Exact form factors of the $SU(N)$ Gross-Neveu model and $1/N$ expansion

Hrachya M. Babujian[*][†], Angela Foerster[‡][§], and Michael Karowski[¶]
Institut für Theoretische Physik, Freie Universität Berlin,
Arnimallee 14, 14195 Berlin, Germany

July 3, 2009

Dedicated to the memory of Alexey Zamolodchikov

Abstract

The general $SU(N)$ form factor formula is constructed. Exact form factors for the field, the energy momentum and the current operators are derived and compared with the $1/N$-expansion of the chiral Gross-Neveu model and full agreement is found. As an application of the form factor approach the equal time commutation rules of arbitrary local fields are derived and in general anyonic behavior is found.

PACS: 11.10.-z; 11.10.Kk; 11.55.Ds
Keywords: Integrable quantum field theory, Form factors

1 Introduction

Quantum chromodynamics, the theory of the strong interactions, is a non abelian gauge theory based on the gauge group $SU(3)$. It was first pointed out by 't Hooft [1, 2] that many features of QCD can be understood by studying a gauge theory based on the gauge group $SU(N)$ in the limit $N \to \infty$. One might think that letting $N \to \infty$ would make the analysis more complicated because of the larger gauge group and consequently increase in the number of dynamical degrees of freedom. Also one can think that $SU(N)$ gauge theory has very little to do with QCD because $N \to \infty$ is not close to $N = 3$. However it is well known that the $1/N$ expansion provides good results which can be compared with experiments [3].

---

[*]Address: Alikhanian Brothers 2, Yerevan, 375036Armenia
[†]E-mail: babujian@yerphi.am
[‡]Address: Instituto de Física da UFRGS, Av. Bento Gonçalves 9500, Porto Alegre, RS - Brazil
[§]E-mail: angela@if.ufrgs.br
[¶]E-mail: karowski@physik.fu-berlin.de

One of the most important trends in theoretical physics in the last decades is the development of exact methods which are completely different from perturbation theory. Resolution of the strong coupling problem would give us a full understanding of the structure of interactions in nonabelian gauge theory. One promising possibility of overcoming the limitations of perturbation theory is the application of exact integrability. From this point of view the two dimensional integrable quantum field theories are in a sense a laboratory for investigations of those properties of quantum field theories, which cannot be described via perturbation theory.

The chiral $SU(N)$ Gross-Neveu [4] model given by the Lagrangian

$$\mathcal{L} = \sum_{i=1}^{N} \bar{\psi}_i \, i\gamma\partial \, \psi_i + \frac{g^2}{2} \left(\left(\sum_{i=1}^{N} \bar{\psi}_i \psi_i \right)^2 - \left(\sum_{i=1}^{N} \bar{\psi}_i \gamma^5 \psi_i \right)^2 \right) \tag{1}$$

is an interesting 1+1 dimensional field theory that can be studied using the $1/N$ expansion. The model is asymptotically free with a spontaneously broken chiral symmetry, and so shares some dynamical features with QCD. Gross and Neveu [4] investigated the model using an $1/N$ expansion. Apparently a chiral $U(1)$-symmetry is spontaneously broken, the fermions acquire mass and a Goldstone boson seems to appear. This is of course not possible in two space-time dimensions and severe infrared divergences appear due the "would-be-Goldstone boson". However, it has been argued by Witten [5] that dynamical mass generation can be reconciled with the absence of spontaneous symmetry breaking. There exist further (different) approaches to overcome these problems and to formulate a $1/N$ expansion [6, 7, 8] (see also [9]). On shell they all agree and are consistent with the exact S-matrix (2). We follow here the approach of Swieca et al. [8] where additional fields are introduced in order to compensate the infrared divergences. The authors claim that since the physical fermions have lost not only the chiral $U(1)$ but also the charge $U(1)$ symmetry, they transform accordingly to pure $SU(N)$. They propose an interpretation of the antiparticles as a bound state of $N-1$ particles. Furthermore this means that the particles satisfy neither Fermi nor Bose statistics, but rather carry "spin" $s = \frac{1}{2}(1 - 1/N)$. As a consequence there are unusual crossing relations and Klein factors.

In this article we will focus on the $SU(N)$ Gross-Neveu <u>form factors</u> using the "bootstrap program" [10, 11]. We provide here some examples, calculate the form factors exactly and compare the results with field theoretical $1/N$ expansions. We emphasize that in addition to the operators in the vacuum sector, such as the energy momentum tensor and the current, we also consider anyonic operators as the fundamental fields. We also derive the equal time commutation rules for local operators which are in particular complicated due to the unusual crossing formulae related to the Klein factors.

The general form factor of an operator $\mathcal{O}(x)$ for n-particles, which is a co-vector valued function and can be written as [12]

$$F_{\underline{\alpha}}^{\mathcal{O}}(\underline{\theta}) = K_{\underline{\alpha}}^{\mathcal{O}}(\underline{\theta}) \prod_{1 \leq i < j \leq n} F(\theta_{ij})$$

where $\underline{\theta} = (\theta_1, \ldots, \theta_n)$ is the set of rapidities of the particles $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$. The scalar function $F(\theta)$ is the minimal form factor function and the K-function $K_{\underline{\alpha}}^{\mathcal{O}}(\underline{\theta})$ contains

the entire pole structure and its symmetry is determined by the form factor equations (i) to (v) [13]. To construct the K-function we must apply the nested off-shell Bethe ansatz to capture the vectorial content of the form factors. This solves the missing link of Smirnov's [14] formula for the $SU(N)$ form factors, where the vectors were given by an "indirect definition" characterized by necessary properties but not provided explicitly. We note that $SU(N)$ form factors were also calculated in [14, 15, 16] using other techniques, see also the related papers [17, 18]. Our results apply not only to chargeless operators such as the energy momentum and the current operators but also to more general ones with anyonic behavior. We believe that our integral representation, besides of being appropriate for a comparison with field theoretical $1/N$ expansions, may also shed some light for a better understanding on the correlation functions of models with more general (anyonic) statistics.

The paper is organized as follows: In section 2 we present the general setting concerning the $SU(N)$ S-matrix, the nested off-shell Bethe ansatz and the chiral Gross-Neveu Lagrangian field theory. We review known results and derive some further formulae which we need in the following. In section 3 we construct the general form factor formula and present some examples in detail, such as the form factors of the energy-momentum tensor, the Dirac field and the $SU(N)$ current. In section 4 we compare our exact results against $1/N$ perturbation theory of the $SU(N)$ Gross-Neveu model. In section 5 we present the commutation rules of the fields. Our conclusions are stated in section 6. In appendix A we provide the general proof of the bound state form factor formula and in appendix B the commutation rule of two fields (in general anyonic) is proved.

2   General setting

The particle spectrum of the chiral $SU(N)$ Gross-Neveu model consists of $N-1$ multiplets of particles of mass $m_r = m_1 \sin\left(r\pi/N\right)/\sin\left(\pi/N\right)$, which correspond to all fundamental $SU(N)$ representations of rank $r = 1, \ldots, N-1$ with representation spaces $V^{(r)}$ of dimension $\binom{N}{r}$. Let $(\alpha) = (\alpha_1, \ldots, \alpha_r)$, $(1 \le \alpha_1 < \cdots < \alpha_r \le N)$ be a particle of rank r. We write

$$(\alpha) \in V = \bigoplus_{r=1}^{N-1} V^{(r)}, \ V^{(r)} \simeq \mathbb{C}^{\binom{N}{r}}$$

where the (α) form a basis of V. A particle of rank r is a bound state of r particles of rank 1. The antiparticle corresponding to (α) is $(\bar\alpha) = (\bar\alpha_1, \ldots, \bar\alpha_{N-r})$, $(1 \le \bar\alpha_1 < \cdots < \bar\alpha_{N-r} \le N)$ (of rank $N-r$) such that the union of the set of indices satisfies $\{\alpha_1, \ldots, \alpha_r\} \cup \{\bar\alpha_1, \ldots, \bar\alpha_{N-r}\} = \{1, \ldots, N\}$.

2.1   The S-matrix

The S-matrix for the scattering of two particles α, β (of rank 1) [19, 20, 6, 8] is

$$S_{\alpha\beta}^{\delta\gamma}(\theta) = \delta_\alpha^\gamma \delta_\beta^\delta b(\theta) + \delta_\alpha^\delta \delta_\beta^\gamma c(\theta) \tag{2}$$

where $\theta = \theta_1 - \theta_2$ is the rapidity difference and $p_{1,2}^\mu = m\,(\cosh\theta_{1,2}, \sinh\theta_{1,2})$. The amplitudes satisfy

$$c(\theta) = -\frac{i\eta}{\theta}b(\theta)\,, \quad \eta = \frac{2\pi}{N}\,,$$

$$a(\theta) = b(\theta) + c(\theta) = -\frac{\Gamma\left(1 - \frac{\theta}{2\pi i}\right)\Gamma\left(1 - \frac{1}{N} + \frac{\theta}{2\pi i}\right)}{\Gamma\left(1 + \frac{\theta}{2\pi i}\right)\Gamma\left(1 - \frac{1}{N} - \frac{\theta}{2\pi i}\right)}\,.$$

We also need the S-matrix for the scattering of a bound state $(\rho) = (\rho_1, \ldots, \rho_{N-1})$, $(1 \le \rho_1 < \cdots < \rho_{N-1} \le N)$ (of rank $N-1$) and a particle α (of rank 1)

$$S_{(\rho)\alpha}^{\beta(\sigma)}(\theta) = (-1)^{N-1}\left(\delta_{(\rho)}^{(\sigma)}\delta_\alpha^\gamma\,b(\pi i - \theta) + \mathbf{C}^{\beta(\sigma)}\mathbf{C}_{(\rho)\alpha}\,c(\pi i - \theta)\right) \tag{3}$$

where the charge conjugation matrices are defined by

$$\mathbf{C}_{(\alpha_1\ldots\alpha_{N-1})\alpha_N} = \mathbf{C}_{\alpha_1(\alpha_2\ldots\alpha_N)} = \epsilon_{\alpha_1\alpha_2\ldots\alpha_N}$$
$$\mathbf{C}^{\alpha_1(\alpha_2\ldots\alpha_N)} = \mathbf{C}^{(\alpha_1\ldots\alpha_{N-1})\alpha_N} = (-1)^{N-1}\epsilon^{\alpha_1\alpha_2\ldots\alpha_N}$$

with $\epsilon_{\alpha_1\ldots\alpha_N}$ and $\epsilon^{\alpha_1\ldots\alpha_N}$ totally anti-symmetric and $\epsilon_{1\ldots N} = \epsilon^{1\ldots N} = 1$. Formula (3) is obtained by applying iteratively the bound state fusion method [21] to (2).

For later convenience we extract the factors $a(\theta)$ and $(-1)^{N-1}b(i\pi - \theta)$, respectively

$$S_{\alpha\beta}^{\delta\gamma}(\theta) = a(\theta)\tilde{S}_{\alpha\beta}^{\delta\gamma}(\theta) \tag{4}$$

$$S_{(\rho)\alpha}^{\beta(\sigma)}(\theta) = (-1)^{N-1}b(i\pi - \theta)\tilde{S}_{(\rho)\alpha}^{\beta(\sigma)}(\theta) \tag{5}$$

such that

$$\tilde{S}_{\alpha\beta}^{\delta\gamma}(\theta) = \delta_\alpha^\gamma\delta_\beta^\delta\tilde{b}(\theta) + \delta_\alpha^\delta\delta_\beta^\gamma\tilde{c}(\theta) \tag{6}$$

$$\tilde{S}_{(\rho)\alpha}^{\beta(\sigma)}(\omega) = \delta_{(\rho)}^{(\sigma)}\delta_\alpha^\beta + \mathbf{C}^{\beta(\sigma)}\mathbf{C}_{(\rho)\alpha}\tilde{d}(\omega) \tag{7}$$

$$\tilde{b}(\theta) = \frac{\theta}{\theta - i\eta}\,, \quad \tilde{c}(\theta) = \frac{-i\eta}{\theta - i\eta}\,, \quad \tilde{d}(\omega) = \frac{c(i\pi - \omega)}{b(i\pi - \omega)} = \frac{-i\eta}{i\pi - \omega}\,, \quad \eta = \frac{2\pi}{N}\,.$$

where $\delta_{(\rho)}^{(\sigma)} = \delta_{\rho_1}^{\sigma_1}\delta_{\rho_2}^{\sigma_2}\ldots\delta_{\rho_{N-1}}^{\sigma_{N-1}}$. Below we will also use for the matrices (6) and (7) the notations $\tilde{S}_{12}(\theta)$ and $\tilde{S}_{\bar{1}2}(\theta)$, respectively.

2.2 Nested "off-shell" Bethe ansatz

The "off-shell" Bethe ansatz is used to construct vector valued functions which have symmetry properties according to a representation of the permutation group generated by a factorizing S-matrix. In addition they satisfy matrix differential [22] or difference [23] equations. For the application to form factors we use the co-vector version $K_{1\ldots n}(\underline{\theta}) \in V_{1\ldots n} = \left(\bigotimes_{i=1}^n V\right)^\dagger$, $(\theta_i \in \mathbb{C},\ i = 1, \ldots, n)$

$$K_{\ldots ij\ldots}(\ldots, \theta_i, \theta_j, \ldots) = K_{\ldots ji\ldots}(\ldots, \theta_j, \theta_i, \ldots)\,\tilde{S}_{ij}(\theta_{ij})$$
$$K_{1\ldots n}(\underline{\theta}') = K_{1\ldots n}(\underline{\theta})Q_{1\ldots n}(\underline{\theta}, i)$$

where $\underline{\theta}' = (\theta_1, \ldots, \theta_i + 2\pi i, \ldots, \theta_n)$ (see below and e.g. [23, 13]). We write the components of the co-vector $K_{1\ldots n}$ as $K_{\underline{\alpha}}$ where $\underline{\alpha} = ((\alpha_{11}, \ldots, \alpha_{1r_1}), \ldots, (\alpha_{n1}, \ldots, \alpha_{nr_n}))$ is a state of n particles of rank $r_1, \ldots, r_n$.

The nested $SU(N)$ "off-shell" Bethe ansatz for particles of rank 1 has been constructed in [13]. Here we need a more general case.

Nested "off-shell" Bethe ansatz for particles of rank 1 **and** $N-1$ **:** We consider a state with n particles of rank 1 and $\bar{n}$ particles of rank $N-1$ and write the off-shell Bethe ansatz co-vector valued function as

$$\boxed{K_{\underline{\alpha}(\rho)}(\underline{\theta}, \underline{\omega}) = \int_{\mathcal{C}_{\underline{\theta}\underline{\omega}}} dz_1 \cdots \int_{\mathcal{C}_{\underline{\theta}\underline{\omega}}} dz_m \, k(\underline{\theta}, \underline{\omega}, \underline{z}) \, \tilde{\Psi}_{\underline{\alpha}(\rho)}(\underline{\theta}, \underline{\omega}, \underline{z})} \tag{8}$$

where $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $(\underline{\rho}) = ((\rho_1), \ldots, (\rho_{\bar{n}})) = ((\rho_{11}, \ldots, \rho_{1N-1}), \ldots, (\rho_{\bar{n}1}, \ldots, \rho_{\bar{n}N-1}))$, $\underline{\theta} = (\theta_1, \ldots, \theta_n)$, $\underline{\omega} = (\omega_1, \ldots, \omega_{\bar{n}})$ and $\underline{z} = (z_1, \ldots, z_m)$. This ansatz transforms the complicated matrix equations to simple equations for the scalar function $k(\underline{\theta}, \underline{\omega}, \underline{z})$ (see [13] and below). The integration contour $\mathcal{C}_{\underline{\theta}\underline{\omega}}$ can in general be characterized as follows: there is a finite number of complex numbers $a_i(\underline{\theta})$, $b_j(\underline{\theta})$ such that the positions of all poles of the integrand are of the form

$$\begin{aligned} &(1) : a_i(\underline{\theta}) + 2\pi i k, \quad k \in \mathbb{N} \\ &(2) : b_j(\underline{\theta}) - 2\pi i l, \quad l \in \mathbb{N} \end{aligned} \tag{9}$$

and $\mathcal{C}_{\underline{\theta}\underline{\omega}}$ runs from $-\infty$ to $+\infty$ such that all poles (1) are above and all poles (2) are below the contour. This contour is just the same as the one used for the definition of Meijer's G-function. It will turn out that for the examples considered below the form factors can be expressed in terms of Meijer's G-functions.

The state $\tilde{\Psi}_{\underline{\alpha}(\rho)}$ in (8) is a linear combination of the basic Bethe ansatz co-vectors

$$\tilde{\Psi}_{\underline{\alpha}(\rho)}(\underline{\theta}, \underline{\omega}, \underline{z}) = L_{\underline{\beta}(\sigma)}(\underline{z}, \underline{\omega}) \tilde{\Phi}_{\underline{\alpha}(\rho)}^{\underline{\beta}(\sigma)}(\underline{\theta}, \underline{\omega}, \underline{z}), \quad \text{with } 1 < \beta_i, \ \sigma_{1j} = 1. \tag{10}$$

As usual in the context of the algebraic Bethe ansatz [24, 25] the basic Bethe ansatz co-vectors are obtained from the monodromy matrix

$$\tilde{T}_{1\ldots n, \bar{1}\ldots\bar{n}, 0}(\underline{\theta}, \underline{\omega}, \theta_0) = \tilde{S}_{10}(\theta_1 - \theta_0) \cdots \tilde{S}_{n0}(\theta_n - \theta_0) \tilde{S}_{\bar{1}0}(\omega_1 - \theta_0) \cdots \tilde{S}_{\bar{n}0}(\omega_{\bar{n}} - \theta_0)$$

$$\equiv \begin{pmatrix} \tilde{A}_{1\ldots n, \bar{1}\ldots\bar{n}}(\underline{\theta}, \underline{\omega}, z) & \tilde{B}_{1\ldots n, \bar{1}\ldots\bar{n}, \beta}(\underline{\theta}, \underline{\omega}, z) \\ \tilde{C}_{1\ldots n, \bar{1}\ldots\bar{n}}^{\beta}(\underline{\theta}, \underline{\omega}, z) & \tilde{D}_{1\ldots n, \bar{1}\ldots\bar{n}, \beta}^{\beta'}(\underline{\theta}, \underline{\omega}, z) \end{pmatrix}, \quad 2 \le \beta, \beta' \le N.$$

where the S-matrices $\tilde{S}_{i0}$ and $\tilde{S}_{\bar{i}0}$ are given by (6) and (7). As usual the Yang-Baxter algebra relation for the S-matrix yields the typical TTS-relation which implies the basic algebraic properties of the sub-matrices A, B, C, D.

