_1.$$  

(4)

The amplitudes of the outgoing waves are the elements of $\xi_{\ell}$. Now $\xi_{\ell}$ can be related to $\xi_{\ell'}$ through repeated use of (Y2):

$$\xi_{\ell'} = \left[ Y_{21}^{p_1} Y_{22}^{p_2} Y_{23}^{p_3} \cdots Y_{N1}^{N-1} \right] \times \left[ Y_{12}^{p_1} Y_{13}^{p_2} \cdots Y_{N2}^{N-2} \cdots \right] \left[ Y_{N1}^{N-1} \right] \xi_{\ell}.  

(5)

Thus the scattering amplitude for $Q_1 \rightarrow Q_0$ is

$$\langle Q_0 | S^\dagger | Q_1 \rangle,$$

(6)

where $S'$ is the right-hand side of (5) with $\xi_{\ell}$ deleted.

4. OPERATOR: (ij)

We did not call the matrix $S'$ in (6) the $S$ matrix because it differs from the usual one in that the labeling of the columns is not in accordance with the usual rules. So this is the incoming wave in $Q_1$, represented by the $[Q_1 I']$ term, describes particle Q1 with momentum $p_{N}$, Q2 with momentum $p_{N-1}$, etc. Thus the correct $S$ matrix is

$$S = S'[[P_{1}^{N} P_{(N-1)}^{N-2}], \ldots]$$
$$= S'[[P_{1}^{N}] [P_{N}^{N} P_{(N-1)}^{N-2}] [P_{N-2}^{N-2} \ldots] [P_{1}^{N} \ldots P_{i}^{N}]].$$  

(7)

If in (7) one explicitly writes $S'$, as given in (5), one observed that the superscripts for the $Y$'s are the same as those for the $P$'s, but in reverse order. One now permutes the last factor $P_{i}$ through to just behind the first factor $Y_{21}^{p_1}$; then the new last factor $P_{i}$ through to just behind the second factor $Y_{N1}^{N}$, etc. The final

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result is
\[ S = [(21)(31)(41) \cdots (N1)] \]
\[ \times [(22)(42) \cdots (N2)] \cdots [(N(N-1))] , \]
(8)
where
\[ \{ij\} = x_{ij} = P_{ii} Y_{ii} - (1 - P_{ii}) x_{ij}^{-1} . \]
(9)

5. S MATRIX

In (8) we have an explicit formula for the S matrix (for both c ≥ 0 and c < 0, p₁ < p₂ < \cdots < pₙ being all real). S is an N! × N! matrix. The scattering only exchanges particle momenta. The elements of S have the following meaning:
\[ \langle A'B'C' \cdots | S | ABC \cdots \rangle \]

= matrix element of S for 
\[ \text{[State: particle } A \text{ with } p₁, B \text{ with } p₂, \text{ etc.] } \rightarrow \]
\[ \text{[State: particle } A' \text{ with } p₁, B' \text{ with } p₂, \text{ etc.] } \]

In (9) the permutation operator \( P_{ii} \) is defined so that, e.g.,
\[ P_{ii}(CDBA) = |BDCA| = P_{ii}(ADCB) . \]

It is easy to verify that each \( \{ij\} \) is unitary. Hence S is unitary. S is a symmetrical matrix, as required by the time-reversal invariance of the interaction we have, because each \( \{ij\} \) is symmetrical and the order of the operators \( \{ij\} \) in (8) can be reversed by repeated application of Eq. (Y12). For example, for \( N = 4 \),
\[ S = (21)(31)(41)(22)(32)(42)(43) \]
\[ = (21)(31)(41)(42)(32) \]
\[ = (21)(43)(31)(32)(32) \]
\[ = (43)(21)(42)(31)(32) \]
\[ = (43)(42)(21)(31)(32) \]
\[ = (43)(42)(32)(41)(31)(21) = \bar{S} . \]

6. ATTRACTIVE CASE

For the case \( c < 0 \), there are bound states for the system of \( N \) particles. The wave function for the bound state is
\[ \psi = \exp[\frac{1}{4} \sum_{i<j} x_i - x_j] . \]
(10)

It is easy to show directly that (10) satisfies the Schrödinger equation.

