Low dimensional integrable models and their applications in field theory and statistical physics

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Contents *

* For multi-author contributions, the underlined name indicates the speaker.

Preface

The INTAS project 99-01459 involves four research teams working in the field of mathematical physics: from France (Theoretical Physics Laboratory, Annecy), from Germany (Institute for Theoretical Physics, Free University, Berlin), from Russia (Physics Department, State University and Steklov Institute, St.Petersburg), and from Armenia (Physics Institute, Yerevan). The main aim of the project is development of the theory of low-dimensional integrable models, in particular, further development of the technique of computation of correlation functions and form-factors and investigation of quantum symmetries of integrable models. More detailed program can be found at http://www.physik.fu-berlin.de/~karowski/intas.html

This volume contains contributions to the workshop on *Low dimensional integrable models and their applications* that took place as a part of the project. The workshop was organized by the Annecy team and was held at the Theoretical Physics Laboratory of Annecy-Le-Vieux (LAPTH) from May 21 to 25, 2001.

Symmetries in staggered models

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Abstract

We construct integrable models with $\mathbb{Z}_2$ grading of both auxiliary (chain) and quantum (time) spaces. These models have a staggered disposition of the anisotropy parameter. The corresponding Yang–Baxter Equations are written down and their solution for the $gl(N)$ case are found. We analyze in details the $N = 2$ case and find the corresponding quantum group behind this solution. It can be regarded as quantum $\mathcal{U}_{q\mathcal{B}}(gl(2))$ group with a matrix deformation parameter $q\mathcal{B}$ with $(q\mathcal{B})^2 = q^2$. The symmetry behind these models can also be interpreted as the tensor product of the (-1)-Weyl algebra by an extension of $\mathcal{U}_q(gl(N))$ with a Cartan generator related to deformation parameter -1.

1 Description of the model

This presentation is based on the articles [1, 2, 3, 4]. Detailed notations and many references may be found in these papers.

We define the double row monodromy matrix :

$$T(\lambda, u) = T_0(\lambda, u) T_1(\lambda, u)$$

with

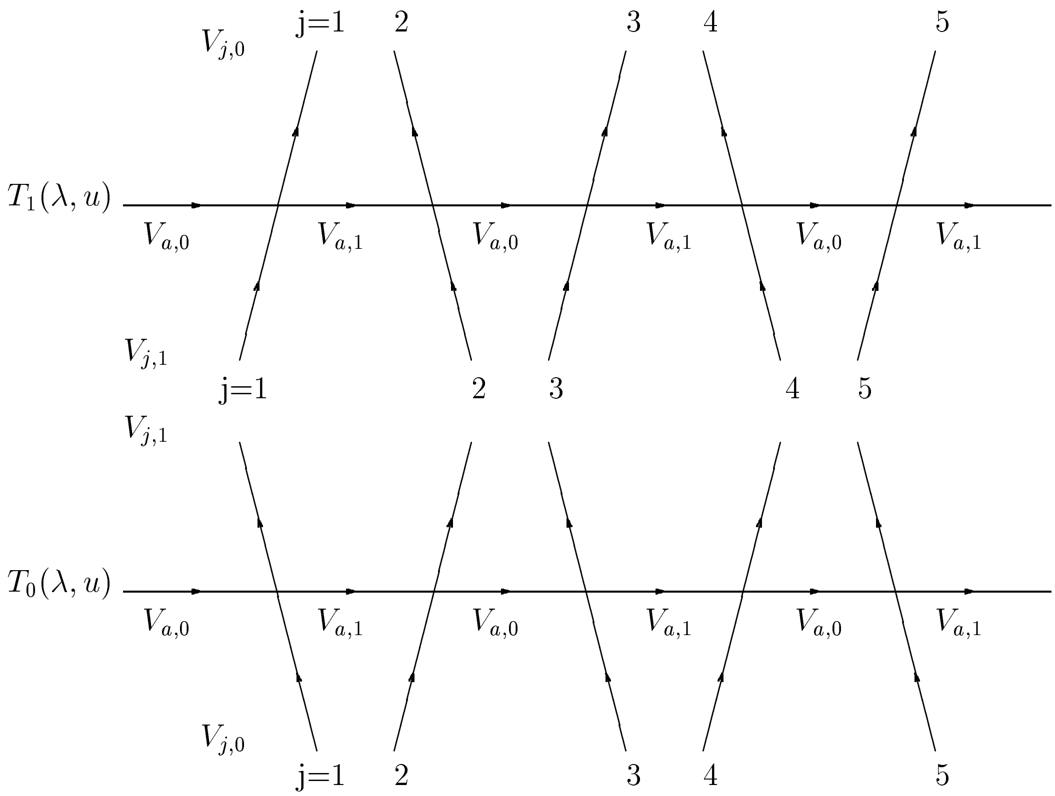
$$T_1(\lambda, u) = \prod_{j=1}^{N} R_{a,2j-1}(\lambda, u) \bar{R}^{\iota_2}_{a,2j}(\lambda, u)\,,$$
$$T_0(\lambda, u) = \prod_{j=1}^{N} \bar{R}^{\iota_1}_{a,2j-1}(\lambda, u) R^{\iota_1\iota_2}_{a,2j}(\lambda, u)\,,$$

where the R-matrices act in the following spaces

$$\begin{array}{rcl} R_{aj}\,(u,v) & : & V_{a,0}(u) \otimes V_{j,0}(v) \longrightarrow V_{j,1}(v) \otimes V_{a,1}(u) \\ R^{\iota_1}_{aj}\,(u,v) & : & V_{a,0}(u) \otimes V_{j,1}(v) \longrightarrow V_{j,0}(v) \otimes V_{a,1}(u) \\ R^{\iota_2}_{aj}\,(u,v) & : & V_{a,1}(u) \otimes V_{j,0}(v) \longrightarrow V_{j,1}(v) \otimes V_{a,0}(u) \\ R^{\iota_1\iota_2}_{aj}\,(u,v) & : & V_{a,1}(u) \otimes V_{j,1}(v) \longrightarrow V_{j,0}(v) \otimes V_{a,0}(u) \end{array}$$

The monodromy matrices T_0 and T_1 can be presented graphically as follows:

This work was partially supported by the INTAS project 99-01459



2 Integrability

In order to have a integrable model with commuting transfer matrices for different spectral parameters

$$[tr\, T(\lambda, u),\; tr\, T(\lambda, v)] = 0\,,$$

it is enough to have the following relations for the $\tau_\sigma(u) = trT_\sigma(\lambda, u)$, $(\sigma = 0, 1)$:

$$\tau_\sigma(\lambda, u)\; \tau_{1-\sigma}(\lambda, v) = \bar{\tau}_\sigma(\lambda, v)\; \bar{\tau}_{1-\sigma}(\lambda, u)\,.$$

This is satisfied if two Yang–Baxter like equation are satisfied

$$\begin{aligned} R_{12}(u,v)\; \bar{R}_{13}^{\iota_1}(u)\; R_{23}(v) &= R_{23}^{\iota_1}(v)\; \bar{R}_{13}(u)\; \tilde{R}_{12}(u,v)\,, \\ \tilde{R}_{12}(u,v)\; R_{13}^{\iota_1\iota_2}(u)\; \bar{R}_{23}^{\iota_2}(v) &= \bar{R}_{23}^{\iota_1\iota_2}(v)\; R_{13}^{\iota_2}(u)\; R_{12}(u,v)\,. \end{aligned}$$

The following R-matrices are solutions of this set of Yang–Baxter equations :

$$R(u) = \begin{pmatrix} \sin(\lambda+u) & 0 & 0 & 0 \\ 0 & \sin(u) & e^{-iu}\sin(\lambda) & 0 \\ 0 & e^{iu}\sin(\lambda) & \sin(u) & 0 \\ 0 & 0 & 0 & \sin(\lambda+u) \end{pmatrix}$$

$$R^{\iota_1}(u) = \begin{pmatrix} \sin(\lambda+u) & 0 & 0 & 0 \\ 0 & -\sin(u) & e^{-iu}\sin(\lambda) & 0 \\ 0 & e^{iu}\sin(\lambda) & -\sin(u) & 0 \\ 0 & 0 & 0 & \sin(\lambda+u) \end{pmatrix}$$

with furthermore

$$\bar{R}(u) \equiv R(\theta - u)\,, \qquad R^{\iota_2}(u) = R^{\iota_1}(-u)\,, \qquad \tilde{R}(u) = R^{\iota_1}(u)\,,$$

where θ is a new parameter.

3 Constant R-matrices:

Let $z = e^{iu}$ and

$$R_{12}(u) = \frac{1}{2i}\left(zR_{12} - z^{-1}R_{21}^{-1}\right)\,.$$

Then the Yang–Baxter equations for the spectral parameter dependent R-matrix $R(u)$ and $R^{\iota_1}(u)$ are equivalent to the following equations for the constant R-matrices R and R^{ι_1}

$$\begin{aligned} R_{12}R^{\iota_1}_{13}R_{23} &= R^{\iota_1}_{23}R_{13}R^{\iota_1}_{12}\,, & R^{\iota_1}_{12}R_{13}R^{\iota_1}_{23} &= R_{23}R^{\iota_1}_{13}R_{12}\,, \end{aligned}$$
$$R_{12}\left(R^{\iota_1}_{31}\right)^{-1}R_{23} - \left(R_{21}\right)^{-1}R^{\iota_1}_{13}\left(R_{32}\right)^{-1} = R^{\iota_1}_{23}\left(R_{31}\right)^{-1}R^{\iota_1}_{12} - \left(R^{\iota_1}_{32}\right)^{-1}R_{13}\left(R^{\iota_1}_{21}\right)^{-1}\,.$$

From the spectral parameter dependent solution we deduce that a solution of the constant equations is given by

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \qquad R^{\iota_1} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & q-q^{-1} & -1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix},$$

where the R is the usual R-matrix of $\mathcal{U}_q(sl(2))$.

4 Algebra

These two R-matrices and the constant YBE equations lead to the following algebra, defined by the generators $L^\pm$, $(L^\pm)^{\iota_1}$

$$\begin{aligned} R_{12}L_1^{\pm\iota_1}L_2^{\pm} &= L_2^{\pm\iota_1}L_1^{\pm}R^{\iota_1}_{12}\,, & R_{12}L_1^{+\iota_1}L_2^{-} &= L_2^{-\iota_1}L_1^{+}R^{\iota_1}_{12}\,, \\ R^{\iota_1}_{12}L_1^{\pm}L_2^{\pm\iota_1} &= L_2^{\pm}L_1^{\pm\iota_1}R_{12}\,, & R^{\iota_1}_{12}L_1^{+}L_2^{-\iota_1} &= L_2^{-}L_1^{+\iota_1}R_{12}\,, \end{aligned}$$
$$\begin{aligned} R_{12}L_1^{-\iota_1}L_2^{+} - \left(R_{21}\right)^{-1}L_1^{+\iota_1}L_2^{-} &= L_2^{+\iota_1}L_1^{-}R^{\iota_1}_{12} - L_2^{-\iota_1}L_1^{+}\left(R^{\iota_1}_{21}\right)^{-1}\,, \\ R^{\iota_1}_{12}L_1^{-}L_2^{+\iota_1} - \left(R^{\iota_1}_{21}\right)^{-1}L_1^{+}L_2^{-\iota_1} &= L_2^{+}L_1^{-\iota_1}R_{12} - L_2^{-}L_1^{+\iota_1}\left(R_{21}\right)^{-1}\,. \end{aligned}$$

Writing the operators $L^\pm$ in the form

$$L^+ = \begin{pmatrix} K_{+1} & 0 \\ E & K_{+2} \end{pmatrix}, \qquad L^- = \begin{pmatrix} K_{-1} & F \\ 0 & K_{-2} \end{pmatrix}$$

and similarly for $L^{\pm\iota_1}$, we get the relations

$$\begin{aligned} K^{\iota_1}_{+1}K_{-1} &= K^{\iota_1}_{-1}K_{+1}\,, & K^{\iota_1}_{+2}K_{-2} &= K^{\iota_1}_{-2}K_{+2}\,, \\ K^{\iota_1}_{+1}K_{+2} &= -K^{\iota_1}_{+2}K_{+1}\,, & K^{\iota_1}_{-1}K_{-2} &= -K^{\iota_1}_{-2}K_{-1}\,, \\ K^{\iota_1}_{+1}K_{-2} &= -K^{\iota_1}_{-2}K_{+1}\,, & K^{\iota_1}_{+2}K_{-1} &= -K^{\iota_1}_{-1}K_{+2}\,, \end{aligned}$$

$$\begin{aligned}
K_{+1}^{\iota_1}E &= q\ E^{\iota_1}K_{+1}\,, & K_{-1}^{\iota_1}E &= q^{-1}\ E^{\iota_1}K_{-1}\\
K_{+2}^{\iota_1}E &= -q^{-1}\ E^{\iota_1}K_{+2}\,, & K_{-2}^{\iota_1}E &= -q\ E^{\iota_1}K_{-2}\,,\\
K_{+1}^{\iota_1}F &= -q^{-1}\ F^{\iota_1}K_{+1}\,, & K_{-1}^{\iota_1}F &= -q\ F^{\iota_1}K_{-1}\,,\\
K_{+2}^{\iota_1}F &= q\ F^{\iota_1}K_{+2}\,, & K_{-2}^{\iota_1}F &= q^{-1}\ F^{\iota_1}K_{-2}\,,
\end{aligned}$$

$$E^{\iota_1}F + F^{\iota_1}E = (q-q^{-1})\left(K_{-1}^{\iota_1}K_{+2} - K_{+1}^{\iota_1}K_{-2}\right)\,.$$

5 An interpretation with an operator-valued deformation parameter $q\mathcal{B}$

We gather the operators in matrices as

$$\begin{aligned}
\mathcal{K}_{\pm1} &= \begin{pmatrix} 0 & K_{\pm1}\\ K_{\pm1}^{\iota_1} & 0\end{pmatrix}, & \mathcal{K}_{\pm2} &= \begin{pmatrix} 0 & K_{\pm2}\\ -K_{\pm2}^{\iota_1} & 0\end{pmatrix},\\
\mathcal{E} &= \begin{pmatrix} 0 & E\\ -E^{\iota_1} & 0\end{pmatrix}, & \mathcal{F} &= \begin{pmatrix} 0 & F\\ F^{\iota_1} & 0\end{pmatrix},\\
\mathcal{B} &= \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}, & \mathcal{B}^2 &= 1\,.
\end{aligned}$$

With this definition

$$\begin{aligned}
\mathcal{K}_{\pm1}\mathcal{K}_{\pm2} &= \mathcal{K}_{\pm2}\mathcal{K}_{\pm1} & \mathcal{B}\mathcal{K}_i &= -\mathcal{K}_i\mathcal{B}\\
\mathcal{B}\mathcal{E} &= -\mathcal{E}\mathcal{B} & \mathcal{B}\mathcal{F} &= -\mathcal{F}\mathcal{B}\\
\mathcal{K}_{\pm1}\mathcal{E} &= q^{\pm1}\mathcal{B}\mathcal{E}\mathcal{K}_{\pm1} & \mathcal{K}_{\pm2}\mathcal{E} &= q^{\mp1}\mathcal{B}\mathcal{E}\mathcal{K}_{\pm2}\\
\mathcal{K}_{\pm1}\mathcal{F} &= q^{\mp1}\mathcal{B}\mathcal{F}\mathcal{K}_{\pm1} & \mathcal{K}_{\pm2}\mathcal{F} &= q^{\pm1}\mathcal{B}\mathcal{F}\mathcal{K}_{\pm2}\\
[\mathcal{E},\mathcal{F}] &= (q-q^{-1})\mathcal{B}(\mathcal{K}_2\mathcal{K}_{-1} - \mathcal{K}_1\mathcal{K}_{-2})
\end{aligned}$$

This is $gl_q(2)$ with, instead of the ordinary deformation parameter q, a deformation matrix $q\mathcal{B}$ with the property $(q\mathcal{B})^2 = q^2$. This algebra can be denoted as $gl_{q\mathcal{B}}(2)$.

