Towards the Construction of Wightman Functions of Integrable Quantum Field Theories*

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Abstract

The purpose of the "bootstrap program" for integrable quantum field theories in 1+1 dimensions is to construct a model in terms of its Wightman functions explicitly. In this article, this program is mainly illustrated in terms of the sine-Gordon and the sinh-Gordon model and (as an exercise) the scaling Ising model. We review some previous results on sine-Gordon breather form factors and quantum operator equations. The problem to sum over intermediate states is attacked in the short distance limit of the two point Wightman function for the sinh-Gordon and the scaling Ising model.

1 Introduction

The 'bootstrap program for integrable quantum field theories in 1+1-dimensions' [1, 2, 3] does <u>*not*</u> start with any classical Lagrangian. Rather this program classifies integrable quantum field theoretic models and in addition it provides their explicit exact solutions in term of all Wightman functions. We have contact with the classical models only, when at the end we compare our exact results with Feynman graph expansions which are based on the Lagrangians.

One of the authors (M.K.) et al. [4] formulated the on-shell program i.e. the exact determination of the scattering matrix using the Yang-Baxter equations. Off-shell quantities, namely form factors were first investigated by Vergeles and Gryanik [5] in the sinh-Gordon model and by Weisz [6] in the sine-Gordon model. The concept of generalized form factors was introduced by one of the authors (M.K.) et al. [7]. In this article

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consistency equations were formulated which are expected to be satisfied by these objects. Thereafter this approach was developed further and studied in the context of several explicit models by Smirnov [8] who proposed the form factor equations $(i)-(v)$ (see below) as extensions of similar formulae in the original article [7]. The formulae were proven by the authors et al. [9].

Finally the Wightman functions are obtained by taking integrals and sums over intermediate states. The explicit evaluation of all these integrals and sums remains an open challenge for almost all models, except the Ising model [10, 11, 12]. In this article we attack this problem for the examples of the sinh-Gordon model and, as an exercise, of the scaling Ising model. We investigate the short distance behavior of the two-point Wightman function of the exponential of the field.

2 The "bootstrap program"

As the final result the 'bootstrap program' for integrable quantum field theories in 1+1-dimensions provides a model in term of all Wightman functions. The result is obtained in three steps:

1. The S-matrix is calculated by means of general properties such as unitarity and crossing, the Yang-Baxter equations (which are a consequence of integrability) and the additional assumption of 'maximal analyticity'. This means that the two-particle S-matrix is an analytic function in the physical plane (of the Mandelstam variable $(p_1+p_2)^2$) and possesses only those poles there which are of physical origin. The only input which depends on the model is the assumption of a particle spectrum. Usually it belongs to representations of a symmetry. Typically there is a correspondence of fundamental representations with multiplets of particles. A *classification* of all S-matrices obeying the given properties is obtained.

2. Generalized form factors which are matrix elements of local operators
$$^{out}\langle\, p'_m,\dots,p'_1\,|\mathcal{O}(x)|\,p_1,\dots,p_n\,\rangle^{in}$$
are calculated by means of the S-matrix. More precisely, the equations $(i)-(v)$ as listed in section 3 are solved. These equations follow from LSZ-assumptions and again the additional assumption of 'maximal analyticity' [9].

3. The Wightman functions are obtained by inserting a complete set of intermediate states. In particular the two point function for a hermitian operator $\mathcal{O}(x)$ reads
$$\langle\, 0\,|\mathcal{O}(x)\,\mathcal{O}(0)|\,0\,\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}\int\cdots\int\frac{dp_1\dots dp_n}{(2\pi)^n 2\omega_1\dots 2\omega_n} \\ \times\left|\langle\, 0\,|\mathcal{O}(0)|\,p_1,\dots,p_n\,\rangle^{in}\right|^2 e^{-ix\sum p_i}.$$
Up to now a proof that these sums converge exists only for the scaling Ising model [10] and the 'Yang-Lee' model [13].

Integrability

Integrability in (quantum) field theories means that there exist ∞-many local conservation laws

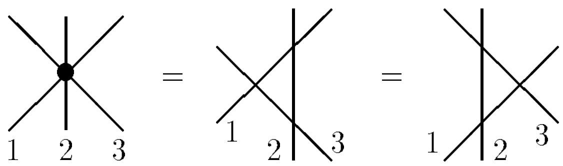
$$\partial_\mu J_L^\mu(t,x) = 0 \quad (L = \pm 1, \pm 3, \dots)\,.$$

A consequence of such conservation laws in 1+1 dimensions is that there is no particle production and the n-particle S-matrix is a product of 2-particle S-matrices

$$S^{(n)}(p_1, \dots, p_n) = \prod_{i<j} S_{ij}(p_i, p_j)\,.$$

If backward scattering occurs the 2-particle S-matrices will not commute and one has to specify the order. In particular for the 3-particle S-matrix there are two possibilities

$$\mathbf{S^{(3)} = S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12}}$$



which yield the **"Yang-Baxter Equation".**

Examples of integrable models in 1+1-dimensions are the **sine-Gordon** model defined by the classical field equation

$$\ddot\varphi(t,x) - \varphi''(t,x) + \frac{\alpha}{\beta}\sin\beta\varphi(t,x) = 0$$

and the massive Thirring model defined by the classical Lagrangian

$$\mathcal{L} = \bar\psi(i\gamma\partial - m)\psi - \tfrac{1}{2}g\ \bar\psi\gamma^\mu\psi\,\bar\psi\gamma_\mu\psi\,.$$

Coleman [14] proved that both models are equivalent on the quantum level.

Further integrable models are: Z_N-Ising models, nonlinear σ-models, Gross-Neveu models, Toda models etc. In the following most of the formulae and explicit solutions are given for the sine-Gordon alias massive Thirring model although often corresponding results exist also for other models.