Here not only one reference co-vector exists. The space of reference co-vectors, defined as usual by

$$\Omega^{(\sigma)} \tilde{B}_\beta = 0,$$

is $(N-1)^{\bar{n}}$ dimensional and is spanned by the co-vectors for all $\underline{(\sigma)} = ((\sigma_{11}, \ldots, \sigma_{1N-1}),$
$\ldots, (\sigma_{\bar{n}1}, \ldots, \sigma_{\bar{n}N-1}))$ with $\sigma_{i1} = 1 < \sigma_{i2} < \cdots < \sigma_{iN-1} \leq N$. They are eigenstates of $\tilde{A}$
and $\tilde{D}_{\beta}^{\beta'}$

$$\Omega^{\underline{(\sigma)}} \tilde{A}(\underline{\theta}, \underline{\omega}, z) = \Omega^{\underline{(\sigma)}}$$
$$\Omega^{\underline{(\sigma)}} \tilde{D}_{\beta}^{\beta'}(\underline{\theta}, \underline{\omega}, z) = \delta_{\beta}^{\beta'} \prod_{i=1}^{n} \tilde{b}(\theta_i - z) \Omega^{\underline{(\sigma)}}.$$

where the indices $1 \ldots n, \bar{1} \ldots \bar{n}$ are suppressed. The basic Bethe ansatz co-vectors in (10)
are defined as

$$\tilde{\Phi}_{\underline{\alpha(\rho)}}^{\underline{\beta(\sigma)}}(\underline{\theta}, \underline{\omega}, \underline{z}) = \left(\Omega^{\underline{(\sigma)}} \tilde{C}^{\beta_m}(\underline{\theta}, \underline{\omega}, z_m) \cdots \tilde{C}^{\beta_1}(\underline{\theta}, \underline{\omega}, z_1) \right)_{\underline{\alpha(\rho)}} \tag{11}$$

where $1 < \beta_i \leq N$.

The technique of the **'nested Bethe ansatz'** means that for the coefficients $L_{\underline{\beta(\sigma)}}(\underline{z}, \underline{\omega})$
in (10) one makes the analogous construction as for $K_{\underline{\alpha(\rho)}}(\underline{\theta}, \underline{\omega})$ where now the indices
$\underline{\beta}, \underline{(\sigma)}$ take only the values $2 \leq \beta_i \leq N$ and $\sigma_{i1} = 1 < \sigma_{i2} < \cdots < \sigma_{iN-1} \leq N$. This
nesting is repeated until the space of the coefficients becomes one dimensional. It is well
known (see [23]) that the 'off-shell' Bethe ansatz states are highest weight states if they
satisfy a certain matrix difference equation. If there are only n particles of rank 1, then
the $SU(N)$ weights are

$$w = (n - n_1, n_1 - n_2, \ldots, n_{N-2} - n_{N-1}, n_{N-1}) \tag{12}$$

where $n_1 = m, n_2, \ldots$ are the numbers of C operators in the various levels of the nesting.
If in addition there are $\bar{n}$ particles of rank $N-1$ the $SU(N)$ weights are

$$w = (n - n_1, n_1 - n_2, \ldots, n_{N-2} - n_{N-1}, n_{N-1} - \bar{n}) + \bar{n}(1, \ldots, 1) \tag{13}$$

because $N-1$ particles of rank 1 yield a bound state of rank $N-1$ and at the lth level
the number of C operators is reduced by $N - l - 1$ (see appendix A).

2.3 Minimal form factors and ϕ-function

To construct the form factors we need the form factor functions $F(\theta)$, $G(\theta)$ and the
function $\phi(\theta)$. The form factor functions $F(\theta)$ and $G(\theta)$ for two particles of rank 1 and
for one particle of rank 1 and one of rank $N-1$, respectively are

$$F(\theta) = c \exp \int_0^\infty \frac{dt}{t \sinh^2 t} e^{\frac{t}{N}} \sinh t \left(1 - \frac{1}{N}\right) \left(1 - \cosh t \left(1 - \frac{\theta}{i\pi}\right)\right) \tag{14}$$

$$G(\theta) = c' \exp \int_0^\infty \frac{dt}{t \sinh^2 t} e^{\frac{t}{N}} \sinh \frac{t}{N} \left(1 - \cosh t \left(1 - \frac{\theta}{i\pi}\right)\right). \tag{15}$$

They are the minimal solutions of the equations

$$F(\theta) = F(-\theta)a(\theta)\,, \quad F(i\pi - \theta) = F(i\pi + \theta)$$
$$G(\theta) = -G(-\theta)b(\pi i - \theta)\,, \quad G(i\pi - \theta) = G(i\pi + \theta)$$

where $a(\theta)$ and $b(\pi i - \theta)$ are the highest weight amplitudes of the corresponding channels of (2) and (3). The ϕ-function

$$\tilde{\phi}(\theta) = \frac{1}{F(-\theta)G(i\pi + \theta)} = \Gamma\left(\frac{-\theta}{2\pi i}\right)\Gamma\left(1 - \frac{1}{N} + \frac{\theta}{2\pi i}\right) \tag{16}$$

is a solution of

$$\prod_{k=0}^{N-2}\tilde{\phi}\left(-\theta - ki\eta\right)\prod_{k=0}^{N-1}F\left(\theta + ki\eta\right) = 1 \tag{17}$$

which follows from the assumption that the antiparticle of a fundamental particle is a bound state of $N - 1$ of them (see below and [13]). The constants c and c' in (14) and (15) follow from (16) and (17).

2.4 Chiral Gross-Neveu model

Swieca et al. [8] wrote the fermion fields $\psi_i(x)$ in the Lagrangian (1) in bosonic form. In order to extract the real particle content of the theory, they introduced in addition the "physical" fields

$$\hat{\psi}_i(x) = \mathcal{K}_i\left(\frac{m}{2\pi}\right)^{1/2} e^{(\pi/4)\gamma^5}\exp\left\{-i\sqrt{\pi}\left(\gamma^5\phi_i(x) + \int_x^{\infty} dy^1 \dot{\phi}_i(y)\right)\right\}$$
$$\phi_i(x) = \left(1 - \frac{1}{N}\right)\varphi_i(x) - \frac{1}{N}\sum_{j\neq i}\varphi_j(x)$$

where $\varphi_i(x)$ are free canonical zero-mass fields and $\mathcal{K}_i$ are Klein factors satisfying

$$\mathcal{K}_i\hat{\psi}_j(x) = \left\{\begin{array}{ll} \hat{\psi}_j(x)\mathcal{K}_i & \text{for} \quad i = j \\ -\hat{\psi}_j(x)\mathcal{K}_i & \text{for} \quad i \neq j\,. \end{array}\right.$$

The fields $\hat{\psi}$ satisfy (with a suitable normal product prescription $\mathcal{N}$)

$$\hat{\psi}_i^{\dagger} = \mathcal{K}\frac{1}{(N-1)!}\sum_{\underline{j}}\epsilon_{ij_1\ldots j_{N-1}}\mathcal{N}\hat{\psi}_{j_1}\ldots\hat{\psi}_{j_{N-1}}\,, \quad \mathcal{K} = \prod_{j=1}^{N}\mathcal{K}_j \tag{18}$$

such that

$$\mathcal{K}\hat{\psi}_j(x) = (-1)^{N-1}\hat{\psi}_j(x)\mathcal{K}\,. \tag{19}$$

Equation (18) means that antiparticles should be identified with bound state of $N-1$ particles and the creation operators of the antiparticle $\hat{b}_\alpha^\dagger$ and of the bound state $\hat{a}_{(\varrho)}^\dagger$ are related by

$$\hat{b}_\alpha^\dagger = \mathcal{K}\epsilon_{\alpha(\varrho)}\hat{a}_{(\varrho)}^\dagger$$
$$\mathcal{K}\hat{a}_\alpha^\dagger\mathcal{K} = (-1)^{N-1}\hat{a}_\alpha^\dagger .$$

The "physical" fields satisfy the anyonic commutation relations

$$\hat{\psi}_i(x)\hat{\psi}_i(y) = \hat{\psi}_i(y)\hat{\psi}_i(x)e^{2\pi is\epsilon(x^1-y^1)}, \text{ for } (x-y)^2 < 0$$

with "spin" $s = \frac{1}{2}(1-1/N)$. This implies that the "physical" S-matrix is related to the one of (2) by [8, 26]

$$S_{\alpha\beta}^{\delta\gamma}(\theta_{12}) = e^{2\pi is\epsilon(\theta_{12})}\hat{S}_{\alpha\beta}^{\delta\gamma}(\theta_{12}) .$$

As a consequence the abnormal crossing relation (3) transforms to a normal one. The bound state S-matrix satisfies

$$S_{(\rho)\beta}^{\delta(\sigma)}(\theta) = (-1)^{N-1}\mathbf{C}_{(\rho)\gamma}S_{\beta\alpha}^{\gamma\delta}(\pi i - \theta)\mathbf{C}^{\alpha(\sigma)}. \tag{20}$$

Therefore the physical crossing relation is

$$\hat{S}_{\bar{\alpha}\beta}^{\delta\bar{\gamma}}(\theta) = \hat{\mathbf{C}}_{\bar{\alpha}\alpha'}\hat{S}_{\beta\gamma'}^{\alpha'\delta}(\pi i - \theta)\hat{\mathbf{C}}^{\gamma'\bar{\gamma}}$$

with $\hat{\mathbf{C}}_{\bar{\alpha}\alpha'} = \delta_{\alpha\alpha'}$, $\hat{\mathbf{C}}^{\gamma'\bar{\gamma}} = \delta^{\gamma'\gamma}$.

3 Form factors

For a state of n particles of rank $r_1, \ldots, r_n$ with rapidities $\underline{\theta} = (\theta_1, \ldots, \theta_n)$ and a local operator $\mathcal{O}(x)$ we define the associated form factor functions $F_{\underline{\alpha}}^{\mathcal{O}}(\underline{\theta})$ by

$$\langle 0 | \mathcal{O}(x) | \theta_1, \ldots, \theta_n \rangle_{\underline{\alpha}}^{in} = e^{-ix(p_1+\cdots+p_n)}F_{\underline{\alpha}}^{\mathcal{O}}(\underline{\theta}) , \quad \text{for } \theta_1 > \cdots > \theta_n.$$

where again $\underline{\alpha} = ((\alpha_{11}, \ldots, \alpha_{1r_1}), \ldots, (\alpha_{n1}, \ldots, \alpha_{nr_n}))$. For all other arrangements of the rapidities the functions $F_{\underline{\alpha}}^{\mathcal{O}}(\underline{\theta})$ are given by analytic continuation. The co-vector valued function $F_{\underline{\alpha}}^{\mathcal{O}}(\underline{\theta})$ satisfies the form factor equations (i) – (v) (see [12, 14, 27, 28, 13]) and can be written as [12]

$$F_{\underline{\alpha}}^{\mathcal{O}}(\underline{\theta}) = K_{\underline{\alpha}}^{\mathcal{O}}(\underline{\theta}) \prod_{1 \le i < j \le n} F_{r_i r_j}(\theta_{ij}) \tag{21}$$

where $F_{r_i r_j}(\theta)$ are the minimal form factor functions. For particles of rank 1 and $N-1$ they are given by $F_{11}(\theta) = F_{N-1N-1}(\theta) = F(\theta)$ and $F_{N-11}(\theta) = F_{1N-1}(\theta) = G(\theta)$ of (14) and (15), respectively. In [13] the form factors of the fundamental particles of rank 1 have been constructed. We shortly recall the results. All other form factors can be obtained from these by applying the bound state fusion procedure which is given by the form factor equation (iv) (see e.g. [13]).

Form factors for particles of rank 1: The K-function in (21) is given by the nested "off-shell" Bethe ansatz (8) for the special case $\bar{n} = 0$ and a special choice of the scalar function $k(\underline{\theta}, \underline{z})$ such that the form factor equations (i) – (v) are satisfied

$$
K_{\underline{\alpha}}^{\mathcal{O}}(\underline{\theta}) = \frac{N_n}{m!} \int_{\mathcal{C}_{\underline{\theta}}} dz_1 \cdots \int_{\mathcal{C}_{\underline{\theta}}} dz_m \, \tilde{h}(\underline{\theta}, \underline{z}) \, p^{\mathcal{O}}(\underline{\theta}, \underline{z}) \, \tilde{\Psi}_{\underline{\alpha}}(\underline{\theta}, \underline{z}) \tag{22}
$$

with

$$
\tilde{h}(\underline{\theta}, \underline{z}) = \prod_{i=1}^{n} \prod_{j=1}^{m} \tilde{\phi}(\theta_i - z_j) \prod_{1 \le i < j \le m} \tau(z_i - z_j) \tag{23}
$$

$$
\tau(z) = \frac{1}{\tilde{\phi}(z)\tilde{\phi}(-z)} \, .
$$

The integration contour $\mathcal{C}_{\underline{\theta}}$ is defined by (9). The dependence on the operator $\mathcal{O}$ enters only through the p-function $p^{\mathcal{O}}(\underline{\theta}, \underline{z})$ which has to satisfy simple equations (see [29, 30, 31, 13]). The K-function is in general a linear combination of the *fundamental building blocks* [29, 30, 31] given by (22). Here we consider only these cases where the sum consists only of one term.

The p-function: The co-vector valued function $\tilde{\Psi}_{\underline{\alpha}}(\underline{\theta}, \underline{z})$ is expressed as in (10) for $\bar{n} = 0$ by an $L_{\underline{\beta}}(\underline{z})$ for which the nesting procedure is applied. The final form is (up to a constant)

$$
F_{\underline{\alpha}}^{\mathcal{O}}(\underline{\theta}) = \prod F(\theta_{ij}) \int d\underline{z}^{(1)} \dots \int d\underline{z}^{(N-1)} \tilde{h}(\underline{\theta}, \underline{\underline{z}}) \, p^{\mathcal{O}}(\underline{\theta}, \underline{\underline{z}}) \, \tilde{\Phi}_{\underline{\alpha}}(\underline{\theta}, \underline{\underline{z}}) \tag{24}
$$

$$
\tilde{h}(\underline{\theta}, \underline{\underline{z}}) = \tilde{h}(\underline{\theta}, \underline{z}^{(1)}) \cdots \tilde{h}(\underline{z}^{(N-2)}, \underline{z}^{(N-1)}) \, .
$$

where $\underline{\underline{z}} = \underline{z}^{(1)}, \dots, \underline{z}^{(N-1)}$. The Bethe state $\tilde{\Phi}_{\underline{\alpha}}(\underline{\theta}, \underline{\underline{z}})$ is obtained by the nesting procedure (see (10), (11) and [13])

$$
\tilde{\Phi}_{\underline{\alpha}}(\underline{\theta}, \underline{\underline{z}}) = \tilde{\Phi}_{\underline{\varsigma}}^{(N-1)}(\underline{z}^{(N-2)}, \underline{z}^{(N-1)}) \dots \tilde{\Phi}_{\underline{\beta}}^{(2),\gamma}(\underline{z}^{(1)}, \underline{z}^{(2)}) \tilde{\Phi}_{\underline{\underline{\alpha}}}^{\underline{\beta}}(\underline{\theta}, \underline{z}^{(1)}) \, .
$$

In general the p-function (see [13]) depends on the rapidities $\underline{\theta}$ and all integration variables $\underline{z}^{(l)}$. Let the operator $\mathcal{O}(x)$ transform as a highest weight $SU(N)$ representation with highest weight vector

$$
w^{\mathcal{O}} = \left(w_1^{\mathcal{O}}, \dots, w_N^{\mathcal{O}} \right) \, .
$$

Because of $SU(N)$ invariance the weight vector of the co-vector $F_{\underline{\alpha}}^{\mathcal{O}}(\underline{\theta})$ is then

$$
\begin{aligned}
w &= \left(w_1^{\mathcal{O}}, \dots, w_N^{\mathcal{O}} \right) + L \left(1, \dots, 1 \right) \\
&= \left(n - n_1, n_1 - n_2, \dots, n_{N-2} - n_{N-1}, n_{N-1} \right)
\end{aligned} \tag{25}
$$

where (12) and the fact, that the weight vector $(1, \dots, 1)$ correspond to the vacuum sector, has been used. In [13] was shown that the p-function has to satisfy a set of equations

in order that the form factor (24) satisfies the form factor equations. In particular to guarantee the transformation properties of the operator the following periodicity relations have to be valid

$$
\begin{aligned}
p^{\mathcal{O}}(\underline{\theta}, \ldots, \underline{z}^{(l)}, \ldots) &= \tilde{\sigma}_1^{\mathcal{O}} p^{\mathcal{O}}(\ldots, \theta_i + 2\pi i, \ldots, \underline{z}^{(l)}, \ldots) \\
&= (-1)^{w_l^{\mathcal{O}} + w_{l+1}^{\mathcal{O}}} p^{\mathcal{O}}(\underline{\theta}, \ldots, z_i^{(l)} + 2\pi i, \ldots)
\end{aligned}
\tag{26}
$$

where

$$
\begin{aligned}
\tilde{\sigma}_1^{\mathcal{O}} &= \sigma_1^{\mathcal{O}} (-1)^{(N-1)\left[\sum_{i=1}^N w_i^{\mathcal{O}}/N\right] - \sum_{i=2}^N w_i^{\mathcal{O}}} \\
\sigma_1^{\mathcal{O}} &= e^{i\pi(1-1/N)Q^{\mathcal{O}}} .
\end{aligned}
\tag{27}
$$

The charge of the operator $\mathcal{O}$ is defined by $Q^{\mathcal{O}} = n \bmod N$ and $\sigma_1^{\mathcal{O}}$ is the statistics factor of $\mathcal{O}$ with respect to the fundamental particle of rank 1. The sign factors $\tilde{\sigma}_1^{\mathcal{O}}/\sigma_1^{\mathcal{O}} = \pm 1$ and $(-1)^{w_l^{\mathcal{O}} + w_{l+1}^{\mathcal{O}}} = \pm 1$ in (26) follow [13] from the sign $(-1)^{(N-1)}$ in the unusual crossing relation (20) (related to the Klein factors of (19)).