6 $\mathcal{U}_q(gl(2))$ subalgebra

We define the quadratic operators

$$\begin{aligned}
e &= K_1^{\iota_1}E\,, & f &= K_2^{\iota_1}F\,,\\
k_1 &= K_1^{\iota_1}K_1\,, & k_2 &= K_2^{\iota_1}K_2\,,\\
l_1 &= K_1^{\iota_1}K_{-1}\,, & l_2 &= K_2^{\iota_1}K_{-2}\,,\\
m &= K_1^{\iota_1}K_2\,.
\end{aligned}$$

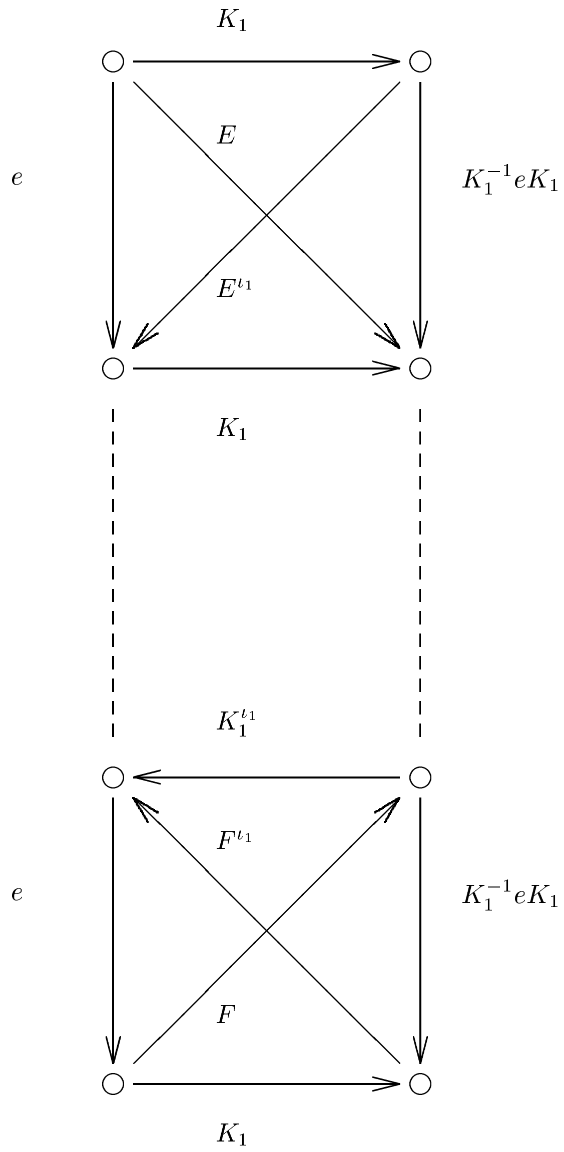
They satisfy the relations

$$\begin{aligned}
k_1e &= -q^2ek_1\,, & k_1f &= -q^{-2}fk_1\,,\\
k_2e &= -q^{-2}ek_2\,, & k_2f &= -q^2fk_2\,,\\
[e,f] &= (q^2-1)(k_1l_2 - k_2l_1)\,, & &\\
l_ie &= -el_i\,, & l_if &= -fl_i\,,\\
me &= -em\,, & mf &= -fm\,.
\end{aligned}$$

The full algebra is obtained for the algebra generated by the quadratic operators by extension by the single linear operator K_1. It is equivalent to $sl(2) \otimes \mathcal{U}_{-q^2,-1}(gl(2))$ where the first $sl(2)$ is taken in the fundamental representation and where $\mathcal{U}_{-q^2,-1}(gl(2))$ is simply $\mathcal{U}_{-q^2}(gl(2))$ with the supplementary operator $l_1 = i^h$.

7 Representations

An irrep of our algebra is the direct sum of the two irrep of $\mathcal{U}_q(gl(2))$ with the same highest weight. These two are connected by the supplementary generator K_1. This is described by the following figure:



8 Coproduct $\Delta^{(3)}$; coaction of $\mathcal{U}_q(gl(2))$

We do not have a coproduct $\Delta : \mathcal{A} \longrightarrow \mathcal{A}\otimes\mathcal{A}$ since the algebra is defined by the use of two different R-matrices. However, one has the following notions. Define

$$\begin{aligned}\Delta^{(3)} \quad : \quad & \mathcal{A} \longrightarrow \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \\ & L \longmapsto L \dot{\otimes} L^{\iota_1} \dot{\otimes} L\end{aligned}$$

i.e.,

$$\begin{aligned}&\Delta^{(3)}(K_1) = K_1\otimes K_1^{\iota_1}\otimes K_1\,, \qquad \Delta^{(3)}(K_2) = K_2\otimes K_2^{\iota_1}\otimes K_2\\ &\Delta^{(3)}(E) = E\otimes K_1^{\iota_1}\otimes K_1 + K_2\otimes E^{\iota_1}\otimes K_1 + K_2\otimes K_2^{\iota_1}\otimes E\,,\\ &\Delta^{(3)}(F) = K_{-1}\otimes K_{-1}^{\iota_1}\otimes F + K_{-1}\otimes F^{\iota_1}\otimes K_{-2} + F\otimes K_{-2}^{\iota_1}\otimes K_{-2}\,.\end{aligned}$$

Proposition. *$\Delta^{(3)}$ is a morphism of algebras.*

Define also $\underline{\Delta} : \mathcal{A} \longrightarrow \mathcal{U}_{q,i}(gl(2))\otimes\mathcal{A}$ by

$$\begin{aligned}&\underline{\Delta}(\mathcal{K}_1) = k_1\otimes\mathcal{K}_1\,, \qquad \underline{\Delta}(\mathcal{K}_2) = k_2\otimes\mathcal{K}_2\,,\\ &\underline{\Delta}(\mathcal{E}) = e\otimes\mathcal{K}_1 + k_2\otimes\mathcal{E}\,,\\ &\underline{\Delta}(\mathcal{F}) = k_1^{\iota_1}\otimes\mathcal{F} + f\otimes\mathcal{K}_2^{\iota_1}\,.\end{aligned}$$

Proposition. *$\underline{\Delta}$ is a morphism of algebras.*

9 Generalization of $\mathcal{U}_q(gl(N))$

Let us consider now the $gl(N)$ algebra. The two constant R-matrices R and R^{ι_1} given by

$$R = \sum_{i=1}^{N} q e_{ii}\otimes e_{ii} + \sum_{\substack{i,j=1\\ i\neq j}}^{N} e_{ii}\otimes e_{jj} + (q-q^{-1})\sum_{\substack{i,j=1\\ i>j}}^{N} e_{ij}\otimes e_{ji}\,,$$

$$R^{\iota_1} = \sum_{i=1}^{N} q e_{ii}\otimes e_{ii} + \sum_{\substack{i,j=1\\ i\neq j}}^{N} b_{ij} e_{ii}\otimes e_{jj} + (q-q^{-1})\sum_{\substack{i,j=1\\ i>j}}^{N} e_{ij}\otimes e_{ji}$$

satisfy the set of YBE equations provided that

$$b_{ij} = b_{ik}b_{kj} \qquad \text{and} \qquad b_{ij}^2 = 1\,.$$

This cocycle condition allows us to write $b_{ij} = b_i b_j^{-1}$, with $b_1 = 1$ and $b_i = \pm 1$ for $i > 1$, thus yielding several solutions.

$$\begin{aligned}&b_j L_{im}^{+} L_{jn}^{-\,\iota_1} + (q-q^{-1})\, b_i L_{jm}^{+} L_{in}^{-\,\iota_1} = b_i L_{jn}^{-} L_{im}^{+\,\iota_1}\,,\\ &b_j L_{im}^{+\,\iota_1} L_{jn}^{-} + (q-q^{-1})\, b_j L_{jm}^{+\,\iota_1} L_{in}^{-} = b_j b_{mn} L_{jn}^{-\,\iota_1} L_{im}^{+}\,.\end{aligned}$$

Compare these relations with ordinary $\mathcal{U}_q(gl(N))$:

$$L_{im}^{+} L_{jn}^{-} + (q-q^{-1})\, L_{jm}^{+} L_{in}^{-} = L_{jn}^{-} L_{im}^{+}$$

By a redefinition of the generators, the previous relations can be written as

$$
\begin{aligned}
L^+_{im} {L^-_{jn}}' + (q - q^{-1})\; L^+_{jm} {L^-_{in}}' &= L^-_{jn} {L^+_{im}}' \,, \\
{L^+_{im}}' L^-_{jn} + (q - q^{-1})\; {L^+_{jm}}' L^-_{in} &= {L^-_{jn}}' L^+_{im} \,.
\end{aligned}
$$

These relations contain no supplementary signs with respect to $\mathcal{U}_q(gl(N))$. However, they still involve alternating L and L' and we have the supplementary relations

$$
l_1^{\iota_1} L^{\pm}_{ij} = b_{ij} L^{\pm}_{ij} l_1 \,, \qquad l_1 L^{\pm\prime}_{ij} = b_{ij} L^{\pm\prime}_{ij} l_1^{\iota_1} \,.
$$

Using the operators $I \equiv K_1 (k_1 l_1)^{-1/2}$ and $I^{\iota_1} \equiv (k_1 l_1)^{-1/2} K'_1$ (these operators are always defined in all finite dimensional representations) satisfying $I^{\iota_1} I + I I^{\iota_1} = 1$, we have

Proposition. *The algebra generated by $L^{\pm}_{ij}$ and $L^{\pm\iota_1}_{ij}$ is equivalent to*

$$
sl(2) \otimes \mathcal{U}_{q,-1}(gl(N))
$$

where the first $sl(2)$ is taken in the fundamental representation, and where $\mathcal{U}_{q,-1}(gl(N))$ corresponds to $\mathcal{U}_q(gl(N))$ extended by adjoining the generator l_1 that anticommutes with some of the L_{ij}'s.

Remark: l_1 can be seen as $\prod_i b_i^{h_{\omega_{i+1}} - h_{\omega_i}}$, where h_{ω_i} is the Cartan generator related with the fundamental weight $\omega_i = \sum (A^{-1})_{ij} h_j$.

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"The bootstrap" program. Integrable QFT in 1+1 dim.

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Contents

1 The "Bootstrap Program"

Construct a *quantum* field theory explicitly by:

1. **Calculate the 2-particle S-matrix** using
   (a) general properties: unitarity, crossing etc.
   (b) symmetry properties
   (c) "bound state bootstrap"
   (d) "Yang-Baxter Equation"
   (e) "maximal analyticity"
2. **Calculate the "form factors"**
   $$\langle\, 0 \,|\, \phi(x) \,|\, p_1, \ldots, p_n \,\rangle^{in} = e^{-ix(p_1+\cdots+p_n)}\, f^{\phi}\,(\theta_1, \ldots, \theta_n)$$
   using
   (a) the S-matrix
   (b) LSZ-assumptions
   (c) "maximal analyticity"
3. **Calculate the "Wightman function"**
   $$\langle\, 0 \,|\, \phi(x)\phi(y) \,|\, 0 \,\rangle = \sum_n \int \langle\, 0 \,|\, \phi(x) \,|\, n \,\rangle^{in}\ {}^{in}\langle\, n \,|\, \phi(y) \,|\, 0 \,\rangle\,.$$

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2 Integrability

Integrability in (quantum) field theories means: there exits infinitely many conservation laws

$$\partial_\mu J_L^\mu(t,x) = 0\,, \qquad L = \pm 1, \pm 3, \ldots$$

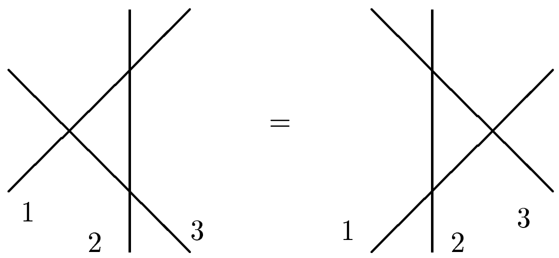
Consequence in 1+1 dim.: the n-particle S-matrix is a product of 2-particle S-matrices

$$S^{(n)}(p_1,\ldots,p_n) = \prod_{i<j} S_{ij}(p_i,p_j)\,.$$

In particular, for the 3-particle S-matrix

$$S^{(3)} = S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12}\,.$$

Yang-Baxter Equation



Examples

The sine-Gordon model:

$$\ddot{\varphi}(t,x) - \varphi''(t,x) + \frac{\alpha}{\beta}\sin\beta\varphi(t,x) = 0\,.$$

The massive Thirring model

$$\mathcal{L}^{MT} = \bar\psi(i\gamma\partial - m)\psi - \frac{1}{2}\,g\;\bar\psi\gamma^\mu\psi\,\bar\psi\gamma_\mu\psi\,.$$

Both models are equivalent on the quantum level [Coleman 75]. In the following explicit solutions are given for the sine-Gordon alias massive Thirring model

Further models:

- Z_N-Ising models
- Nonlinear σ-models
- Gross-Neveu models
- Toda models

3 The S-matrix

Example: Sine-Gordon alias massive Thirring model. Particles: soliton, anti-soliton and breathers (as soliton-anti-soliton bound states).

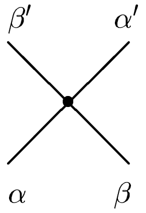
2-particle S-matrix

$$S(\theta) = \begin{pmatrix} u & & & & & & \\ & t & r & & & & \\ & r & t & & & & \\ & & & u & & & \\ & & & & S_{sb} & & \\ & & & & & S_{bb} & \\ & & & & & & \ddots \end{pmatrix},$$

where the rapidity difference $\theta = |\theta_1 - \theta_2|$ is defined by $p_i = m_i(\cosh\theta_i, \sinh\theta_i)$.

We start with the 2-particle S-matrix:

$$S_{\alpha\beta}^{\beta'\alpha'} =$$



$$: \quad S_{ss}^{ss} = u, \quad S_{s\bar{s}}^{\bar{s}s} = t, \quad S_{s\bar{s}}^{s\bar{s}} = r$$

$s =$ soliton, $\bar{s} =$ anti-soliton

1. Unitarity: $S(-\theta)S(\theta) = 1$ implies

$$\begin{aligned} u(-\theta)u(\theta) &= 1\,, \\ t(-\theta)t(\theta) + r(-\theta)r(\theta) &= 1\,, \\ t(-\theta)r(\theta) + r(-\theta)t(\theta) &= 0\,. \end{aligned}$$

2. Crossing

$$u(i\pi - \theta) = t(\theta)\ , \quad r(i\pi - \theta) = r(\theta)\,.$$

3. Yang-Baxter

$$r(\theta_{12})u(\theta_{13})r(\theta_{23}) + t(\theta_{12})r(\theta_{13})t(\theta_{23}) = u(\theta_{12})r(\theta_{13})u(\theta_{23})\,.$$

The "minimal" general solution of these equations is [2]:

$$u(\theta, \nu) = \exp\int_0^\infty \frac{dt}{t}\,\frac{\sinh\frac{t}{2}(1-\nu)}{\sinh\frac{\nu t}{2}\,\cosh\frac{t}{2}}\,\sinh t\frac{\theta}{i\pi}\,,$$

where ν is a free parameter related to the couplings

$$\frac{1}{\nu} = \frac{8\pi}{\beta^2} - 1 = 1 + \frac{2g}{\pi}\,.$$

This S-matrix was first obtained in [1] by other methods. It has been checked in perturbation theory.

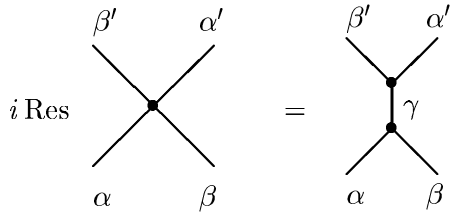
Bound states

Let γ be a bound state of the particles α and β with mass

$$m_\gamma = \sqrt{m_\alpha^2 + m_\beta^2 + 2m_\alpha m_\beta \cos a}\ , \quad (0 < a < \pi)$$

$\Rightarrow$ the 2 particle S-matrix has a pole

$$i\mathrm{Res}_{\theta=ia}\dot{S}_{\alpha\beta}^{\beta'\alpha'}(\theta) \quad = \quad \Gamma_\gamma^{\beta'\alpha'}\,\Gamma_{\alpha\beta}^\gamma$$

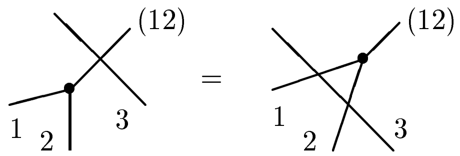


$\Gamma_{\alpha\beta}^\gamma$ is the "bound state intertwiner".

