Exact form factors for the scaling Z_N-Ising and the affine A_{N-1}-Toda quantum field theories

H. Babujian [*†] and M. Karowski [‡]
Institut für Theoretische Physik
Freie Universität Berlin,
Arnimallee 14, 14195 Berlin, Germany

September 12, 2003

Abstract

Previous results on form factors for the scaling Ising and the sinh-Gordon models are extended to general Z_N-Ising and affine A_{N-1}-Toda quantum field theories. In particular result for order, disorder parameters and para-fermi fields $\sigma_Q(x)$, $\mu_{\tilde{Q}}(x)$ and $\psi_Q(x)$ are presented for the Z_N-model. For the A_{N-1}-Toda model form factors for exponentials of the Toda fields are proposed. The quantum field equation of motion is proved and the mass and wave function renormalization are calculated exactly.

PACS: 11.10.-z; 11.10.Kk; 11.55.Ds
Keywords: Integrable quantum field theory, Form factors

In the framework of the bootstrap program the quantum field theories are defined by the S-matrices. However, we may motivate the models as follows: The $Z(N)$-Ising quantum field theory in 1+1 dimensions is considered as the scaling limit of a classical statistical lattice model in 2-dimensions given by the partition function

$$Z=\sum_{\{\sigma\}}\exp\left(-\frac{1}{kT}\sum_{\langle ij\rangle}E(\sigma_i,\sigma_j)\right)\ ;\ \sigma_i\in\left\{1,\omega,\dots,\omega^{N-1}\right\},\ \omega=e^{2\pi i/N}$$

as a generalization of the Ising model. It was conjectured by Köberle and Swieca [1] that there exists a $Z(N)$-invariant interaction $E(\sigma_i,\sigma_j)$ such that the resulting quantum field theory is integrable. This model has also been discussed as a deformation [2, 3] of a conformal Z_N para-fermi field theory [4]. The classical A_{N-1}-Toda model is defined by the Lagrangian

$$\mathcal{L}=\frac{1}{2}\left(\partial_\mu\vec{\varphi}\right)^2+\frac{\alpha}{\beta^2}\sum_{j=0}^{N-1}\exp\left(\beta\vec{\alpha}_j\vec{\varphi}\right)$$

[*]Permanent address: Yerevan Physics Institute, Alikhanian Brothers 2, Yerevan, 375036 Armenia.
[†]e-mail: babujian@lx2.yerphi.am, babujian@physik.fu-berlin.de
[‡]e-mail: karowski@physik.fu-berlin.de

where $\vec{\varphi} = (\varphi^1, \ldots, \varphi^{N-1})$ are real fields, $\vec{\alpha}_j$ ($|\vec{\alpha}_j| = \sqrt{2}$) are the simple positive A_{N-1}-roots and $\vec{\alpha}_0 = -\sum_{j=1}^{N-1} \vec{\alpha}_j$. The field equations are

$$\Box \vec{\alpha}_j \vec{\varphi} + \frac{\alpha}{\beta}\left(2e^{\beta\vec{\alpha}_j\vec{\varphi}} - e^{\beta\vec{\alpha}_{j+1}\vec{\varphi}} - e^{\beta\vec{\alpha}_{j-1}\vec{\varphi}}\right) = 0\,. \tag{1}$$

The Z_N-Ising model in the scaling limit and the affine A_{N-1}-Toda model posses the same particle content. There are $N-1$ types of particles: $a = 1, \ldots, N-1$ of charge a, mass $m_a = M \sin \pi \frac{a}{N}$ and $\bar{a} = N - a$ is the antiparticle of a. In particular for $N = 2$ the scaling Z_2-Ising model is the well investigated model [5, 6] which is equivalent to a massive free Dirac field theory. The affine A_1-Toda model is the sinh-Gordon model which is equivalent to the lowest breather sector of the sine-Gordon model for imaginary couplings. The n-particle S-matrices factorize in terms of two-particle ones since the models are integrable. The two-particle S-matrix for the Z_N-Ising model has been proposed by Köberle and Swieca [1]. The scattering of two particles of type 1 is

$$S^Z(\theta) = \frac{\sinh \frac{1}{2}(\theta + \frac{2\pi i}{N})}{\sinh \frac{1}{2}(\theta - \frac{2\pi i}{N})}\,. \tag{2}$$

The two-particle S-matrix for the A_{N-1}-Toda model has been proposed by Arinshtein, Fateev and Zamolodchikov [7] (see also [8]). The scattering of two particles of type 1 is

$$S^T(\theta) = S^Z(\theta) \frac{\sinh \frac{1}{2}(\theta - \frac{\pi i}{N}B)}{\sinh \frac{1}{2}(\theta + \frac{\pi i}{N}B)} \frac{\sinh \frac{1}{2}(\theta - \frac{\pi i}{N}(2-B))}{\sinh \frac{1}{2}(\theta + \frac{\pi i}{N}(2-B))}\,,\ B = \frac{2\beta^2}{4\pi + \beta^2}. \tag{3}$$

Both S-matrices are consistent with the picture that the bound state of $N-1$ particles of type 1 is the anti-particle of 1. This will be essential also for the construction of form factors below.