3.1 General form factors of particles of rank 1 and $N-1$:

Using the bound state procedure (see appendix A) which means taking residues of (21) or (22) one derives the form factors and K-functions for n particles of rank 1 with rapidities $\underline{\theta}$ and $\bar{n}$ particles of rank $N-1$ with rapidities $\underline{\omega}$. As usual we split off the minimal part

$$
F_{\underline{\alpha}(\rho)}^{\mathcal{O}}(\underline{\theta}, \underline{\omega}) = K_{\underline{\alpha}(\rho)}(\underline{\theta}, \underline{\omega}) \prod_{1 \le i < j \le n} F(\theta_{ij}) \prod_{i=1}^{n} \prod_{j=1}^{\bar{n}} G(\theta_i - \omega_j) \prod_{1 \le i < j \le \bar{n}} F(\omega_{ij}) .
\tag{28}
$$

The K-function is given by a nested 'off-shell' Bethe ansatz (8)

$$
\boxed{
K_{\underline{\alpha}(\rho)}^{\mathcal{O}}(\underline{\theta}, \underline{\omega}) = \frac{N_{n\bar{n}}}{m!} \int_{\mathcal{C}_{\underline{\theta}\underline{\omega}}} dz_1 \cdots \int_{\mathcal{C}_{\underline{\theta}\underline{\omega}}} dz_m \, \tilde{h}(\underline{\theta}, \underline{z}) \, p^{\mathcal{O}}(\underline{\theta}, \underline{\omega}, \underline{z}) \, \tilde{\Psi}_{\underline{\alpha}(\rho)}(\underline{\theta}, \underline{\omega}, \underline{z})
}
\tag{29}
$$

where $\tilde{h}(\underline{\theta}, \underline{z})$ is the scalar function (23). Note that this h-function does not depend on $\underline{\omega}$. For the $SU(N)$ S-matrix the function $\tilde{\phi}(\theta)$ is given by (16). The integration contour $\mathcal{C}_{\underline{\theta}\underline{\omega}}$ (see Fig. 1) has been defined in the context of (8).

Nesting: The state $\tilde{\Psi}_{\underline{\alpha}(\rho)}$ in (29) is a linear combination of the basic Bethe ansatz co-vectors (11)

$$
\tilde{\Psi}_{\underline{\alpha}(\rho)}(\underline{\theta}, \underline{\omega}, \underline{z}) = L_{\underline{\beta}(\sigma)}(\underline{z}, \underline{\omega}) \tilde{\Phi}_{\underline{\alpha}(\rho)}^{\underline{\beta}(\sigma)}(\underline{\theta}, \underline{\omega}, \underline{z}), \quad \text{with } 1 < \beta_i, \ \sigma_{1j} = 1
$$

where $L_{\underline{\beta}(\sigma)}(\underline{z}, \underline{\omega})$ satisfies again a representation like (29). This nesting is iterated until all $\beta_i = N$ and all $(\sigma)_i = (1, 2, \ldots, N-1)$. Only for the highest level Bethe ansatz the
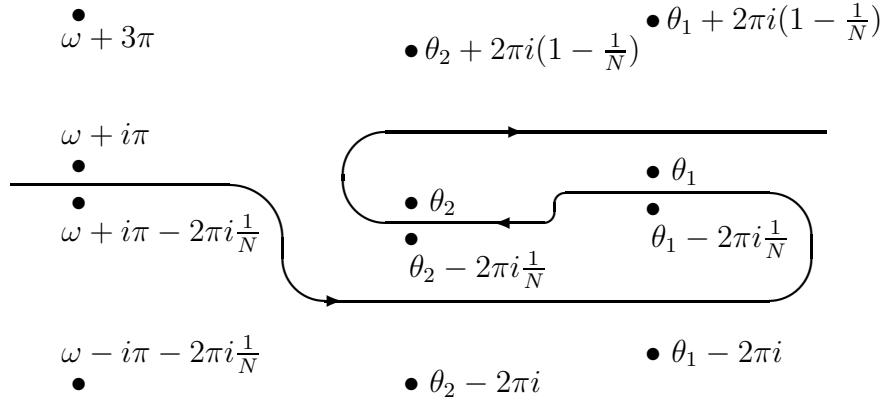
Figure 1: The integration contour $\mathcal{C}_{\theta_1\theta_2\omega}$ for two particles and one bound state.

h-function depends on $\underline{\omega}$. The final result is

$$K^{\mathcal{O}}_{\underline{\alpha}(\rho)}(\underline{\theta},\omega) = \int d\underline{z}^{(1)} \ldots \int d\underline{z}^{(N-1)} \tilde{h}\,(\underline{\theta},\underline{\omega},\underline{z}) p^{\mathcal{O}}(\underline{\theta},\underline{\omega},\underline{z}) \tilde{\Phi}_{\underline{\alpha}(\rho)}(\underline{\theta},\underline{\omega},\underline{z}) \tag{30}$$

$$\tilde{h}\,(\underline{\theta},\underline{\omega},\underline{z}) = \prod_{l=0}^{N-2} \tilde{h}(\underline{z}^{(l)},\underline{z}^{(l+1)}) \prod_{i=1}^{\bar{n}} \prod_{j=1}^{n_{N-1}} \tilde{\chi}(\omega_i - z_j^{(N-1)})$$

$$\tilde{\chi}(\omega) = \Gamma\left(\frac{1}{2} + \frac{\omega}{2\pi i}\right) \Gamma\left(\frac{1}{2} - \frac{1}{N} - \frac{\omega}{2\pi i}\right).$$

The complete Bethe ansatz state is

$$\tilde{\Phi}_{\underline{\alpha}(\rho)}(\underline{\theta},\underline{\omega},\underline{z}) = \tilde{\Phi}^{(N-2)(\eta)}_{\underline{\varsigma}(\lambda)}(\underline{z}^{(N-2)},\underline{\omega},\underline{z}^{(N-1)}) \ldots \tilde{\Phi}^{(1)\gamma(\kappa)}_{\underline{\beta}(\sigma)}(\underline{z}^{(1)},\underline{\omega},\underline{z}^{(2)}) \tilde{\Phi}^{\beta(\sigma)}_{\underline{\alpha}(\rho)}(\underline{\theta},\underline{\omega},\underline{z}^{(1)})$$

where (η) denotes $\bar{n}$ highest weight bound states $(\eta_{i1},\ldots,\eta_{iN-1}) = (1,2,\ldots,N-1)$. The p-functions in (29) and (30) are obtained again by the bound state procedure from a solution of (26) for $\bar{n} = 0$. In particular for $\bar{n} = 1$ (with the replacements in (26) $\underline{\theta} \to \underline{\theta},\underline{\varphi}$ and $\underline{z}^{(l)} \to \underline{z}^{(l)},\underline{y}^{(l)}$ where $\underline{\varphi} = (\varphi_1,\ldots,\varphi_{N-1})$, $\underline{y}^{(l)} = (y_1,\ldots,y_{N-1-l})$, $l = 1,\ldots,N-2$)

$$p^{\mathcal{O}}(\underline{\theta},\omega,\underline{z}) = p^{\mathcal{O}}(\underline{\theta},\underline{\varphi},\underline{z}^{(1)},\underline{y}^{(1)},\ldots,\underline{z}^{(N-1)},\underline{y}^{(N-1)}).$$

Here $y_i^{(l)} = \varphi_i^{(l)}$, $i = 1,\ldots,N-1-l$ and $\varphi_k = \omega + ki\eta - i\pi$. The proofs of the statements of this subsection and more details can be found in appendix A.

3.2   Examples

To illustrate our general results we present some simple examples. In addition, we also derive the $1/N$ expansion of exact form factors for the purpose of later comparison with the $1/N$-perturbation theory of the chiral $SU(N)$ Gross-Neveu model.

The energy momentum tensor: For the local operator $\mathcal{O}(x) = T^{\rho\sigma}(x)$ (where $\rho, \sigma = \pm$ denote the light cone components) the p-function for n particles of rank 1 (as for the sine-Gordon model in [28])

$$p^{T^{\rho\sigma}}(\underline{\theta}, \underline{z}) = \sum_{i=1}^{n} e^{\rho\theta_i} \sum_{i=1}^{m} e^{\sigma z_i} \tag{31}$$

satisfies the equations (26) with $w^T = (0, 0, \dots, 0)$. For the $n = N$ particle form factor the weight vector is $w = (1, 1, \dots, 1, 1)$. Due to (12) there are $n_l = N - l$ integrations in the l-th level of the off-shell Bethe ansatz.

   We calculate the form factor of the particle α and the bound state $(\lambda) = (\lambda_1, \dots, \lambda_{N-1})$ of $N-1$ particles. We apply the bound state formulae (28) and (29) for $n = \bar{n} = 1$. Due to (13) there is just one integration in every level of the nested Bethe ansatz ($l = 1, \dots, N-1$)

$$F^{T^{\rho\sigma}}_{\alpha(\lambda)}(\theta, \omega) = K^{T^{\rho\sigma}}_{\alpha(\lambda)}(\theta, \omega)\, G(\theta - \omega)$$

$$K^{T^{\rho\sigma}}_{\alpha(\lambda)}(\theta, \omega) = N_2^{T^{\rho\sigma}} \left(e^{\rho\theta} + e^{\rho\omega} \right) \int_{\mathcal{C}_{\underline{\theta}}} dz \tilde{\phi}(\theta - z) e^{\sigma z} \tilde{\Psi}_{\alpha(\lambda)}(\theta, \omega, z) \tag{32}$$

$$\tilde{\Psi}_{\alpha(\lambda)}(\theta, \omega, z) = L^{(1)}_{\beta(\mu)}(z, \omega) \tilde{\Phi}^{\beta(\mu)}_{\alpha(\lambda)}(\theta, \omega, z)$$

where $G(\theta)$ defined in (15) is the minimal form factor function of two particles of rank 1 and $N - 1$. The integration in every level of the nested Bethe ansatz ($l = N - 2, \dots, 1$) can be solved iteratively

$$L^{(l)}_{\beta(\mu)}(z, \omega) = \epsilon_{\beta(\mu)} L^{(l)}(\omega - z) \ \text{ with } \beta > l, \ (\mu) = (1, 2, \dots, l, *, \dots, *)$$

$$L^{(l)}(\omega - z) = c_l\, \Gamma\left(\frac{1}{2} + \frac{\omega - z}{2\pi i} \right) \Gamma\left(-\frac{1}{2} + \frac{l}{N} - \frac{\omega - z}{2\pi i} \right). \tag{33}$$

The remaining integral in(32) may be performed (see appendix A) with the result[1]

$$\langle 0 | T^{\rho\sigma}(0) | \theta, \omega \rangle^{in}_{\alpha(\lambda)} = 4m_1^2 \epsilon_{\alpha(\lambda)} e^{\frac{1}{2}(\rho+\sigma)(\theta+\omega+i\pi)} \frac{\sinh\frac{1}{2}(\theta - \omega - i\pi)}{\theta - \omega - i\pi} G(\theta - \omega). \tag{34}$$

Similar as in [28] one can prove the eigenvalue equation

$$\left(\int dx T^{\pm 0}(x) - \sum_{i=1}^{n} p_i^{\pm} \right) | \theta_1, \dots, \theta_n \rangle^{in}_{\underline{\alpha}} = 0$$

for arbitrary states.

---

[1]In [32, 23] this result has been obtained using Jackson type integrals.

The fields $\psi_\alpha(x)$: Because the Bethe ansatz yields highest weight states we obtain the matrix elements of the spinor field $\psi(x) = \psi_1(x)$. The p-function for the local operator $\psi^{(\pm)}(x)$ for n particles of rank 1 (see also [27])

$$p^{\psi^{(\pm)}}(\underline{\theta}, \underline{z}) = \exp \pm \frac{1}{2} \left(\sum_{i=1}^{m} z_i - \left(1 - \frac{1}{N}\right) \sum_{i=1}^{n} \theta_i \right).$$

satisfies the equations (26) with $w^\psi = (1, 0, \ldots, 0)$. For example the 1-particle form factor is

$$\langle 0 | \psi^{(\pm)}(0) | \theta \rangle_\alpha = \delta_{\alpha 1} \, e^{\mp \frac{1}{2}\left(1 - \frac{1}{N}\right)\theta}.$$

The last formula is consistent with the proposal of Swieca et al. [6, 8] that the statistics of the fundamental particles in the chiral $SU(N)$ Gross-Neveu model should be $\sigma = \exp(2\pi i s)$, where $s = \frac{1}{2}\left(1 - \frac{1}{N}\right)$ is the spin (see also (27)). For the $n = N + 1$ particle form factor there are again $n_l = N - l$ integrations in the l-th level of the off-shell Bethe ansatz and the $SU(N)$ weights are $w = (2, 1, \ldots, 1, 1)$. Due to (13) there is again just one integration in every level of the nested Bethe ansatz. Similar as above one obtains the two-particle and one-bound state form factor

$$F_{\alpha\beta(\lambda)}^{\psi^{(\pm)}}(\underline{\theta}) = K_{\alpha\beta(\lambda)}^{\psi^{(\pm)}}(\underline{\theta}) F(\theta_{12}) G(\theta_{13}) G(\theta_{23})$$

$$K_{\alpha\beta(\lambda)}^{\psi^{(\pm)}}(\underline{\theta}) = N^\psi e^{\mp \frac{1}{2}\left(\left(1 - \frac{1}{N}\right)(\theta_1 + \theta_2) + \frac{1}{N}\theta_3\right)} \int_{\mathcal{C}_{\underline{\theta}}} dz \tilde{\phi}(\theta_1 - z) \tilde{\phi}(\theta_2 - z) e^{\pm \frac{1}{2} z} \tilde{\Psi}_{\alpha\beta(\lambda)}(\underline{\theta}, z) \quad (35)$$

$$\tilde{\Psi}_{\alpha\beta(\lambda)}(\underline{\theta}, z) = L_{\gamma(\mu)}(z, \theta_3) \tilde{\Phi}_{\alpha\beta(\lambda)}^{\gamma(\mu)}(\underline{\theta}, z), \quad \text{with } 1 < \gamma, \ \lambda_1 = 1$$

where the function $L_{\gamma(\mu)}(z, \theta_3) = \epsilon_{\gamma(\mu)} L^{(1)}(\theta_3 - z)$ is the same as in (33) above. We were not able to perform this integration, however, the result can be expressed in terms of Meijer's G-functions

$$K_{\alpha\beta(\lambda)}^{\psi_\delta^{(\pm)}}(\underline{\theta}) = \epsilon_{\alpha(\lambda)} \delta_\beta^\delta K_1^{\psi^{(\pm)}}(\underline{\theta}) + \epsilon_{\beta(\lambda)} \delta_\alpha^\delta K_2^{\psi^{(\pm)}}(\underline{\theta})$$

$$K_1^{\psi^{(\pm)}}(\underline{\theta}) = N_1^\psi e^{\mp \frac{1}{2}\left(\left(1 - \frac{1}{N}\right)(\theta_1 + \theta_2) + \frac{1}{N}\theta_3\right)} G_{33}^{33} \left(e^{\pm i\pi} \ \middle| \ \begin{matrix} \frac{\theta_1}{2\pi i} + 1, \frac{\theta_2}{2\pi i} + 1, \frac{\theta_3}{2\pi i} + \frac{3}{2} - \frac{1}{N} \\ \frac{\theta_1}{2\pi i} - \frac{1}{N}, \frac{\theta_2}{2\pi i} - \frac{1}{N} + 1, \frac{\theta_3}{2\pi i} + \frac{1}{2} \end{matrix} \right)$$

and $K_2^{\psi^{(\pm)}}$ is obtained by the form factor equation (i).

1/N expansion of the exact form factor: We consider the connected part of the matrix element

$$^\gamma\langle \theta_3 | \psi_\delta^{(\pm)}(x) | \theta_1, \theta_2 \rangle_{\alpha\beta}^{in,conn.} = \mathbf{C}^{(\lambda)\gamma} F_{\alpha\beta(\lambda)}^{\psi^{(\pm)}}(\theta_1, \theta_2, \theta_3 - i\pi).$$

Instead of the field ψ we consider the operator $\mathcal{O}_\delta = (-i(i\gamma\partial - m)\psi)_\delta$

$$F_{\alpha\beta}^{\mathcal{O}_\delta, \gamma} = F_{(1)}^{\mathcal{O}} \delta_\alpha^\gamma \delta_\beta^\delta - F_{(2)}^{\mathcal{O}} \delta_\beta^\gamma \delta_\alpha^\delta, \quad F_{(2)}^{\mathcal{O}}(\theta_1, \theta_2, \theta_3) = F_{(1)}^{\mathcal{O}}(\theta_2, \theta_1, \theta_3).$$

For $N \to \infty$ we expand the minimal form factors

$$F(\theta) = \frac{-i}{\pi} \sinh \tfrac{1}{2}\theta + O(1/N) , \quad G(\theta) = 1 + O(1/N) ,$$

perform the integration in (35) and obtain (after a lengthy calculation)

$$F^{\mathcal{O}}_{(1)} = -\frac{2mi\pi}{N} \frac{\sinh \theta_{13}}{\theta_{13}} \left(\frac{1}{\cosh \tfrac{1}{2}\theta_{13}} - \gamma^5 \frac{1}{\sinh \tfrac{1}{2}\theta_{13}} \right) u(\theta_2) + O(N^{-2}) . \tag{36}$$

We use the following conventions for the γ-matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \quad \gamma^5 = \gamma^0 \gamma^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and for the spinors

$$u(p) = \sqrt{m} \begin{pmatrix} e^{-\theta/2} \\ e^{\theta/2} \end{pmatrix} , \quad v(p) = \sqrt{m}\, i \begin{pmatrix} e^{-\theta/2} \\ -e^{\theta/2} \end{pmatrix} . \tag{37}$$

In section 4 below we compare this result with the $1/N$-expansion of the chiral $SU(N)$ Gross-Neveu model in terms of Feynman graphs.