The bound state S-matrix

The scattering of (12) with 3 is obtained by the "bootstrap" equation

$$S_{(12)3}(\theta_{(12)3})\,\Gamma_{12}^{(12)} \quad = \quad \Gamma_{12}^{(12)}\,S_{13}(\theta_{13})S_{23}(\theta_{23})$$



Example: sine-Gordon alias massive Thirring model. The pole of $S_{s\bar{s}}(\theta)$ at $\theta = i\pi(1-\nu)$ corresponds to the bound state

$$\begin{aligned} \text{soliton} \ + \ \text{anti}-\text{soliton} \quad &\longrightarrow \quad \text{breather} \\ s + \bar{s} \quad &\longrightarrow \quad (s\bar{s}) = b \end{aligned}$$

Using the "bootstrap" equations one can derive

$$S_{ss} \to S_{bs} \to S_{bb}$$

$\longrightarrow$ the breather-soliton S-matrix

$$S_{bs}(\theta) = t(\theta + \frac{1}{2}i\pi(1-\nu))\,u(\theta - \frac{1}{2}i\pi(1-\nu)) = \frac{\sinh\theta + i\sin\frac{1}{2}\pi(1+\nu)}{\sinh\theta - i\sin\frac{1}{2}\pi(1+\nu)}$$

$\longrightarrow$ the 2-breather S-matrix

$$S_{bb}(\theta) = S_{bs}(\theta + \frac{1}{2}i\pi(1-\nu))\,S_{bs}(\theta - \frac{1}{2}i\pi(1-\nu)) = \frac{\sinh\theta + i\sin\pi\nu}{\sinh\theta - i\sin\pi\nu}$$

The pole of $S_{bb}(\theta)$ at $\theta = i\pi\nu$ correspond to the bound state (Karowski and Thun [3]):

$$b + b \longrightarrow (bb) = b_2 = \text{second breather}$$

The pole of $S_{bs}(\theta)$ at $\theta = i\frac{1}{2}\pi(1+\nu)$ correspond to the bound state

$$b + s \longrightarrow (bs) = s$$

Using the "bootstrap" equations

$$\begin{aligned} S_{bs}(\theta) &= S_{bs}(\theta + i\pi\nu)\, S_{bb}(\theta - \frac{1}{2}i\pi(1-\nu))\,, \\ u(\theta) &= u(\theta + i\pi\nu)\, S_{sb}(\theta - \frac{1}{2}i\pi(1-\nu))\,, \end{aligned}$$

one can derive inversely

$$S_{bb} \to S_{bs} \to S_{ss}\,.$$

4 Form factors

Definition: Let $\mathcal{O}(x)$ be a local operator: the generalized form factor is

$$f^{\mathcal{O}}_{\alpha_1\dots\alpha_n}(\theta_1,\dots,\theta_n) = \langle\, 0\,|\,\mathcal{O}(0)\,|\,p_1,\dots,p_n\,\rangle^{in}_{\alpha_1\dots\alpha_n}$$

for $\theta_1 > \dots > \theta_n$ (elsewhere by analytic continuation)

Properties (Karowski and Weisz [6], Smirnov [8], Babujian, Fring, Karowski and Zapletal [10]):

LSZ-assumptions+ 'maximal analyticity' $\Longrightarrow$ form factor equations

(i) Watson's equation: $f^{\mathcal{O}}_{\dots ij\dots}(\dots,\theta_i,\theta_j,\dots) = f^{\mathcal{O}}_{\dots ji\dots}(\dots,\theta_j,\theta_i,\dots)\, S_{ij}(\theta_i-\theta_j)$

(ii) Crossing:

$$_{\bar\alpha_1}\langle\, p_1\,|\,\mathcal{O}(0)\,|\,\dots,p_n\,\rangle^{in,conn.}_{\dots\alpha_n} = f^{\mathcal{O}}_{\alpha_1\dots\alpha_n}(\theta_1+i\pi,\dots,\theta_n) = f^{\mathcal{O}}_{\dots\alpha_n\alpha_1}(\dots,\theta_n,\theta_1-i\pi)$$

(iii) Recursion relation: $\operatorname{Res}_{\theta_{12}=i\pi} f^{\mathcal{O}}_{1\dots n}(\theta_1,\dots) = 2i\,\mathbf{C}_{12}\; f^{\mathcal{O}}_{3\dots n}(\theta_3,\dots)\,(\mathbf{1} - S_{2n}\dots S_{23})$
where $\mathbf{C}_{12}$ is the charge conjugation matrix

(iv) Bound state form factors: $\operatorname{Res}_{\theta_{12}=\theta_{12}^{(12)}} f^{\mathcal{O}}_{123\dots n}(\underline{\theta}) = f^{\mathcal{O}}_{(12)3\dots n}(\theta_{(12)},\underline{\theta}')\,\sqrt{2}\,\Gamma^{(12)}_{12}$

(v) Lorentz invariance: $f^{\mathcal{O}}_{1\dots n}(\theta_1+u,\dots,\theta_n+u) = e^{su}\, f^{\mathcal{O}}_{1\dots n}(\theta_1,\dots,\theta_n)$
where s is the "spin" of $\mathcal{O}$

Locality

Theorem: (i) – (iv) and a general crossing formula $\Rightarrow$

$$^{in}\langle\,\phi\,|\;[\mathcal{O}(x),\mathcal{O}(y)]\;|\,\psi\,\rangle^{in} = 0$$

for all matrix elements, if $(x-y)$ is space like.
Proof: For the case without bound states: Smirnov [8], Lashkevich 94. General case: Quella [12].

Solutions of the form factor equations

2-particle form factors

$$\langle\, 0 \,|\, \mathcal{O}(0) \,|\, p_1, p_2 \rangle^{in/out} = F\left((p_1+p_2)^2 \pm i\varepsilon\right) = F\left(\pm\theta_{12}\right)$$

$$\text{Particle number conservation} \Rightarrow \quad \langle\, 0 \,|\, \mathcal{O}(0) \,|\, p_1, p_2 \rangle^{in} = \langle\, 0 \,|\, \mathcal{O}(0) \,|\, p_2, p_1 \rangle^{out}\, \dot{S}\left(\theta_{12}\right)$$

$$\text{Crossing} \Rightarrow \quad \langle\, p_1 \,|\, \mathcal{O}(0) \,|\, p_2 \rangle = F\left(i\pi - \theta_{12}\right)$$

$$\Longrightarrow \quad \text{Watson's equations} \quad \boxed{\begin{array}{l} F\left(\theta\right) = F\left(-\theta\right) \dot{S}\left(\theta\right) \\ F\left(i\pi-\theta\right) = F\left(i\pi+\theta\right) \end{array}}$$

"maximal analyticity" $\Rightarrow$ unique solution

EXAMPLE: sine-Gordon alias massive Thirring model. 2-soliton form factor (Weisz 77, Karowski and Weisz [6]):

$$F(i\pi-\theta) = \cosh\frac{1}{2}\theta \,\exp\int_0^\infty \frac{dt}{t}\,\frac{\sinh\frac{t}{2}(1-\nu)}{\sinh\frac{\nu t}{2}\,\cosh\frac{t}{2}\sinh t}\,\sin^2 t\frac{\theta}{2\pi}\,.$$

The general solution for $f^{\mathcal{O}}_{\alpha_1\ldots\alpha_n}(\theta_1,\ldots,\theta_n)$

($\alpha_i =$ soliton, anti-soliton)

THEOREM ([10], [13], and Babujian and Karowski (in preparation)):

$$f^{\mathcal{O}}_{\alpha_1\ldots\alpha_n}(\underline{\theta}) = \int_{\mathcal{C}_{\underline{\theta}}} dz_1 \cdots \int_{\mathcal{C}_{\underline{\theta}}} dz_m\, h(\underline{\theta},\underline{z})\, p^{\mathcal{O}}(\underline{\theta},\underline{z})\, \Psi_{\alpha_1\ldots\alpha_n}(\underline{\theta},\underline{z})$$

satisfies the properties (i) – (v), **if** the scalar function $h(\underline{\theta},\underline{z})$ is

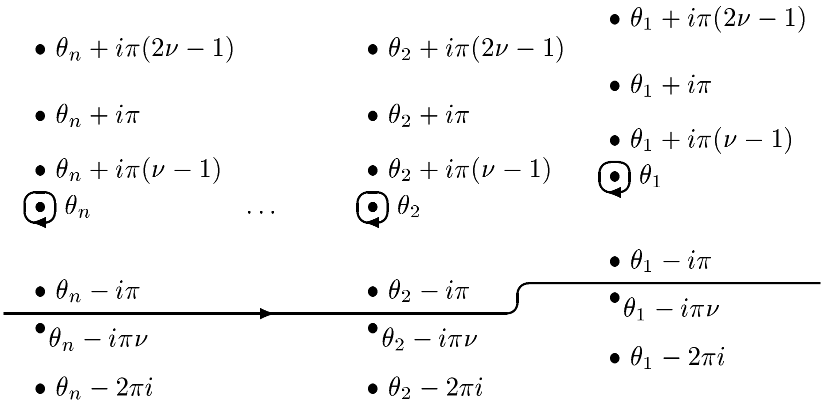
$$h(\underline{\theta},\underline{z}) = \prod_{1\le i<j\le n} F(\theta_{ij}) \prod_{i=1}^{n}\prod_{j=1}^{m} \phi(\theta_i - z_j) \prod_{1\le i<j\le m} \tau(z_i - z_j)\,,$$

$$\tau(z) = \frac{1}{\phi(z)\,\phi(-z)}\,, \qquad \phi(z) = \frac{1}{F(z)\,F(z+i\pi)}\,,$$

where $F(z)$ is the 2-soliton form factor

$$\text{and } \textbf{if} \left\{ \begin{array}{l} \text{the integration contour } \mathcal{C}_{\underline{\theta}}, \\ \text{the “Bethe ansatz'' state } \Psi_{\alpha_1\ldots\alpha_n}(\underline{\theta},\underline{z}), \\ \text{and the p}-\text{function } p^{\mathcal{O}}(\underline{\theta},\underline{z}) \end{array} \right\} \text{ are as follows :}$$

The integration contour $\mathcal{C}_{\underline{\theta}}$

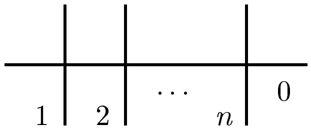


"Off-shell Bethe ansatz" state : $\Psi_{\alpha_1\ldots\alpha_n}(\underline{\theta}, \underline{z})$

Define the "monodromy matrix"

$$T_{1\ldots n,0}(\underline{\theta}, \theta_0) = \dot{S}_{10}(\theta_1 - \theta_0)\, \dot{S}_{20}(\theta_2 - \theta_0) \cdots \dot{S}_{n0}(\theta_n - \theta_0)$$

$=$



acting in $V^{1\ldots n} \otimes V_0$ with $V^{1\ldots n} = V_1 \otimes \cdots \otimes V_n$
(all $V_i \cong \mathbb{C}^2$ for $sl_q(2)$).
Define the sub-matrices A, B, C, D by

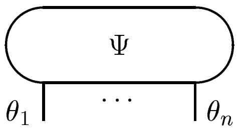
$$T_{1\ldots n,0}(\underline{\theta}, z) \equiv \begin{pmatrix} A_{1\ldots n}(\underline{\theta}, z) & B_{1\ldots n}(\underline{\theta}, z) \\ C_{1\ldots n}(\underline{\theta}, z) & D_{1\ldots n}(\underline{\theta}, z) \end{pmatrix} .$$

The "pseudo-vacuum" consisting only of solitons

$$\Omega_{1\ldots n} = s \otimes \cdots \otimes s \, .$$

A Bethe ansatz co-vector in $V_{1\ldots n}$ is defined by

$$\Psi_{1\ldots n}(\underline{\theta}, \underline{z}) = \Omega_{1\ldots n} C_{1\ldots n}(\underline{\theta}, z_1) \cdots C_{1\ldots n}(\underline{\theta}, z_m)$$



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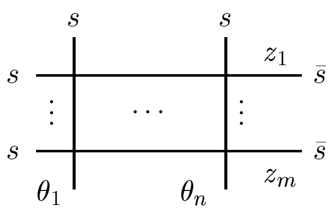


The p-function depends on the operator $\mathcal{O}(x)$.
$p_n^{\mathcal{O}}(\underline{\theta}, \underline{z})$ is a polynomial in $\exp\{\pm z_j\}$ $(j = 1, \ldots, m)$.
The following conditions guaranty the properties (i) – (v) of the form factors.

a) $p_n^{\mathcal{O}}(\underline{\theta}, \underline{z})$ symmetric with respect to the θ's and the z's.

b) $p_n^{\mathcal{O}}(\dots, \theta_n + 2\pi i, \underline{z}) = \sigma_{\mathcal{O}n}\, p_n^{\mathcal{O}}(\underline{\theta}, \underline{z})$

c) $p_n^{\mathcal{O}}(\theta_1 = \theta_n + i\pi, \underline{\tilde{\theta}}, \theta_n; \underline{\tilde{z}}, z_m = \theta_n) = p_{n-2}^{\mathcal{O}}(\underline{\tilde{\theta}}, \underline{\tilde{z}}) + \tilde{p}^{(1)}(\underline{\theta})$
$p_n^{\mathcal{O}}(\theta_1 = \theta_n + i\pi, \underline{\tilde{\theta}}, \theta_n; \underline{\tilde{z}}, z_m = \theta_1) = \sigma_{\mathcal{O}1}\, p_{n-2}^{\mathcal{O}}(\underline{\tilde{\theta}}, \underline{\tilde{z}}) + \tilde{p}^{(2)}(\underline{\theta})$

d) $p_n^{\mathcal{O}}(\underline{\theta} + \mu, \underline{z} + \mu) = e^{s\mu}\, p_n^{\mathcal{O}}(\underline{\theta}, \underline{z})$

REMARK: the number of integrations m depends on the charge of the operator $q = n - 2m$.

EXAMPLES: for the sine-Gordon alias massive Thirring-model.
Operators and their p-functions (up to normalizations)

$$\mathcal{O}(x) \leftrightarrow p_n^{\mathcal{O}}(\underline{\theta}, \underline{z})$$

The fundamental Fermi field ($q = n - 2m = 1$)

$$\psi(x) \leftrightarrow \exp \pm \left(\sum_{j=1}^{m} z_j - \frac{1}{2} \sum_{i=1}^{n} \theta_i \right)$$

The fundamental breather field ($q = n - 2m = 0$)

$$\varphi(x) \leftrightarrow \frac{1}{\sum e^{\theta} \sum e^{-\theta}} \left(\sum e^{-\theta_i} \sum e^{z_j} + \sum e^{\theta_i} \sum e^{-z_j} \right)$$

The current $j^\mu = \overline{\psi}\gamma^\mu\psi(x)$

$$\overline{\psi}\gamma^\pm\psi(x) \leftrightarrow \frac{\pm 1}{\sum e^{\mp\theta_i}} \left(\sum e^{-\theta_i} \sum e^{z_j} + \sum e^{\theta_i} \sum e^{-z_j} \right)$$

The energy momentum tensor (with $\rho, \sigma = +, -$)

$$T^{\rho\sigma}(x) \leftrightarrow \rho \frac{\sum e^{\rho\theta_i}}{\sum e^{-\sigma\theta_i}} \left(\sum e^{-\theta_i} \sum e^{z_j} - \sum e^{\theta_i} \sum e^{-z_j} \right)$$

The ∞-many conserved currents

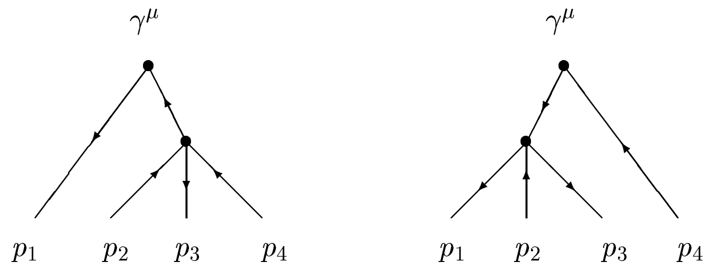
$$J_L^\pm \leftrightarrow \sum_{i=1}^{n} e^{\pm\theta_i} \sum_{j=1}^{m} e^{L z_j} \quad (L = \pm 1, \pm 3, \dots)$$

The higher charges fulfill the eigenvalue equation

$$\left(\int dx\, J_L^0(x) - \sum_{i=1}^{n} \left(p_i^+\right)^L \right) |\, p_1, \dots, p_n\rangle^{in} = 0\,.$$

Feynman graph expansion:

Example: 4-particle form factor of the φ-field



(Coleman 75): $\epsilon^{\mu\nu}\partial_\nu\varphi = -\frac{2\pi}{\beta}\overline{\psi}\gamma^\mu\psi$

$$\langle\, 0 \,|\, \varphi\,(0) |\, p_1, p_2, p_3, p_4\,\rangle^{in}_{\bar{s}s\bar{s}s} = \frac{2\pi i g}{\beta}\frac{\sinh\frac{1}{2}\theta_{13}\sinh\frac{1}{2}\theta_{24}\sinh\frac{1}{2}(\theta_{12}+\theta_{34})}{\prod_{i<j}\cosh\frac{1}{2}\theta_{ij}} + O(g^2)$$

agrees with the exact formula.