The form factor bootstrap program has been applied in [6] to the Z_2-model. Form factors for the Z_3-model were investigated by one of the present authors in [9]. There the form factors of the order parameter σ_1 were proposed up to four particles. Kirilov and Smirnov [10] proposed all form factors of the Z_3-model in terms of determinants. For general N form factors for chargeless states (n particles of type 1 and n particles of type $N-1$) were calculated in [11]. Low particle number form factors of A_{N-1}-Toda models[1] where investigated by Destri and De Vega [12], Oota [13] and Lukyanov [14]. In the present letter we present integral representations for all matrix elements of field operators for the Z_N-Ising and the A_{N-1}-Toda models.

For the Z_N-model we consider the fields $\psi_{Q\tilde{Q}}(x)$, $(Q, \tilde{Q} = 0, \ldots, N-1)$ with charge Q, spin $Q\tilde{Q}/N$ and statistics factor[2] (with respect to the particle $a = 1$) $e^{2\pi i\tilde{Q}/N}$. There are in particular the order parameters $\sigma_Q(x) = \psi_{Q0}(x)$, the disorder parameters $\mu_{\tilde{Q}}(x) = \psi_{0\tilde{Q}}(x)$ and the para-fermi fields $\psi_Q(x) = \psi_{QQ}(x)$ (for $Q = 1, \ldots, N-1$). They satisfy the space like commutation rules:

$$\begin{aligned}
\sigma_Q(x)\sigma_{Q'}(y) &= \sigma_{Q'}(y)\sigma_Q(x) \\
\mu_{\tilde{Q}}(x)\mu_{\tilde{Q}'}(y) &= \mu_{\tilde{Q}'}(y)\mu_{\tilde{Q}}(x) \\
\sigma_Q(x)\mu_{\tilde{Q}}(y) &= \mu_{\tilde{Q}}(y)\sigma_Q(x)e^{\theta(x^1-y^1)2\pi i Q\tilde{Q}/N} \\
\psi_Q(x)\psi_Q(y) &= \psi_Q(y)\psi_Q(x)e^{\epsilon(x^1-y^1)2\pi i Q^2/N}\,.
\end{aligned} \tag{4}$$

[1] For other Toda models see [26, 27] and for sinh-Gordon also [28, 29].

[2] Another model with nontrivial statistics is the Federbusch model (see e.g. [15]).

For the A_{N-1}-Toda models we present integral representations for all matrix elements of normal ordered exponentials of the fields

$$:\exp\left(\gamma_1\varphi^1+\cdots+\gamma_{N-1}\varphi^{N-1}\right):$$

where the φ^a are the fundamental Toda fields.

The generalized form factors $\mathcal{O}_n(\theta_1,\dots,\theta_n)$ are defined by the vacuum - n-particle matrix elements

$$\langle\, 0\,|\,\mathcal{O}(x)\,|\,p_1,\dots,p_n\,\rangle^{in}_{a_1\dots a_n}=e^{-ix(p_1+\cdots+p_n)}\,\mathcal{O}_{a_1\dots a_n}(\theta_1,\dots,\theta_n)$$

where the a_i denote the type (charge) and the θ_i are the rapidities of the particles ($p_i=m(\cosh\theta_i,\sinh\theta_i)$). This definition is meant for $\theta_1>\cdots>\theta_n$, in the other sectors of the variables the functions $\mathcal{O}_{a_1\dots a_n}(\theta_1,\dots,\theta_n)$ are given by analytic continuation with respect to the θ_i. General matrix elements are obtained from $\mathcal{O}_{\underline{a}}(\underline{\theta})$ by crossing which means in particular the analytic continuation $\theta_i\to\theta_i\pm i\pi$. The form factor equations which have to be solved are[3]:

(o) The form factor function $\mathcal{O}_{\underline{a}}(\underline{\theta})$ is meromorphic with respect to all variables $\theta_1,\dots,\theta_n$.

(i) It satisfies Watson's equations

$$\mathcal{O}_{\dots a_ia_j\dots}(\dots,\theta_i,\theta_j,\dots)=\mathcal{O}_{\dots a_ja_i\dots}(\dots,\theta_j,\theta_i,\dots)\,S_{a_ia_j}(\theta_{ij}).$$

(ii) The crossing relation means for the connected part (see e.g. [19]) of the matrix element

$$\begin{aligned}{}_{\bar a_1}\langle\, p_1\,|\,\mathcal{O}(0)\,|\,p_2,\dots,p_n\,\rangle^{in,conn.}_{a_2\dots a_n}&=\sigma_{\mathcal{O}1}\,\mathcal{O}_{a_1a_2\dots a_n}(\theta_1+i\pi,\theta_2,\dots,\theta_n)\\&=\mathcal{O}_{a_2\dots a_na_1}(\theta_2,\dots,\theta_n,\theta_1-i\pi)\end{aligned}$$

where $\sigma_{\mathcal{O}1}$ is the statistics factor of the operator $\mathcal{O}$ with respect to the particle 1.