The current $J^\mu_{\alpha\beta}(x)$:   The $SU(N)$ current $J^\mu_{\alpha(\rho)}(x)$ transforms as the adjoint representation with the weight vector $w^J = (2, 1, \ldots, 1, 0)$. Again, because the Bethe ansatz yields highest weight states we obtain the matrix elements of the highest weight component

$$J^\mu_{\alpha(\rho)} = \delta_{\alpha 1} \epsilon_{(\rho)N} \epsilon^{\mu\nu} \partial_\nu \varphi$$

where we have introduced the pseudo-potential $\varphi(x)$. We start from

$$F^\varphi_{\underline{\alpha}}(\underline{\theta}) = K^\varphi_{\underline{\alpha}}(\underline{\theta}) \prod F(\theta_{ij})$$

$$K^\varphi_{\underline{\alpha}}(\underline{\theta}) = \int d\underline{z}^{(1)} \ldots \int d\underline{z}^{(N-1)} h(\underline{\theta}, \underline{\underline{z}})\, p^\varphi(\underline{\theta}, \underline{\underline{z}})\, \Psi_{\underline{\alpha}}(\underline{\theta}, \underline{\underline{z}})$$

with $\underline{\underline{z}} = \underline{z}^{(1)}, \ldots, \underline{z}^{(N-1)}$. The proposal for the p-function for n particles of rank 1 (see also [27])

$$p^\varphi(\underline{\theta}, \underline{\underline{z}}) = N^\varphi \left(\sum_{i=1}^{n} \exp \theta_i \right)^{-1} \exp \frac{1}{2} \left(\sum_{i=1}^{n} \theta_i - \sum_{i=1}^{n_1} z_i^{(1)} - \sum_{i=1}^{n_{N-1}} z_i^{(N-1)} \right)$$

satisfies the equations (26) with $w^\varphi = w^J$.

   We calculate the form factor of the particle α and the bound state $(\lambda) = (\lambda_1, \ldots, \lambda_{N-1})$ of $N - 1$ particles with weight vector $w = (2, 1, \ldots, 1, 0)$. We apply the bound state

formulae (28) and (29) for $n = \bar{n} = 1$. Due to (13) there is no integration in each level of the nested Bethe ansatz $(l = 1, \ldots, N - 1)$ and

$$F^{\varphi}_{\alpha(\lambda)}(\theta, \omega) = K^{\varphi}_{\alpha(\lambda)}(\theta, \omega)\, G(\theta - \omega)$$

$$K^{\varphi}_{\alpha(\lambda)}(\theta, \omega) = N_2^{\varphi} \delta_{\alpha 1}\epsilon_{(\lambda)N}\frac{e^{\frac{1}{2}(\theta+\omega)}}{e^{\theta} + e^{\omega}}$$

The form factor for the $SU(N)$ current is therefore

$$F^{J^{\pm}_{\beta(\rho)}}_{\alpha(\lambda)}(\theta, \omega) = \langle\, 0 \,|\, J^{\pm}_{\beta(\rho)}(0) \,|\, \theta, \omega \,\rangle^{in}_{\alpha(\lambda)} = \pm N_2 \delta^{\beta}_{\alpha}\delta^{(\rho)}_{(\lambda)}\left(e^{\pm\theta} + e^{\pm\omega}\right)\frac{e^{\frac{1}{2}(\theta+\omega)}}{e^{\theta} + e^{\omega}}G(\theta - \omega)$$

$$= \delta^{\beta}_{\alpha}\delta^{(\rho)}_{(\lambda)}\, \bar{v}(\omega)\gamma^{\pm}u(\theta)\, G(\theta - \omega)/G(i\pi)\,. \tag{38}$$

Also here we calculate the $1/N$-expansion of the exact form factor for later comparison with the $1/N$-perturbation theory of the chiral $SU(N)$ Gross-Neveu model. Using the expansion of the minimal form factor function

$$G(\theta) = c'\left(1 - \frac{1}{N}\left(1 - \frac{1}{2}\frac{i\pi - \theta}{\tanh\frac{1}{2}\theta}\right)\right) + O(N^{-2})$$

we obtain the $1/N$ expansion of the exact the $SU(N)$ current form factor as

$$F^{J^{\pm}_{\beta(\rho)}}_{\alpha(\lambda)}(\theta, \omega) = \langle\, 0 \,|\, J^{\pm}_{\beta(\rho)}(0) \,|\, \theta, \omega \,\rangle^{in}_{\alpha(\lambda)} = \delta^{\beta}_{\alpha}\delta^{(\rho)}_{(\lambda)}\, \bar{v}(\omega)\gamma^{\pm}u(\theta)\, G(\theta - \omega)/G(i\pi)$$

$$= \delta^{\beta}_{\alpha}\delta^{(\rho)}_{(\lambda)}\, \bar{v}(\omega)\gamma^{\pm}u(\theta)\left(1 - \frac{1}{N}\left(1 - \frac{1}{2}\frac{i\pi - (\theta - \omega)}{\tanh\frac{1}{2}\left(i\pi - (\theta - \omega)\right)}\right)\right) + O(N^{-2})\,.$$

4   The chiral $SU(N)$ Gross-Neveu model

Let the fermi fields $\psi_{\alpha}(x)$, $(\alpha = 1, \ldots, N)$ form an $SU(N)$-multiplet. The field theory is defined by the Lagrangian [4]

$$\mathcal{L}(\psi, \bar{\psi}) = \bar{\psi}\, i\gamma\partial\, \psi + \tfrac{1}{2}g^2\left((\bar{\psi}\psi)^2 - (\bar{\psi}\gamma^5\psi)^2\right)$$

or equivalently

$$\mathcal{L}(\psi, \bar{\psi}, \sigma, \pi) = \bar{\psi}(i\gamma\partial - \sigma - i\gamma^5\pi)\psi - \tfrac{1}{2}g^{-2}(\sigma^2 + \pi^2)$$

where $\sigma(x)$ is scalar and $\pi(x)$ a pseudoscalar field. The field equations for these fields are

$$\sigma = -g^2\bar{\psi}\psi,\ \ \pi = -ig^2\bar{\psi}\gamma^5\psi\,.$$

4.1   The 1/N perturbation theory

Using the bootstrap program and the results of [19], the S-matrix i.e. the on-shell solution of the model has been proposed in [7, 8]. It is well known [4, 5, 7, 8] that the naive $1/N$-expansion of the chiral Gross-Neveu model suffers on severe infrared problems. In [7, 8] two different approaches to overcome these problems were proposed and it was shown that the exact S-matrix was consistent with both. We will show that an off-shell quantity as our solution for the three particle form factor of the field $\psi(x)$ is also consistent with the $1/N$-expansion of [8]. Without presenting details we note that we do not obtain consistency with the approach of [7]. Since in the literature (see e.g. [8, 33, 9]) there are some errors and misprints we present a detailed derivation of the approach of Swieca et al.

The generation functional of Greens's functions for the chiral Gross-Neveu model is

$$Z(\xi, \bar{\xi}) = \int d\psi \, d\bar{\psi} \, d\sigma \, d\pi \, \exp i \left(\mathcal{A}(\psi, \bar{\psi}, \sigma, \pi) + \bar{\xi}\psi + \bar{\psi}\xi \right) \tag{39}$$

with the action $\mathcal{A}(\psi, \bar{\psi}, \sigma, \pi) = \int d^2x \, \mathcal{L}(\psi, \bar{\psi}, \sigma, \pi)$. In eq. (39) and in the following we use a short notation of the x-integrations e.g. $\bar{\xi}\psi = \int d^2x \bar{\xi}(x)\psi(x)$.

When quantizing the model, severe infrared divergences appear due to the "would-be Goldstone boson" π. Following Kurak, Köberle and Swieca [8] we introduce two additional bosonic fields $A(x)$ and $B(x)$ quantized with negative norm. The A-field compensates the infrared divergences. In fact as we will see below that together with the infrared divergences of π it decouples from the rest of the model. We replace the fermi fields by

$$\psi(x) \to \psi'(x) = \exp i \left(\gamma^5 A(x) + B(x) \right) \, \psi(x).$$

The B-field is introduced, in order not to change the statistics of the ψ-fields. Finally we have the Lagrangian

$$\mathcal{L} = \bar{\psi}'(i\gamma\partial - \mu)\psi' - \tfrac{1}{2}g^{-2}(\sigma^2 + \pi^2) + \tfrac{1}{2}N(\alpha^{-2}A\Box A + \beta^{-2}B\Box B)$$
$$\text{with} \quad \mu = \sigma + i\gamma^5\pi - \gamma^5\gamma\partial A + \gamma\partial B \, .$$

The couplings α and β are unrenormalized, their renormalized values are $\sqrt{\pi}$. Performing the ψ'-integrations in the generation functional we obtain

$$Z(\xi, \bar{\xi}) = \int d\sigma \, d\pi \, dA \, dB \, \exp \left(i\mathcal{A}_{eff}(\sigma, \pi, A, B) - \bar{\xi}S\xi \right)$$

with the fermi propagator $S = i(i\gamma\partial - \mu)^{-1}$ and the effective action

$$\mathcal{A}_{eff}(\sigma, \pi, A, B)$$
$$= -iN \, \text{Tr} \ln(i\gamma\partial - \mu) - \tfrac{1}{2} \int d^2x \, \left(g^{-2} \left(\sigma^2 + \pi^2 \right) - N(\alpha^{-2}A\Box A + \beta^{-2}B\Box B) \right) \, .$$

The symbol Tr means the trace with respect to x-space and spinor space. The trace with respect to $SU(N)$-isospin has been taken and gives the factor N. We define the vertex functions Γ by

$$\mathcal{A}_{eff}(\varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^2 x_1 \ldots d^2 x_n \, \Gamma_{\varphi}^{(n)}(x_1, \ldots, x_n) \, \varphi_1(x_1) \ldots \varphi_n(x_n) \,,$$

where $\varphi_i \in \{\sigma - \sigma_0, \pi - \pi_0, A - A_0, B - B_0\}$. The values σ_0 etc. are defined by the condition that $\mathcal{A}_{eff}(\varphi)$ is stationary at this point. This means that the one-point vertex functions $\Gamma_{\sigma}^{(1)}(x) = \delta \mathcal{A}_{eff}/\delta\sigma = 0$ etc. vanish

$$\Gamma_{\sigma}^{(1)}(x) = N \text{ tr } S(x,x) - g^{-2}\sigma_0 = 0$$
$$\Gamma_{\pi}^{(1)}(x) = N \text{ tr } \left(i\gamma^5 S(x,x)\right) - g^{-2}\pi_0 = 0$$
$$\Gamma_{A}^{(1)}(x) = N \text{ tr } \left(-\gamma^5\gamma\partial S(x,x)\right) + N\alpha^{-2}\Box A_0 = 0$$
$$\Gamma_{B}^{(1)}(x) = N \text{ tr } \left(\gamma\partial S(x,x)\right) + N\beta^{-2}\Box B_0 = 0.$$

The three last equations mean $\pi_0 = A_0 = B_0 = 0$ and the first one implies $\sigma_0 = m$ with

$$N \int \frac{d^2 p}{(2\pi)^2} \text{ tr } \frac{i}{\gamma p - m} - g^{-2}\sigma_0 = 0 \Rightarrow \sigma_0 = m = M e^{-\frac{\pi}{Ng^2}} \,,$$

where M is an UV-cutoff. There is the effect of mass generation and dimensional transmutation: the dimensionless coupling g is replaced by the mass m. The $1/N$-expansion is obtained by expanding the effective action at this stationary point. The resulting Feynman rules are given by the simple vertices

$$V_{\sigma}(k) = (-i)\,, \quad V_{\pi}(k) = \gamma^5\,, \quad V_A(k) = \gamma^5\gamma k\,, \quad V_B(k) = -\gamma k \tag{40}$$

and the propagators in momentum space

$$\tilde{\Delta}_{\sigma\sigma}(k) = -\frac{i\pi}{N} \frac{1}{\cosh^2 \frac{1}{2}\phi} \frac{\sinh\phi}{\phi} \,, \tag{41}$$

$$\tilde{\Delta}_{\pi\pi}(k) = -\frac{i\pi}{N} \frac{1}{\sinh^2 \frac{1}{2}\phi} \left(\frac{\sinh\phi}{\phi} - 1\right) \,,$$

$$\tilde{\Delta}_{AA}(k) = -\frac{i\pi}{Nk^2} \,,$$

$$\tilde{\Delta}_{BB}(k) = -\frac{i\pi}{Nk^2} \,,$$

$$\tilde{\Delta}_{\pi A}(k) = \tilde{\Delta}_{A\pi}(k) = -2m\frac{i\pi}{Nk^2} \,.$$

where $k^2 = -4m^2 \sinh^2 \frac{1}{2}\phi$. To obtain the propagators one calculates the two point vertex functions $\Gamma_{ij}^{(2)}$ from the bubble graph of Fig.2 with the various vertices and uses
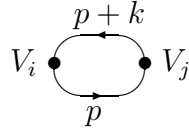
Figure 2: *The bubble graph.*

$\Delta = i\Gamma^{(2)^{-1}}$. In [8] it was argued that the unrenormalized values of α and β are to be replaced by $\alpha \to \infty$ and $\beta \to \sqrt{\pi}$. In that limit the propagators are those of (41). One observes that the A- and B-propagators remain free and the infrared singularity in the π-propagator disappears.

As an example we consider the four point vertex function

$$\tilde{\Gamma}^{(4)DC\delta\gamma}_{AB\alpha\beta}(-p_3, -p_4, p_1, p_2) = \delta^\delta_\alpha \delta^\gamma_\beta \, \Gamma^{DC}_{AB}(p_2 - p_3) - \delta^\gamma_\alpha \delta^\delta_\beta \, \Gamma^{CD}_{AB}(p_3 - p_1)$$

where A, B, C, D are spinor indices, $\alpha, \beta, \gamma, \delta$ are isospin indices and Γ is given by the Feynman graph of Fig. 3. Taking into account the contributions from all vertices (40)
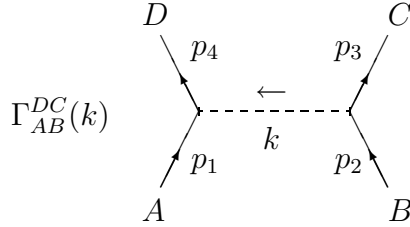
Figure 3: *The four point vertex*

and all the propagators (41) we obtain

$$\Gamma(k) = \sum_{i,j} V_i(k)\tilde{\Delta}_{ij}(k)V_j(-k) \tag{42}$$

$$= -1 \otimes 1 \, \tilde{\Delta}_{\sigma\sigma}(k) + \gamma^5 \otimes \gamma^5 \, \tilde{\Delta}_{\pi\pi}(k) - \gamma^5\gamma k \otimes \gamma^5\gamma k \, \tilde{\Delta}_{AA}(k)$$
$$- \gamma^5 \otimes \gamma^5\gamma k \, \tilde{\Delta}_{\pi A}(k) + \gamma^5\gamma k \otimes \gamma^5 \, \tilde{\Delta}_{A\pi}(k) - \gamma k \otimes \gamma k \, \tilde{\Delta}_{BB}(k) \ .$$

Inserting the expressions for the propagators we finally obtain

$$\Gamma(k) = \frac{i\pi}{N} \left\{ 1 \otimes 1 \, \frac{1}{\cosh^2(\phi/2)} \frac{\sinh\phi}{\phi} - \gamma^5 \otimes \gamma^5 \, \frac{1}{\sinh^2(\phi/2)} \left(\frac{\sinh\phi}{\phi} - 1 \right) \right.$$
$$\left. + \frac{1}{k^2} \left(\gamma^5\gamma k \otimes \gamma^5\gamma k + 2m\gamma^5 \otimes \gamma^5\gamma k - 2m\gamma^5\gamma k \otimes \gamma^5 + \gamma k \otimes \gamma k \right) \right\} \tag{43}$$

where the tensor product structure of the spinor matrices is obvious from Fig. 3. We now apply these results to the examples of section 3 and investigate the three particle form factor of the fundamental fermi field and the two particle form factor of the $SU(N)$ current in $1/N$-expansion in lowest nontrivial order.

The three particle form factor of the fundamental fermi field: For convenience we multiply the field with the Dirac operator, take

$$\mathcal{O}_{D\delta}(x) = \left(-i\left(i\gamma\partial - m\right)\psi(x)\right)_{D\delta}$$

and define

$$_{out}^{\gamma}\langle\, p_3\,|\,\mathcal{O}_{D\delta}(0)\,|\,p_1, p_2\,\rangle_{\alpha\beta}^{in} = F^{\mathcal{O}_{D\delta}\gamma}{}_{\alpha\beta}(\theta_{12}, \theta_{13}, \theta_{23})\,.$$

By means of LSZ-techniques one can express the connected part in terms of the 4-point vertex function

$$F_{conn.\alpha\beta}^{\mathcal{O}_{D\delta}\,\gamma}(\theta_{12}, \theta_{13}, \theta_{23}) = \bar{u}_C(p_3)\,\Gamma_{AB\,\alpha\beta}^{DC\,\delta\gamma}(-p_3, p_3 - p_1 - p_2, p_1, p_2)\,u_A(p_1)u_B(p_2).$$

The lowest order contributions are given by the Feynman graphs of Fig. 4
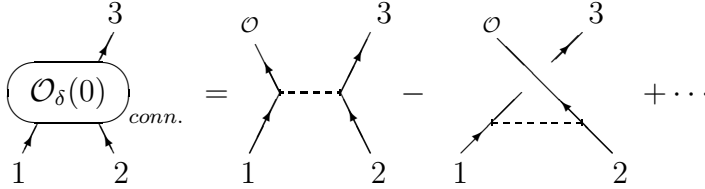
Figure 4: *The connected part of the three particle form factor of the fundamental fermi field in $1/N$-expansion.*

$$F_{conn.\alpha\beta}^{\mathcal{O}_{D\delta}\,\gamma} = \bar{u}_C(p_3)\left\{\delta_{\alpha\delta}\delta_{\beta\gamma}\,\Gamma_{AB}^{DC}(p_2 - p_3) - \delta_{\alpha\gamma}\delta_{\beta\delta}\,\Gamma_{AB}^{CD}(p_3 - p_1)\right\}u_A(p_1)u_B(p_2).$$

where Γ is given by Fig. 3 and eq. (43) and the spinor $u(p)$ by eq. (37). It turns out that for p_1, p_2 and p_3 on-shell several terms vanish or cancel and we obtain up to order $1/N^2$

$$F_{conn.\alpha\beta}^{\mathcal{O}_{D\delta}\,\gamma} = \frac{2mi\pi}{N}\left\{\delta_{\alpha\delta}\delta_{\beta\gamma}\frac{\sinh\theta_{23}}{\theta_{23}}\left(\frac{1}{\cosh\frac{1}{2}\theta_{23}} - \gamma^5\frac{1}{\sinh\frac{1}{2}\theta_{23}}\right)u_D(p_1)\right.$$
$$\left. - \delta_{\alpha\gamma}\delta_{\beta\delta}\frac{\sinh\theta_{13}}{\theta_{13}}\left(\frac{1}{\cosh\frac{1}{2}\theta_{13}} - \gamma^5\frac{1}{\sinh\frac{1}{2}\theta_{13}}\right)u_D(p_2)\right\} \quad (44)$$

which agrees with the result for the exact form factor (36). In [8] was shown that if the momentum $p_4 = p_1 + p_2 - p_3$ is also on-shell then the expression (44) is consistent with the exact S-matrix (2).