It is clear that (10) is of Bethe’s form (Y2) with
\[ p₁ = \frac{1}{2} i c (N - 1) , \]
\[ p₂ = \frac{1}{2} i c (N - 3) , \]
\[ \cdots , \]
\[ pₙ = - \frac{1}{2} i c (N - 1) , \]
(11)
and with
\[ \xi_p = (\text{a column with all elements equal}) , \]
(12a)
\[ \xi_p = 0 \text{ for all } p \neq I . \]
(12b)
that
\[ p_3 - p_1 = -i \xi, \quad p_5 - p_4 = p_1 - p_3 = -i \xi. \] (14)

The operators \( Y_{ij} \) are all defined and have nonzero eigenvalues, except for the following:
\[ Y_{12} = Y_{34} = Y_{15} = \frac{1}{2} (P_{8} - 1) ; \]
\[ Y_{21}, Y_{43}, Y_{45} \] are not defined, \( \xi_3 = \xi_4 = 0 \).
(15) \( y_{31} = y_{43} = y_{54} = 0 \). \( y_{31} = y_{43} = y_{54} = 0 \). \( y_{31} = y_{43} = y_{54} = 0 \).

For the wave function (Y2) to be bounded, such columns as \( \xi_3 \cdot \xi_5 \) must be zero; for otherwise as \( x_{01} \to \infty \),
the terms in (Y2) with the elements of \( \xi_3 \cdot \xi_5 \) as coefficients will diverge exponentially. Considerations like this and a reexamination of (Y2), which remains valid except for the cases where \( Y_{ij} \) is not defined, finally lead to
\[ \xi_P = 0 \quad \text{if} \quad P \text{ is of type } A , \] (17a)
\[ \xi_P = 0 \quad \text{if} \quad P \text{ is not of type } A , \] (17b)
where \( P \) is defined to be of type \( A \) if in
\[ P = [P_1, P_2, P_3, P_4, P_5] \]
and 1 and 2 are in that order and 3, 4, 5 are in that order (e.g., [23145] is not in \( A \), [34152] is in \( A \)). Furthermore,
\[ \xi_7 = \xi_{12345} \]
satisfies
\[ \xi_7 = P^{12}_8 \xi_7 = P^{45}_8 \xi_7 = P^{45}_8 \xi_7 = P^{45}_8 \xi_7 = \xi_7 . \] (18)

Because of (18), we have, e.g.,
\[ \xi_{13245} = Y_{13}^{12} \xi_{13245} = \frac{1}{2} (P^{12}_8 - 1) \xi_{13245} = 0 , \]
\[ \xi_{42315} = Y_{42}^{31} Y_{43}^{21} Y_{45}^{21} Y_{45}^{21} \xi_{42315} = 0 . \] (19)
\[ \xi_{12345} \text{ still gives the outgoing waves, but the incoming waves are not given by } \xi_{42315}, \text{ which is zero by (19).} \]
Instead, it is given by \( \xi_{42315} \). Thus, instead of the \( S' \) of (6), we have now
\[ \xi_{12345} = S' \xi_{42315} , \]
\[ S' = (Y_{12}^{13} Y_{43}^{21} Y_{45}^{21}) (Y_{12}^{13} Y_{43}^{21} Y_{45}^{21}) . \] (20)

Again, the \( S \) matrix is obtained by a permutation of the columns of \( S' \):
\[ S = \{32\} \{42\} \{52\} \{31\} \{41\} \{51\} . \] (22)