Breather form factors

[11] and Babujian and Karowski (in preparation):
The general soliton formula + fusion (iv) $\Rightarrow$

$$f_n^{\mathcal{O}}(\theta_1,\ldots,\theta_n) = K_n^{\mathcal{O}}(\underline{\theta})\prod_{1\le i<j\le n} F_{bb}(\theta_{ij})\,,$$

where

$$K_n^{\mathcal{O}}(\underline{\theta}) = \sum_{l_1=0}^{1}\cdots\sum_{l_n=0}^{1}(-1)^{\sum l_i}\prod_{i<j}\left(1+(l_i-l_j)\frac{i\sin\pi\nu}{\sinh\theta_{ij}}\right)p_n^{\mathcal{O}}(\underline{\theta},\underline{z})$$

with $z_i = \theta_i - \frac{i\pi}{2}\,(1+(2l_i-1)\nu)$. $F_{bb}(\theta)$ fulfills Watson's equations

$$F_{bb}(\theta) = F_{bb}(-\theta)S_{bb}(\theta) = F_{bb}(2\pi i-\theta)$$

with the 2-breather S-matrix

$$S_{bb}(\theta) = \frac{\sinh\theta + i\sin\pi\nu}{\sinh\theta - i\sin\pi\nu}\,.$$

The p-function $p_n^{\mathcal{O}}(\underline{\theta},\underline{z})$ again depends on the operator.

5 Some results

Babujian and Karowski [11]

The quantum sine-Gordon equation

$$\Box\varphi(t,x) + \frac{\alpha}{\beta} : \sin\beta\varphi : (t,x) = 0\,.$$

The "bare" mass $\sqrt{\alpha}$ is related to the renormalized mass by

$$\alpha = m^2 \frac{\pi\nu}{\sin\pi\nu}\,, \qquad \left(\nu = \beta^2/(8\pi - \beta^2)\right)\,,$$

where m is the physical mass of the fundamental boson. The factor $\frac{\pi\nu}{\sin\pi\nu}$ modifies the classical equation and has to be considered as a quantum correction. For fixed bare mass square α and $\nu \to 1,2,3,\ldots$ the physical mass goes to zero. At these values of the coupling the model is related to Baxter's RSOS-models which correspond to minimal conformal models with central charge $c = 1 - 6/(\nu(\nu+1))$.

The trace of the energy momentum tensor

$$T^\mu_{\ \mu}(t,x) = -2\frac{\alpha}{\beta^2}\left(1 - \frac{\beta^2}{8\pi}\right)(: \cos\beta\varphi : (t,x) - 1)$$

is the anomalous dimension of $\cos\beta\varphi$.

Again this operator equation is modified by a quantum correction $(1 - \beta^2/8\pi)$. Obviously for fixed bare mass square α and $\beta^2 \to 8\pi$ the model will be conformal invariant which is related to a Berezinski-Kosterlitz-Thouless phase transition.

The operator equations are equations of all their matrix elements.

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Off-shell decomposition of $SO(N)$ and $SP(2N)$ Yang-Mills fields

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Abstract

We extend the decomposition of Yang–Mills field introduced by Faddeev and Niemi to the case of $sp(2N)$ and $so(N)$ algebras.

1 Introduction

L. Faddeev and A. Niemi proposed in [1] a new decomposition of the $SU(N)$ Yang–Mills field:

$$A_\mu = C^i_\mu\, h_i + \rho^X_i\, X_i + \rho^Y_{ij}\, Y_{ij} + \rho^Z_{ij}\, Z_{ij} + \tilde\rho^X_i\, \overline{X}_i + \tilde\rho^Y_{ij}\, \overline{Y}_{ij} + \tilde\rho^Z_{ij}\, \overline{Z}_{ij}\,, \tag{1}$$

where h_i is the basis of the Cartan subalgebra, such that

$$h_i h_j = -\frac{1}{2N}\delta_{ij} + d_{ijk}h_k, \qquad d_{ijk} \text{ – fully symmetric}\,,$$

and X, Y, Z are built with the help of the Maurer–Cartan one–form

$$R = dg\, g^{-1}\,, \qquad g \in SU(N)/U(1)^{N-1}$$

in the following way:

$$\begin{aligned}
X_i &= Rh_i - h_i R \\
Y_{ij} &= i(h_i R h_j - h_j R h_i) && \text{antisymmetric} \\
Z_{ij} &= h_i R h_j + h_j R h_i - h_i h_j R - R h_j h_i && \text{symmetric.}
\end{aligned}$$

The bar over X, Y, Z in (1) denotes complex conjugation, which for the algebra $su(N)$ is equivalent to the minus transposition operation $(-T)$.

The algebra $su(N)$ has the dimension $d = N^2 - 1$ and the rank $r = N - 1$; the corresponding Yang-Mills field has $3d = 3N^2 - 3$ times {function over Minkowski space} degrees of freedom. One can count the number of parameters in (1) and check that it is exactly $3N^2 - 3$, i.e., A_μ is an off-shell Yang-Mills field in a certain gauge.

Below we will extend the decomposition (1) to the case of $sp(2N)$ and $so(N)$ algebras. It may be appropriate for investigating the structure of the theory in the infrared limit, since the emerging variables can be considered as the relevant order parameters. Another approach to generalization of (1) is given in [2].

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2 $sp(2N)$ algebra

Let us start our generalization with the case of the $sp(2N)$ algebra. In the defining representation, its elements obey the condition

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} A^T \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = -A. \tag{2}$$

We choose the basis of the Cartan subalgebra in the usual way:

$$h_i = \mathrm{diag}\,(0,\dots,0,1,0,\dots,0,-1,0,\dots,0)\,, \qquad i = 1,\dots N,$$

and introduce additional matrices

$$p_i = h_i^2 \quad \text{such that} \quad p_i p_i = p_i = p_i^T\,.$$

Now, since the dimension of $sp(2N)$ is almost two times bigger than that of the algebra $su(N{+}1)$ (that has the same rank), we have to introduce more one–forms:

$$\begin{aligned} Z_{ij} &= h_i R h_j + h_j R h_i && \text{symmetric} \quad i \neq j \\ W_{ij} &= p_i R p_j + p_j R p_i && \text{symmetric} \quad i \neq j \\ Y_{ij} &= h_i R p_j - p_j R h_i && \\ U_i &= Z_{ii} - W_{ii}. && \end{aligned}$$

It can be proven that these one–forms are linearly independent and belong to $sp(2N)$, i.e., obey the defining condition (2).

To double the number of independent one–forms, it is possible to use the minus transposition again. But now it is equivalent to the matrix $\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ rather than to complex conjugation.

The decomposition of $sp(2N)$ Yang-Mills field now takes the form:

$$A_\mu = C_\mu^i\, h_i + \rho_{ij}^Z\, Z_{ij} + \rho_{ij}^W\, W_{ij} + \rho_{ij}^Y\, Y_{ij} + \rho_i^U\, U_i + \tilde{\rho}_{ij}^Z\, Y_{ij}^{-T} + \dots\,. \tag{3}$$

This expression has the following number of the degrees of freedom: C_μ yields $3r$, R yields $(d-r)$, $\rho^{Z,W}$ yield $1/2r(r-1)$, ρ^U yields r and ρ^Y yields r^2. For the algebra $sp(2N)$ we have

$$r = N\,, \qquad d = 2N^2 + N\,.$$

Hence the number of parameters in the decomposition (3) is

$$3N + 2N^2 + 2(1/2N(N-1) + 1/2N(N-1) + N + N^2) = 3N + 2N^2 + 4N^2 = 3d\,,$$

which is exactly the number of the degrees of freedom for the $sp(2N)$ off-shell Yang-Mills field.

3 $so(N)$ algebras

The defining representation of $so(2N)$, $so(2N+1)$ is described by the condition:

$$A^T = -A,$$

which means that the operation $-T$ acts trivially on A_μ and cannot produce additional one-forms in order to build off-shell Yang-Mills field. But we can find another conjugation operation that works for so algebra by fixing its action on the Cartan subalgebra (it will be a change of the sign, as for $su(N)$) and assuming that it is associative with respect to the matrix multiplication. From these two conditions one can derive that the conjugation should act on the element $A^{\alpha\beta}$ of the matrix one–form A in a very simple way:

$$A^{\alpha\beta} \to (-1)^{\alpha\beta} A^{\alpha\beta}$$

or it can be realized as an inner isomorphism produced by the matrix

$$J = \mathrm{diag}\,(1, -1, 1, -1, \ldots)\,.$$

As the basis of the Cartan subalgebra, we take the following matrices:

$$h_i = \begin{pmatrix} 0 & 0 & & & & & & \\ 0 & \ddots & \ddots & & & & & \\ & \ddots & 0 & -1 & & & & \\ & & 1 & 0 & \ddots & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & \ddots & \ddots & 0 \\ & & & & & 0 & 0 \end{pmatrix}, \quad i = 1, \ldots N.$$

We introduce again the matrices

$$p_i = h_i^2 \quad \text{such that} \quad p_i^T = p_i = -p_i p_i.$$

The set of one–forms now depends on whether the algebra in question is $so(2N)$ or $so(2N+1)$. The first three forms

$$\begin{aligned} Z_{ij} &= h_i R h_j + h_j R h_i && \text{symmetric} \quad i \neq j \\ W_{ij} &= p_i R p_j + p_j R p_i && \text{symmetric} \quad i \neq j \\ Y_{ij} &= h_i R p_j - p_j R h_i && \qquad\qquad\quad i \neq j \end{aligned}$$

work for both cases, while for $so(2N+1)$ two more forms

$$\begin{aligned} X_i &= R h_i - h_i R \\ V_i &= R p_i + p_i R - 2 p_i R p_i \end{aligned}$$

should be employed.

The resulting decomposition looks as follows

$$\begin{aligned} A_\mu = C^i_\mu\, h_i + \rho^Z_{ij}\, Z_{ij} + \rho^W_{ij}\, W_{ij} + \rho^Y_{ij}\, Y_{ij} + \big(\rho^X_i\, X_i + \rho^V_i\, V_i + \tilde\rho^X_i\, J\, X_i\, J + \\ + \tilde\rho^V_i\, J\, V_i\, J\big) + \tilde\rho^Z_{ij}\, J\, Z^{-T}_{ij}\, J + \tilde\rho^W_{ij}\, J\, W^{-T}_{ij}\, J + \tilde\rho^Y_{ij}\, J\, Y^{-T}_{ij}\, J\,, \quad (4) \end{aligned}$$

where the terms in the brackets are present only for the $so(2N+1)$ case. Counting the number of independent parameters gives us $3d$ for both series of algebras, which implies that the decomposition (4) is off–shell complete.

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Thermodynamic Bethe ansatz and exotic statistics

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Abstract

We derive and investigate the thermodynamic Bethe ansatz equations for a system of relativistic particles obeying a general extensive statistics. The conformal limit of such a system is studied. As a particular example, we consider the Gentile intermediate statistics.

1 TBA

Let us consider a relativistic multi-particle system of l different species of particles confined to a finite region of the size L. We denote by n_a the number of particles and by N_a the dimension of the Fock-space related to the species a. In the thermodynamic limit N_a and L approach infinity but n_a/L remain finite. The description of this limit is based on the thermodynamic Bethe ansatz (TBA) [1]. For the fraction of particles with rapidities between $\theta - \Delta\theta/2$ and $\theta + \Delta\theta/2$ it is convenient to introduce densities

$$\Delta N_a = \rho_a(\theta)\, \Delta\theta\, L\,, \qquad \Delta n_a = \rho^r_a(\theta)\, \Delta\theta\, L\,. \tag{1}$$

The total energy of the system is given by

$$E\,[\rho^r\,] = L \sum_{a=1}^{l} \int_{-\infty}^{\infty} \mathrm{d}\theta\, \rho^r_a(\theta)\, m_a \cosh\theta\,. \tag{2}$$

In order to find the equilibrium state of the system, we minimize the free energy $F\,[\rho, \rho^r\,] = E\,[\rho^r\,] - TS\,[\rho, \rho^r\,]$ with respect to ρ^r at constant temperature T:

$$\frac{\delta F}{\delta \rho^r_a} = \frac{\delta E}{\delta \rho^r_a} - T\frac{\delta S}{\delta \rho^r_a} - T\sum_{b=1}^{l} \frac{\delta S}{\delta \rho_b}\frac{\delta \rho_b}{\delta \rho^r_a} = 0\,. \tag{3}$$

Assuming that the scattering of particles in the system is factorizable and diagonal, the boundary conditions read

$$\exp(iLm_i \sinh\theta_i) \prod_{j\neq i}^{N} S_{ij}(\theta_i - \theta_j) = \kappa_i\,, \qquad i = 1, \ldots, N\,, \tag{4}$$

where the exact values of the constant phases κ_i are irrelevant for our purposes. Taking the logarithmic derivative of (4) and employing densities as in (1), we obtain

$$m_a \cosh\theta + 2\pi \sum_{b=1}^{l} \big(\varphi_{ab} * \rho^r_b\big)(\theta) = 2\pi\, \rho_a(\theta)\,, \tag{5}$$

where $\varphi_{ab}(\theta) = -i\partial_\theta(\ln S_{ab}(\theta))$ and $(u * v)(\theta) \equiv 1/(2\pi) \int \mathrm{d}\theta' u(\theta - \theta')v(\theta')$.

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2 Haldane-Wu statistics

In order to proceed with deriving the TBA, we need to specify the exclusion statistics which governs the behaviour of particles in the system. Since we want to discuss an exotic statistics, we recall in this section the derivation [2] of the TBA for the Haldane-Wu statistics.

A generalization of the Pauli exclusion principle suggested by Haldane in [3] looks as follows

$$\frac{\Delta D_a}{\Delta n_b} = -g_{ab}\,, \tag{6}$$

where D is the number of available states (holes) before the n-th particle has been added to the system. The parameter (a matrix) g_{ab} is called the statistical interaction. Eq. (6) provides a motivation to consider a system of particles obeying the so-called Haldane-Wu statistics. For such a system, the total dimension of the Hilbert space is postulated [4] to be

$$\hat{W}_g(N,n) = \frac{(N+(1-g)n+g-1)!}{n!\,(N-gn+g-1)!}\,. \tag{7}$$

This formula leads, with the help of the Stirling formula, to the following expression for the entropy

$$\begin{aligned} S &= kL\sum_{a=1}^{l}\int_{-\infty}^{\infty} d\theta \left[(\rho_a - \sum_b g_{ab}\rho_b^r)\ln\left(\frac{\rho_a+\rho_a^r-\sum_b g_{ab}\rho_b^r}{\rho_a-\sum_b g_{ab}\rho_b^r}\right)\right. \\ &\quad \left. +\rho_a^r\ln\left(\frac{\rho_a+\rho_a^r-\sum_b g_{ab}\rho_b^r}{\rho_b^r}\right)\right]. \end{aligned} \tag{8}$$

Substituting (2), (5) and (8) into (3) and using the variables

$$y_a(\theta) = \frac{\rho(\theta)-\sum_b g_{ab}\rho_b^r(\theta)}{\rho_a^r(\theta)}\,,$$

we obtain the TBA equations [2]:

$$\frac{1}{kT}m_a\cosh\theta = \ln(1+y_a(\theta)) + \sum_{b=1}^{l}\big(\Phi_{ab}*\ln(1+y_b^{-1})\big)(\theta)\,, \tag{9}$$

where $\Phi_{ab}(\theta) = \varphi_{ab}(\theta) - 2\pi g_{ab}\delta(\theta)$. Substituting this back into the expression for the free energy and introducing $c(T) = -\frac{6F(T)}{\pi L T^2}$, we obtain the finite-size scaling function. As usual we use a variable $r = 1/T$ and set the Boltzmann constant to be one. Then the finite-size scaling function acquires the form

$$c_g(r) = \sum_{a=1}^{l}\frac{6m_a r}{\pi^2}\int_0^\infty \mathrm{d}\theta\,\cosh\theta\,\ln(1+y_a^{-1}(\theta))\,. \tag{10}$$

In the small r limit the scaling function $c(r)$ becomes the effective central charge of a conformal field theory which describes the ultraviolet limit of a given massive model. That is, $\lim_{r\to 0} c(r) = c_{\mathrm{eff}} = c - 24h_{\min}$, where c is the conformal anomaly and h' is the lowest conformal weight in the corresponding conformal field theory.