The 1/N expansion of the $SU(N)$ current form factor: We check the proposed exact form factor (38) in $1/N$ expansion. Fig. 5 shows the diagrams contributing to $F_{\alpha(\lambda)}^{J_{\beta(\rho)}^{\pm}}$
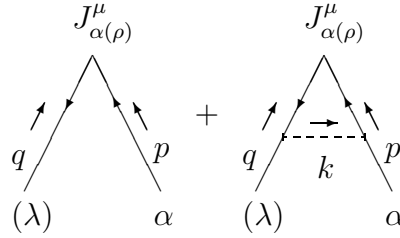
Figure 5: Diagrams contributing to the form factor of the $SU(N)$ current in the Gross-Neveu models up to order N^{-2}.

in order N^0 and N^{-1} which give

$$F_{\alpha(\lambda)}^{J_{\beta(\rho)}^{\pm}} = \delta_\alpha^\beta \delta_{(\lambda)}^{(\rho)} \bar{v}(q) \gamma^\mu u(p) F^J(\theta) \tag{45}$$

$$= \delta_\alpha^\beta \delta_{(\lambda)}^{(\rho)} \bar{v}(q) \gamma^\pm u(p) + \delta_\alpha^\beta \delta_{(\lambda)}^{(\rho)} \sum_{i,j} \int \frac{d^2 k}{(2\pi)^2} \Delta_{ij}(k)$$

$$\times \left\{ \bar{v}(q) V_i(k) \frac{i}{\gamma q + \gamma k - m} \gamma^\pm \frac{i}{\gamma p - \gamma k - m} V_j(-k) u(p) - \text{substr.} \right\} + O(N^{-2}) \, . \tag{46}$$

The k integration can be performed using the propagators (41) and the vertices (40). For convenience we write the total 4-point vertex function (42) which is a part of (46) as

$$\Gamma = \sum_{i,j} \Delta_{ij}(k) V_i(k) \otimes V_j(-k) = \Gamma_\sigma + \Gamma_\pi + \Gamma_V + \Gamma_{rest}$$

with

$$\Gamma_\sigma = \frac{i\pi}{N} 1 \otimes 1 \frac{1}{\cosh^2(\phi/2)} \frac{\sinh\phi}{\phi}$$

$$\Gamma_\pi = -\frac{i\pi}{N} \gamma^5 \otimes \gamma^5 \frac{1}{\sinh^2(\phi/2)} \left(\frac{\sinh\phi}{\phi} - 1 \right)$$

$$\Gamma_V = \frac{i\pi}{N} \gamma^\mu \otimes \gamma_\mu$$

$$\Gamma_{rest} = \frac{i\pi}{N} \frac{1}{k^2} \left(\gamma^5 \gamma k \otimes \gamma^5 (\gamma k - 2m) + \gamma^5 (\gamma k + 2m) \otimes \gamma^5 \gamma k \right)$$

where $\gamma k \otimes \gamma k = \gamma^5 \gamma k \otimes \gamma^5 \gamma k + k^2 \gamma^\mu \otimes \gamma_\mu$ has been used. Correspondingly we decompose the form factor function $F^J(\theta)$ in (45) (in order to avoid infra-red problems in the calculation) as

$$F^J(\theta) = 1 + (F_\sigma(\theta) + F_\pi(\theta) + F_V(\theta) + F_{rest}(\theta)) + O(N^{-2}).$$

The first contribution $F_\sigma(\theta)$ is given by the $O(2N)$-Gross-Neveu form factor $F_-^{GN}(\theta)$ which has been calculated in [12]

$$F_\sigma(\theta) = F_-^{GN}(\theta) - 1$$

$$= \frac{1}{2N}\left(\int_0^\infty \frac{\sinh^2 \frac{1}{2}\phi}{\phi^2 + \pi^2}\left(\frac{\phi\coth\frac{1}{2}\phi - \hat{\theta}\coth\frac{1}{2}\hat{\theta}}{\cosh^2\frac{1}{2}\phi - \cosh^2\frac{1}{2}\hat{\theta}} - \left(\hat{\theta}\to 0\right)\right)d\phi\right) + \frac{1}{N}\left(\frac{\hat{\theta}}{\sinh\hat{\theta}} - 1\right)$$

$$= \frac{1}{2N}\left(1 - \frac{1}{2}\hat{\theta}\left(\coth\frac{1}{2}\hat{\theta} - \tanh\frac{1}{2}\hat{\theta}\right) - \frac{1}{2}\psi\left(\frac{1}{2} + \frac{\hat{\theta}}{2\pi i}\right) - \frac{1}{2}\psi\left(\frac{1}{2} - \frac{\hat{\theta}}{2\pi i}\right) + \psi\left(\frac{1}{2}\right)\right)$$

$$+ \frac{1}{N}\left(\frac{\hat{\theta}}{\sinh\hat{\theta}} - 1\right)$$

where $\hat{\theta} = i\pi - \theta$ and $\psi(z) = (\ln\Gamma(z))'$. Similarly we obtain

$$F_\pi(\theta) = -\frac{1}{2N}\left(\int_0^\infty \frac{\sinh^2\frac{1}{2}\phi}{\phi^2 + \pi^2}\left(\frac{\phi\coth\frac{1}{2}\phi - \hat{\theta}\coth\frac{1}{2}\hat{\theta}}{\cosh^2\frac{1}{2}\phi - \cosh^2\frac{1}{2}\hat{\theta}} - \left(\hat{\theta}\to 0\right)\right)d\phi\right)$$

$$= -\frac{1}{2N}\left(1 - \frac{1}{2}\hat{\theta}\left(\coth\frac{1}{2}\hat{\theta} - \tanh\frac{1}{2}\hat{\theta}\right) - \frac{1}{2}\psi\left(\frac{1}{2} + \frac{\hat{\theta}}{2\pi i}\right) - \frac{1}{2}\psi\left(\frac{1}{2} - \frac{\hat{\theta}}{2\pi i}\right) + \psi\left(\frac{1}{2}\right)\right),$$

and

$$F_V(\theta) = \frac{1}{2N}\frac{\hat{\theta}}{\sinh\hat{\theta}}\left(\cosh\hat{\theta} - 1\right)$$

$$F_{rest}(\theta) = 0$$

and therefore

$$F^J(\theta) = 1 - \frac{1}{N}\left(1 - \frac{1}{2}\frac{\hat{\theta}}{\tanh\frac{1}{2}\hat{\theta}}\right) + O(N^{-2})$$

which agrees with the $1/N$-expansion of the exact result for form factor of the the current derived in section 3.

5 Commutation rules

In [31] commutation rules were derived for the $Z(N)$ scaling Ising models. The results for the $SU(N)$ Gross-Neveu model are similar, however, the proof is much more complicated because of the unusual crossing relations (20) and (related to this) the Klein factors (19).

Let $|\underline{\theta}\rangle_{\underline{\alpha}}^{in}$ with $\underline{\alpha} = ((\alpha_{11}, \ldots, \alpha_{1r_1}), \ldots, (\alpha_{\alpha 1}, \ldots, \alpha_{\alpha r_\alpha}))$ be a state of α particles of rank $r_1, \ldots, r_\alpha$ $(1 \le r_j \le N - 1)$ (or bound states of r_j particles of rank 1). We define the charge of a state to be the sum of all ranks of the particles in the state

$$Q_\alpha = \sum_{j=1}^\alpha r_j\,.$$

The weight $w_i(\underline{\alpha})$ $(1 \le i \le N)$ of the state $\underline{\alpha}$ is equal to the number of $\alpha_{jk} = i$. Therefore the total charge of the state $\underline{\alpha}$ is

$$Q_\alpha = \sum_{i=1}^{N} w_i(\underline{\alpha})$$

(see appendix B). If $\underline{\alpha}$ is a state for which the form factor $F_{\underline{\alpha}}^\psi(\underline{\theta})$ does not vanish we use (25) and define the charge of the operator ψ by

$$Q_\psi = Q_\alpha \bmod N = \sum_{i=1}^{N} w_i^\psi \bmod N$$

with $0 \le Q_\psi < N$.

Examples: For the energy momentum tensor $T^{\mu\nu}(x)$ (which is a $SU(N)$ scalar) the fundamental field $\psi_\alpha(x)$ (which is a $SU(N)$ vector) and the $SU(N)$ current $J_{\alpha\beta}^\mu(x)$ (which transforms as the adjoint representation) the weights and the charges are

$$
\begin{aligned}
w^T &= (0, 0, \ldots, 0, 0)\,, & Q_T &= 0 \\
w^\psi &= (1, 0, \ldots, 0, 0)\,, & Q_\psi &= 1 \\
w^J &= (2, 1, \ldots, 1, 0)\,, & Q_J &= 0\,.
\end{aligned}
$$

Theorem 1 *The equal time commutation rule of two fields $\phi(x)$ and $\psi(y)$ with charge Q_ϕ and Q_ψ, respectively, is (in general anyonic)*

$$\phi(x)\psi(y) = \psi(y)\phi(x) \exp\left(2\pi i \epsilon(x^1 - y^1)\tfrac{1}{2}\left(1 - 1/N\right) Q_\phi Q_\psi\right)\,. \tag{47}$$

The proof of this theorem can be found in appendix B.

6 Conclusions

In this paper the general $SU(N)$ form factor formula is constructed. As an application of this result exact $SU(N)$ form factors for the field, the energy momentum tensor and the current operators are derived in detail. In the large N limit these form factors are compared with the $1/N$-expansion of the Gross-Neveu model and full agreement is found. The commutation rules of arbitrary fields are derived and in general anyonic behavior is found. We believe that our results may be relevant for the computation of correlation functions in fermionic ladders [34]. In addition the series of the $1/N$-expansion of our exact form factors could hopefully help to understand the same series in QCD.

Acknowledgments

Appendix

A   Bound state form factors

Proof of formulae (29) – (30):   Here we present a sketch of the proof for the bound
state form factors formula. For simplicity several formulae will be written only up to
constants, the normalization can be fixed at the end by the physical properties of the
operator. The form factor formula for particles of rank 1 was proved in [13]. Applying
the bound state procedure to this result we derive the formula for $\bar{n} = 1$ bound state of
rank $N - 1$, the general case $\bar{n} > 1$ follows easily.

The bound state intertwiner [28] is defined by

$$i \operatorname*{Res}_{\varphi_{N-1N-2}=i\eta} \ldots i \operatorname*{Res}_{\varphi_{21}=i\eta} S^{\lambda}_{\underline{\mu}}(\underline{\bar{\varphi}}) = \Gamma^{\lambda}_{(\rho)} \Gamma^{(\rho)}_{\underline{\mu}}$$

where the S-matrix $S^{\lambda}_{\underline{\mu}}(\underline{\bar{\varphi}})$ exchanges all particles with rapidities $\underline{\bar{\varphi}} = \varphi_{N-1}, \ldots, \varphi_1 \to \underline{\varphi} = \varphi_1, \ldots, \varphi_{N-1}$. It satisfies the bound state fusion equation

$$\Gamma^{(\sigma)}_{\underline{\mu}} S^{\beta\underline{\mu}}_{\underline{\lambda}\alpha}(\underline{\theta}, \theta) = S^{\beta(\sigma)}_{(\rho)\alpha}(\omega, \theta) \Gamma^{(\rho)}_{\underline{\lambda}} \, . \tag{48}$$

Lemma 2 *The form factor for n particles $\underline{\alpha} = \alpha_1, \ldots, \alpha_n$ of rank 1 and one bound state*
$(\rho) = (\rho, \ldots, \rho_{N-1})$ *(with $\rho_1 < \cdots < \rho_{N-1}$) of rank $N - 1$ may be written as*

$$F_{\underline{\alpha}(\rho)}(\underline{\theta}, \omega) = \left(\sqrt{2}i\right)^{2-N} F_{\underline{\alpha}\underline{\lambda}}(\underline{\theta}\underline{\varphi}) \Gamma^{\underline{\lambda}}_{(\rho)}, \quad with \ \omega = \frac{1}{N-1}(\varphi_1 + \cdots + \varphi_{N-1})$$

for $\varphi_{N-1N-2} = \cdots = \varphi_{21} = i\eta$, i.e. $\varphi_j = \omega + ji\eta - i\pi$.

Proof. We start with a form factor $F_{\underline{\alpha}\underline{\mu}}(\underline{\theta}\underline{\bar{\varphi}})$ for $n + N - 1$ particles of rank 1 with
rapidities $\underline{\theta} = \theta_1, \ldots, \theta_n, \underline{\bar{\varphi}} = \varphi_{N-1}, \ldots, \varphi_1$ and quantum numbers $\underline{\alpha} = \alpha_1, \ldots, \alpha_n, \underline{\mu} = \mu_1, \ldots, \mu_{N-1}$ (for convenience we use for $\underline{\bar{\varphi}}$ an inverse numbering). Applying iteratively
the bound state fusion procedure (see e.g. [28, 13]) we obtain the bound state form factor

$$F_{\underline{\alpha}(\rho)}(\underline{\theta}, \omega)\left(\sqrt{2}i\right)^{N-2} \Gamma^{(\rho)}_{\underline{\mu}} = i \operatorname*{Res}_{\varphi_{N-1N-2}=i\eta} \ldots i \operatorname*{Res}_{\varphi_{21}=i\eta} F_{\underline{\alpha}\underline{\mu}}(\underline{\theta}\underline{\bar{\varphi}}) \tag{49}$$

$$= F_{\underline{\alpha}\underline{\lambda}}(\underline{\theta}\underline{\varphi}) i \operatorname*{Res}_{\varphi_{N-1N-2}=i\eta} \ldots i \operatorname*{Res}_{\varphi_{21}=i\eta} S^{\lambda}_{\underline{\mu}}(\underline{\bar{\varphi}})$$

$$= F_{\underline{\alpha}\underline{\lambda}}(\underline{\theta}\underline{\varphi}) \Gamma^{\underline{\lambda}}_{(\rho)} \Gamma^{(\rho)}_{\underline{\mu}}$$

where the form factor equation (i) $F_{\underline{\alpha}\underline{\mu}}(\underline{\theta}\underline{\bar{\varphi}}) = F_{\underline{\alpha}\underline{\lambda}}(\underline{\theta}\underline{\varphi}) S^{\lambda}_{\underline{\mu}}(\underline{\bar{\varphi}})$ (see e.g. [13]) has been used.
∎

We start from the K-function $K^{\mathcal{O}}_{\underline{\alpha}\underline{\lambda}}(\underline{\theta}\underline{\varphi})$ for particles of rank 1 given by the general
formula (22) where we replace $\underline{\theta} \to \underline{\theta}\underline{\varphi}$, $(\underline{\varphi} = \varphi_1, \ldots, \varphi_{N-1})$ and integration variables
$\underline{z} \to \underline{z}\underline{y}$, $(\underline{y} = y_1, \ldots, y_{N-2})$

$$K^{\mathcal{O}}_{\underline{\alpha}\underline{\lambda}}(\underline{\theta}\underline{\varphi}) = \int d\underline{z} \int d\underline{y} \tilde{h}(\underline{\theta}\underline{\varphi}, \underline{z}\underline{y}) \, p(\underline{\theta}\underline{\varphi}, \underline{z}\underline{y}) \, \tilde{\Psi}_{\underline{\alpha}}(\underline{\theta}\underline{\varphi}, \underline{z}\underline{y}) \, .$$

The state $\tilde\Psi_{\underline\alpha}$ is a linear combination of the basic Bethe ansatz co-vectors (11) (for $\bar n = 0$)

$$\tilde\Psi_{\underline\alpha}(\underline{\theta\varphi}, \underline{zy}) = L_{\underline\beta}^{(1)}(\underline{zy})\tilde\Phi_{\underline\alpha}^{\underline\beta}(\underline{\theta\varphi}, \underline{zy}), \quad \text{with } 1 < \beta_i$$

where the $L_{\underline\beta}^{(1)}(\underline{zy})$ again satisfy a representation like the $K_{\underline{\alpha\lambda}}^{\mathcal O}(\underline{\theta\varphi})$. Iterating this nesting procedure we arrive at

$$K_{\underline{\alpha\lambda}}^{\mathcal O}(\underline{\theta\varphi}) = \int d\underline{z}^{(1)} \int d\underline{y}^{(1)} \dots \int d\underline{z}^{(N-1)} \int \underline{y}^{(N-1)} \tilde h \, p^{\mathcal O} \, \tilde\Phi_{\underline{\alpha\lambda}}$$

where the functions $\tilde h$, $p^{\mathcal O}$ and $\tilde\Phi_{\underline{\alpha\lambda}}$ depend on the variables $\underline{\theta\varphi}, \underline{zy}$, ($\underline z = \underline z^{(1)}, \dots, \underline z^{(N-1)}$, $\underline y = \underline y^{(1)}, \dots, \underline y^{(N-2)}$, $\underline y^{(l)} = y_1^{(l)}, \dots, y_{N-1-l}^{(l)}$, ($l = 1, \dots, N-2$)). If we take the residues in (49) at $\varphi_{i+1,i} = i\eta$ the pinching phenomenon (see [13] and Fig. 1) appears at $y_1^{(1)} = \varphi_1, \dots, y_{N-2}^{(1)} = \varphi_{N-2}$. This propagates to the higher level integrations such that we may replace $\underline y^{(l)} \to \underline\varphi^{(l)} = \varphi_1, \dots, \varphi_{N-1-l}$ which are related to ω by $\varphi_j = \omega + ji\eta - i\pi$. The h-function (23) for the lowest level Bethe ansatz then takes the form (up to a constant)

$$\tilde h(\underline\theta, \underline\varphi, \underline z^{(1)}, \underline\varphi^{(1)}) = \tilde\phi(\underline\theta - \underline z)\tilde\phi(\underline\theta - \underline\varphi^{(1)})\tilde\phi(\underline\varphi - \underline z)\tau(\underline z)\tau(\underline z - \underline\varphi^{(1)})$$
$$= \tilde h(\underline\theta, \underline z)\tilde\phi(\underline\theta - \underline\varphi^{(1)})\tilde\phi(\underline\varphi - \underline z)\tau(\underline z - \underline\varphi^{(1)}).$$