(iii) The function $\mathcal{O}_n(\underline{\theta})$ has poles determined by one-particle states in each sub-channel. In particular, if 1 is the antiparticle of 2, it has the so-called annihilation pole at $\theta_{12}=i\pi$ which implies the recursion formula

$$\operatorname*{Res}_{\theta_{12}=i\pi}\mathcal{O}_n(\theta_1,\dots,\theta_n)=2i\,\mathcal{O}_{n-2}(\theta_3,\dots,\theta_n)\left(\mathbf{1}-\sigma_{\mathcal{O}1}S(\theta_{2n})\dots S(\theta_{23})\right).$$

(iv) Bound state form factors yield another recursion formula

$$\operatorname*{Res}_{\theta_{12}=iu}\mathcal{O}_{ab\dots}(\underline{\theta})=\sqrt{2}\mathcal{O}_{c\dots}(\theta_c,\underline{\theta}')\,\Gamma^c_{ab}$$

where Γ^c_{ab} is the bound state intertwiner (see e.g. [19]) defined by

$$i\operatorname*{Res}_{\theta_{12}=iu}S_{ab}(\theta)=\Gamma^{ba}_c\Gamma^c_{ab}$$

if iu is the position of the bound state pole.

[3]These formulae have been proposed in [17] as a generalization of formulae in [16] and they has been proven in [18] using LSZ assumptions.

(v) Since we are dealing with relativistic quantum field theories Lorentz covariance in the form

$$\mathcal{O}_n(\theta_1,\dots,\theta_n)=e^{s\mu}\,\mathcal{O}_n(\theta_1+\mu,\dots,\theta_n+\mu)$$

holds if the local operator transforms as $\mathcal{O}\to e^{s\mu}\mathcal{O}$ where s is the "spin" of $\mathcal{O}$.

We investigate generalized form factors of an operator $\mathcal{O}(x)$ and n particles of type 1 and for simplicity we write $\mathcal{O}_n(\underline{\theta})=\mathcal{O}_{1\dots1}(\underline{\theta})$. Note that all further matrix elements with different particle states of the field operator $\mathcal{O}(x)$ are obtained by the crossing formula (ii) and the bound state formula (iv). The form factors $\mathcal{O}_n(\underline{\theta})$ are of the form [16]

$$\mathcal{O}_n(\underline{\theta})=K_n^{\mathcal{O}}(\underline{\theta})\prod_{1\le i<j\le n}F(\theta_{ij}),\quad(\theta_{ij}=\theta_i-\theta_j) \tag{5}$$

where $F(\theta)$ is the 'minimal' form factor function. It is the solution of Watsons equation [20] and the crossing relation for $n=2$

$$\begin{aligned}F(\theta)&=F(-\theta)S(\theta)\\F(i\pi-\theta)&=F(i\pi+\theta)\end{aligned} \tag{6}$$

with no poles and zeros in the physical strip $0<\operatorname{Im}\theta\le\pi$ (and a simple zero at $\theta=0$). We obtain the solutions (see [16] for the procedure to solve (6))

$$F^Z(\theta)=c_Z\exp\int_0^\infty\frac{dt}{t}\,\frac{2\sinh t\frac{N-1}{N}\cosh t\frac{1}{N}}{\sinh^2 t}\left(1-\cosh t\left(1-\frac{\theta}{i\pi}\right)\right) \tag{7}$$

$$F^T(\theta)=\exp\int_0^\infty\frac{dt}{t}\,\frac{-4\sinh t\frac{N-1}{N}\sinh t\frac{B}{2N}\sinh t\frac{2-B}{2N}}{\sinh^2 t}\cosh t\left(1-\frac{\theta}{i\pi}\right) \tag{8}$$

for the Z_N and A_{N-1} S-matrices given by (2) and (3), respectively. The normalization constant c_Z will be fixed below and F^T is normalized by $F^T(\infty)=1$. In eq. (5) $K_n^{\mathcal{O}}(\underline{\theta})$ is a rational function of the e^{θ_i} and has the 'physical poles'[4] in $0<\operatorname{Im}\theta_{ij}\le\pi$ corresponding to the form factor properties (iii) and (iv). The form factor equations (i) and (ii) hold because of the equations (6). We propose $K_n^{\mathcal{O}}(\underline{\theta})$ as linear combinations of the integrals