In the ultraviolet limit Eq. (9) acquire the form of the so-called constant TBA equations:

$$\ln(1+y_a) = \sum_{b=1}^{l} (N_{ab} + g_{ab}) \ln(1 + y_a^{-1}) \,, \tag{11}$$

whereas (10) yields [2] the value of the effective central charge

$$c_g^{\text{eff}} = \frac{6}{\pi^2} \sum_{a=1}^{l} L\left(\frac{1}{1+y_a}\right) . \tag{12}$$

Here

$$N_{ab} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta \varphi_{ab}(\theta)$$

and $L(t)$ denotes the Rogers dilogarithm:

$$L(t) = \sum_{k\geq 1} \frac{t^k}{k^2} + \frac{1}{2} \ln t \ln(1-t) \,.$$

3 Generalized extensive statistics

Now we want to implement a generalized extensive exclusion statistics into the thermodynamic Bethe ansatz. Our exposition is based on [5].

Let $W(N,n)$ denote the total dimension of the Hilbert space for a system of n indistinguishable particles that can occupy N different states in the Fock-space. Then the statistics is called *extensive* if there exists a generating function for the dimensions $W(N,n)$, i.e.

$$\big(f(t)\big)^N = \sum_{k\geq 0} W(N,k)\, t^k \,. \tag{13}$$

An example is provided by the Gentile statistics [6] where

$$f_G(t) = 1 + t + t^2 + \ldots + t^G \,.$$

Consider a general system that possesses the extensivity property (13) with

$$f(t) = \sum_{k=0}^{d} P_k t^k \,. \tag{14}$$

We assume the following properties of (14):

a) POSITIVITY: $P_0 = 1$ and $P_k \geq 0$ for $k \geq 1$;

b) CONVERGENCE: the series (14) converges for $0 \leq |z| < R \leq \infty$;

c) NON-DEGENERACY: there exists no integer $p > 1$ such that $f(t) = \sum_k P_{pk} t^{pk}$.

This leads to the following possible types of $f(t)$:

I.	$d < \infty,$	$R = \infty$
II.	$d = \infty,$	$R = \infty$
IIIa.	$d = \infty,$	$1 < R < \infty$
IIIb.	$d = \infty,$	$0 < R \leq 1\,.$

For instance, the Gentile statistics belongs to the type I; the Boltzmann statistics with $f = e^t$ belongs to the type II; the bosonic statistics, $f = \frac{1}{1-t}$, belongs to the type IIIb.

It is worth remarking that we can replace the positivity property with
a') WEAK POSITIVITY: the first negative coefficient in the expansion of $(f(t))^N$ appears only at sufficiently large power n_0 (of order $O(N)$).

This is exactly the case of the Haldane-Wu statistics with statistical interaction g, where $n_0 \approx N/g$. For instance, for $g = 1/2$, we have

$$f(t) = 1 + \frac{t^2}{2} + t\sqrt{1 + t^2/4} = 1 + t + \frac{t^2}{2} + \frac{t^3}{8} - \frac{t^5}{128} + \ldots$$

which corresponds to $n_0 = 2N + 3$.

In order to find the entropy of the system, $S = k \ln W(N, n)$, in the thermodynamic limit ($N \to \infty$ and n/N is fixed), we express $W(N, n)$ as the following contour integral

$$W(N,n) = \frac{1}{2\pi i} \oint_{|z|=\rho} dz \, \big(f(z)\big)^N z^{-n-1} = \frac{1}{2\pi i} \oint_{|z|=\rho} \frac{dz}{z} \, e^{N(\ln f(z) - \mu \ln z)} \, . \tag{15}$$

The saddle point of the exponent in (15), denote it x, is found as the root of the equation (prime stands for a derivative)

$$x \, f'(t)|_{t=x} = \mu \, f(x) \, . \tag{16}$$

Now, an application of the saddle point method yields the asymptotics of (15),

$$W(N, \mu N) = \frac{\exp\{N(\ln f(x) - \mu \ln x)\}}{\sqrt{2\pi N h_\mu(x)}} \Big(\frac{1}{x} + O(\frac{1}{N})\Big) \, , \tag{17}$$

where $h_\mu(t) = \frac{f''(t)}{\mu f(t)} + \frac{1-\mu}{t^2}$. Note that, according to the Hadamard theorem, $h_\mu(x) > 0$ if $f(t)$ satisfies the positivity condition. Eq. (17) gives the entropy density:

$$s(\mu) = \lim_{N \to \infty} \frac{1}{N} \ln W(N, \mu N) = \ln f(x) - \mu \, \ln x \, . \tag{18}$$

Let us list some properties of the entropy density:

a) $s(\mu)$ is defined for $0 \le \mu \le d$;

b) $P_0 = 1$ implies that $s(0) = 0$;

c) $\partial_x \mu = x \mu h_\mu(x) > 0$ for $x > 0$ and $\partial_\mu s(\mu) = -\ln x$ imply that $\partial^2_\mu s(\mu) < 0$;

d) $s(d) = \ln P_d$ for type I; $s(\infty) = \infty$ for type IIIb; $s(\infty) = -\infty$ for types II and IIIa.

The last property shows that dealing with the types II and IIIa may be problematic. However, let us notice that in the TBA framework the variable x depends on the rapidity θ. The typical picture is that $x(\theta)$ falls off as $|\theta|$ grows (for instance, $x(\theta) = \exp\{-mr \cosh \theta\}$ for the trivial S-matrix). In this case $x(\theta) < x(0)$ and we can consider $f(t)$ of the types II and IIIa if $s(\mu)$ remains non-negative for $\mu \le \mu(x(0))$.

The total entropy is given by

$$S[\rho, \rho^r\,] = kL \sum_{a=1}^{l} \int_{-\infty}^{\infty} \mathrm{d}\theta \, \rho_a(\theta) s(\mu_a) \, . \tag{19}$$

With the help of (16), we find

$$\frac{\delta S}{\delta \rho_a^r} = -\ln x_a(\theta)\,, \qquad \frac{\delta S}{\delta \rho_a} = \ln f_a\big(x_a(\theta)\big)\,. \tag{20}$$

Substituting this into (3) and employing (5), we obtain the TBA equations for a generalized extensive statistics,

$$\frac{1}{kT} m_a \cosh\theta = -\ln x_a(\theta) + \sum_{b=1}^{l} \big(\varphi_{ab} * \ln f_b(x_b)\big)(\theta)\,. \tag{21}$$

The corresponding finite-size scaling function reads

$$c(r) = \sum_{a=1}^{l} \frac{6 m_a r}{\pi^2} \int_0^\infty \mathrm{d}\theta \, \cosh\theta \, \ln f_a(x_a(\theta))\,.$$

Now we are interested in the small r limit of the scaling function $c(r)$. Assume that $\varphi_{ab}(\theta) = \varphi_{ab}(-\theta)$ and that $\varphi_{ab}(\theta)$ falls off sufficiently fast as $\theta \to \infty$. Then in the ultra-violet limit we obtain [5]

$$c_{\text{eff}} = \frac{6}{\pi^2} \sum_{a=1}^{l} c_a\,, \tag{22}$$

$$c_a = -\frac{1}{2} \ln x_a \ln f_a(x_a) + \int_0^{x_a} \frac{\mathrm{d}t}{t} \ln f_a(t)\,, \tag{23}$$

where $x_a \equiv x_a(\theta = 0)$ are determined from the constant TBA:

$$\ln x_a = -\sum_{b=1}^{l} N_{ab} \ln f_b(x_b)\,, \qquad a = 1, \ldots, l\,. \tag{24}$$

Here, as before,

$$N_{ab} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\theta \varphi_{ab}(\theta)\,.$$

Notice that Eq. (23) implies that $c_a(0) = 0$. Moreover, taking a derivative of (23) and using (16), we find

$$\partial_{x_a} c_a = \frac{s\big(\mu_a(x_a)\big)}{2 x_a}\,. \tag{25}$$

Thus, $c_a(x_a)$ is positive and strictly increasing function. Furthermore, if all $N_{ab} \geq 0$, then it has the upper bound

$$c_a \leq \int_0^1 \frac{\mathrm{d}t}{t} \ln f_a(t)\,.$$

For the Gentile statistics (if the order G is the same for all species) Eq. (23) acquires the following form

$$c_a = L\left(x_a\right) - \frac{1}{G+1} L\left(x_a^{G+1}\right)\,. \tag{26}$$

Introducing the pseudo-energies, $\epsilon_a = -\ln x_a$, we can rewrite (26) in the form that recovers the well-known values for the bosonic and fermionic statistics,

$$c_a = L(e^{-\epsilon_a})\,, \tag{27}$$

$$c_a = L(e^{-\epsilon_a}) - \tfrac{1}{2} L(e^{-2\epsilon_a}) = L(\frac{1}{1+e^{\epsilon_a}})\,, \tag{28}$$

for $G=\infty$ and $G=1$, respectively. In the latter case we used the the Abel identity:

$$L(t)-\frac{1}{2}L(t^2)=L(\frac{t}{1+t})\,.$$

Finally, let us consider the simplest example with $S(\theta)=-1$ (the Ising model) or with $S(\theta)=1$ (the Klein-Gordon model). In both cases equation (21) is solved trivially, $x(\theta)=\exp\{-rm\cosh\theta\}$. Then for the Gentile statistics we can compute the entire scaling function:

$$c_{G}(r)=\frac{6rm}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k}\Big(K_1(krm)-K_1((G+1)krm)\Big)\,. \tag{29}$$

Observe that

$$c_{G}(r)=c_{\rm b}(r)-\tfrac{1}{G+1}c_{\rm b}((G+1)r)\,.$$

where $c_{\rm b}(r)$ is the bosonic scaling function. This formula shows, in particular, that the effective central charge corresponding to (29) is

$$c_{G}^{\rm eff}=\frac{G}{G+1}\,.$$

It is worth noticing that this $c_{G}^{\rm eff}$ can be identified for every positive integer G as the effective central charge of a minimal conformal model $\mathcal{M}(s,t)$ (such that $st=6(G+1)$).

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BCS model of superconductivity and the algebraic Bethe ansatz

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Abstract

This talk is based on the work [1]. It is shown that Richardson's exact solution of the discrete-state BCS (DBCS) model of superconductivity can be recovered as a special case of algebraic Bethe ansatz solution of the inhomogeneous six-vertex model with twisted boundary condition using Sklyanin's K-matrix construction and taking the quasiclassical limit.

1 The DBCS model

The DBCS model that is commonly used (see [3, 4] for recent reviews) to describe superconducting pairing correlations in nanoscale metallic grains is defined as follows: one considers a reduced BCS Hamiltonian,

$$H = \sum_{j,\sigma=\pm} \varepsilon_j c^\dagger_{j\sigma} c_{j\sigma} - g \sum_{jj'} c^\dagger_{j+} c^\dagger_{j-} c_{j'-} c_{j'+} \ , \tag{1}$$

for electrons in a set of pairs of time-reversed single-particle states $|j,\pm\rangle$ with energies ε_j, which are scattered pairwise from level j' to j, with interaction strength g. Richardson managed to solve this model exactly, for an arbitrary set of levels ε_j (degenerate levels are allowed, but are to be distinguished by distinct j-labels, i.e., they have $\varepsilon_i = \varepsilon_j$ for $i \neq j$): Since any level occupied by only a single electron does not participate in and remains "blocked" to the pair-scattering described by H, the labels of all such single-occupied levels are good quantum numbers. The eigenstates $|\alpha\rangle$ and corresponding eigenenergies $\mathcal{E}_\alpha$ of H thus have the following general form:

$$|\alpha\rangle \ = \ \prod_{i\in B} c^\dagger_{i\sigma_i} |\Psi_P\rangle \ , \qquad \mathcal{E}_\alpha = \mathcal{E}_P + \sum_{i\in B} \varepsilon_i \ . \tag{2}$$

Here B is the set of singly-occupied, blocked levels, and $|\psi_P\rangle$ is an eigenstate, with eigenvalue $\mathcal{E}_P$ and containing precisely P pairs of electrons, of a Hamiltonian H_U which has precisely the same form as the H of Eq. (1), except that now the j-sums are restricted to run only over the set U of all *unblocked* or non-singly-occupied levels. It is now convenient to introduce the pseudospin variables

$$S^z_j = \tfrac{1}{2}\left(1 - c^\dagger_{j+}c_{j+} - c^\dagger_{j-}c_{j-}\right) \ , \qquad S^+_j = c_{j-}c_{j+} \ , \qquad S^-_j = c^\dagger_{j+}c^\dagger_{j-} \ , \tag{3}$$

which satisfy the standard $SU(2)$ relations

$$[S^+_i, S^-_j] = 2\,\delta_{ij}\,S^z_j \ , \qquad [S^z_i, S^\pm_j] = \pm\,\delta_{ij}\,S^\pm_j \ , \tag{4}$$

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and in terms of which H_U takes the form

$$H_U \;=\; \sum_j^U 2\varepsilon_j(1/2 - S_j^z) \;-\; g\sum_{ij} S_i^- S_j^+ \;. \tag{5}$$

Richardson [5] has shown that the P-pair eigenstates (unnormalized) and eigenenergies have the general form

$$|\Psi_P\rangle \;=\; \prod_{l=1}^P S^-(\mu_l)|0\rangle\,, \qquad \text{with} \quad S^-(\mu_l) = \sum_i^U \frac{S_i^-}{2\varepsilon_i - \mu_l}\,, \tag{6}$$

$$\mathcal{E}_P \;=\; \sum_{l=1}^P \mu_l \;. \tag{7}$$

Here the P parameters μ_l $(l = 1, \ldots, P)$ are a solution of a set of P coupled algebraic equations, which we shall call the "Richardson equations",

$$\frac{1}{g} - \sum_i^U \frac{1}{2\varepsilon_i - \mu_l} + \sum_{\substack{l'=1\\ l'\neq l}}^P \frac{2}{\mu_{l'} - \mu_l} = 0\,, \qquad \text{for} \quad l = 1, \ldots, P\;. \tag{8}$$

These are to be solved (numerically, see Appendix B of [4]) subject to the restrictions $\mu_{l'} \neq \mu_l$ if $l' \neq l$. A simple proof of this result may be found in Appendix B of [4]; its strategy is to verify that $(H_U - \mathcal{E}_P)|\Psi_P\rangle = 0$ by simply commuting H_U to the very right past all of the S_l^- operators in (6).

2 The quasiclassical limit of "twisted" inhomogeneous XXZ model and DBSC

In this section we show that taking the quasiclassical limit of the inhomogeneous 6-vertex with specially twisted boundary conditions one exactly recovers above described DBSC model in the case of rational Boltzmann weights. It appears[1] that in the case of trigonometric Boltzmann weights one arrives at a more general class of models recently discussed in the context of superconductivity in small grains[2].