8. ALLOWED STATES

Equation (22) gives explicitly the \( S \) matrix for a two-particle bound state scattered by a three-particle bound state. Because of (18), \( S \) should only operate between states \( \Phi \), satisfying
\[ \Phi = P^{35}_8 \Phi = P^{35}_8 \Phi = P^{35}_8 \Phi . \] (23)

We shall call such states “allowed” states. Among the \( 5! = 120 \) components of the column \( \Phi \), there are only
\[ 5! = 120 \] independent allowed ones. For example,
\[ \langle ABCDE | \Phi \rangle = \langle ABCED | \Phi \rangle = \langle ABEDC | \Phi \rangle = \langle ABDEC | \Phi \rangle = etc. = \langle ABEDC | \Phi \rangle = \langle BAEDC | \Phi \rangle = \langle BADEC | \Phi \rangle = etc. \]
\[ \langle BADEC | \Phi \rangle = 1 / (12)^{3 / 2} \]
together describe the allowed incoming state
\[ AB + CDE , \] (24)

where \( AB \) is the symmetrical bound state of \( A \) and \( B \) with momentum \( p_1 + p_3 \), and \( CDE \) is the symmetrical bound state of \( C, D \), and \( E \) with momentum \( p_3 + p_4 + p_5 \).

9. SOME IDENTITIES

We shall prove in Sec. 10 three important properties of the \( S \) of Eq. (22). A few mathematical preliminaries will be given in this section.

We note that
\[ \{12\} = \frac{1}{2} (1 - P^{12}_8) , \{34\} = \frac{1}{2} (1 - P^{34}_8) , \{45\} = \frac{1}{2} (1 - P^{45}_8) , \] (25)
so that Eq. (23) is equivalent to
\[ 0 = \{12\} \Phi = \{34\} \Phi = \{45\} \Phi . \] (26)

(Y12) remains valid, or rather the following hold true:
\[ \{ij\} \{ji\} = 1 , \] (27a)
\[ \{ij\} \{kj\} \{ki\} = \{ki\} \{kj\} \{ij\} , \] (27b)
\[ \{ij\} \{kl\} = \{kl\} \{ij\} \]
if \( i, j, k, l \) are all different, \( \{ij\} \{kl\} = \{kl\} \{ij\} \)
provided the undefined \( \{21\}, \{43\}, \) and \( \{54\} \) do not appear. [Note that \( \{35\} \) and \( \{53\} \) are defined.]

Although \( \{21\} \) is not defined, we can try to define \( y_{31} \) (21) so that \( y_{31} = 0 \) does not appear any more in the denominator. In other words, we define
\[ \{21\} = (1 - y_{31}) P^{31}_8 + 1 = P^{31}_8 + 1 , \]
\[ \{43\} = P^{43}_8 + 1 , \]
\[ \{54\} = P^{54}_8 + 1 . \] (28)

With this definition, (27b) is true also for those cases where \( \{21\}, \{43\}, \) and/or \( \{54\} \) appear, provided we replace them by \( \{21\}, \{43\}, \) and \( \{54\} \). For example,
\[ \{43\} \{53\} \{54\} = \{54\} \{53\} \{43\} , \]
\[ \{21\} \{51\} \{52\} = \{52\} \{51\} \{21\} . \] (29)

\( \Phi \) is allowed if, and only if,
\[ 2 \Phi = \{21\} \Phi = \{43\} \Phi = \{54\} \Phi . \] (30)

Equations (23), (26), and (30) are equivalent.
10. UNITARITY AND SYMMETRY OF S

We now first prove that if $\Phi$ is allowed, so is $S\Phi$. This follows from

$$
\{12\}S\Phi = \{12\}\{(32)\}(31)\{(42)\}(41)\{(52)\}(51)\Phi \\
= \{(31)\}(32)\{(41)\}(42)\{(51)\}(52)\{(12)\}.
$$