$$\begin{aligned}I_{n\underline{m}}(\underline{\theta},p_n^{\mathcal{O}})=&\frac{1}{m_1!\dots m_{N-1}!}\left(\prod_{k=1}^{N-1}\prod_{j=1}^{m_k}\int_{\mathcal{C}_{\underline{\theta}}}\frac{dz_{kj}}{R}\right)\\&\times\prod_{k=1}^{N-1}\left(\prod_{j=1}^{m_k}\prod_{i=1}^{n}\phi(z_{kj}-\theta_i)\prod_{1\le i<j\le m_k}\tau(z_{ki}-z_{kj})\right)\\&\times\prod_{1\le k<l\le N-1}\prod_{i=1}^{m_k}\prod_{j=1}^{m_l}\kappa(z_{ki}-z_{lj})p_{n\underline{m}}^{\mathcal{O}}(\underline{\theta},\underline{z})\end{aligned} \tag{9}$$

with

$$\kappa(z)=\frac{1}{\phi(-z)},\ \tau(z)=\frac{1}{\phi(z)\phi(-z)},\ R=2\pi i\operatorname*{Res}_{z=0}\phi(z)\,.$$

[4] For bound state form factors there are also higher order 'physical poles' (see e.g. [21, 8, 26, 27]).

The integration contour $\mathcal{C}_{\underline{\theta}}$ encloses the points $z = \theta_i$. The p-functions $p^{\mathcal{O}}_{n\underline{m}}(\underline{\theta}, \underline{z})$ are symmetric with respect to the θ_i and the z_{kj} for fixed j. They depend on the operator $\mathcal{O}(x)$ whereas the other functions are determined by the S-matrix only. For both models the bound state of $N-1$ particles of type 1 is the anti-particle of 1. Together with the form factor recursion relations (iii) and (iv) this property implies the following equation for the function $\phi(z)$

$$\prod_{k=1}^{N-1} \phi\left(\theta + 2\pi i k/N\right) \prod_{k=0}^{N-1} F\left(\theta + 2\pi i k/N\right) = 1 \tag{10}$$

where $F(\theta)$ is the 'minimal' form factor function, here given by eqs. (7,8) for the Z_N- and the A_{N-1}-model, respectively. One easily verifies that the functions

$$\begin{aligned}
\phi^Z(z) &= \frac{1}{\sinh \frac{1}{2} z \sinh \frac{1}{2}(z - 2\pi i/N)} \\
\phi^T(z) &= \phi^Z(z) \sinh \tfrac{1}{2}\left(z - i\pi B/N\right) \sinh \tfrac{1}{2}\left(z - i\pi\left(2 - B\right)/N\right)
\end{aligned}$$

solve the equation (10) for the Z_N- and the A_{N-1}-model, respectively, and moreover they satisfy the 'Jost-function property'

$$\frac{\phi(\theta)}{\phi(-\theta)} = S(\theta)$$

Equation (10) also determines the normalization constant c_Z. The following lemma will be used to construct solutions of the form factor equations (i) – (v).

Lemma 1 *The functions $I_{n\underline{m}}(\underline{\theta}, p^{\mathcal{O}}_{n\underline{m}})$ satisfy the recursion relation*

$$\begin{aligned}
\operatorname*{Res}_{\theta_{N-1N}=iu} \cdots \operatorname*{Res}_{\theta_{12}=iu} I_{n\underline{m}}(\underline{\theta}, p^{\mathcal{O}}_{n\underline{m}}) &= c \left(\prod_{j=N+1}^{n} \prod_{i=1}^{N} F^{\min}_{11}(\theta_{ij}) \right)^{-1} \\
&\quad \times I_{n-N\underline{m}-1}(\underline{\theta}', p^{\mathcal{O}}_{n-N\underline{m}-1}) \left(1 - \sigma_{\mathcal{O}} \prod_{i=N+1}^{n} S(\theta_{Ni})\right)
\end{aligned} \tag{11}$$

if eq. (10) holds, the p-functions are analytic and satisfy for $\theta_{12} = \cdots = \theta_{N-1N} = iu = 2\pi i/N$

$$p^{\mathcal{O}}_{n\underline{m}}(\underline{\theta}, \underline{z}') = c' p^{\mathcal{O}}_{n-N\underline{m}-1}(\underline{\theta}', \underline{z}''') \tag{12}$$

$$p^{\mathcal{O}}_{n\underline{m}}(\underline{\theta}, \underline{z}'') = \sigma_{\mathcal{O}}\, p^{\mathcal{O}}_{n\underline{m}}(\underline{\theta}, \underline{z}') \tag{13}$$

where c and c' are a normalization constants, $\sigma_{\mathcal{O}}$ is the statistics factor of the operator $\mathcal{O}$ with respect to the particle of type 1 and

$$\begin{aligned}
\underline{z}' &= \underline{z}'' = \underline{z}\,;\ \ \text{except } z'_{i1} = \theta_i\,,\ z''_{11} = \theta_N\,,\ z''_{i+11} = \theta_i \\
\underline{z}''' &= \left(z_{12}, \ldots, z_{1m_1}; \ldots; z_{N-12}, \ldots, z_{N-1m_{N-1}}\right) \\
\underline{\theta}' &= \left(\theta_{N+1}, \theta_{N+2}, \ldots, \theta_n\right)\,.
\end{aligned}$$