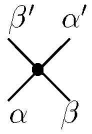
The S-matrix

For the Sine-Gordon alias massive Thiring model the particle spectrum consists of: soliton, anti-soliton and breathers (as soliton anti-soliton bound states). Since backward scattering can only appear for particles with the same mass, the two-particle S-matrix is of the form

$$
S(\theta_{12}) = \begin{pmatrix} u & & & & & & \\ & t & r & & & & \\ & r & t & & & & \\ & & & u & & & \\ & & & & S_{sb} & & \\ & & & & & S_{bb} & \\ & & & & & & \ddots \end{pmatrix}
$$

where the rapidity difference $\theta_{12} = |\theta_1 - \theta_2|$ is defined by $p_i = m_i(\cosh\theta_i, \sinh\theta_i)$.

We start with the soliton (anti-soliton) S-matrix:

$S^{\beta'\alpha'}_{\alpha\,\beta} =$ 

$: \quad S^{ss}_{ss} = u, \quad S^{\bar{s}s}_{s\bar{s}} = t, \quad S^{s\bar{s}}_{s\bar{s}} = r$

$s =$ soliton, $\bar{s} =$ anti-soliton. As input conditions we have:

1. Unitarity: $S(-\theta)S(\theta) = 1$

$$
\begin{aligned}
u(-\theta)u(\theta) &= 1 \\
t(-\theta)t(\theta) + r(-\theta)r(\theta) &= 1 \\
t(-\theta)r(\theta) + r(-\theta)t(\theta) &= 0
\end{aligned}
$$

2. Crossing:

$$
u(i\pi - \theta) = t(\theta)\ , \quad r(i\pi - \theta) = r(\theta)
$$

3. Yang-Baxter:

$$
r(\theta_{12})u(\theta_{13})r(\theta_{23}) + t(\theta_{12})r(\theta_{13})t(\theta_{23}) = u(\theta_{12})r(\theta_{13})u(\theta_{23})
$$

4. *Maximal analyticity:*

   $S(\theta)$ is meromorphic in the 'physical strip' $0 \le \mathrm{Im}\,\theta \le \pi$ and all poles there have a physical interpretation, in particular all bound states correspond to simple poles. An S-matrix satisfying this condition is also called 'minimal'. For couplings $g < 0$ (in the language of the massive Thirring model) there are no soliton anti-soliton bound states. Therefore in this region of the coupling constant $S(\theta)$ is holomorphic in the 'physical strip' $0 \le \mathrm{Im}\,\theta \le \pi$.

The S-matrix bootstrap using the Yang-Baxter relations was proposed by Karowski, Thun, Truong and Weisz [4]. It was shown in this article that the 'minimal' general solution to these equations is

$$
u(\theta, \nu) = \exp \int_0^\infty \frac{dt}{t} \frac{\sinh \frac{t}{2}(1-\nu)}{\sinh \frac{\nu t}{2}\, \cosh \frac{t}{2}} \sinh t \frac{\theta}{i\pi}\,.
$$

This S-matrix was first obtained by Zamolodchikov [15] from the extrapolation of semi-classical expressions. It has been checked in perturbation theory. The free parameter ν is related to the coupling constants by

$$\frac{1}{\nu} = \frac{8\pi}{\beta^2} - 1 = 1 + \frac{2g}{\pi}\,.$$

The second equation is due to Coleman and the first one is obtained by analyzing the pole structure of the amplitude $u(\theta,\nu)$. The assumption of 'maximal analyticity' and comparison with the known semi-classical bound state spectrum provides the identification of the parameter ν.

The two-breather S-matrix

$$S_{bb}(\theta) = \frac{\sinh\theta + i\sin\pi\nu}{\sinh\theta - i\sin\pi\nu} \tag{1}$$

is obtained by the bound state fusion method [16].

3 Form factors

Definition 1 *For a local operator $\mathcal{O}(x)$ the generalized form factors [7] are defined as*

$$\mathcal{O}_{\alpha_1\dots\alpha_n}(\theta_1,\dots,\theta_n) = \langle\, 0\,|\,\mathcal{O}(0)\,|\,p_1,\dots,p_n\,\rangle^{in}_{\alpha_1\dots\alpha_n}$$

for $\theta_1 > \dots > \theta_n$. For other orders of the rapidities they are defined by analytic continuation. The index α_i denotes the type of the particle with momentum p_i. We also use the short notations $\mathcal{O}_{\underline{\alpha}}(\underline{\theta})$ or $\mathcal{O}_{1\dots n}(\underline{\theta})$.

For the sine Gordon model α denotes the soliton, anti-soliton or breathers. Similar as for the S-matrix, 'maximal analyticity' for generalized form factors means again that they are meromorphic and all poles in the 'physical strips' $0 \le \operatorname{Im}\theta_i \le \pi$ have a physical interpretation. Together with the usual LSZ-assumptions [17] of local quantum field theory the following form factor equations can be derived

(i) Watson's equations [18]:

$$\mathcal{O}_{\dots ij\dots}(\dots,\theta_i,\theta_j,\dots) = \mathcal{O}_{\dots ji\dots}(\dots,\theta_j,\theta_i,\dots)\,S_{ij}(\theta_i-\theta_j)\,. \tag{2}$$

(ii) Crossing relation (for the connected part of the matrix element):

$$\begin{aligned} {}_{\bar\alpha_1}\langle\, p_1\,|\,\mathcal{O}(0)\,|\,p_2,\dots,p_n\,\rangle^{in,conn.}_{\alpha_2\dots\alpha_n} &= \mathcal{O}_{\alpha_1\alpha_2\dots\alpha_n}(\theta_1+i\pi,\theta_2,\dots,\theta_n) \\ &= \mathcal{O}_{\alpha_2\dots\alpha_n\alpha_1}(\theta_2,\dots,\theta_n,\theta_1-i\pi)\,. \end{aligned} \tag{3}$$

(*iii*) Recursion relation:

$$\operatorname*{Res}_{\theta_{12}=i\pi} \mathcal{O}_{1\dots n}(\theta_1,\dots) = 2i\,\mathbf{C}_{12}\;\mathcal{O}_{3\dots n}(\theta_3,\dots)\,(\mathbf{1}-S_{2n}\dots S_{23})\;, \tag{4}$$

where $\mathbf{C}_{12}$ is the charge conjugation matrix.