Here and in the following we use the short notation

$$\tilde\phi(\underline\theta - \underline\varphi^{(1)}) = \prod_{i=1}^{n}\prod_{j=1}^{N-2}\tilde\phi(\theta_i - \varphi_j^{(1)}), \quad \tau(\underline z) = \prod_{1 \le i < j \le n_1}^{n}\tau(z_i - z_j)$$

et cetera, where the product is taken over all indices. The Bethe ansatz states defined by (11) are related by

$$\tilde\Phi_{\underline{\alpha\lambda}}^{\underline{\beta\mu}}(\underline\theta, \underline\varphi, \underline z, \underline\varphi^{(1)})\Gamma_{(\rho)}^{\lambda} = \tilde b(\underline\theta - \underline\varphi^{(1)})\frac{b(i\pi - \omega + \underline z)}{a(\underline\varphi - \underline z)}\Gamma_{(\sigma)}^{\mu 1}\tilde\Phi_{\underline{\alpha}(\rho)}^{\beta(\sigma)}(\underline\theta, \omega, \underline z)$$

where the bound state relation (48) together with (5) and (4) has been used. These equations together imply

$$\tilde h(\underline\theta, \underline\varphi, \underline z, \underline\varphi^{(1)})\tilde\Phi_{\underline{\alpha\lambda}}^{\underline{\beta\mu}}(\underline\theta, \underline\varphi, \underline z, \underline\varphi^{(1)})\Gamma_{(\rho)}^{\lambda} = \frac{\tilde\phi(\underline\varphi^{(1)} + i\eta - \underline\theta)}{\tilde\phi(\underline\varphi^{(2)} + i\eta - \underline z)}\tilde h(\underline\theta, \underline z)\Gamma_{(\sigma)}^{\mu 1}\tilde\Phi_{\underline{\alpha}(\rho)}^{\beta(\sigma)}(\underline\theta, \omega, \underline z).$$

The equations for bound state rapidity $\omega = \varphi_1 - i\eta + i\pi = \varphi_{N-1} + i\eta - i\pi$ and the relations $\tilde b(z - \varphi_j)\tilde\phi(z - \varphi_j) = -\tilde\phi(\varphi_{j+1} - z)$ and $\tilde\phi(\varphi_1 - z)/\tilde\phi(z - \varphi_{N-1}) = -b(i\pi - \omega + z)$ have been used. Therefore we obtain the integral representation

$$K_{\underline\alpha(\rho)}(\underline\theta, \omega) = \int d\underline z \tilde h(\underline\theta, \underline z)p(\underline\theta, \omega, \underline z)\tilde\Psi_{\underline\alpha(\rho)}(\underline\theta, \omega, \underline z)$$

$$\tilde\Psi_{\underline\alpha(\rho)}(\underline\theta, \omega, \underline z) = L_{\underline\beta(\sigma)}^{(1)}(\underline z, \omega)\tilde\Phi_{\underline\alpha(\rho)}^{\beta(\sigma)}(\underline\theta, \omega, \underline z), \quad \text{with } 1 < \beta_i, \ \sigma_1 = 1 < \sigma_2 < \dots < \sigma_{N-1}$$

with the new K-, L- and p-functions given in terms of the old ones

$$K_{\underline{\alpha}(\rho)}(\underline{\theta},\omega) = \frac{1}{\tilde{\phi}(\underline{\varphi}^{(1)} + i\eta - \underline{\theta})}K_{\underline{\alpha}\underline{\lambda}}(\underline{\theta},\underline{\varphi})\Gamma^{\underline{\lambda}}_{(\rho)}$$

$$L^{(1)}_{\underline{\beta}(\sigma)}(\underline{z},\omega) = \frac{1}{\tilde{\phi}(\underline{\varphi}^{(2)} + i\eta - \underline{z})}L^{(1)}_{\underline{\beta}\underline{\mu}}(\underline{z}\underline{\varphi}^{(1)})\Gamma^{\underline{\mu}1}_{(\sigma)}$$

$$p(\underline{\theta},\omega,\underline{z}) = p(\underline{\theta}\underline{\varphi},\underline{z}\underline{\varphi}^{(1)}), \quad (\varphi_j = \omega + ji\eta - i\pi)\,.$$

Correspondingly we obtain for a higher level Bethe ansatz $l = 1,\ldots,N-3$

$$\tilde{h}(\underline{z}^{(l)},\underline{\varphi}^{(l)},\underline{z}^{(l+1)},\underline{\varphi}^{(l+1)}) = \tilde{h}(\underline{z}^{(l)},\underline{z}^{(l+1)})\tilde{\phi}(\underline{z}^{(l)} - \underline{\varphi}^{(l+1)})\tilde{\phi}(\underline{\varphi}^{(l)} - \underline{z}^{(l+1)})\tau(\underline{z}^{(l+1)} - \underline{\varphi}^{(l+1)})$$

and

$$\tilde{\Phi}^{(l)\underline{\gamma}\underline{\nu}}_{\underline{\beta}\underline{\mu}}(\underline{z}^{(l)},\underline{\varphi}^{(l)},\underline{z}^{(l+1)},\underline{\varphi}^{(l+1)})\Gamma^{\underline{\mu}l\cdots1}_{(\sigma)}$$

$$= \prod_{i=N-l}^{N-1}\frac{1}{b(\varphi_i - \underline{z}^{(l+1)})}\frac{\tilde{b}(\underline{z}^{(l)} - \underline{\varphi}^{(l+1)})b(i\pi - \omega + \underline{z}^{(l+1)})}{a(\underline{\varphi}^{(l)} - \underline{z}^{(l+1)})}\Gamma^{\underline{\nu}l+1\cdots1}_{(\varsigma)}\Phi^{(l)\underline{\gamma}(\varsigma)}_{\underline{\beta}(\sigma)}(\underline{z}^{(l)},\omega,\underline{z}^{(l+1)})$$

such that

$$\tilde{h}(\underline{z}^{(l)},\underline{\varphi}^{(l)},\underline{z}^{(l+1)},\underline{\varphi}^{(l+1)})\tilde{\Phi}^{(l)\underline{\gamma}\underline{\nu}}_{\underline{\beta}\underline{\mu}}(\underline{z}^{(l)},\underline{\varphi}^{(l)},\underline{z}^{(l+1)},\underline{\varphi}^{(l+1)})\Gamma^{\underline{\mu}l\cdots1}_{(\sigma)}$$

$$= \frac{\tilde{\phi}(\underline{\varphi}^{(l+1)} + i\eta - \underline{z}^{(l)})}{\tilde{\phi}(\underline{\varphi}^{(l+2)} + i\eta - \underline{z}^{(l+1)})}\tilde{h}(\underline{z}^{(l)},\underline{z}^{(l+1)})\Gamma^{\underline{\nu}l+1\cdots1}_{(\varsigma)}\Phi^{(l)\underline{\gamma}(\varsigma)}_{\underline{\beta}(\sigma)}(\underline{z}^{(l)},\omega,\underline{z}^{(l+1)})$$

and

$$L^{(l)}_{\underline{\beta}(\sigma)}(\underline{z}^{(l)},\omega) = \int d\underline{z}^{(l+1)}\tilde{h}(\underline{z}^{(l)},\underline{z}^{(l+1)})p(\underline{z}^{(l)},\underline{z}^{(l+1)})\tilde{\Psi}^{(l)}_{\underline{\beta}(\sigma)}(\underline{z}^{(l)},\omega,\underline{z}^{(l+1)})$$

$$\tilde{\Psi}^{(l)}_{\underline{\beta}(\sigma)}(\underline{z}^{(l)},\omega,\underline{z}^{(l+1)}) = L^{(l+1)}_{\underline{\gamma}(\varsigma)}(\underline{z}^{(l+1)},\omega)\tilde{\Phi}^{(l)\underline{\gamma}(\varsigma)}_{\underline{\beta}(\sigma)}(\underline{z}^{(l)},\omega,\underline{z}^{(l+1)})$$

$$\text{with } l < \gamma_i,\ \varsigma_1 = 1,\ldots,\varsigma_l = l < \varsigma_{l+1} < \cdots < \varsigma_{N-1}\,.$$

Note that for $l = N-3$ the relation $\tilde{\phi}(\underline{\varphi}^{(l+2)} + i\eta - \underline{z}^{(l+1)}) = 1$ holds because $\underline{\varphi}^{(N-1)} = \emptyset$.
For the highest level $l = N-2$ we have

$$\tilde{h}(\underline{z}^{(N-2)},\varphi_1,\underline{z}^{(N-1)}) = \tilde{h}(\underline{z}^{(N-2)},\underline{z}^{(N-1)})\tilde{\phi}(\varphi_1 - \underline{z}^{(N-1)})$$

$$\tilde{\Phi}^{(N-2)}_{\underline{\beta}\underline{\mu}}(\underline{z},\varphi_1,\underline{u})\Gamma^{\underline{\mu}N-2\cdots1}_{(\sigma)} = \Gamma^{N-1\cdots1}_{(\varsigma)}\tilde{\Phi}^{(N-2)(\varsigma)}_{\underline{\beta}(\sigma)}(\underline{z}^{(N-2)},\omega,\underline{z}^{(N-1)})$$

such that

$$\tilde{h}(\underline{z}^{(N-2)},\varphi_1,\underline{z}^{(N-1)})\tilde{\Phi}^{(N-2)}_{\underline{\beta}\underline{\mu}}(\underline{z},\varphi_1,\underline{u})\Gamma^{\underline{\mu}N-2\cdots1}_{(\sigma)}$$

$$= \tilde{h}(\underline{z}^{(N-2)},\underline{z}^{(N-1)})\tilde{\chi}(\omega - \underline{z}^{(N-1)})\Gamma^{N-1\cdots1}_{(\eta)}\tilde{\Phi}^{(N-2)(\eta)}_{\underline{\beta}(\sigma)}(\underline{z}^{(N-2)},\omega,\underline{z}^{(N-1)})$$

and

$$L_{\underline{\beta}(\sigma)}^{(N-2)}(\underline{z}^{(l)},\omega) = \int d\underline{z}^{(N-1)}\tilde{h}(\underline{z}^{(N-2)},\underline{z}^{(N-1)})p(\underline{z}^{(N-2)},\underline{z}^{(N-1)})\tilde{\Psi}_{\underline{\beta}(\sigma)}^{(N-2)}(\underline{z}^{(N-2)},\omega,\underline{z}^{(N-1)})$$

$$\tilde{\Psi}_{\underline{\beta}(\sigma)}^{(N-2)}(\underline{z}^{(N-2)},\omega,\underline{z}^{(N-1)}) = L^{(N-1)}(\omega - \underline{z}^{(N-1)})\tilde{\Phi}^{(N-2)\underline{\gamma}(\eta)}_{\underline{\beta}(\sigma)}(\underline{z}^{(l)},\omega,\underline{z}^{(l+1)})$$

$$\text{with } \gamma_i = N, \ \sigma_1 = 1,\ldots,\sigma_{N-1} = N-1\,.$$

Here

$$L^{(N-1)}(\omega) = \tilde{\chi}(\omega) = \tilde{\phi}(\varphi_1) = \Gamma\left(\frac{1}{2} + \frac{\omega}{2\pi i}\right)\Gamma\left(\frac{1}{2} - \frac{1}{N} - \frac{\omega}{2\pi i}\right)\,.$$

Finally we combine the minimal form factors in formula (24) for $\varphi_{N-1N-2} = \cdots = \varphi_{21} = i\eta$

$$F_{\underline{\alpha}(\rho)}(\underline{\theta},\omega) = F_{\underline{\alpha}\underline{\lambda}}(\underline{\theta}\underline{\varphi})\Gamma_{(\rho)}^{\lambda} = F(\underline{\theta})G(\underline{\theta} - \omega)\frac{1}{\tilde{\phi}(\underline{\varphi} + i\eta - \underline{\theta})}K_{\underline{\alpha}\underline{\lambda}}(\underline{\theta}\underline{\varphi})\Gamma_{(\rho)}^{\lambda}$$

$$= F(\underline{\theta})G(\underline{\theta} - \omega)K_{\underline{\alpha}(\rho)}(\underline{\theta},\omega)\,.$$

The relations (16) for the minimal form factor function G for one particle of rank 1 and one of rank $N-1$ and (17) have been used. Therefore the final result is

$$K_{\underline{\alpha}(\rho)}^{\mathcal{O}}(\underline{\theta},\omega) = \int d\underline{z}^{(1)}\ldots\int d\underline{z}^{(N-1)}\tilde{h}\,(\underline{\theta},\omega,\underline{z})p^{\mathcal{O}}(\underline{\theta},\omega,\underline{z})\tilde{\Phi}_{\underline{\alpha}(\rho)}(\underline{\theta},\omega,\underline{z})$$

$$\tilde{h}\,(\underline{\theta},\omega,\underline{z}) = \prod_{l=0}^{N-2}\tilde{h}(\underline{z}^{(l)},\underline{z}^{(l+1)})\prod_{i=1}^{n_{N-1}}\tilde{\chi}(\omega - z_i^{(N-1)})$$

$$p^{\mathcal{O}}(\underline{\theta},\omega,\underline{z}) = p^{\mathcal{O}}(\underline{\theta}\underline{\varphi},\underline{z}\underline{y})\text{ with }\underline{y}^{(l)} = \underline{\varphi}^{(l)}$$

where $p^{\mathcal{O}}(\underline{\theta}\underline{\varphi},\underline{z}\underline{y})$ is the p-function for particles of rank 1 only. The complete Bethe ansatz state is

$$\tilde{\Phi}_{\underline{\alpha}(\rho)}(\underline{\theta},\omega,\underline{z}) = \tilde{\Phi}_{\underline{\varsigma}(\lambda)}^{(N-2)(\eta)}(\underline{z}^{(N-2)},\omega,\underline{z}^{(N-1)})\ldots\tilde{\Phi}_{\underline{\beta}(\sigma)}^{(1)\underline{\gamma}(\kappa)}(\underline{z}^{(1)},\omega,\underline{z}^{(2)})\tilde{\Phi}_{\underline{\alpha}(\rho)}^{\underline{\beta}(\sigma)}(\underline{\theta},\omega,\underline{z}^{(1)})$$

where (η) is the highest weight bound state $(\eta) = (1,2,\ldots,N-1)$.

The energy momentum tensor: We apply the results above to the example of the energy momentum tensor and prove (34). In this case $n = \bar{n} = 1$, and the p-function is that of (31)

$$p^{T^{\rho\sigma}}(\theta,\omega,\underline{z}) = \left(e^{\rho\theta} + e^{\rho\omega}\right)e^{\sigma z}\,.$$

Lemma 3 *The functions $L_{\beta(\mu)}^{(l)}(z,\omega)$ (for all $l = 1,\ldots,N-3$) are explicitly given as*

$$L_{\beta(\mu)}^{(l)}(z,\omega) = \epsilon_{\beta(\mu)}L^{(l)}(\omega - z)\ \ with\ \beta > l,\ (\mu) = (1,2,\ldots,l,*,\ldots,*)$$

$$L^{(l)}(\omega - z) = c_l\,\Gamma\left(\frac{1}{2} + \frac{\omega - z}{2\pi i}\right)\Gamma\left(-\frac{1}{2} + \frac{l}{N} - \frac{\omega - z}{2\pi i}\right) \tag{50}$$

Proof. Again some equations are given up to unessential constants. We use induction, start with

$$L^{(N-1)}(z,\omega) = \epsilon_{N(1\ldots N-1)}\tilde{\chi}(\omega - z) = (-1)^{N-1}\Gamma\left(\frac{1}{2} + \frac{\omega - z}{2\pi i}\right)\Gamma\left(\frac{1}{2} - \frac{1}{N} - \frac{\omega - z}{2\pi i}\right),$$

and then calculate iteratively for $l = N - 1, \ldots, 2$ the integrals

$$L^{(l-1)}_{\beta(\mu)}(z,\omega) = \int_{\mathcal{C}_{z\omega}} du\,\tilde{\phi}(z - u)L^{(l)}(\omega - u)\epsilon_{\gamma(\nu)}\tilde{\Phi}^{(l-1)\gamma(\nu)}_{\beta(\mu)}(z,\omega,u)$$

$$\epsilon_{\gamma(\nu)}\tilde{\Phi}^{(l-1)\gamma(\nu)}_{\beta(\mu)}(z,\omega,u) = \epsilon_{\gamma(\nu)}\tilde{S}^{\gamma l}_{\beta\delta}(z - u)\tilde{S}^{\delta(\nu)}_{(\mu)l}(\omega - u)$$

$$= \epsilon_{\beta(\mu)}\left((N - l)\,\delta^l_\beta \tilde{b}(z - u)\tilde{d}(\omega - u) + \delta^{>l}_\beta \tilde{c}(z - u)\right)$$

where $\delta^{>l}_\beta = 1$ for $\beta > l$ and 0 else. Both integrals

$$I_1 = \int_{\mathcal{C}_{z\omega}} du\tilde{\phi}(z - u)L^{(l)}(\omega - u)(N - l)\tilde{b}(z - u)\tilde{d}(\omega - u)$$

$$I_2 = \int_{\mathcal{C}_{z\omega}} du\tilde{\phi}(z - u)L^{(l)}(\omega - u)\tilde{c}(z - u)$$

can be calculated by means of the formula

$$\int_{-\infty}^{\infty} dz\Gamma\left(a + \frac{z}{2\pi i}\right)\Gamma\left(b + \frac{z}{2\pi i}\right)\Gamma\left(c - \frac{z}{2\pi i}\right)\Gamma\left(d - \frac{z}{2\pi i}\right)$$

$$= (2\pi)^2\,\frac{\Gamma(a + c)\,\Gamma(a + d)\,\Gamma(b + c)\,\Gamma(b + d)}{\Gamma(a + b + c + d)}$$

and yield the result (50). ∎

Finally we have to calculate

$$K_{\alpha(\lambda)}(\theta,\omega) = \left(e^{\rho\theta} + e^{\rho\omega}\right)\int_{\mathcal{C}_{\theta\omega}} dz\,\tilde{\phi}(\theta - z)L^{(1)}(\omega - z)e^{\sigma z}\epsilon_{\delta(\mu)}\tilde{\Phi}^{\delta(\mu)}_{\alpha(\lambda)}(\theta,\omega,z)$$

$$\epsilon_{\delta(\mu)}\tilde{\Phi}^{\delta(\mu)}_{\alpha(\lambda)}(\theta,\omega,z) = \epsilon_{\alpha(\lambda)}\left((N - 1)\,\delta^1_\alpha \tilde{b}(\theta - z)\tilde{d}(\omega - z) + \delta^{>1}_\alpha \tilde{c}(\theta - z)\right)$$

which yields the result (34) using the formula

$$\int_{\mathcal{C}}\left(\Gamma(a + \frac{z}{2\pi i})\Gamma(b + \frac{z}{2\pi i})\Gamma(c - \frac{z}{2\pi i})\Gamma(d - \frac{z}{2\pi i})\right)e^{\sigma z}dz = \frac{\sigma(2\pi i)^3}{ab - cd}\exp\left(\sigma i\pi(c + d)\right)$$

for $a + b + c + d = 0$.