The proof of this lemma will be published elsewhere [22]. We propose the following solutions of (i) – (v) and identify the operators by means of the quantum numbers as charge, spin and statistics factor. For the Toda model we also consider the asymptotic behavior and field equations.

The scaling Z_N-Ising model: The scaling Z_2-Ising model is well known [5, 6], it is equivalent to a free fermi field theory. As a generalization we propose for the general Z_N-model the form factors for n particles of type 1 (up to normalizations) as

$$\mathcal{O}_n(\underline{\theta}) = I^Z_{n\underline{m}}(\underline{\theta}) \prod_{1\le i<j\le n} F^Z(\theta_{ij})$$

where $I^Z_{n\underline{m}}(\underline{\theta})$ is defined by eq. (9) in terms of ϕ^Z. The p-functions and $\underline{m}$ are given by the following correspondences to operators. For $n = Nm + Q$ particles of type 1 and $Q, \tilde{Q} = 0, \dots, N-1$ the p-functions

$$p^{Q\tilde{Q}}_{n\underline{m}}(\underline{\theta}, \underline{z}) = \exp\left(\frac{\tilde{Q}}{N}\sum_{i=1}^{n}\theta_i - \sum_{k=1}^{\tilde{Q}}\sum_{j=1}^{m_k} z_{kj}\right) \tag{14}$$

(with $m_k = m+1$ for $1 \le k < Q,\ m_k = m$ for $Q \le k$) belong to operators

$$\psi_{Q,\tilde{Q}}(x) \quad \text{with} \left\{\begin{array}{ll} \text{charge} & Q \\ \text{spin } \bmod 1 & Q\tilde{Q}/N \\ \text{statistics factor} & e^{2\pi i\tilde{Q}/N} \end{array}\right. .$$

This follows from (v) and (13). Also (12) holds which means that the form factor equations (i) – (v) are satisfied. In particular we have for

$$\begin{array}{lll} \tilde{Q} = 0 & \text{the order parameters} & \sigma_Q(x) \\ Q = 0 & \text{the disorder parameter} & \mu_{\tilde{Q}}(x) \\ Q = \tilde{Q} & \text{the para-fermi fields} & \psi_Q(x) \end{array}$$

with the commutation rules (4). One can show [22] that for $N = 2$ the formulae (9) and (14) reproduce the well known results for the scaling Ising model as mentioned above. Note also that the properties of the fields for the Z_N Ising model are consistent with the results for the conformal Z_N para-fermi field theory [4]. For the higher conserved currents $J^{\pm}_L(x)$ see equations (18) and (19) below.

Examples: Up to normalization constants we calculate for general N for the order parameters

$$\begin{aligned} \langle\, 0\,|\,\sigma_1(0)\,|\,p\rangle &= 1 \\ \langle\, 0\,|\,\sigma_2(0)\,|\,p_1, p_2\rangle^{in} &= \frac{F^Z(\theta_{12})}{\sinh\frac{1}{2}(\theta_{12} - 2\pi i/N)\sinh\frac{1}{2}(\theta_{12} + 2\pi i/N)} \end{aligned}$$

and for $N = 3$

$$\begin{aligned} \langle\, 0\,|\,\sigma_1(0)\,|\,p_1, p_2, p_3, p_4\rangle^{in} = &\left(\sum e^{\theta_i}\sum e^{-\theta_i} - 1\right) \\ &\times \prod_{1\le i<j\le 4} \frac{F^Z(\theta_{ij})}{\sinh\frac{1}{2}(\theta_{ij} - 2\pi i/3)\sinh\frac{1}{2}(\theta_{ij} + 2\pi i/3)} \end{aligned}$$

which agrees with the results of [9, 10]. For disorder parameters we have

$$\langle\, 0\,|\,\mu_{\tilde{Q}}(0)\,|\,0\rangle = 1$$

and for $N = 3$ we obtain

$$\langle\, 0\,|\,\mu_{1,2}(0)\,|\,p_1,p_2,p_3\rangle^{in} = \left(e^{\pm\theta_1}+e^{\pm\theta_2}+e^{\pm\theta_3}\right)\; e^{\mp\frac{1}{3}\sum_{i=1}^{3}\theta_i} \\ \times \prod_{1\le i<j\le 3} \frac{F^Z(\theta_{ij})}{\sinh\frac{1}{2}(\theta_{ij}-2\pi i/3)\sinh\frac{1}{2}(\theta_{ij}+2\pi i/3)}\,.$$