(*iv*) Bound state form factors equations:

$$\operatorname*{Res}_{\theta_{12}=ia} \mathcal{O}_{123\dots n}(\underline{\theta}) = \mathcal{O}_{(12)3\dots n}(\theta_{(12)},\underline{\theta}')\,\sqrt{2}\,\Gamma^{(12)}_{12}\,, \tag{5}$$

where a is the fusion angle and $\Gamma^{(12)}_{12}$ is the fusion intertwiner [19, 20].

(*v*) Lorentz invariance:

$$\mathcal{O}_{1\dots n}(\theta_1+u,\dots,\theta_n+u) = e^{su}\,\mathcal{O}_{1\dots n}(\theta_1,\dots,\theta_n)\,, \tag{6}$$

where s is the "spin" of $\mathcal{O}$.

These equations have been proposed by Smirnov [8] as generalizations of equations derived in the original articles [7, 11, 3]. They have been proven [9] by means of the LSZ-assumptions and 'maximal analyticity'. They hold in this form for bosons; for fermions or more generally for anyons there are some additional phase factors.

Two-particle form factors

For the two-particle form factors the form factor equations are easily understood. The usual assumptions of local quantum field theory yield

$$\langle\, 0\,|\,\mathcal{O}(0)\,|\,p_1\,,p_2\rangle^{in/out} = F\left((p_1+p_2)^2 \pm i\varepsilon\right) = \;F\left(\pm\theta_{12}\right)$$

where the rapidity difference is defined by $p_1 p_2 = m^2\cosh\theta_{12}$. For integrable theories one has particle number conservation which implies (for any eigenstate of the two-particle S-matrix)

$$\langle\, 0\,|\,\mathcal{O}(0)\,|\,p_1\,,p_2\rangle^{in} = \langle\, 0\,|\,\mathcal{O}(0)\,|\,p_2\,,p_1\rangle^{out}\,S\left(\theta_{12}\right)\;.$$

Crossing (3) means

$$\langle\, p_1\,|\,\mathcal{O}(0)\,|\,p_2\rangle = F\left(i\pi-\theta_{12}\right)$$

where for one-particle states in- and out-states coincide. Therefore Watson's equations follow

$$\begin{aligned} F\left(\theta\right) &= F\left(-\theta\right)S\left(\theta\right) \\ F\left(i\pi-\theta\right) &= F\left(i\pi+\theta\right)\;. \end{aligned}$$

For general theories Watson's [18] equations only hold below the particle production thresholds. However, for integrable theories there is no particle production and therefore they hold for all complex values of θ. It has been shown [7] that these equations together with "maximal analyticity" have a unique solution.

As an example we write the sine-Gordon two-breather form factor [7]

$$F(\theta) = \exp\int_0^\infty \frac{dt}{t\sinh t}\left(\frac{\cosh(\frac{1}{2}+\nu)t}{\cosh\frac{1}{2}t}-1\right)\cosh t\left(1-\frac{\theta}{i\pi}\right) \tag{7}$$

A formula for sine-Gordon breather form factors

We are looking for solutions of the form factor equations $(i)-(v)$. The generalized form factors for arbitrary numbers of breathers are of the form [7]

$$\langle\, 0 \,|\, \mathcal{O}(0)\, | p_1, \ldots , p_n \,\rangle^{in} = K_n^{\mathcal{O}}(\underline{\theta}) \prod_{1\leq i<j\leq n} F(\theta_{ij}) \tag{8}$$

where $F(\theta)$ is the two-breather form factor above and the 'K-function' satisfies Watsons equations with $S = 1$. We make the *Ansatz*[1]

$$K_n^{\mathcal{O}}(\underline{\theta}) = \sum_{l_1=0}^{1} \cdots \sum_{l_n=0}^{1} (-1)^{\sum l_i} \prod_{i<j} \left(1 + (l_i - l_j) \frac{i \sin \pi\nu}{\sinh \theta_{ij}}\right) p_n^{\mathcal{O}}(\underline{\theta}, \underline{l})\,. \tag{9}$$

The dependence of form factors on the operator $\mathcal{O}(x)$ enters only through the p-functions $p_n^{\mathcal{O}}(\underline{\theta}, \underline{z})$. This Ansatz transforms the form factor equations $(i)-(v)$ to simpler equations for the p-function $p_n^{\mathcal{O}}(\underline{\theta}, \underline{l})$ [23]. The p-function $p_n^{\mathcal{O}}(\underline{\theta}, \underline{l})$ is holomorphic with respect to all variables $\theta_1, \ldots, \theta_n$. It is symmetric with respect to the exchange of the variables θ_i and l_i at the same time and it is periodic with period $2\pi i$.

$$p_n^{\mathcal{O}}(\ldots, \theta_i, \theta_j, \ldots, l_i, l_j, \ldots) = p_n^{\mathcal{O}}(\ldots, \theta_j, \theta_i, \ldots, l_j, l_i, \ldots) \tag{10}$$

$$p_n^{\mathcal{O}}(\underline{\theta}, \underline{l}) = p_n^{\mathcal{O}}(\theta_1 - 2\pi i, \theta_2, \ldots, \theta_n, \underline{l}) \tag{11}$$