The R-matrix of XXZ model can be written as

$$\mathbf{R_{mi}} = \mathbf{I_m} \otimes \mathbf{I_i} + \frac{\tanh\eta}{\tanh(\lambda_m - 2\varepsilon_i)} \boldsymbol{\sigma}_\mathbf{m}^z \otimes \boldsymbol{\sigma}_\mathbf{i}^z + \frac{2\sinh\eta}{\sinh(\lambda_m - 2\varepsilon_i)} \left(\boldsymbol{\sigma}_\mathbf{m}^+ \otimes \boldsymbol{\sigma}_\mathbf{i}^- + \boldsymbol{\sigma}_\mathbf{m}^- \otimes \boldsymbol{\sigma}_\mathbf{i}^+\right), \tag{9}$$

where η is the so called quasiclassicality parameter, λ_m is the spectral parameter associated to the m's row and ε_i is the inhomogeneity parameter corresponding to the i's vertical line. We introduce also Sklyanin's k-matrix acting on the horizontal spaces as follows:

$$\mathbf{K} = \mathbf{I} + \frac{\eta}{g}\boldsymbol{\sigma}^z. \tag{10}$$

The transfer matrix of the model, which is the trace of the monodromy matrix over the horizontal (auxiliary) space as usual is defined by:

$$\mathbf{T}(\lambda_m) \equiv \mathrm{Tr}_m\{\mathbf{K_m R_{mN}} \ldots \mathbf{R_{m1}}\}. \tag{11}$$

In the quasiclassical limit $\eta \to 0$ we have:

$$\mathbf{T}(\lambda_m) = 2\mathbf{I} + \frac{4}{g}\eta^2 \mathbf{P}(\lambda_m) + O(\eta^3), \tag{12}$$

where

$$\begin{aligned} \mathbf{P}(\lambda) &= \frac{1}{2}\sum_{i=1}^{N} \frac{\boldsymbol{\sigma}_{\mathbf{i}}^{z}}{\tanh(\lambda - 2\varepsilon_i)} \\ &+ g\sum_{\substack{i,j\\ i<j}}^{N} \left(\frac{\boldsymbol{\sigma}_{\mathbf{i}}^{z}\boldsymbol{\sigma}_{\mathbf{j}}^{z}}{2\tanh(\lambda - 2\varepsilon_i)\tanh(\lambda - 2\varepsilon_j)} + \frac{\boldsymbol{\sigma}_{\mathbf{i}}^{+}\boldsymbol{\sigma}_{\mathbf{j}}^{-} + \boldsymbol{\sigma}_{\mathbf{i}}^{-}\boldsymbol{\sigma}_{\mathbf{j}}^{+}}{\sinh(\lambda - 2\varepsilon_i)\sinh(\lambda - 2\varepsilon_j)} \right). \end{aligned} \tag{13}$$

Due to the commutativity of transfer matrices for different spectral parameters, we have

$$\big[\mathbf{P}(\lambda), \mathbf{P}(\lambda')\big] = 0. \tag{14}$$

$\mathbf{P}(\lambda)$ can be viewed as the generating operator for all possible conserved operators of the model.

A convenient way of obtaining a *complete* set of *commuting* conserved operators, is to take the residues of $\mathbf{P}(\lambda)$ at the points $\lambda = 2\varepsilon_i$ for $i = 1 \ldots N$,

$$\mathbf{H}_{\mathbf{i}} = \mathrm{Res}[\mathbf{P}(\lambda); \lambda \to 2\varepsilon_i] \; = \; \oint_{\mathcal{C}_i} \frac{d\lambda}{2\pi i}\mathbf{P}(\lambda) \;, \tag{15}$$

where $\mathcal{C}_i$ is a small contour in the complex λ-plane, encircling the point $2\varepsilon_i$. Explicitly evaluating the residues for the present model, one obtains

$$\mathbf{H}_{\mathbf{i}} = \frac{\boldsymbol{\sigma}_{\mathbf{i}}^{z}}{2} + g\sum_{\substack{j=1\\ j\neq i}}^{N} \left(\frac{\boldsymbol{\sigma}_{\mathbf{i}}^{z}\boldsymbol{\sigma}_{\mathbf{j}}^{z}}{2\tanh(2\varepsilon_i - 2\varepsilon_j)} + \frac{\boldsymbol{\sigma}_{\mathbf{i}}^{+}\boldsymbol{\sigma}_{\mathbf{j}}^{-} + \boldsymbol{\sigma}_{\mathbf{i}}^{-}\boldsymbol{\sigma}_{\mathbf{j}}^{+}}{\sinh(2\varepsilon_i - 2\varepsilon_j)} \right) \;. \tag{16}$$

Then Eq. (14) immediately implies that all of these operators commute: $[\mathbf{H}_{\mathbf{i}}, \mathbf{H}_{\mathbf{j}}] = 0$. Now it is straightforward to recover the DBCS model and Richardson's solution thereof, by considering the rational case of XXZ model, which technically means that in all above formulae one should simply replace

$$\tanh x \to x, \qquad \sinh x \to x. \tag{17}$$

It is not difficult to see, that the DBSC Hamiltonian (6) can be expressed in terms of $\mathbf{H}_{\mathbf{i}}$ as follows:

$$H_U = \sum_{i=1}^{N} (g - 2\varepsilon_i)\, \mathbf{H}_{\mathbf{i}} + g\left(\sum_{i=1}^{N} \mathbf{H}_{\mathbf{i}}\right)^2 + \sum_{i=1}^{N}\left(\varepsilon_i - \frac{3g}{4}\right) \tag{18}$$

The usual ABA technic of finding eigenstates and eigenvalues of (18) reproduces Richardson's solution of DBSC model described in Section 1.

3 Summary

Our work fully clarifies the origin of the integrability of the DBCS model by explicitly constructing an integrable twisted 6-vertex model from which the DBSC model can be derived. Moreover, by this construction we pave the way for using the powerful algebraic Bethe ansatz machinery to calculate various quantities that have not yet been studied for the DBCS model. For example, there has recently been great progress in using the ABA to calculate matrix elements (or form factors) and correlation functions in vertex models, which, by building upon our work, could now fruitfully be applied to the DBCS model, too.

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An overview on the phase diagram of the frustrated two-leg ladder model

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Abstract

Using Density-Matrix Renormalization Group, we investigate the general phase diagram of the frustrated two-leg ladder with Heisenberg interactions along legs, rungs and diagonals. We confirm that all antiferromagnetic gapped states belong to the same universality class as the Haldane phase. In a three-dimensional phase-diagram, we determine a continuous surface with singularities in the string-order parameter or its first derivative, corresponding to a transition between two Haldane phases with different topological order. Some parts of this transition surface are critical with zero gap and vanishing string-order parameter. In the complementary parts, the transition is first order with finite gap and string order.

This talk is based on the recent authors article [1].

1 Introduction

$\mathcal{N}$-legged spin ladders are formed by assembling $\mathcal{N}$ spin-$\frac{1}{2}$ chains one next to the other. Many weakly-coupled ladder systems have now been synthesized. Among them, the family $Sr_{n-1}Cu_{n+1}O_{2n}$ consists in weakly coupled $\frac{1}{2}(n+1)$-legged ladders which are obtained from the CuO_2 planes of the parent compound $SrCuO_2$. After the suggestion that mechanisms similar to those occurring in the CuO_2 planes of cuprate ceramics may also lead to superconductivity[2], an intense theoretical and experimental activity has been developed on these systems. The doped spin-ladder compound $Sr_{14-x}Ca_xCu_{24}O_{41}$ is superconducting up to 10 K under pressure. Some theoretical tools which are specific to one-dimensional systems make the theoretical understanding of ladders easier than that of the CuO_2 planes responsible for the high-temperature superconductivity in cuprate ceramics. It is believed that the mechanisms leading to superconductivity are similar in both systems and the understanding of spin ladders should give some insight in the physics of the more complex two-dimensional superconducting systems.

A first step in the theoretical understanding of these systems is the study of un-doped ladders in which the spin interactions are of Heisenberg type. The exact solution of the spin-$\frac{1}{2}$ Heisenberg chain has been known for seventy years from the Bethe ansatz[3]. The quantum fluctuations prevent long-range antiferromagnetic order, there is no gap in the excitation spectrum and spin-spin correlations decay in power law as a function of the distance. However, the crossover from ladders to the square lattice, by increasing the number of coupled chains, is far from being smooth. While ladders with an odd number of legs exhibit properties similar to those of a single chain, i.e. gapless excitations and a power-law fall-off of spin-spin correlations, ladders with an even number of legs have a finite energy gap to the lowest spin-1 excitation and exponential decay of spin-spin correlations. These remarkable quantum properties are reminiscent

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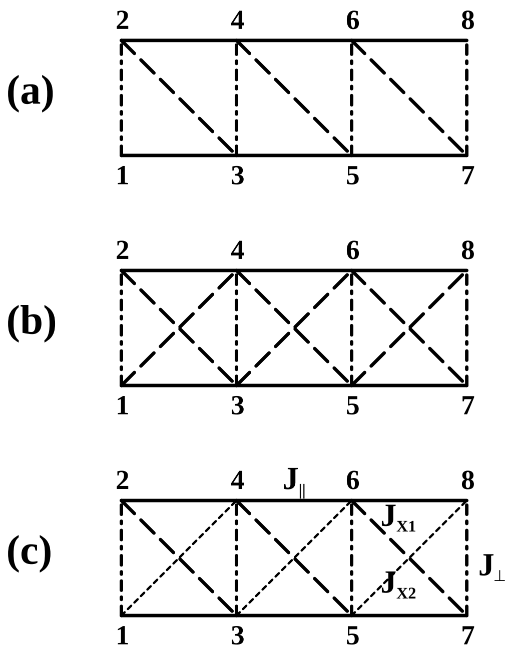

Figure 1: Frustrated-ladder models: (a) The "Zig-Zag chain". (b) Frustrated ladder with equal diagonal interactions. (c) The more general model studied in this paper

of those, first conjectured by Haldane[4], concerning the spin-$\frac{n}{2}$ Heisenberg chain with, respectively, n odd and n even. Therefore, a natural question arises: how are the ladder phases related to the phases in the spin chain? For the two-leg ladder, this question has been partly answered by White[5]. He considered a two-leg ladder with additional interaction along one diagonal of each plaquette [Fig. 1 (a)] and showed by varying this interaction that there is a continuous path from the Haldane to the ladder phase which does not cross any phase boundary or critical points. Some controversial arguments were published recently by Wang[6]. He considered a ladder with additional equal interactions along both diagonals of each plaquette [Fig. 1 (b)]. Using Density-Matrix Renormalization Group (DMRG) calculations[7] he proved the existence of a first-order transition line and suggested the occurrence of a phase-transition from the Haldane-phase to a gapped singlet phase with different sensitivity with respect to boundary conditions.

It is therefore interesting to study the general "frustrated two-leg ladder model" [Fig. 1 (c)], including two different diagonal interactions in the plaquettes, which contains both previous works as particular cases and to draw a general phase diagram. We thus consider the following Hamiltonian:

$$\begin{aligned} H = 2\sum_{i=1}^{L}[\quad & J_{\|}(\mathbf{S_i}\cdot\mathbf{S_{i+2}} + \mathbf{S_{2i-1}}\cdot\mathbf{S_{2i+1}}) + \\ & + J_{\perp}\mathbf{S_{2i-1}}\cdot\mathbf{S_{2i}} + J_{X1}\mathbf{S_{2i}}\cdot\mathbf{S_{2i+1}} + J_{X2}\mathbf{S_{2i-1}}\cdot\mathbf{S_{2i+2}}\,] \end{aligned} \tag{1}$$

(L is the ladder length and the number of spins is $N = 2L$) and propose a general three-dimensional phase diagram for this model. There is an obvious symmetry of this Hamiltonian by exchanging J_{X1} and J_{X2} and it is more convenient to use:

$$\mathcal{S} = J_{X1} + J_{X2} \qquad \text{and} \qquad \mathcal{D} = J_{X1} - J_{X2}$$

as parameters. We choose the following energy scaling:

$$|2J_{\|}| + |J_{\perp}| + |\mathcal{S}| + |\mathcal{D}| = 1\,. \tag{2}$$

Due to the symmetry, it is sufficient to consider $\mathcal{D} > 0$. We choose as independent variables $x = 2J_{\|}$, $y = \mathcal{S}$, $z = J_{\perp}$ and $\mathcal{D}$ is given by the previous relation. The variables $\{x, y, z\}$ obey: $|x| + |y| + |z| < 1$ and are thus contained in the regular octahedron represented in Fig. 2.

Using density–matrix renormalization group (DMRG), conformal field theory and some exact analytical results, we draw transition surfaces separating various phases and identify their critical parts.

Due to the time restriction, in this talk only the region, where all parameters x, y, z are positive, is considered. The general case, which incorporates the 8 regions with different signs of the parameters, is presented in our article in Phys. Rev. B [1].

2 Identification of different phases of the phase-diagram: Evidence for two topologically distinct spin-1 Haldane phases

The points Z' (all interactions null, except $J_\perp = -1$) and A' (all interactions null except $J_{X2} = -1/2$) represent independent pairs of spins 1/2 coupled ferromagnetically along rungs and along plaquette diagonals respectively. The 3^L degenerate ground-state is identical to that of L independent spins-1: $\tilde{\mathbf{S}}_\mathbf{i} = (\mathbf{S_{2i}} + \mathbf{S_{2i-1}})$ at Z' and $\tilde{\mathbf{S}}_\mathbf{i} = (\mathbf{S_{2i}} + \mathbf{S_{2i-3}})$ at A'. In the neighborhood of both points, degenerate perturbation theory is relevant as far as the other coupling parameters are much smaller than the gap between the 3^L degenerate ground-state and the first excited state (i.e. $<< 1$). The split 3^L lowest levels can be mapped onto the states of a chain of L interacting spins-1 with effective Hamiltonian (at first order in the coupling parameters):

$$H_{eff} = \frac{2J_{\|} + J_{X1} + J_{X2}}{2} \sum_{i=1}^{L} \tilde{\mathbf{S}}_\mathbf{i} \cdot \tilde{\mathbf{S}}_\mathbf{i+1} \quad (3)$$

in the neighborhood of point Z' and

$$\begin{aligned} H_{eff} &= \frac{2J_{\|} + J_\perp}{2} \sum_{i=1}^{L} \tilde{\mathbf{S}}_\mathbf{i} \cdot \tilde{\mathbf{S}}_\mathbf{i+1} \\ &+ \frac{J_{X1}}{2} \sum_{i=1}^{L} \tilde{\mathbf{S}}_\mathbf{i} \cdot \tilde{\mathbf{S}}_\mathbf{i+2} \end{aligned} \quad (4)$$

in the neighborhood of point A'.

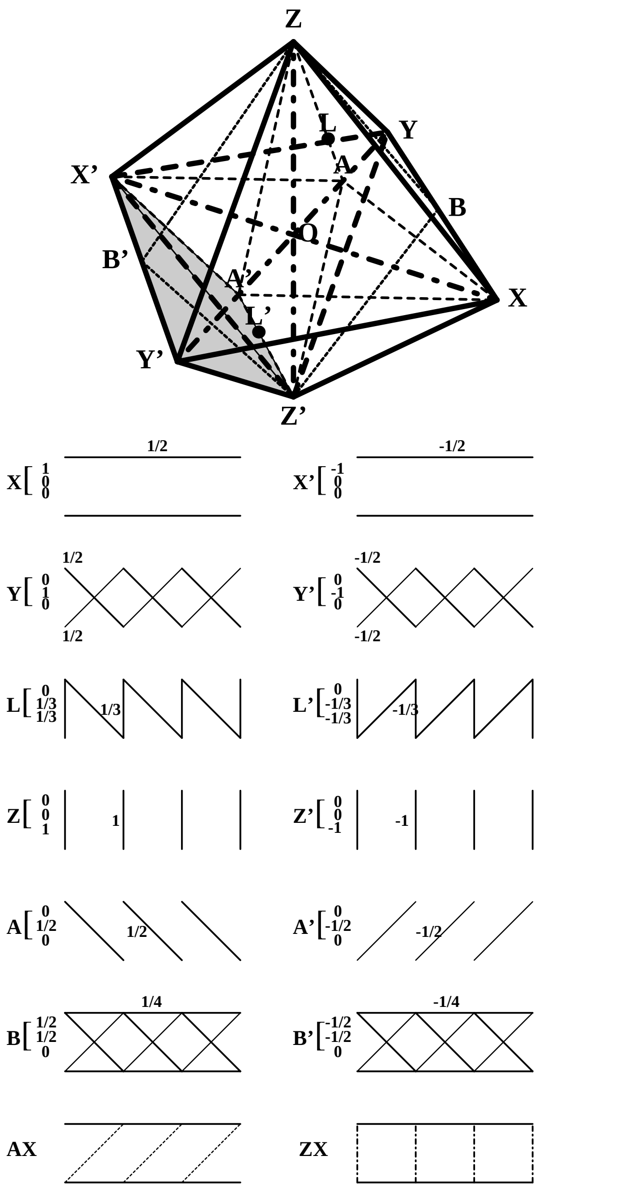

Figure 2: The regular octahedron inscribed in a sphere of radius 1, representing the parameter-space of our Hamiltonian [Eq. (1)] with the energy scaling of Eq. (3). Some remarkable points with their $\{x, y, z\}$ coordinates are listed: the spin-$\frac{1}{2}$ Heisenberg antiferromagnetic (X,Y,L) or ferromagnetic (X',Y',L') chain, products of independent dimers (A,Z) or triplet pairs (A',Z'), the usual ladder (AX and ZX segments) etc... The "Zig-Zag" chain represented in Fig. 1 (a) corresponds to the surface of the irregular octahedron (AA',XX',ZZ'). In the gray volume, all interactions are negative (ferromagnetic).