For the para-fermi fields we have for example

$$\langle\, 0\,|\,\psi_1(0)\,|\,p\rangle = e^{\frac{1}{N}\theta}$$

$$\langle\, 0\,|\,\psi_2(0)\,|\,p_1,p_2\rangle^{in} = \frac{\left(e^{-\theta_1}+e^{-\theta_2}\right)e^{\frac{2}{N}(\theta_1+\theta_2)}F^Z(\theta_{12})}{\sinh\frac{1}{2}(\theta_{12}-2\pi i/N)\sinh\frac{1}{2}(\theta_{12}+2\pi i/N)}$$

and for $N = 3$

$$\langle\, 0\,|\,\psi_1(0)\,|\,p_1,p_2,p_3,p_4\rangle^{in} = e^{\frac{2}{3}\sum_{i=1}^{4}\theta_i}\sum_{i<j}e^{-\theta_i-\theta_j} \\ \times \prod_{1\le i<j\le 4} \frac{F^Z(\theta_{ij})}{\sinh\frac{1}{2}(\theta_{ij}-2\pi i/3)\sinh\frac{1}{2}(\theta_{ij}+2\pi i/3)}\,.$$

The affine A_{N-1}-Toda model: This model possesses only bosonic fields. Therefore we consider constant p-functions and generalize results of our investigations [23] on the sinh-Gordon model which is the A_1-Toda model. Again we consider first form factors for n particles of type 1. We write the K-function in (5) as a linear combination of the integrals $I^T_{n\underline{m}}(\underline{\theta},1)$ given by eq. (9) in terms of ϕ^T

$$K_n^{\mathcal{O}}(\underline{\theta}) = N_n \sum_{m_1=0}^{n}\cdots\sum_{m_{N-1}=0}^{n}\prod_{k=1}^{N-1} q_k^{m_k}\, I^T_{n\underline{m}}(\underline{\theta},1) \tag{15}$$

for $N-1$ parameters q_k. In particular $K_1(\underline{\theta}) = N_1\left(1+\sum_{k=1}^{N-1}q_k\right)$. We propose that this K-function yields the n-particle (charge 1) form factors of the general exponential of the fields

$$\mathcal{O}(x) = :\exp\vec{\gamma}\vec{\varphi}:(x)\,.$$

This is motivated as follows: In [23] (see also [16]) we argued that the form factors of exponentials of bose fields satisfy the momentum space 'clustering' property for $\operatorname{Re}\theta_1 \to \infty$

$$\left[e^{\gamma\varphi}\right]_n(\theta_1,\theta_2,\ldots,\theta_n) = \left[e^{\gamma\varphi}\right]_1(\theta_1)\;\left[e^{\gamma\varphi}\right]_{n-1}(\theta_2,\ldots,\theta_n) + O(e^{-\theta_1})\,.$$

The form factors given by (5) and (15) show just this asymptotic behavior because of

$$\lim_{\theta\to\infty} F^T(\theta) = 1$$

$$\lim_{\theta_1\to\infty} I^T_{n\underline{m}}(\underline{\theta}) = I^T_{n-1\underline{m}}(\underline{\theta}') + \sum_{k=1}^{N-1} I^T_{n-1\underline{m}_k}(\underline{\theta}')$$

where $\underline{\theta}' = (\theta_2, \ldots, \theta_n)$ and $\underline{m}_k = m_1, \ldots, m_k - 1, \ldots, m_{N-1}$. The normalization constants are obtained from this equation and (11) as $N_n = N_1^n$ and

$$N_1 = \left(\frac{2i}{r}\right)^{1/2} \left(\frac{1}{F^T(iu)} \prod_{k=2}^{N-1} \frac{\left(\phi^T(kiu)\right)^{1/2}}{\left(\phi^T(kiu) F^T(kiu)\right)^k} \prod_{k=1}^{N-1} q_k^{-1} \right)^{1/N} \tag{16}$$

where $u = \frac{2\pi}{N}$, $r = \operatorname*{Res}_{z=iu} \phi^T(z)$ and the intertwiner $\Gamma_{k1}^{k+1} = i \left(\frac{r}{i\phi^T(-kiu)}\right)^{1/2}$ have been used.