With the short hand notation $\underline{\theta}' = \theta_2, \ldots, \theta_n$, $\underline{\theta}'' = \theta_3, \ldots, \theta_n$ and $\underline{l}'' = l_3, \ldots, l_n$ the recursion relation

$$p_n^{\mathcal{O}}(\theta_2 + i\pi, \underline{\theta}', \underline{l}) = g(l_1, l_2) p_{n-2}^{\mathcal{O}}(\underline{\theta}'', \underline{l}'') + h(l_1, l_2) \tag{12}$$

holds where $g(0,1) = g(1,0) = 2/(F(i\pi) \sin \pi\nu)$ and $h(l_1, l_2)$ is independent of $\underline{l}''$. Lorentz covariance reads as

$$p_n^{\mathcal{O}}(\theta_1 + \mu, \ldots, \theta_n + \mu, \underline{l}) = e^{s\mu}\, p_n^{\mathcal{O}}(\theta_1, \ldots, \theta_n, \underline{l}). \tag{13}$$

These conditions of the p-function are sufficient to guarantee the properties of the form factors.

Theorem 2 *If the p-function $p_n^{\mathcal{O}}(\underline{\theta}, \underline{l})$ satisfies the conditions (10–13) the form factor function $\mathcal{O}_n(\underline{\theta})$ satisfies the properties (2–6).*

This theorem has been proven in [23].

[1] Using an integral representation [9, 21] for general soliton form factors we derived this formula in [21] for several local operators (see also [22]). Here we consider it as an Ansatz also for operators such as the general exponential of the breather field which is nonlocal with respect to the soliton field.

Examples of operators and their p-functions: For several cases the correspondence between local operators and their p-functions have been proposed in [23]. Here we provide three examples:

1. The normal ordered exponential of the field (see also [24])

$$\mathcal{O}(x) = :e^{i\gamma\varphi(x)}: \quad \leftrightarrow p(\underline{\theta}, \underline{l}) = \left(\frac{2}{F(i\pi)\sin\pi\nu}\right)^{\frac{n}{2}} \prod_{i=1}^{n} e^{i\pi\nu\frac{\gamma}{\beta}(-1)^{l_i}} \tag{14}$$

2. Expanding the last relation with respect to γ one obtains the p-functions for normal ordered powers $:\varphi(x)^N:$ in particular for $N = 1$

$$:\varphi(x): \quad \leftrightarrow p(\underline{\theta}, \underline{l}) = \frac{\pi\nu}{\beta}\left(\frac{2}{F(i\pi)\sin\pi\nu}\right)^{\frac{n}{2}} \sum_{i=1}^{n}(-1)^{l_i}$$

   which yields (for $n = 1$) the 'wave function renormalization constant'

$$Z^{\varphi} = \langle\, 0 \,|\, \varphi(0) \,|p\,\rangle^2 = \frac{8\pi^2\nu^2}{F(i\pi)\beta^2\sin\pi\nu}$$

   (see also [7]).

3. The higher conserved currents (which are typical for integrable quantum field theories)

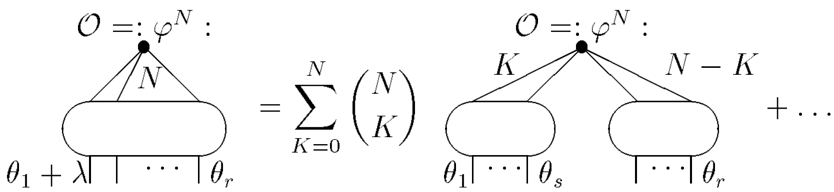
$$J_L^{\pm}(x) \leftrightarrow \pm N_n^{(J_L)} \sum_{i=1}^{n} e^{\pm\theta_i} \sum_{i=1}^{n} e^{L\left(\theta_i - \frac{i\pi}{2}(1-(-1)^{l_i}\nu)\right)}.$$

Asymptotic behavior of the form factors for $:e^{i\gamma\varphi}:$

Let $\mathcal{O} =: \varphi^N :$ be the normal ordered power of a bosonic field. Write the rapidities as $\underline{\theta} = \lambda\theta'_1, \ldots, \lambda\theta'_m, \theta''_1, \ldots, \theta''_{n-m}$ and consider the limit $\lambda \to \infty$. Then the asymptotic behavior of the n-boson form factor is

$$\begin{aligned}\left[\varphi^N\right]_n(\underline{\theta}) &= \langle\, 0 \,|\ :\varphi^N:(0)\,|p_1, \ldots, p_n\,\rangle^{in} \\ &= \sum_{K=0}^{N}\binom{N}{K}\left[\varphi^K\right]_m(\underline{\theta}')\,\left[\varphi^{N-K}\right]_{n-m}(\underline{\theta}'') + O(e^{-\lambda}).\end{aligned}$$

This can be proven in any order of perturbation theory as follows. The matrix element on the left hand side may be written in terms of Feynman graphs as



where all other graphs not drawn have lines which connect both parts directly. Weinberg's power counting theorem for bosonic Feynman graphs implies that these contributions decrease for $\lambda \to \infty$ as $O(\lambda^k e^{-\lambda})$. This behavior is also assumed to hold for the exact form factors (the fact is that the 'logarithmic terms' λ^k do not show up for the exact expressions since the K-functions are meromorphic in the e^{θ_i}). Therefore for the exponentials of the boson field $: e^{i\gamma\varphi} :$ we have the asymptotic behavior

$$\left[e^{i\gamma\varphi}\right]_n (\underline{\theta}) = \left[e^{i\gamma\varphi}\right]_m (\underline{\theta}') \left[e^{i\gamma\varphi}\right]_{n-m} (\underline{\theta}'') + O(e^{-\lambda}).$$

It is easy to see [20, 23] that our proposal (14) together with (8) and (9) satisfies this asymptotic behavior[2]. The asymptotic behavior of other form factors is more complicated [21] in particular if fermions are involved.