B   Commutation rules

In this appendix we use the short notation for form factors i.e. matrix elements of the field $\psi(x)$ at $x = 0$

$$\psi_{\underline{\alpha}}^{\underline{\beta}}(\underline{\theta}'_\beta, \underline{\theta}_\alpha) = F^{\psi\,\underline{\beta}}_{\underline{\alpha}}(\underline{\theta}'_\beta, \underline{\theta}_\alpha) = {}^{\underline{\beta},out}\langle\, \underline{\theta}'_\beta|\, \psi(0)|\, \underline{\theta}_\alpha\,\rangle^{in}_{\underline{\alpha}}\,. \tag{51}$$

To proof the general commutation rules of fields (47) we have to consider $SU(N)$ sum rules and general crossing relations.

B.1 $SU(N)$ sum rules

Particles and anti-particles: Let $(\alpha) = (\alpha_1, \ldots, \alpha_r)$, $(1 \le \alpha_1 < \cdots < \alpha_r \le N)$ a particle of rank (and charge) r. The corresponding anti-particle is $(\bar\alpha) = (\bar\alpha_1, \ldots, \bar\alpha_{N-r})$, $(1 \le \bar\alpha_1 < \cdots < \bar\alpha_{N-r} \le N)$ (of rank $N - r$) such that the union of the set of indices satisfies $\{\alpha_1, \ldots, \alpha_r\} \cup \{\bar\alpha_1, \ldots, \bar\alpha_{N-r}\} = \{1, \ldots, N\}$. Therefore

$$\sum_{k=1}^{r} \alpha_k + \sum_{k=1}^{N-r} \bar\alpha_k = \sum_{k=1}^{N} k = \frac{1}{2}N(N+1)\,. \tag{52}$$

Charges: Let $\underline{\alpha} = ((\alpha_{11}, \ldots, \alpha_{1r_1}), \ldots, (\alpha_{\alpha 1}, \ldots, \alpha_{\alpha r_\alpha}))$ be a state of α particles of rank $r_1, \ldots, r_\alpha$ $(1 \le r_j \le N-1)$ (or bound states of r_j particles of rank 1). We define the charge of a state as the sum of all ranks of the particles in the state $\underline{\alpha}$

$$Q_\alpha = \sum_{j=1}^{\alpha} r_j\,.$$

The charge of anti-particles (bound states) we define as

$$Q_{\bar\alpha} = \sum_{j=1}^{\alpha} r_j(N-1) = (N-1)Q_\alpha$$

$$Q_\alpha + Q_{\bar\alpha} = NQ_\alpha\,.$$

Weights: Let $\underline{\alpha} = ((\alpha_{11}, \ldots, \alpha_{1r_1}), \ldots, (\alpha_{\alpha 1}, \ldots, \alpha_{\alpha r_\alpha}))$ be a state of α particles. The weight $w_i(\underline{\alpha})$ of the state $\underline{\alpha}$ is equal to the number of $\alpha_{jk} = i$, $(1 \le i \le N)$

$$w_i(\underline{\alpha}) = \sum_{j=1}^{\alpha}\sum_{k=1}^{r_j} \delta_{i\alpha_{jk}}\,, \quad (1 \le i \le N).$$

Therefore the total charge of the state $\underline{\alpha}$ is

$$Q_\alpha = \sum_{j=1}^{\alpha} r_j = \sum_{j=1}^{\alpha}\sum_{k=1}^{r_j} 1 = \sum_{j=1}^{\alpha}\sum_{k=1}^{r_j}\sum_{i=1}^{N} \delta_{i\alpha_{jk}} = \sum_{i=1}^{N} w_i(\underline{\alpha})$$

Similarly, we consider $\underline{\gamma} = ((\gamma_{11}, \ldots, \gamma_{1s_1}), \ldots, (\gamma_{\gamma 1}, \ldots, \gamma_{\gamma s_\gamma}))$.

Sum rules: Because of $SU(N)$ invariance

$$\psi_{\underline{\alpha}}^{\underline{\gamma}}(\underline{\theta}'_\gamma, \underline{\theta}_\alpha) = {}^{\gamma, out}\langle \underline{\theta}'_\gamma | \psi(0) | \underline{\theta}_\alpha \rangle_{\underline{\alpha}}^{in} \neq 0$$

(or $\psi_{\underline{\alpha}\bar{\gamma}} \neq 0$) implies for the weights

$$w(\underline{\alpha}) = w(\underline{\gamma}) + w^\psi + L\,(1,\dots,1)\,,\ \ L \in \mathbb{Z}$$

where w^ψ is the weight vector of the operator ψ and $(1,\dots,1)$ are weights of a state in the vacuum sector. Therefore

$$Q_\alpha = \sum_{i=1}^N w_i(\underline{\alpha}) = Q_\gamma + \sum_{i=1}^N w_i^\psi + NL\,.$$

The charge of the operator ψ is defined by

$$Q_\psi = (Q_\alpha - Q_\gamma)\,\mathrm{mod}\,N,\ \ 0 \le Q_\psi < N \tag{53}$$

$$= \sum_{i=1}^N w_i^\psi \,\mathrm{mod}\,N\,.$$

For a particle $(\alpha_{j1}, \dots, \alpha_{jr_j})$ of rank r_j we use the short notation $(\alpha_j) = (\alpha_{j1}, \dots, \alpha_{jr_j})$ and $\alpha_j = \sum_{k=1}^{r_j} \alpha_{jk}$ and correspondingly, $\gamma_j = \sum_{k=1}^{s_j} \gamma_{jk}$. Then $SU(N)$ invariance implies

$$\sum_{j=1}^{\alpha} \alpha_j - \sum_{j=1}^{\gamma} \gamma_j = \tfrac{1}{2}(N+1)\,(Q_\alpha - Q_\gamma - Q_\psi) + R_\psi$$

$$\text{with}\qquad R_\psi = \sum_{i=1}^N i\,w_i^\psi - \tfrac{1}{2}(N+1)\left(\sum_{i=1}^N w_i^\psi - Q_\psi\right) \tag{54}$$

which can be straightforwardly proved using the above definitions.

Examples:

$$\begin{array}{llll}
T^{\mu\nu}: & w^T = (0,0,\dots,0) & Q_T = 0 & R_T = 0 \\
\psi_\alpha: & w^\psi = (1,0,\dots,0) & Q_\psi = 1 & R_\psi = 1 \\
j^{\mu\nu}: & w^J = (2,1,\dots,1,0) & Q_j = 0 & R_j = 1 - N
\end{array}$$

B.2   Crossing

B.2.1   A partial S-matrix

Definition 4   Let $\underline{\theta}_\beta = (\theta_{\pi(1)},\dots,\theta_{\pi(\alpha)})$ be a permutation of $\underline{\theta}_\alpha = (\theta_1,\dots,\theta_\alpha)$. Then $S_{\underline{\alpha}}^{\beta}(\underline{\theta}_\beta; \underline{\theta}_\alpha)$ is the matrix representation of the permutation group $\mathcal{S}_\alpha$ generated by the simple transpositions $\sigma_{ij} : i \leftrightarrow j$ for any pair of nearest neighbor indices $1 \le i, j = i+1 \le \alpha$ as[2]

$$\sigma_{ij} \rightarrow S(\theta_{ij})$$

---

[2]Note that this definition is quite analogous to that of representations of the braid group by means of spectral parameter independent R-matrices.

Because of the Yang-Baxter relation and unitarity of the S-matrix the representation is well defined. We will also use the notation

$$S_{\underline{\alpha}}^{\underline{\mu\lambda}}\left(\underline{\theta}_\mu\underline{\theta}_\lambda;\underline{\theta}_\alpha\right)$$

if π is that permutation which reorders the array $\underline{\theta}_\alpha$ such that it coincides with the combined arrays of $\underline{\theta}_\mu$ and $\underline{\theta}_\lambda$.

As an example consider the case $\underline{\theta}_\alpha = (\theta_1,\theta_2,\theta_3,\theta_4)$, $\underline{\theta}_\mu = (\theta_2,\theta_3)$ and $\underline{\theta}_\lambda = (\theta_1,\theta_4)$

$$S_{\underline{\alpha}}^{\underline{\mu\lambda}}(\theta_2\theta_3\theta_1\theta_4;\theta_1\theta_2\theta_3\theta_4) = S_{\alpha'_1\alpha_3}^{\mu_2\lambda_1}(\theta_{13})S_{\alpha_1\alpha_2}^{\mu_1\alpha'_1}(\theta_{12})\delta_{\alpha_4}^{\lambda_2}$$

If the permutation inverts the rapidities completely $S_{\underline{\alpha}}^{\underline{\beta}}\left(\underline{\theta}_\beta;\underline{\theta}_\alpha\right) = S_{\underline{\alpha}}^{\underline{\beta}}(\underline{\theta}_\alpha)$ is the full S-matrix.

B.2.2   Crossing for SU(N)

As was argued by Swieca et al. [8] the particles of the chiral $SU(N)$ Gross-Neveu model posses anyonic statistics and due to the unusual crossing property of the S-matrix, Klein factors are needed. The crossing relations of the form factors for normal fields and particles were derived in [28] by means of LSZ-assumptions and maximal analyticity. They have to be modified for the chiral $SU(N)$ Gross-Neveu model.

We propose crossing relations

$$\psi_{\underline{\alpha}}^{\underline{\gamma}}(\underline{\theta}'_\gamma;\underline{\theta}_\alpha) = \sigma_{(\gamma)}^\psi \sum_{\substack{\theta_\eta\cup\theta_\nu=\theta_\gamma\\\theta_\lambda\cup\theta_\mu=\theta_\alpha}} \zeta_{(\gamma,\alpha,\eta)}^\psi S_{\underline{\nu\eta}}^{\underline{\gamma}}(\underline{\theta}_\gamma;\underline{\theta}_\nu\underline{\theta}_\eta)\, \mathbf{1}_{\underline{\mu}}^{\underline{\nu}}\, \mathbf{C}^{\underline{\eta\bar\eta}}\psi_{\underline{\bar\eta\lambda}}(\underline{\theta}'_{\bar\eta}+i\pi_-,\underline{\theta}_\lambda)S_{\underline{\alpha}}^{\underline{\mu\lambda}}\left(\underline{\theta}_\mu\underline{\theta}_\lambda;\underline{\theta}_\alpha\right) \quad (55)$$

$$= \sum_{\substack{\theta_\eta\cup\theta_\nu=\theta_\gamma\\\theta_\lambda\cup\theta_\mu=\theta_\alpha}} \xi_{(\gamma,\alpha,\eta)}^\psi S_{\underline{\eta\nu}}^{\underline{\gamma}}(\underline{\theta}_\gamma;\underline{\theta}_\eta\underline{\theta}_\nu)\, \mathbf{C}^{\underline{\bar\eta\eta}}\psi_{\underline{\lambda\bar\eta}}\mathbf{1}_{\underline{\mu}}^{\underline{\nu}}S_{\underline{\alpha}}^{\underline{\lambda\mu}}\left(\underline{\theta}_\lambda\underline{\theta}_\mu;\underline{\theta}_\alpha\right) \quad (56)$$

which, compared to the formulae in [28], are modified by the factors $\zeta_{(\gamma,\alpha,\eta)}^\psi$ and $\xi_{(\gamma,\alpha,\eta)}^\psi$

$$\zeta_{(\gamma,\alpha,\eta)}^\psi = \rho_{(\gamma,\alpha)}^\psi e^{i\pi(N-1)\frac{1}{2}Q_\eta(Q_\eta+N)}e^{\frac{2\pi i}{N}\left(R_\psi Q_\alpha - Q_\psi\sum\bar\gamma_j\right)}$$

$$\xi_{(\gamma,\alpha,\eta)}^\psi = e^{i\pi(N-1)\left(\frac{1}{2}Q_\eta(Q_\eta+N)+Q_\nu Q_\psi\right)}e^{\frac{2\pi i}{N}\left(R_\psi Q_\alpha - Q_\psi\sum\bar\gamma_j\right)}$$

$$\rho_{(\gamma,\alpha)}^\psi = (-1)^{\left(N-1+(1-1/N)\left(Q_\alpha+Q_{\bar\gamma}-Q_\psi\right)\right)Q_{\bar\gamma}}$$

$$\sigma_{(\gamma)}^\psi = e^{i\pi(1-1/N)Q_\psi Q_{\bar\gamma}}$$

with $\bar\gamma_j = \frac{1}{2}N(N+1) - \gamma_j$, due to (52). The sign factor $\rho_{(\gamma,\alpha)}^\psi$ and the statistics factor $\sigma_{(\gamma)}^\psi$ were introduced in [13]. The charge Q_ψ of the operator ψ and the number R_ψ are defined in (53) and (54).

B.3   Commutation rules

In [31] commutation rules were derived for the $Z(N)$ scaling Ising models. The results for the $SU(N)$ Gross-Neveu model are very similar, however the proof is more complicated because of the unusual crossing relations and the presence of the Klein factors.

Theorem 5 *The equal time commutation rule of two fields $\phi(x)$ and $\psi(y)$ with charge Q_ϕ and Q_ψ, respectively, is (in general anyonic)*

$$\phi(x)\psi(y) = \psi(y)\phi(x) \exp\left(2\pi i\epsilon(x^1 - y^1)\tfrac{1}{2}\left(1 - 1/N\right)Q_\phi Q_\psi\right) .$$

Proof. We consider an arbitrary matrix element of products of fields

$$(\phi(x)\psi(y))^{\underline{\beta}}_{\underline{\alpha}}(\underline{\theta}'_\beta, \underline{\theta}_\alpha) = {}^{\beta, out}\langle\, \underline{\theta}'_\beta|\, \phi(x)\psi(y)|\, \underline{\theta}_\alpha\,\rangle^{in}_{\underline{\alpha}} .$$

Inserting a complete set of intermediate states $|\,\underline{\tilde{\theta}}_\gamma\,\rangle^{in}_{\underline{\gamma}}$ we obtain

$$(\phi(x)\psi(y))^{\underline{\beta}}_{\underline{\alpha}}(\underline{\theta}'_\beta, \underline{\theta}_\alpha) = e^{iP'_\beta x - iP_\alpha y}\frac{1}{\gamma!}\int_{\underline{\tilde{\theta}}_\gamma} \phi^{\underline{\beta}}_{\underline{\gamma}}(\underline{\theta}'_\beta, \underline{\tilde{\theta}}_\gamma)\psi^{\underline{\gamma}}_{\underline{\alpha}}(\underline{\tilde{\theta}}_\gamma, \underline{\theta}_\alpha)e^{-i\tilde{P}_\gamma(x-y)} \qquad (57)$$

where P_α = the total momentum of the state $|\,\underline{\theta}_\alpha\,\rangle^{in}_{\underline{\alpha}}$ etc. and $\int_{\underline{\tilde{\theta}}_\gamma} = \prod_{k=1}^{\gamma}\int\frac{d\tilde{\theta}_k}{4\pi}$. Einstein summation convention over all sets $\underline{\gamma}$ is assumed. We also define $\gamma! = \prod_{r=1}^{N} n_r!$ where n_r is the number of particles of rank r in $\underline{\gamma}$. We apply the general crossing formulae (55,56). Strictly speaking, we apply the second version (56) of the crossing formula to the matrix element of ϕ

$$\phi^{\underline{\beta}}_{\underline{\gamma}}(\underline{\theta}'_\beta, \underline{\tilde{\theta}}_\gamma) = \sum_{\substack{\underline{\theta}'_\rho \cup \underline{\theta}'_\tau = \underline{\theta}'_\beta \\ \underline{\tilde{\theta}}_\varsigma \cup \underline{\tilde{\theta}}_\sigma = \underline{\tilde{\theta}}_\gamma}} \xi^{\phi}_{(\beta,\gamma,\rho)} S^{\underline{\beta}}_{\underline{\rho}\underline{\tau}}\, \phi_{\underline{\varsigma}\underline{\bar{\rho}}}(\underline{\tilde{\theta}}_\varsigma, \underline{\theta}'_{\bar{\rho}} - i\pi_-)\mathbf{C}^{\underline{\rho}\underline{\bar{\rho}}}\mathbf{1}^{\underline{\tau}}_{\underline{\sigma}}\, S^{\underline{\varsigma}\underline{\sigma}}_{\underline{\gamma}}$$

where $\underline{\bar{\rho}} = (\bar{\rho}_\rho, \ldots, \bar{\rho}_1)$ with $\bar{\rho}$ = antiparticle of ρ and $\underline{\theta}'_{\bar{\rho}} - i\pi_-$ means that all rapidities taken the values $\theta' - i\,(\pi - \epsilon)$. The matrix $\mathbf{1}^{\underline{\tau}}_{\underline{\sigma}}(\underline{\theta}'_{\tau}, \underline{\tilde{\theta}}_{\varkappa})$ is defined by (51) with $\mathcal{O} = \mathbf{1}$ the unit operator. Summation is over all decompositions of the sets of rapidities $\underline{\theta}'_\beta$ and $\underline{\tilde{\theta}}_\gamma$. To the matrix element of ψ we apply the first version of the crossing formula (55)

$$\psi^{\underline{\gamma}}_{\underline{\alpha}}(\underline{\tilde{\theta}}_\gamma, \underline{\theta}_\alpha) = \sigma^{\psi}_{(\gamma,\alpha)} \sum_{\substack{\underline{\theta}_\nu \cup \underline{\theta}_\eta = \underline{\theta}_\gamma \\ \underline{\theta}_\mu \cup \underline{\theta}_\lambda = \underline{\theta}_\alpha}} \zeta^{\psi}_{(\gamma,\alpha,\eta)} S^{\underline{\gamma}}_{\underline{\nu}\underline{\eta}}\, \mathbf{1}^{\underline{\nu}}_{\underline{\mu}}\, \mathbf{C}^{\underline{\eta}\underline{\bar{\eta}}}\psi_{\underline{\bar{\eta}}\underline{\lambda}}(\underline{\theta}'_{\bar{\eta}} + i\pi_-, \underline{\theta}_\lambda)S^{\underline{\mu}\underline{\lambda}}_{\underline{\alpha}} .$$