The exponentials of the special linear combination of fields $\vec{\gamma}\vec{\varphi}(x) = \gamma \vec{\alpha}_j \vec{\varphi}(x)$ are obtained for the choice

$$q_k = \omega^{-k} \omega^{N - \hat{\gamma}\left(\delta_{kj} - \delta_{kj+1}\right)}. \tag{17}$$

(For $j = 0$ and $N - 1$ there are extra factors $\omega^{\pm\hat{\gamma}}$). As in [24, 23] we obtain the fields $\vec{\alpha}_j \vec{\varphi}(x)$ by expansion in terms of γ. The quantum version of the field equation (1) is satisfied for all matrix elements if we take $\hat{\gamma} = \gamma \frac{B}{2\beta}$ and relate the renormalized and the bare mass by

$$m^2 = 4\,\alpha\, \sin^2 \frac{\pi}{N} \frac{2N \sin \frac{\pi}{2N} B \sin \frac{\pi}{2N}(2 - B)}{\pi B \sin \frac{\pi}{N}}$$

which agrees with the results of [14, 25]. The proof is analog to the one in [23]. The proposal (17) is in addition motivated by the fact that the vacuum 1-particle (of charge b) matrix element for the field with charge a turns out to be proportional to δ_{ab} [5]

$$\langle\, 0 \,|\, \varphi^a(0) \,|\, p \,\rangle_b = \frac{-1}{\sqrt{N} 2 \sin \frac{a\pi}{N}} \sum_{j=0}^{N-1} \omega^{ja} \langle\, 0 \,|\, \vec{\alpha}_j \vec{\varphi}(0) \,|\, p \,\rangle_b = \delta_{ab} \sqrt{Z^a}\,.$$

The form factors of the bound states $|\, p \,\rangle_b$ are obtained from (15) with (17) by applying iteratively (iv) as

$$\begin{aligned} \langle\, 0 \,|\, :\!\exp \gamma \vec{\alpha}_j \vec{\varphi}\!:(0) \,|\, p \,\rangle_b &= \frac{\beta}{\pi B} \sqrt{N Z^b} \chi_b(\gamma) \\ \chi_b(\gamma \vec{\alpha}_j) &= \prod_{k=1}^{N-1} q_k^{-b/N} \sum_{0 \le k_1 < \cdots < k_b < N} q_{k_1} \ldots q_{k_b} \\ &= -4 \omega^{-bj} \frac{\sin \frac{b\pi}{N}}{\sin \frac{\pi}{N}} \sin\left(\frac{\pi}{N} \frac{B\gamma}{2\beta}\right) \sin \frac{\pi}{N} \left(1 - \frac{B\gamma}{2\beta}\right). \end{aligned}$$

The function χ_b may be written as a character $\chi_b(\gamma \vec{\alpha}_j) = \operatorname{tr}_{\pi_b} \omega^{\left(\vec{\rho} - \frac{B}{2\beta}\gamma \vec{\alpha}_j\right)\vec{H}}$ [14]. By expansion with respect to γ we obtain

$$\langle\, 0 \,|\vec{\alpha}_j \vec{\varphi}(0) \,|\, p \,\rangle_b = -\sqrt{\frac{Z^b}{N}} \omega^{-bj}\, 2 \sin \frac{\pi b}{N}\,.$$

[5] This corresponds to the complex representation of the roots $\alpha_j^a = \frac{-1}{\sqrt{N}} 2 \sin \frac{a\pi}{N} \omega^{-ja}$ and fields with $(\varphi^a)^\dagger = \varphi^{N-a}$.

The wave function renormalization constants Z^a are calculated from (vi) and (16) as

$$Z^a = \frac{\pi B(2-B)}{4\sin\frac{\pi}{2}B} \exp \int_0^\infty \frac{dt}{t} \frac{2\sinh\frac{t}{2}\frac{B}{N}\sinh\frac{t}{2}\frac{2-B}{N}}{\sinh^2 t \sinh t\frac{1}{N}} \left(\sinh^2 t\frac{a}{N} + \sinh^2 t\frac{\bar a}{N}\right)$$

which satisfies the charge conjugation symmetry $a \leftrightarrow \bar a = N - a$ and agree with the results of [12, 14].

The higher conserved currents $J_L^\pm(x)$ which are characteristic for integrable models are obtained by (5) and (9) and the K- and p-function

$$K_n^{J_L^\pm}(\underline{\theta}) = \sum_{m_1=0}^{n} \cdots \sum_{m_{N-1}=0}^{n} \prod_{k=1}^{N-1} \omega^{km_k} I_{n\underline{m}}^T(\underline{\theta}, p_n^{J_L^\pm}) \tag{18}$$

$$p_{n\underline{m}}^{J_L^\pm}(\underline{\theta}, \underline{z}) = \sum_{i=1}^{n} e^{\pm\theta_i} \sum_{k=1}^{N-1} \sum_{j=1}^{m_k} e^{Lz_{kj}}\,. \tag{19}$$

The higher charges $Q_L = \int dx J_L^0(x)$ satisfy the eigenvalue equation

$$\left(Q_L - \sum_{i=1}^{n} e^{L\theta_i}\right) | \, p_1, \ldots, p_n \rangle^{in} = 0\,.$$

Since the currents are Z_N-chargeless the number n of particles of charge 1 is $0 \bmod N$. Obviously we get the energy momentum tensor from $J_{\pm1}^\pm(x)$. The higher conserved currents $J_L^\pm(x)$ for the Z_N-Ising model are obtained by same p-function with the additional factor $\prod_{k=1}^{N-1} \delta_{m_k m}$ for $n = Nm$. In a more detailed version of this letter [22], we will present the proofs and further results.