Quantum field operator equations

The sine-Gordon equation

We start with the local operator $:\sin\gamma\varphi:(x) = \frac{1}{2i} :(e^{i\gamma\varphi} - e^{-i\gamma\varphi}):(x)$. For the exceptional value $\gamma = \beta$ we find [21, 23] that also $\Box^{-1} :\sin\beta\varphi:(x)$ is local. Moreover the quantum sine-Gordon field equation

$$\Box\varphi(x) + \frac{\alpha}{\beta} :\sin\beta\varphi:(x) = 0 \tag{15}$$

holds for all matrix elements, if the "bare" mass $\sqrt{\alpha}$ is related to the renormalized mass by[3]

$$\alpha = m^2 \frac{\pi\nu}{\sin\pi\nu} \tag{16}$$

where m is the physical mass of the fundamental boson. The result may be checked in perturbation theory by Feynman graph expansions. In particular in lowest order the relation between the bare and the renormalized mass (16) had already been calculated in the original article [7]. The result is

$$m^2 = \alpha \left(1 - \frac{1}{6}\left(\frac{\beta^2}{8}\right)^2 + O(\beta^6)\right)$$

which agrees with the exact formula above.

Here is a sketch of the proof of the field equation [23] which uses induction and Liouville's theorem. Consider the K-functions of the left hand side of eq. (15)

$$f_n(\underline{\theta}) = -\sum e^{\theta_i} \sum e^{-\theta_i} K_n^{(1)}(\underline{\theta}) + \frac{\pi\nu}{\beta\sin\pi\nu}\frac{1}{2i}\left(K_n^{(q)}(\underline{\theta}) - K_n^{(1/q)}(\underline{\theta})\right).$$

The results of the previous section imply $f_1(\theta) = f_2(\underline{\theta}) = 0$ for $q = e^{i\pi\nu}$. As an induction assumption we take $f_{n-2}(\underline{\theta}'') = 0$. The function $f_n(\underline{\theta})$ is meromorphic in terms of the

[2] This type of arguments has been also used before [7, 25, 26, 27].

[3] Before such a formula was found [28, 29] by different methods.

$x_i = e^{\theta_i}$ with at most simple poles at $x_i = \pm x_j$ since $\sinh\theta_{ij} = (x_i + x_j)(x_i - x_j)/(2x_i x_j)$. The residues of the poles at $x_i = x_j$ vanish because of the symmetry under the exchange of $x_i \leftrightarrow x_j$. The residues at $x_i = -x_j$ are proportional to $f_{n-2}(\underline{\theta}'')$ because of the recursion relation (iii). Furthermore it can be shown [23] that $f_n(\underline{\theta}) \to 0$ for $x_i \to \infty$. Therefore $f_n(\underline{\theta})$ vanishes identically by Liouville's theorem.

The factor $\frac{\pi\nu}{\sin\pi\nu}$ in (16) modifies the classical equation and has to be considered as a quantum correction. For the sin- and sinh-Gordon model an analogous quantum field equation has been discussed previously [22, 27]. Note that in particular at the 'free fermion point' $\nu \to 1$ ($\beta^2 \to 4\pi$) this factor diverges, a phenomenon which is to be expected by investigations of the short distance behavior [30]. For fixed bare mass square α and $\nu \to 2, 3, 4, \ldots$ the physical mass goes to zero. These values of the coupling are known to be specific: 1) the Bethe Ansatz vacuum in the language of the massive Thirring model shows phase transitions [31] and 2) the model at these points is related [32, 33, 34] to Baxters RSOS-models which correspond to minimal conformal models with central charge $c = 1 - 6/(\nu(\nu+1))$.

The trace of the energy momentum tensor

As a further operator equation we find [20, 23] (see also [22]) that the trace of the energy momentum tensor satisfies

$$T^{\mu}_{\ \mu}(x) = -2\frac{\alpha}{\beta^2}\left(1 - \frac{\beta^2}{8\pi}\right)(:\cos\beta\varphi:(x) - 1)\,. \tag{17}$$

Again this operator equation is to be understood as a relation of all its matrix elements. The equation is modified compared to the classical one by a quantum correction $(1 - \beta^2/8\pi)$. As a consequence of this fact the model will be conformal invariant in the limit $\beta^2 \to 8\pi$ for fixed bare mass square α. This is related to a Berezinski-Kosterlitz-Thouless [35] phase transition.

This results may be checked again in perturbation theory by Feynman graph expansions. The quantum corrections of the trace of the energy momentum tensor (17) yields

$$\langle\, p \,|:\cos\beta\varphi:(0) - 1|\, p\,\rangle = -\beta^2\left(1 + \frac{\beta^2}{8\pi}\right) + O(\beta^6).$$

This again agrees with the exact formula above since the usual normalization for the energy momentum given by $\int dx\, T^{0\mu}(x)|\,p\,\rangle = p^\mu|\,p\,\rangle$ implies $\langle\, p\,|\,T^{\mu}_{\ \mu}|\,p\,\rangle = 2m^2$.