The Hamiltonian (4) for $(2J_{\|} + J_{X1} + J_{X2}) > 0$ and the Hamiltonian (5) for $2J_{\|} > 0$ and $J_{X1} = 0$ ($J_{X1} = 0$ is satisfied in the planes A'XZ, A'XZ', A'X'Z) represent the spin-1 Haldane chain. For $J_{X1} = 0$ both spin-1 Hamiltonians (4) and (5) are formally identical. However, they represent different topological configurations of the original ladder: the spin-1 operators $\tilde{\mathbf{S}}_{\mathbf{i}}$ represent the projection on the triplet state of the sum of a pair of spins-$\frac{1}{2}$ along one rung in Hamiltonian (4) and along one diagonal of a plaquette in Hamiltonian (5).

The most important breakthrough for the understanding of the antiferromagnetic spin-1 Heisenberg chain has been the discovery by Affleck-Kennedy-Lieb-Tasaki (AKLT)[8] of an exactly soluble Hamiltonian differing from the Heisenberg model only by addition to the Hamiltonian of a biquadratic term:

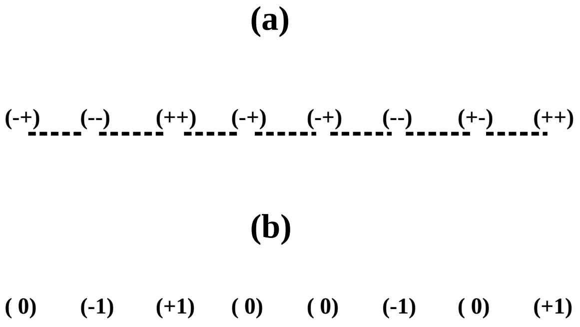

Figure 3: Schematic view of a typical component of the VBS state described by Eq. 8. (a) The dashed links represent Valence-Bonds between two fictive neighboring spins-$\frac{1}{2}$. (b) Sequence of spin-1 S^z values corresponding to (a): the non-zero spins are in a Néel order (-1,1,-1,1,....)

$$H_{AKLT} = \sum_i \left[\tilde{\mathbf{S}}_{\mathbf{i}} \cdot \tilde{\mathbf{S}}_{\mathbf{i+1}} + \frac{1}{3} (\tilde{\mathbf{S}}_{\mathbf{i}} \cdot \tilde{\mathbf{S}}_{\mathbf{i+1}})^2 \right] \tag{5}$$

Its exact ground-state called "Valence-Bond Solid" (VBS) is constructed out of valence bonds[9], it is nondegenerate and breaks no symmetry. It can be represented as shown in Fig. 3 (a).

- The spin-1 variables are expressed as a sum of two spin-$\frac{1}{2}$ variables
- Let $\psi_{\alpha,\beta}$ denote a spin-1 state in terms of the symmetrized spin-$\frac{1}{2}$ variables:

$$\begin{array}{rcl} \psi_{++} = & & |++>, \\ \psi_{+-} = & \psi_{-+} = (|+-> + |-+>)/\sqrt{2} \\ \psi_{--} = & & |-->. \end{array} \tag{6}$$

- Each spin-$\frac{1}{2}$ is contracted with one spin-$\frac{1}{2}$ at a neighbor site into a singlet state: a "valence bond". Contraction of two spin-$\frac{1}{2}$ variables into a singlet state can be written as $\epsilon^{\alpha\beta}|\alpha\beta>/\sqrt{2}$, where $\epsilon^{\alpha,\beta}$ represents the antisymmetric tensor: $\epsilon^{++} = \epsilon^{--} = 0$ and $\epsilon^{+-} = -\epsilon^{-+} = 1$
- The contractions are such that every spin-1 is connected to both its neighbors by a valence bond and the VBS state can be written:

$$|\Psi_{VBS}> = 2^{-N/2} \psi_{\alpha_1\beta_1} \epsilon^{\beta_1\alpha_2} \psi_{\alpha_2\beta_2} \epsilon^{\beta_2\alpha_3} \cdots \psi_{\alpha_i\beta_i} \epsilon^{\beta_i\alpha_{i+1}} \cdots \psi_{\alpha_L\beta_L} \epsilon^{\beta_L\alpha_1} \tag{7}$$

The VBS state, used as variational state for the Heisenberg spin-1 chain, gives an upper bound of the energy which is only a few percent higher than the best approximates through DMRG and is thought to capture the essentials of the physics of the this model. The VBS state has some "hidden" topological long-range order[10].

It is easy to see that each term of the sum in the previous relation has S_i^z sequences ordered in a way shown schematically in Fig. 3 (b). If we discard the sites with $S_i^z = 0$, the remaining $S_i^z = \pm 1$ sites have Néel order: $|1, -1, 1, -1, 1, -1, \cdots >$. This topological order is characterized by the non-local "string-order parameter":

$$\mathcal{O}^z(|m-l|) = \left\langle \tilde{S}_l^z \exp\left(i\pi \sum_{k=l+1}^{k=m-1} \tilde{S}_k^z\right) \tilde{S}_m^z \right\rangle \tag{8}$$

A normalized string-order parameter:

$$\tilde{\mathcal{O}}^z(d) = \frac{-\mathcal{O}^z(d)}{< (\tilde{S}^z)^2 >^2} \tag{9}$$

has been later introduced by White[7]. Its value for $d \to \infty$ is 1 for the VBS state and 0.84 for the Heisenberg spin-1 chain.

From the previous discussion about the neighborhoods of points Z' and A' in the phase diagram, it is natural to introduce two string-order parameters:

- $\mathcal{O}_\perp(d)$ obtained with $\tilde{S}_i^z = (S_{2i}^z + S_{2i-1}^z)$ in Eq. 9, with triplet pairs of spins along rungs
- $\mathcal{O}_X(d)$ obtained with $\tilde{S}_i^z = (S_{2i}^z + S_{2i-3}^z)$ in Eq. 9, with triplet pairs of spins along one diagonal of each plaquette

These two topologically distinct Haldane phases, that we now call $HD_\perp$ and HD_X, characterized respectively by the string-order parameters $\mathcal{O}_\perp$ and $\mathcal{O}_X$ have been introduced in a recent work by Kim et al.[11], with slightly different notations. They are depicted in Fig. 4. Using field-theoretical methods, Kim et al. have found evidences for some transition lines between these two Haldane phases around some particular points of the phase diagram corresponding to two-decoupled Heisenberg chains (see sections IV and V).

With J_{X1} strictly positive, the full Hamiltonian (5) represents the "frustrated" antiferromagnetic Heisenberg S=1 chain, with interactions $J_1 = (2J_{||} + J_\perp)/4$ and $J_2 = J_{X1}/4$ between first and second neighbors. This model has been studied by Kolezhuk et al.[12] They find a sharp discontinuity in the string-order parameter $\mathcal{O}_X$ at $J_2/J_1 \approx 0.7444$, suggesting a first-order transition. The significance of this first-order transition will appear in section V, when placed in the framework of the general phase-diagram.

3 DMRG with suitable boundary conditions

Our purpose is to determine transition lines or surfaces separating different phases, in particular the two topologically distinct Haldane phases $HD_\perp$ and HD_X defined in section II. We use the Density-Matrix Renormalization Group (DMRG) method with open boundary conditions which gives much more accurate results than with periodic boundary conditions[7]. However, a well-known inconvenience for the Haldane spin-1 chain with open boundary conditions is the four-fold degeneracy of the ground-state. This degeneracy is easily understood through the description in Fig. 3 (a) and Eq. (8) of the VBS state, which is thought to capture the essential of the physics of the Haldane chain. In Fig. 3 (a), with open boundary conditions, there are two spin-$\frac{1}{2}$ at each end of the chain which are not contracted through a valence bond to any other half-spin. The interaction between these extra half-spins fall off exponentially with L and in the long-chain limit, there is a four-fold degeneracy of the ground-state due to these two free spins.

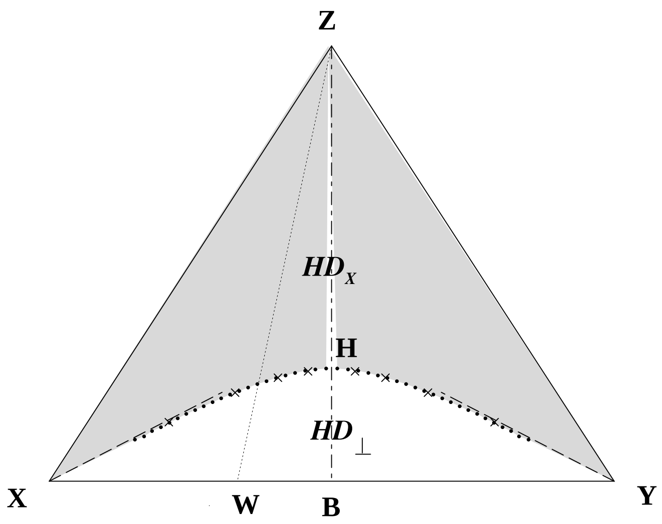

Figure 4: The XYZ face with equal diagonal interactions and all J's positive. The symmetry axis ZB corresponds to the "composite spin" model with a first-order transition from the Haldane $HD_\perp$ phase to the dimer HD_X phase at H. The dashed lines are exact first order transition lines between the $HD_\perp$ and HD_X phase in the neighborhoods of X and Y (two independent spin-$\frac{1}{2}$ chains) obtained through field-theoretical methods). The dots are our first order transition points obtained through DMRG. They are compared to earlier results of Wang (crosses). The behavior of nearest-neighbor pair correlations, gap and string-order parameter along the lines ZB and ZW corresponding to $J_\parallel = J_{X1} + J_{X2}$ are shown in Fig. 9

In their DMRG treatment of the spin-1 chain, White and Huse remove this inconvenient degeneracy by adding at each end an extra spin-$\frac{1}{2}$ which forms a singlet with half of the last spin-1 in the chain[13]. When applying DMRG to the frustrated ladder problem, the adaptation of this trick is straightforward and already built in the ladder topology. In the HD_X phase, we use the boundary conditions shown in Fig. 5 (a) which are generally chosen in treating the ladder problem[6]. In the $HD_\perp$ phase, we use other boundary conditions shown in Fig. 5 (b). If the ground state is viewed as a VBS state, in either case Figs. 5 (a) and (b) show that there is no "free spin" ("free" means not linked to any another spin with a "valence bond") at either end. Consequently, the ground-state is non degenerate in the long chain limit. In either case our DMRG algorithm for growing the chain length proceeds by adding one rung to each of the right and left Blocks as shown in Fig. 5. What differs is the shape of the initial Block: it is a rectangle in case (a) for the HD_X phase and a trapezoïd in case (b) for the $HD_\perp$ phase.

Using the same algorithm for any point of the phase-diagram (most previous DMRG studies on ladders have proceeded in such a way[6]) generally leads to numerical difficulties when crossing the phase-boundary. Suppose that we start from a point in the HD_X phase with the suitable algorithm [Fig. 5 (a)] and follow a path to the $HD_\perp$ phase, at the transition to the $HD_\perp$ phase, the ground-state, which was non-degenerate becomes four-fold degenerate and we need to keep a much larger number of states to obtain a reasonable approximation of the transition point. To determine numerically many transition points in a three-dimensional phase diagram, we need a "cheap" algorithm for calculating a transition point with a reasonable accuracy in a minimum of computer time.

Our algorithm uses the fact that at a phase transition, there are some singularities in the first or second derivatives of the energy. There is a discontinuity in the first derivative if the transition is first order. Here, if the transition is second order the second derivative is generally infinite at the critical point. We thus proceed as follows:

- We choose a path P_0P_0' going from the HD_X phase (P_0) to the $HD_\perp$ phase (P_0') and parameterized by the variable u. Using the algorithm depicted in Fig. 5 (a), we calculate at P_0 the energy and its first and second derivatives. The first derivative of the energy, with respect to $x = 2J_{\|}$, $y = \mathcal{S}$ and $z = J_{\perp}$ is obtained by calculating the correlations:

$$\begin{aligned} C_{\|} &= < 1 + 4\mathbf{S_i} \cdot \mathbf{S_{i+2}} > /2\,, & C_{\perp} &= < 1 + 4\mathbf{S_{2i-1}} \cdot \mathbf{S_{2i}} > /2\,, \\ C_{X1} &= < 1 + 4\mathbf{S_{2i}} \cdot \mathbf{S_{2i+1}} > /2\,, & C_{X2} &= < 1 + 4\mathbf{S_{2i-1}} \cdot \mathbf{S_{2i+2}} > /2\,. \end{aligned} \tag{10}$$

The derivative of the energy $\partial E_0/\partial u$ along the path is obtained from the preceding correlations. The second derivative $\partial^2 E_0/\partial u^2$ is obtained numerically through the same calculation with an infinitesimal variation on u (chosen at the square root of the numerical precision). An approximate second order polynomial expression of the energy near P_0 is deduced:

$$E = E_0 + s\partial E_0/\partial s + (s^2/2)\partial^2 E_0/\partial s^2 \tag{11}$$

- Using the other algorithm depicted in Fig. 5 (b), a polynomial approximation of the energy around P_0' is obtained in the same way.
- A first approximation P_0^C of the critical point is given by the intersection of both polynomials.
- A new starting point P_1 is chosen half way between P_0 and P_0^C, a new end point P_1' is chosen half way between P_0^C and P_0' and the process is reiterated.

An accuracy of 4 to 5 digits is generally obtained for the critical point after 8 to 10 iterations. We keep 30 to 60 states and use chain length $L \approx 50$. Some accuracy tests we be given in section V.

4 The phase-diagram in $[+,+,+]$ sector, with $J_{\|}$, $J_{\perp}$ and $\mathcal{S}$ positive

The boundary face XYZ

The boundary face XYZ, with equal antiferromagnetic diagonal interactions $J_{X1} = J_{X2} = J_X$ [Fig. 1 (b)], has been studied by Wang[6]. There is an obvious symmetry which exchanges $J_{\|}$ and J_X by twisting every other rung by an angle π around the axis of the ladder. ZB is a symmetry line. It is thus sufficient to study the half triangle XBZ. From the recent field-theoretical results of Kim et al.[11], near X, which represents two decoupled spin-$\frac{1}{2}$ chains, there is a first order transition between the two Haldane phases HD_X and $HD_{\perp}$ along the line $J_{\perp} = \mathcal{S}$. From the previous paragraph there is a first order transition between the same phases along the symmetry axe ZB at H. Using the DMRG algorithm described in section III, we have calculated the full transition line from X to H. It is show in Fig. 8. It is in good agreement with earlier results obtained by Wang[6]. However, in contrast to the conclusions of Wang, our results show that all along this line we have a first-order transition with finite gap from the HD_X to the $HD_{\perp}$ Haldane phase, in agreement with theoretical-field results[11].