Acknowledgments

We thank V.A. Fateev, A. Fring, A. Nersesyan, R. Schrader, B. Schroer, J. Teschner, A. Tsvelik, Al.B. Zamolodchikov and A.B. Zamolodchikov for discussions. In particular we thank V.A. Fateev for bringing the preprint [10] to our attention and F.A. Smirnov for sending a copy. H.B. was supported by DFG, Sonderforschungsbereich 288 'Differentialgeometrie und Quantenphysik' and partially by the grants INTAS 99-01459 and INTAS 00-561. This work is also supported by the EU network EUCLID, 'Integrable models and applications: from strings to condensed matter', HPRN-CT-2002-00325.

References

[1] R.Köberle and J.A. Swieca, *Phys. Lett.* **B** 86 (1979) 209.

[2] A.B. Zamolodchikov, *Int. Journ. Mod. Phys.,* **A 3** (1988) 743-750.

[3] V.A. Fateev, *Int. Journ. Mod. Phys.,* **A 6** (1991) 2109-2132.

[4] V.A. Fateev and A.B. Zamolodchikov, *Sov. Phys. JETP,* **62** (1985) 215.

[5] R.Z. Bariev, *Phys. Lett.* **55A** (1976) 456;
B. McCoy, C.A. Tracy and T.T. Wu, *Phys. Rev. Lett.* **38** (1977) 783;
M. Sato, T. Miwa and M. Jimbo, *Proc. Japan Acad.* **53A** (1977) 6.

[6] B. Berg, M. Karowski and P. Weisz, *Phys. Rev.* **D19** (1979) 2477.

[7] A.E. Arinshtein, V.A. Fateev and A.B. Zamolodchikov, *Phys. Lett.* **B 87** (1979) 389.

[8] H.W Braden, E. Corrigan, P.E. Dorey and R. Sasaki, *Nucl. Phys.* **B338** (1990) 689-746.

[9] M. Karowski, *Field Theories in 1+1 Dimensions with Soliton Behaviour: Form Factors and Green's Functions*, in 'Lecture Notes in Physics 126' (Springer) (1979) p. 344.

[10] A.N. Kirilov and F.A. Smirnov, Kiev preprint ITF-88-13P (1988) (in Russian).

[11] M. Jimbo, H. Konno, S. Odake, Y. Pugai and J. Shiraishi, *Free field construction for the ABF models in Regime II,* (2000) math.QA/0001071.

[12] C. Destri and H.J. De Vega, *Nucl. Phys.* **B358** (1991) 251-294.

[13] T. Oota, *Nucl. Phys.* **B466** [FS] (1996) 361-382.

[14] S. Lukyanov, Phys. Lett. **B408** (1997) 182-200.

[15] O. Castro-Alvaredo and A. Fring, *Nucl. Phys.* **B618** (2001) 437.

[16] M. Karowski and P. Weisz, *Nucl. Phys.* **B139** (1978) 445.

[17] F.A. Smirnov *'Form Factors in Completely Integrable Models of Quantum Field Theory', Adv. Series in Math. Phys.* **14**, World Scientific 1992.

[18] H. Babujian, A. Fring, M. Karowski and A. Zapletal, *Nucl. Phys.* **B538** [FS] (1999) 535-586.

[19] H. Babujian and M. Karowski, *Nucl. Phys.* **B620** (2002) 407.

[20] K.M. Watson, *Phys. Rev.* **95** (1954) 228.

[21] S. Coleman and H.J. Thun, *Commun. Math. Phys.* **61** (1978) 31.

[22] H. Babujian and M. Karowski, in preparation.

[23] H. Babujian and M. Karowski, *Journ. Phys. A: Math. Gen.* **35** (2002) 9081-9104.

[24] H.M. Babujian and M. Karowski, *Phys. Lett.* **B 411** (1999) 53-57.

[25] C. Ahn, P. Baseilhac, V. Fateev, C. Kim and C. Rim, *Phys. Lett.* **B481** (2000).

[26] G. Delfino and G. Mussardo, *Nucl. Phys.* **B455** (1995) 724-758.

[27] C. Acerbi, G. Mussardo and A. Valleriani, *Int. J. Mod. Phys.* **A11** (1996) 5327-5364.

[28] A. Fring, G. Mussardo and P. Simonetti, *Nucl. Phys.* **B393** (1993) 413-441.

[29] A. Koubek and G. Mussardo, *Phys. Lett.* **B311** (1993) 193-201.