4 Wightman functions

As the simplest case we consider the two-point function of two local scalar operators $\mathcal{O}(x)$ and $\mathcal{O}'(x)$

$$w(x) = \langle\, 0\,|\,\mathcal{O}(x)\,\mathcal{O}'(0)\,|\,0\,\rangle.$$

Summation over all intermediate states

Inserting a complete set of in-states we may write

$$
\begin{aligned}
w(x) &= \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{dp_1}{2\pi 2\omega_1} \cdots \int \frac{dp_n}{2\pi 2\omega_n} e^{-ix(p_1+\cdots+p_n)} \\
&\quad \times \langle\, 0 \,|\, \mathcal{O}(0) \,|p_1, \ldots, p_n\,\rangle^{in\;in}\langle p_n, \ldots, p_1 \,|\, \mathcal{O}'(0) \,|\, 0\,\rangle \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \int d\theta_1 \cdots \int d\theta_n e^{-ix\sum p_i} g_n(\underline{\theta})\,.
\end{aligned}
$$

We have introduced the functions

$$
\begin{aligned}
g_n(\underline{\theta}) &= \frac{1}{(4\pi)^n} \langle\, 0 \,|\, \mathcal{O}(0) \,|p_1, \ldots, p_n\,\rangle^{in\;in}\langle p_n, \ldots, p_1 \,|\, \mathcal{O}'(0) \,|\, 0\,\rangle \\
&= \frac{1}{(4\pi)^n} \mathcal{O}(\theta_1, \ldots, \theta_n\,)\, \mathcal{O}'(\theta_n + i\pi, \ldots, \theta_1 + i\pi)\,.
\end{aligned}
$$

where crossing has been used. In particular we consider exponentials of a scalar bose field

$$
\mathcal{O}^{(\prime)}(x) = :e^{i\gamma^{(\prime)}\varphi(x)}:
$$

where $:\cdots:$ means normal ordering with respect to the physical vacuum which means that

$$
\langle\, 0 \,|\, :e^{i\gamma^{(\prime)}\varphi(x)}: \,|\, 0\,\rangle = 1
$$

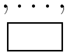
and therefore $g_0 = 1$.

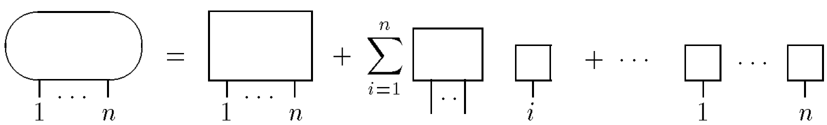
The Log of the two-point function For $g_0 = 1$ we may write (see also [22])

$$
\begin{aligned}
w(x) &= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int d\theta_1 \cdots \int d\theta_n e^{-ix\sum p_i} g_n(\underline{\theta}) \\
&= \exp \sum_{n=1}^{\infty} \frac{1}{n!} \int d\theta_1 \cdots \int d\theta_n e^{-ix\sum p_i} h_n(\underline{\theta})
\end{aligned}
$$

It is well known that the functions g_n and h_n are related by

$$
g_I = \sum_{I_1 \cup \cdots \cup I_k = I} h_{I_1} \ldots h_{I_k}
$$

where we use the short hand notation $g_I = g_n(\theta_1, \ldots, \theta_n)$ with $I = \{1, \ldots, n\}$. The relations of the g's and the h's may be depicted with $g =$  and $h =$  as



For example

$$
\begin{aligned}
g_1 &= h_1 \\
g_{12} &= h_{12} + h_1 h_2 \\
g_{123} &= h_{123} + h_{12}h_3 + h_{13}h_2 + h_{23}h_1 + h_1 h_2 h_3 \\
&\dots
\end{aligned}
$$

Due to Lorentz invariance it is sufficient to consider $x = (-i\tau, 0)$. Let $\mathcal{O}(x)$ and $\mathcal{O}'(x)$ be scalar operators. Then the functions $h_n(\underline{\theta})$ depend on the rapidity differences only. We use the formula for the modified Bessel function of the third kind

$$
i\Delta_+(x) = \langle\, 0 \,|\, \varphi(x)\, \varphi(0) \,|\, 0 \,\rangle = \frac{1}{4\pi} \int d\theta e^{-\tau m \cosh\theta} = \frac{1}{2\pi} K_0(m\tau)
$$

to perform one integration

$$
\begin{aligned}
\ln w(x) &= \sum_{n=1}^{\infty} \frac{1}{n!} \int d\theta_1 \cdots \int d\theta_n e^{-\tau m \sum \cosh\theta_i} h_n(\underline{\theta}) \\
&= 2 \sum_{n=1}^{\infty} \frac{1}{n!} \int d\theta_1 \cdots \int d\theta_{n-1} h_n(\theta_1, \dots, \theta_{n-1}, 0) K_0(m\tau\xi)
\end{aligned}
$$

with

$$
\xi^2 = \left(\sum_{i=1}^{n-1} \cosh\theta_i + 1 \right)^2 - \left(\sum_{i=1}^{n-1} \sinh\theta_i \right)^2 .
$$

Short distance behavior $\tau \to 0$

In order to perform the conformal limit of masive models one investigates the short distance behavior (see e.g. [36, 37, 22, 38, 39]). For small τ we use the expansion of the modified Bessel function of the third kind and obtain

$$
\begin{aligned}
\ln w(x) = -2 \sum_{n=1}^{\infty} \frac{1}{n!} &\int d\theta_1 \cdots \int d\theta_{n-1} h_n(\theta_1, \dots, \theta_{n-1}, 0) \\
&\times \left(\ln m\tau + \ln\xi + \gamma_E - \ln 2 + O\left(\tau^2 \ln\tau\right)\right)
\end{aligned}
$$

where $\gamma_E = 0.5772\ldots$ is Euler's or Mascheroni's constant. Therefore the two-point Wightman function has power like behavior for short distances

$$
w(x) \approx C\, (m\tau)^{-4\Delta} \quad \text{for } \tau \to 0
$$

where the dimension is given by

$$
\Delta = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n!} \int d\theta_1 \cdots \int d\theta_{n-1} h_n(\theta_1, \dots, \theta_{n-1}, 0)
$$