Now

$$
S(21') = \{(32)\}(31)\{(42)\}(41)\{(52)\}(51)\{21'\} = \{(31)\}(32)\{(41)\}(42)\{(51)\}(52).
$$

$$
S(21')\{54'\} = \{21'\}\{54'\}\{(31)\}\{32\}\{(41)\}\{(52)\}\{42\},
$$

$$
S(21')\{54'\}\{53\} = \{21'\}\{54'\}\{53\}\{(31)\}\{32\}\{(51)\}\{52\} \{(42)\},
$$

$$
S(21')\{54'\}\{53\}\{43'\} = \{21'\}\{54'\}\{53\}\{43'\}\{(31)\}\{32\}\{(51)\}\{52\}\{(42)\} = \{21'\}\{54'\}\{53\}\{43'\}S.
$$

But

$$
\{21'\}\{54'\}\{53\}\{43'\}\Phi_3 = 8(2y_{54}-1)\Phi_3 = 24\Phi_2,
$$

$$
\Phi_1\{21'\}\{54'\}\{53\}\{43'\} = 8(2y_{54}-1)\Phi_1 = 24\Phi_4.
$$

Thus (33) yields directly (31).

Last we shall prove that $S$ is unitary for allowed states, i.e., if $\Phi_2$ is allowed,

$$
\Phi_3 S^* S \Phi_2 = \Phi_4 S \Phi_2.
$$

To prove this we find that

$$
S^* = \{(51)\}{\{41\}}{\{31\}}{\{52\}}{\{42\}}{\{32\}}.
$$

Now

$$
P_{12}P_{34}P_{52} = \{(23)\} \{24\}\{(25)\}\{13\}\{14\}\{15\}
$$

Thus

$$
P_{12}P_{34}P_{52} S^* = P_{12}P_{34} P_{52}.
$$

Put

$$
\Phi_3 = S \Phi_2.
$$

Thus $\Phi_1$ is allowed, and $\Phi_1 = P_{12}P_{34}P_{52} \Phi_1$. Equations (35) and (31) give

$$
\Phi_3 \Phi_3 = \Phi_1 S \Phi_2 = \Phi_1 S \Phi_2 = \Phi_1 S^* S \Phi_2.
$$

11. GENERAL CASE

The results of Secs. 7–10 can be generalized in a straightforward way to the scattering between any number of particles or compounds, each of which may be a bound state of any number of particles. The $S$ matrix can be easily written down. For example, we write down the $S$ matrix for a scattering between a single particle of momentum $p_1$, a bound triplet of momentum $p_2 + p_3 + p_4$, and a bound double of momentum $p_5 + p_6$. These $p$'s are plotted in Fig. 2(b). We have, like Eq. (22),

$$
S = \{(21)\}\{(23)\}\{(24)\}\{(25)\}\{(26)\} \times (63)\{(52)\}  \times \{(51)\}.
$$

For the case where the $p$'s are given by Fig. 2(c), we have

$$
S = \{(32)\}\{(34)\}\{(35)\}\{41\}\{(42)\}\{(43)\} \times (63)\{(52)\}  \times (64)\{(51)\}.
$$

For the case where the $p$'s are given by Fig. 2(d), we have

$$
S = \{(21)\}\{(23)\}\{(24)\}\{(25)\}\{(26)\} \times (63)\{(52)\}  \times (64)\{(51)\}.
$$

All these $S$ matrices are unitary and symmetrical for the allowed $\Phi$ in each case.

12. REDUNDANT POLES

The $S$ matrix discussed above has evidently matrix elements that are rational functions of the relative momenta of the particles involved. For real values of these relative momenta, $S$ is regular. But for complex values of these relative momenta, $S$ may have poles. For example, in the reaction $AB + CDE$ discussed in Secs. 7–10, for which the $S$ matrix is given by (22), there are poles when $y_{23}, y_{34}, y_{45}, y_{24}, y_{25}, y_{45}$ or $y_{25}$ vanishes. However, only the pole $y_{25} = 0$ corresponds to a bound state (the $S$ particle bound state). The others are redundant poles. This point was already realized by McGuire.

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