We insert (55) and (56) in (57) and use the product formula $S^{\underline{\varsigma}\underline{\sigma}}_{\underline{\gamma}}(\underline{\tilde{\theta}}_\varsigma\underline{\tilde{\theta}}_\sigma; \underline{\tilde{\theta}}_\gamma)S^{\underline{\gamma}}_{\underline{\nu}\underline{\eta}}(\underline{\tilde{\theta}}_\gamma; \underline{\tilde{\theta}}_\nu\underline{\tilde{\theta}}_\eta) = S^{\underline{\varsigma}\underline{\sigma}}_{\underline{\nu}\underline{\eta}}(\underline{\tilde{\theta}}_\varsigma\underline{\tilde{\theta}}_\sigma; \underline{\tilde{\theta}}_\nu\underline{\tilde{\theta}}_\eta)$. Let us first assume that the sets rapidities in the initial state $\underline{\theta}_\alpha$ and the ones of the final state $\underline{\theta}'_\beta$ have no common elements. This also implies $\underline{\tilde{\theta}}_\nu \cap \underline{\tilde{\theta}}_\sigma = \emptyset$. Then we may use (ii) to get $S^{\underline{\varsigma}\underline{\sigma}}_{\underline{\nu}\underline{\eta}}(\underline{\tilde{\theta}}_\varsigma\underline{\tilde{\theta}}_\sigma; \underline{\tilde{\theta}}_\nu\underline{\tilde{\theta}}_\eta) = \delta^{\underline{\varsigma}\underline{\sigma}}_{\underline{\nu}\underline{\pi}}$ and then we can perform the $\underline{\tilde{\theta}}_\nu$- and $\underline{\tilde{\theta}}_\sigma$-integrations. The remaining $\tilde{\theta}$-integration variables are $\underline{\tilde{\theta}}_\omega = \underline{\tilde{\theta}}_\varsigma \cap \underline{\tilde{\theta}}_\eta$. Then we may

write for the sets of particles $\underline{\varsigma} = \underline{\mu\omega}$, $\underline{\eta} = \underline{\omega\tau}$ and $\underline{\gamma} = \underline{\mu\omega\tau}$ and similar for rapidities and momenta. Equation (57) simplifies to

$$(\phi(x)\psi(y))^{\beta}_{\underline{\alpha}}(\underline{\theta}'_{\beta},\underline{\theta}_{\alpha}) = \sum_{\substack{\underline{\theta}'_{\rho}\cup\underline{\theta}'_{\tau}=\underline{\theta}'_{\beta} \\ \underline{\theta}_{\mu}\cup\underline{\theta}_{\lambda}=\underline{\theta}_{\alpha}}} \frac{\mu!\tau!}{\mu\omega\tau!}S^{\beta}_{\underline{\rho\tau}}(\underline{\theta}'_{\rho},\underline{\theta}'_{\tau})\int_{\underline{\tilde{\theta}}_{\omega}} X^{\underline{\rho\tau}}_{\underline{\mu\lambda}}$$

$$\times\, S^{\underline{\mu\lambda}}_{\underline{\alpha}}(\underline{\theta}_{\alpha})e^{i\left(P'_{\rho}-P_{\mu}\right)x-i(P_{\lambda}-P'_{\tau})y} \quad (58)$$

where

$$X^{\underline{\rho\tau}}_{\underline{\mu\lambda}} = \sigma^{\psi}_{(\gamma,\alpha)}\zeta^{\psi}_{(\gamma,\alpha,\eta)}\xi^{\phi}_{(\beta,\gamma,\rho)}\phi_{\underline{\mu\omega\bar{\rho}}}(\underline{\theta}_{\mu},\underline{\tilde{\theta}}_{\omega},\underline{\theta}'_{\bar{\rho}}-i\pi_{-})$$

$$\times\, \mathbf{C}^{\underline{\bar{\rho}\rho}}\mathbf{C}^{\underline{\tau\bar{\tau}}}\mathbf{C}^{\underline{\omega\bar{\omega}}}\psi_{\underline{\bar{\tau}\bar{\omega}\lambda}}(\underline{\theta}'_{\bar{\tau}}+i\pi_{-},\underline{\tilde{\theta}}_{\bar{\omega}}+i\pi_{-},\underline{\theta}_{\lambda})e^{-i\tilde{P}_{\omega}(x-y)}. \quad (59)$$

Similarly, if we apply for the operator product $\psi(y)\phi(x)$ and

$$(\psi(y)\phi(x))^{\beta}_{\underline{\alpha}}(\underline{\theta}'_{\beta},\underline{\theta}_{\alpha}) = e^{iP'_{\beta}x-iP_{\alpha}y}\frac{1}{\delta!}\int_{\underline{\tilde{\theta}}_{\delta}} \psi^{\beta}_{\underline{\delta}}(\underline{\theta}'_{\beta},\underline{\tilde{\theta}}_{\delta})\phi^{\delta}_{\underline{\alpha}}(\underline{\tilde{\theta}}_{\delta},\underline{\theta}_{\alpha})e^{-i\tilde{P}_{\delta}(y-x)},$$

use the second crossing formula to the matrix element of ϕ

$$\phi^{\delta}_{\underline{\alpha}}(\underline{\tilde{\theta}}_{\delta},\underline{\theta}_{\alpha}) = \sum_{\substack{\underline{\tilde{\theta}}_{\varphi}\cup\underline{\tilde{\theta}}_{\kappa}=\underline{\tilde{\theta}}_{\delta} \\ \underline{\theta}_{\mu}\cup\underline{\theta}_{\lambda}=\underline{\theta}_{\alpha}}} \xi^{\phi}_{(\delta,\alpha,\varphi)}S^{\delta}_{\underline{\varphi\kappa}}\,\phi_{\underline{\mu\bar{\rho}}}(\underline{\theta}_{\mu},\underline{\tilde{\theta}}_{\bar{\rho}}-i\pi_{-})\mathbf{C}^{\underline{\varphi\bar{\rho}}}\mathbf{1}^{\kappa}_{\underline{\lambda}}S^{\underline{\mu\lambda}}_{\underline{\alpha}}$$

and the first one to the matrix element of ψ

$$\psi^{\beta}_{\underline{\delta}}(\underline{\theta}'_{\beta},\underline{\tilde{\theta}}_{\delta}) = \dot{\sigma}^{\psi}_{(\beta,\delta)}\sum_{\substack{\underline{\theta}'_{\rho}\cup\underline{\theta}'_{\tau}=\underline{\theta}'_{\beta} \\ \underline{\tilde{\theta}}_{\xi}\cup\underline{\tilde{\theta}}_{\chi}=\underline{\tilde{\theta}}_{\delta}}} \zeta^{\psi}_{(\beta,\delta,\tau)}S^{\beta}_{\underline{\rho\tau}}\mathbf{1}^{\rho}_{\underline{\xi}}\mathbf{C}^{\underline{\tau\bar{\tau}}}\psi_{\underline{\bar{\tau}\chi}}(\underline{\theta}'_{\bar{\tau}}+i\pi_{-},\underline{\tilde{\theta}}_{\chi})S^{\underline{\xi\chi}}_{\underline{\delta}}.$$

Similarly we obtain (with $\underline{\chi} = \underline{\bar{\omega}\lambda}$, $\underline{\varphi} = \underline{\bar{\omega}\rho}$, $\underline{\delta} = \underline{\mu\bar{\omega}\tau}$) equation (58) where $X^{\underline{\rho\tau}}_{\underline{\mu\lambda}}$ replaced by

$$Y^{\underline{\rho\tau}}_{\underline{\mu\lambda}} = \sigma^{\psi}_{(\beta,\delta)}\zeta^{\psi}_{(\beta,\delta,\tau)}\xi^{\phi}_{(\delta,\alpha,\varphi)}\phi_{\underline{\mu\omega\bar{\rho}}}(\underline{\theta}_{\mu},\underline{\tilde{\theta}}_{\omega}-i\pi_{-},\underline{\theta}'_{\bar{\rho}}-i\pi_{-})$$

$$\times\, \mathbf{C}^{\underline{\bar{\rho}\rho}}\mathbf{C}^{\underline{\tau\bar{\tau}}}\mathbf{C}^{\underline{\omega\bar{\omega}}}\psi_{\underline{\bar{\tau}\bar{\omega}\lambda}}(\underline{\theta}'_{\bar{\tau}}+i\pi_{-},\underline{\tilde{\theta}}_{\bar{\omega}},\underline{\theta}_{\lambda})e^{i\tilde{P}_{\omega}(x-y)} \quad (60)$$

which means that only $\sigma^{\psi}_{(\gamma,\alpha)}\zeta^{\psi}_{(\gamma,\alpha,\eta)}\xi^{\phi}_{(\beta,\gamma,\rho)}$ is replaced by $\sigma^{\psi}_{(\beta,\delta)}\zeta^{\psi}_{(\beta,\delta,\tau)}\xi^{\phi}_{(\delta,\alpha,\varphi)}$ and the integration variables $\underline{\tilde{\theta}}_{\omega}$ by $\underline{\tilde{\theta}}_{\bar{\omega}}-i\pi_{-}$, i.e. $\tilde{P}_{\omega}$ by $-\tilde{P}_{\omega}$.

If there were no bound states, there would be no singularities in the physical strip and we could shift in the matrix element of $\psi(y)\phi(x)$ (58) with (60) for equal times and $x^1 < y^1$ the integration variables by $\tilde{\theta}_i \to \tilde{\theta}_i + i\pi_{-}$. Note that the factor $e^{i\tilde{P}_{\omega}(x-y)}$ decreases

for $0 < \mathrm{Re}\,\tilde{\theta}_i < \pi$ if $x^1 < y^1$. Because $\tilde{P}_\omega \to -\tilde{P}_\omega$ (if $\underline{\tilde{\theta}}_\omega \to \underline{\tilde{\theta}}_{\bar{\omega}} - i\pi_-$) we get the matrix element of $\phi(x)\psi(y)$ (58) with (59) up to the factor

$$\frac{\sigma^\psi_{(\gamma,\alpha)}\zeta^\psi_{(\gamma,\alpha,\eta)}\xi^\phi_{(\beta,\gamma,\rho)}}{\sigma^\psi_{(\beta,\delta)}\zeta^\psi_{(\beta,\delta,\tau)}\xi^\phi_{(\delta,\alpha,\varphi)}} = e^{-2\pi i\frac{1}{2}\left(1-\frac{1}{N}\right)Q_\phi Q_\psi} \;.$$

This equality follows after a long and cumbersome but straightforward calculation. In [31] was shown that we obtain the same result if there are bound states. ∎

References

[1] G. 't Hooft, A planar diagram theory for strong interactions, Nuclear Physics B **72**, 461–473 (1974).

[2] G. 't Hooft, A two dimensional model for mesons, Nuclear Physics B **75**, 461–470 (1974).

[3] A. V. Manohar, Large N QCD, (1998).

[4] D. J. Gross and A. Neveu, Dynamical symmetry breaking in asymptotically free field theories, Phys. Rev. **D10**, 3235 (1974).

[5] E. Witten, Chiral symmetry, the 1/N expansion, and the SU(N) Thirring model, Nucl. Phys. **B145**, 110 (1978).

[6] V. Kurak and J. A. Swieca, Anti-particles as bound states of particles in the factorized S-matrix framework, Phys. Lett. **B82**, 289–291 (1979).

[7] E. Abdalla, B. Berg, and P. Weisz, More about the S-matrix of the chiral SU(N) Thirring model, Nucl. Phys. **B157**, 387–391 (1979).

[8] R. Koberle, V. Kurak, and J. A. Swieca, Scattering theory and 1/N expansion in the chiral Gross- Neveu model, Phys. Rev. **D20**, 897–902 (1979).

[9] E. Abdalla, M. C. B. Abdalla, and K. D. Rothe, Nonperturbative methods in two-dimensional quantum field theory, Singapore, Singapore: World Scientific (2001) p. 832.

[10] M. Karowski, Exact S matrices and form-factors in (1+1)-dimensional field theoretic models with soliton behavior, Phys. Rept. **49**, 229–237 (1979).

[11] M. Karowski, The bootstrap program for 1+1 dimensional field theoretic models with soliton behavior, in: W. Rühl (Ed.), Field theoretic methods in particle physics, Plenum, New York, (1980) , 307–324, Presented at Kaiserslautern NATO Inst. 1979.

[12] M. Karowski and P. Weisz, Exact form factors in (1+1)-dimensional field theoretic models with soliton behavior, Nucl. Phys. **B139**, 455–476 (1978).

[13] H. M. Babujian, A. Foerster, and M. Karowski, The nested SU(N) off-shell Bethe ansatz and exact form factors, J. Phys. **A41**, 275202 (2008).

[14] F. Smirnov, Form Factors in Completely Integrable Models of Quantum Field Theory, Adv. Series in Math. Phys. **14**, World Scientific (1992).

[15] A. Nakayashiki and Y. Takeyama, On form factors of SU(2) invariant Thirring model, math-ph/0105040 (2001).

[16] Y. Takeyama, Form factors of SU(N) invariant Thirring model, Publ. Res. Inst. Math. Sci. Kyoto **39**, 59–116 (2003).

[17] E. Mukhin, V. Tarasov, and A. Varchenko, Bethe Eigenvectors of Higher Transfer Matrices, math.QA/0605015 (2006).

[18] S. Pakuliak, Weight Functions and Nested Bethe Ansatz, Ann. Hernri Poincare **7**, 1541–1554 (2007).

[19] B. Berg, M. Karowski, V. Kurak, and P. Weisz, Factorized U(n) symmetric S matrices in two-dimensions, Nucl. Phys. **B134**, 125–132 (1978).

[20] B. Berg and P. Weisz, Exact S-matrix of the chiral invariant SU(N) Thirring model, Nucl. Phys. **B146**, 205–214 (1978).

[21] M. Karowski, On the bound state problem in (1+1) dimensional field theories, Nucl. Phys. **B153**, 244–252 (1979).

[22] H. M. Babujian, Correlation function in WZNW model as a Bethe wave function for the Gaudin magnetics, In 'Gosen 1990, Proceedings, Theory of elementary particles' 12-23. (see high energy physics index 29 (1991) No. 12257) , 12–23 (1990).

[23] H. Babujian, M. Karowski, and A. Zapletal, Matrix Difference Equations and a Nested Bethe Ansatz, J. Phys. **A30**, 6425–6450 (1997).

[24] L. D. Faddeev, E. K. Sklyanin, and L. A. Takhtajan, The quantum inverse problem method. 1, Theor. Math. Phys. **40**, 688–706 (1980).

[25] L. A. Takhtajan and L. D. Faddeev, The Quantum method of the inverse problem and the Heisenberg XYZ model, Russ. Math. Surveys **34**, 11–68 (1979).

[26] M. Karowski and H. J. Thun, Complete S matrix of the O(2N) Gross-Neveu model, Nucl. Phys. **B190**, 61–92 (1981).

[27] H. M. Babujian, A. Fring, M. Karowski, and A. Zapletal, Exact form factors in integrable quantum field theories: The sine-Gordon model, Nucl. Phys. **B538**, 535–586 (1999).

[28] H. Babujian and M. Karowski, Exact form factors in integrable quantum field theories: The sine-Gordon model. II, Nucl. Phys. **B620**, 407–455 (2002).

[29] H. Babujian and M. Karowski, Sine-Gordon form factors and quantum field equations, J. Phys. **A35**, 9081–9104 (2002).

[30] H. Babujian and M. Karowski, Exact form factors for the scaling Z(N)-Ising and the affine A(N-1) Toda quantum field theories, Phys. Lett. **B575**, 144–150 (2003).

[31] H. Babujian, A. Foerster, and M. Karowski, Exact form factors in integrable quantum field theories: The scaling Z(N)-Ising model, Nucl. Phys. **B736**, 169–198 (2006).

[32] H. Babujian, M. Karowski, and A. Zapletal, U(N) Matrix Difference Equations and a Nested Bethe Ansatz, hep-th/9611006 (1996).

[33] R. Koberle, V. Kurak, and J. A. Swieca, Erratum: Scattering theory and 1/N expansion in the chiral Gross- Neveu model, Phys. Rev. **D20**, 2638 (1979).

[34] R. Konik and A. W. W. Ludwig, Exact zero temperature correlation functions for two leg Hubbard ladders and carbon nanotubes, cond-mat/9810332 (1998).