if the integrals exist. This is true for the exponentials of bose fields $\mathcal{O} =: e^{i\gamma\varphi(x)}:$ because of the asymptotic behavior for $\operatorname{Re}\theta_1 \to \infty$

$$\begin{aligned}\mathcal{O}_n(\theta_1,\theta_2,\dots) &= \mathcal{O}_1(\theta_1)\mathcal{O}_{n-1}(\theta_2,\dots) + O(e^{-\theta_1})\\ g_n(\theta_1,\theta_2,\dots\theta_n) &= g_1 g_{n-1}(\theta_2,\dots\theta_n) + O(e^{-|\theta_1|})\end{aligned}$$

as shown above. Therefore the functions h_n satisfy (see also [22])

$$h_n(\underline{\theta}) = O(e^{-|\theta_i|}) \quad \text{for } \operatorname{Re}\theta_i \to \pm\infty\,.$$

This follows when we distinguish in the relation of the g's and the h's above the variable θ_1 and reorganize the terms on the right hand side as follows

$$g_I = \sum_{1\in J\subseteq I} h_J g_{I\setminus J}\,.$$

The constant C is obtained as

$$C = \exp\left(-2\sum_{n=1}^{\infty}\frac{1}{n!}\int d\theta_1\cdots\int d\theta_{n-1} h_n(\theta_1,\dots,\theta_{n-1},0)\left(\ln\tfrac{1}{2}\xi + \gamma_E\right)\right)$$

and it should be related to the vacuum expectation value $G = \langle\, 0\,|\mathcal{O}(x)|\,0\,\rangle_C$ in the 'conformal normalization'

$$C = m^{4\Delta}G^{-2}\,.$$

Such vacuum expectation value was calculated in [40] for the sine-Gordon model.

Examples

For simplicity let us consider the case that both operator are the exponential of the field at the special value $\gamma = \gamma' = \beta$

$$\mathcal{O}(x) = \mathcal{O}'^{\dagger}(x) = :e^{i\beta\varphi(x)}:$$

1. The free case: For the free case $h_1 = \frac{1}{4\pi}|\beta|^2$ and all $h_n = 0$ for $n > 1$

$$\begin{aligned}w(x) &\approx e^{-\frac{1}{2\pi}(\gamma_E - \ln 2)|\beta|^2}(m\tau)^{-\frac{1}{2\pi}|\beta|^2} \quad \text{for } \tau \to 0\\ \Delta &= \frac{1}{8\pi}|\beta|^2\\ C &= e^{-\frac{1}{2\pi}(\gamma_E - \ln 2)|\beta|^2}\end{aligned}$$

which is a well known result.

2. The sinh-Gordon model The 2-particle S-matrix is obtained from the sine-Gordon breather S-matrix (1) for imaginary couplings β

$$S(\theta) = \frac{\sinh\theta + i\sin\pi\nu}{\sinh\theta - i\sin\pi\nu} \quad \text{with} \; -1 \le \nu = \frac{\beta^2}{8\pi - \beta^2} \le 0\,.$$

The 2-particle form factor function is given by (7) for $-1 \le \nu \le 0$. The sinh Gordon model has the self-dual point

$$\nu = -\frac{1}{2} \quad \text{or} \quad \beta^2 = -4\pi\,.$$

The dimension of the exponential of the field $\mathcal{O}(x) = :e^{i\beta\varphi(x)}:$ for the sinh-Gordon model is in the 1- and 1+2-particle intermediate state approximation (see Fig. 1)

$$\begin{aligned}\Delta_{1+2} &= \frac{1}{2}\left(h_1 + \frac{1}{2!}\int d\theta\, h_2(\theta,0) + \dots\right)\\ &= -\frac{\sin\pi\nu}{\pi F(i\pi)} + \left(\frac{\sin\pi\nu}{\pi F(i\pi)}\right)^2 \int_{-\infty}^{\infty} d\theta\, \left(F(\theta)F(-\theta) - 1\right) + \dots\end{aligned}$$

The integral may be calculated exactly with the result [41]

$$-\frac{\pi}{2}\sin\pi\nu F^2(i\pi) - \pi\frac{\cos\pi\nu - 1}{\sin\pi\nu} + 2\left(1 - \frac{\pi\nu\cos\pi\nu}{\sin\pi\nu}\right)$$

In principle the higher particle intermediate state integrals may also be calculated, how-

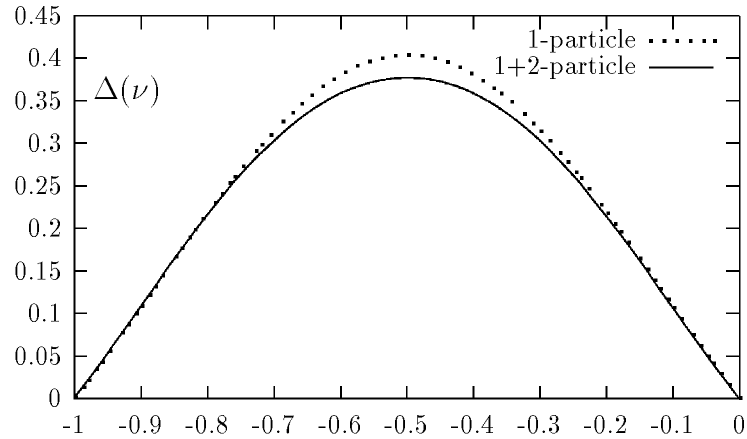


Figure 1: *Dimension of an exponential of the field for the sinh-Gordon model: 1- and 1+2-particle intermediate state contributions*

ever, up to now we could not derive a general formula. This is possible for the scaling Ising model which is considered as the next example. The constant C in the approximation of 1-intermediate states is given as

$$C_1 = \exp\left(-2h_1\left(\gamma_E - \ln 2\right)\right) = \exp\left(4\frac{\sin\pi\nu}{\pi F(i\pi)}\left(\gamma_E - \ln 2\right)\right)$$

We have not calculated the integrals appearing in higher-particle intermediate state contributions.

3. The scaling Ising model [11, 42, 12, 39] The S-matrix is $S = -1$. The form factors of the order parameter σ are non-vanishing for odd n [11]

$$\langle\, 0\,|\,\sigma(0)\,|p_1,\ldots,p_n\,\rangle^{in} = (2i)^{\frac{n-1}{2}} \prod_{i<j} \tanh\frac{1}{2}\theta_{ij}$$

and for the disorder parameter μ for even n

$$\langle\, 0\,|\,\mu(0)\,|p_1,\ldots,p_n\,\rangle^{in} = (2i)^{\frac{n}{2}} \prod_{i<j} \tanh\frac{1}{2}\theta_{ij}\,.$$

We introduce the operator $\mathcal{O}(x) = \mu(x) + \sqrt{2i}\sigma(x)$ normalized such that $\mathcal{O}_0 = 1$ and $\mathcal{O}_1 = \sqrt{2i}$. It obviously satisfies the asymptotic cluster behavior like an exponential of a bose field

$$\mathcal{O}_n(\theta_1,\ldots,\theta_n) = \mathcal{O}_1(\theta_1)\mathcal{O}_{n-1}(\theta_2,\ldots,\theta_n) + O(e^{-\theta_1}) \quad \text{for } \theta_1 \to \infty\,.$$

This allows us to apply the same methods as above

$$\begin{aligned}\langle\, 0\,|\,\mathcal{O}(x)\,\mathcal{O}^\dagger(x)\,|\,0\,\rangle &= \sum_{n=1}^{\infty}\frac{1}{n!}\int d\theta_1\cdots\int d\theta_n e^{-ix\sum p_i} g_n(\underline{\theta}) \\ &\approx C\,(m\tau)^{-4\Delta} \quad \text{for } \tau\to 0\end{aligned}$$

with $g_n(\underline{\theta}) = (2\pi)^{-n}\prod_{i<j}\tanh^2\frac{1}{2}\theta_{ij}$. To obtain dimension the Δ and the constant C we have to calculate

$$\Delta = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n!}\int d\theta_1\cdots\int d\theta_{n-1} h_n(\theta_1,\ldots,\theta_{n-1},0) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{1}{2\pi}\right)^n I_n$$

$$C = \exp\left(-2\sum_{n=1}^{\infty}\frac{1}{n!}\int d\theta_1\cdots\int d\theta_{n-1} h_n(\theta_1,\ldots,\theta_{n-1},0)\,(\ln\xi + \gamma_E - \ln 2)\right)$$

All integrals in the sum over the intermediate states in the formula of the dimension can be performed. The result may be expressed by a recursion relation

$$\begin{aligned} I_1 &= 1 \\ I_2 &= \textstyle\int d\theta\left(\tanh^2\theta - 1\right) = -4 \\ I_3 &= \textstyle\int d\theta_1\int d\theta_2\left\{\tanh^2\frac{1}{2}\theta_{12}\tanh^2\frac{1}{2}\theta_1\tanh^2\frac{1}{2}\theta_2\right. \\ &\qquad \left. -\tanh^2\frac{1}{2}\theta_{12} - \tanh^2\frac{1}{2}\theta_1 - \tanh^2\frac{1}{2}\theta_2 + 2\right\} = 4\pi^2 \\ I_n &= (n-2)^2\, I_3 I_{n-2} \end{aligned}$$

The sum over all intermediate states can be performed by solving the recursion relation which yields

$$\begin{aligned}\Delta &= \frac{1}{4\pi}\sum_{k=0}^{\infty}\frac{\Gamma^2(2k+1)}{\Gamma^2(k+1)\,\Gamma(2k+2)}4^{-k} - \frac{1}{8\pi^2}\sum_{k=1}^{\infty}\frac{\Gamma^2(k)}{\Gamma(2k+1)}4^k \\ &= \frac{1}{8} - \frac{1}{16} = \frac{1}{16}\end{aligned}$$

and which is a well known result (see for example [10]). For the constant C we calculated the 1- and 2-particle intermediate state contributions

$$\begin{aligned} C_{1+2} &= \exp\left(-\frac{1}{\pi}\left(\gamma_E - \ln 2\right)\right) \\ &\times \exp\left(-\frac{1}{(2\pi)^2}\int d\theta\left(\tanh^2\tfrac{1}{2}\theta - 1\right)\left(\ln\cosh\tfrac{1}{2}\theta + \gamma_E\right)\right) \\ &= e^{-(\gamma_E - \ln 2)/\pi} e^{-(\ln 2 - \gamma_E - 1)/\pi^2} = 1.134\,8\,. \end{aligned}$$

Using

$$\begin{aligned} \langle\, 0 \,|\, \mathcal{O}(x)\, \mathcal{O}^\dagger(x) \,|\, 0 \,\rangle &= \langle\, 0 \,|\, \left(\mu + \sqrt{2i}\sigma\right)(x)\, \left(\mu + \sqrt{2i}\sigma\right)^\dagger (0) \,|\, 0 \,\rangle \\ &\approx \left(C^{(\mu)} + 2C^{(\sigma)}\right)(m\tau)^{-4\Delta} \end{aligned}$$

this approximation may be compared with the results of [10]

$$C = C^{(\mu)} + 2C^{(\sigma)} = 1.084\,8\,.$$

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