As pointed out in section III, in contrast to our algorithm which uses different boundary conditions on either side of the transition line to avoid the degeneracy of the ground state, Wang has used everywhere the DMRG algorithm with boundary conditions corresponding to Fig. 5 (a). As he observed, below the transition line, the ground state is degenerate and it is nondegenerate above this line. However, his conclusion suggesting a transition from a Haldane phase to a "Dimer" phase with different properties concerning the behavior with respect to boundary conditions is incorrect. If we use everywhere the DMRG algorithm with the other

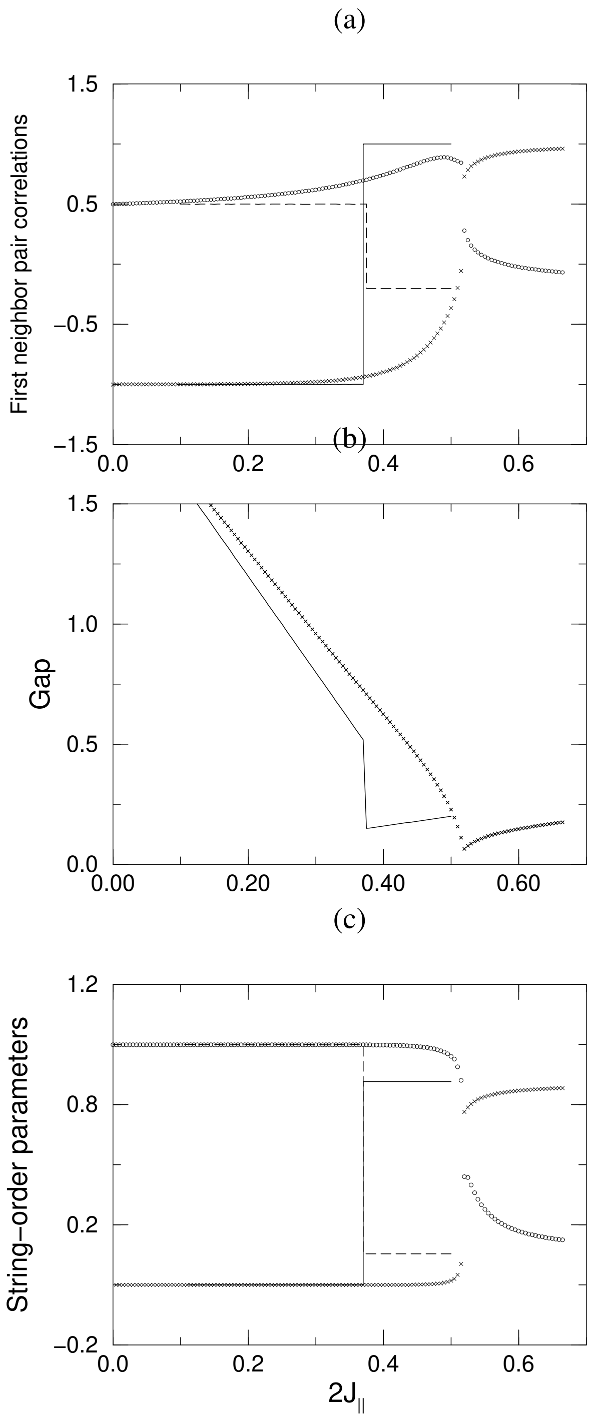

Figure 5: First order transition between Haldane phases with different topological order. (a) Pair correlation $C_\perp$ along ZW (crosses) and ZB (full line); pair correlation C_{X1} along ZW (circles) and ZB (dashed line). (b) Gap along ZW (crosses) and ZB (full line). (c) String order parameter $\tilde{\mathcal{O}}_X$ along ZW (crosses) and ZB (full line); string order parameter $\tilde{\mathcal{O}}_\perp$ along ZW (circles) and ZB (dashed line).

boundary condition corresponding to Fig. 5 (b), we find a degenerate ground state above the transition line and a non-degenerate ground state below! The correct interpretation is, as previously stressed, the transition between two topologically distinct Haldane phases. Typical sets of curves showing the discontinuities in the first-neighbor pair correlations [Eq. (11)], gap and string-order parameters along the dotted line ZW in Fig. 8 corresponding to $J_{\|} = J_{X1} + J_{X2}$ and along the symmetry line ZB are shown in Fig. 9.

The OYZ plane

The OYZ ($J_{\|} = 0$) plane corresponds to coupled dimerized chains. In this plane (Fig. 10) the point L corresponds to a simple antiferromagnetic Heisenberg chain and Y represents two decoupled Heisenberg chains. Strong arguments have been given for the existence of a critical line joining L to Y[17, 18] and a simple analytic approximation of this line has also been conjectured[18].

With our nontrivial energy scaling [Eq. (3)], it corresponds to the real root of the third-order equation:

$$J_\perp^3 - \mathcal{S}(1-J_\perp)^2 - \mathcal{S}^3 + 2\mathcal{S}^2(1-J_\perp) = 0 \quad (12)$$

We have determined through DMRG the transition between the two Haldane phases in this plane (see Fig. 10). From X to L, the transition line follows approximately the analytic form previously proposed but strongly differ beyond L. The critical line extends beyond L up to a point T_1 that we have determined through the same method as that used for the "Zig-Zag" model (cf. section IV). From T_1 to Ω, the transition is first order with finite gap.

The transition surface

We have exhibited four transition lines, between the two Haldane phases with different topological order, in four different planes of the phase diagram: XAZ, XYZ, OBZ, OYZ. We expect that all four lines belong to a same transition surface. Using the DMRG algorithm described in section III, we have investigated this transition surface over a fine grid in the {x,y} plane. As a good compromise, we kept 30 to 60 states in the DMRG scheme and chose a 0.02×0.02 {x,y} grid.

The results are shown in Fig. 11. The cuts of this surface through the planes XALZ, OBHZ, XYZ and OYZ have already been represented in Figs. 6, 7, 8 and 10 respectively. The straight line joining L (simple antiferromagnetic spin-$\frac{1}{2}$ Heisenberg chain) to X (two decoupled Heisenberg chains) is a symmetry line in the AXZ plane (cf. section IV) and lies in the transition surface. We used it for an accuracy test on our DMRG algorithm. For the points of our {x,y} grid which lie on the projection of this line in the XOY plane, we have plotted in Fig. 12 the difference $|z_c^{DMRG} - z_c^{Exact}|$ between the approximate coordinate z_c^{DMRG} of the surface obtained through DMRG and the exact value z_c^{Exact} corresponding to LX.

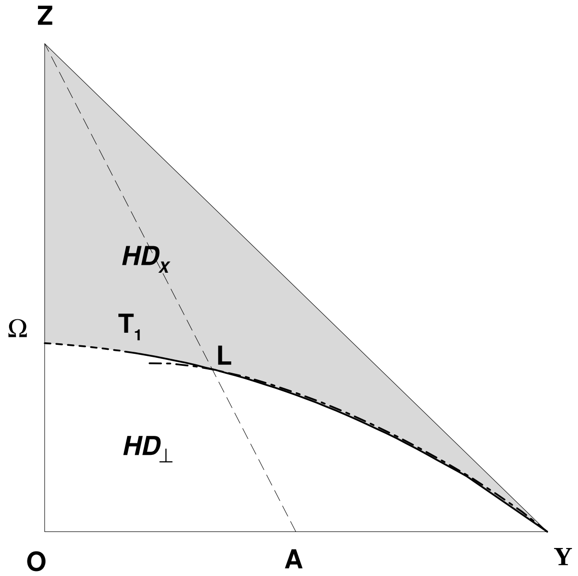

Figure 6: The transition line between the two Haldane phases with different topological order in the $J_\| = 0$ plane. L represents a simple spin-$\frac{1}{2}$ antiferromagnetic Heisenberg chain and Y two decoupled chains. The heavy line YLT$_1$ is critical with zero gap. On T$_1$L (dotted line), the transition is first order. The dash-dotted line is a simple approximation to the critical line.

Except in the neighborhood of X: two decoupled spin-$\frac{1}{2}$ Heisenberg chains, where a much larger number of states $\approx 30^2$ should be retained in the DMRG scheme to obtain precise results, we obtain an accuracy which is better than three to four digits on z_c. The accuracy is the best (7 digits) near the Majumdar-Gosh point G. It decreases drastically to $3 \sim 2$ digits close to X (two decoupled Heisenberg chains) and is of about 5 digits near L, in the critical regime.

Field-theoretical results[11] prove that in both planes XYZ and XALZ the transition line, near X, is first order and correspond to $\mathcal{S} = J_\perp$. In agreement with these exact results, our numerical data indicate that the transition surface is tangent at X to the plane $\mathcal{S} = J_\perp$.

Knowing this transition surface, we now determine which part of it is critical, with zero gap. We already know two intersecting critical lines LT$_1$ and YLT. We expect that they lie in a same critical part of the surface which is limited by a line of tricritical points (*"tricritical line"*) passing through T and T$_1$. To determine this tricritical line, we choose the same methods as these previously used for the "Zig-Zag chain" model[14, 15] (see section IV).

It is proved from field theory that, when umklapp interactions become relevant variables, simultaneously the spectrum develops a gap between the S=0 ground state and the first excited S=1 states and also the degeneracy of the first S=0 excited state and lowest S=1 state is split. Hence in the thermodynamical limit, the difference δ between the energies of these two lowest excited states is zero at the critical value for exchange parameters where the gap vanishes. As a simple illustration, let us consider the curved line ΩLY (Figs. 10 and 11) corresponding to the cut of the transition surface through the OYZ plane.

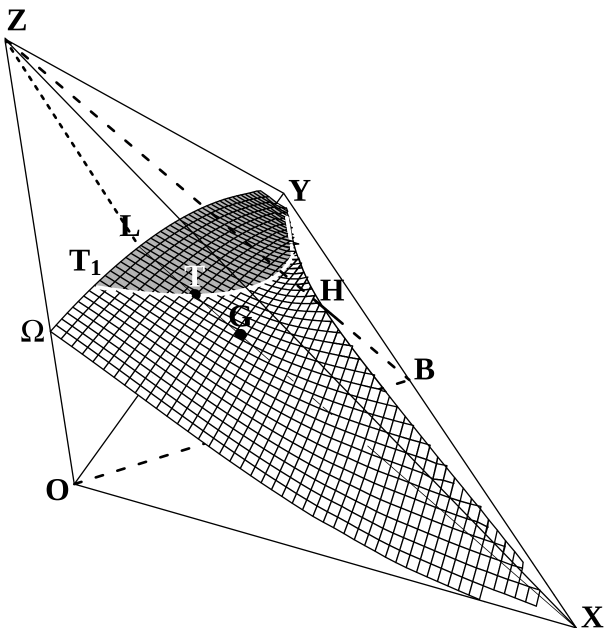

Figure 7: The transition surface separating the two gapped Haldane phases with different topological order in the [+,+,+] sector. Each point of the grid has been obtained through DMRG, keeping 30 states. The dark part of the surface is critical with vanishing gap and string-order parameters. The clear part corresponds to a first-order transition with finite gap and degenerate ground state. The white line which separates both parts of the surface thus appears as a tricritical line. The cuts of the surface through the planes XALZ, OBHZ, XYZ, and OYZ have been represented in Figs. 6, 7, 8 and 10 respectively. The straight line LTGX, joining L (one simple Heisenberg chain) to X (two decoupled Heisenberg chains) lies in the surface. The Majumdar-Ghosh point G corresponds to the intersection of this line with the OBHZ plane. Near X, our DMRG results are consistent with the exact result obtained from field theory (see text), i.e. the transition surface is tangent to the plane $J_\perp = \mathcal{S} = J_{X1} + J_{X2}$.

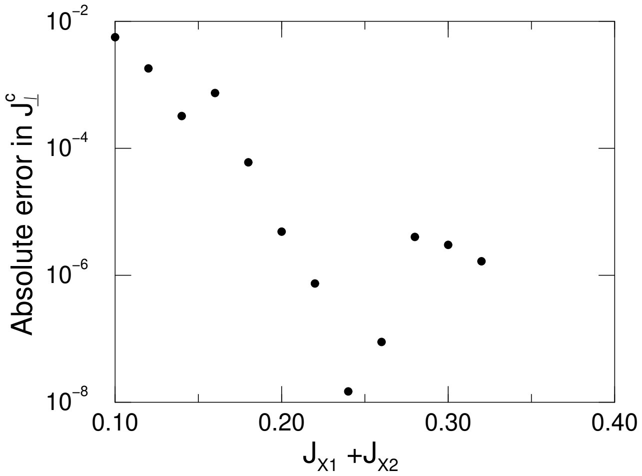

Figure 8: A test of accuracy of our DMRG scheme, with only 30 states kept, along the exact transition line LX. The accuracy is best (7 digits) near the Majumdar-Ghosh point G. It decreases drastically to 3 to 2 digits close to X (two decoupled Heisenberg chains) and also decreases when moving to L in the critical regime as the second-order transition becomes softer.

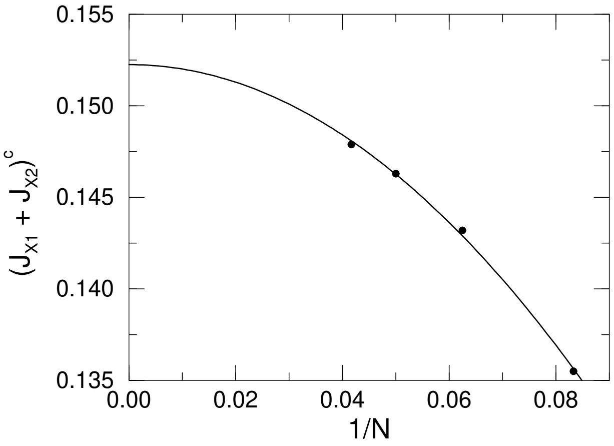

Figure 9: Critical value of $\mathcal{S} = J_{X1} + J_{X2}$ for which the difference δ between the two lowest excited states vanishes, as a function of 1/N, for finite chains with periodic boundary conditions. We follow the transition line ΩLY determined through DMRG in the OYZ plane (Fig. 10). A quadratic extrapolation gives $(J_{X1} + J_{X2})^c \approx 0.152$ in the thermodynamic limit, fixing the position of T_1 on the transition line.

Moving along this curve, we use Lanczös diagonalization on finite systems of length $L = N/2$, with periodic boundary conditions, to calculate the value $\mathcal{S}_c(N)$ of the parameter $\mathcal{S} = J_{X1} + J_{X2}$ at which δ vanishes.

The results are represented in Fig. 13 as a function of $1/N$. A quadratic extrapolation gives $\mathcal{S}_c(N \to \infty) \approx 0.152$ in the thermodynamic limit.

A much better accuracy (six digits) on the position of T on LX, for the "Zig-Zag chain" model has been obtained recently by Eggert[15]. He used the field-theoretical result that, on LX at the critical point T, the marginal operator vanishes and the energy difference $\delta(N)$ between the two first excited levels is exactly proportional to $1/N^3$. Although we have not been able to generalize this field-theoretical result for every point on the tricritical line, we have found numerically that except in the vicinity of the boundary plane XYZ, this $1/N^3$ law works remarkably well for a large part of the tricritical line. And we have used it to improve our accuracy.

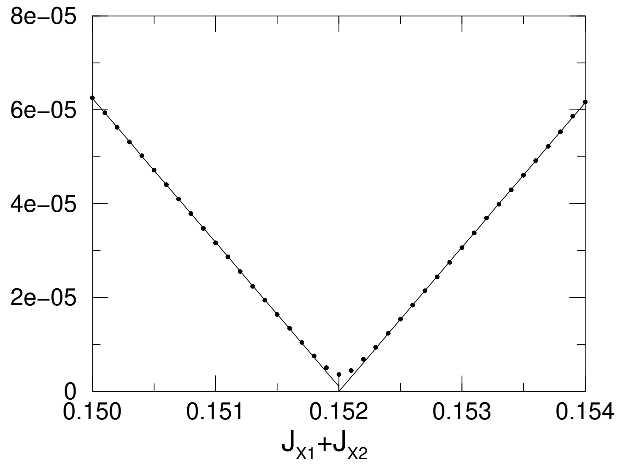

Figure 10: Mean square deviation with respect to a $1/N^3$ law of the difference between the two first excited levels of a finite ladder with periodic boundary conditions. We follow the cut of the transition surface through the plane $2J_{\|} = 0$ determined through DMRG.

Figure 14 illustrates around the critical value $\mathcal{S} = 0.152$ found by the previous method (Fig. 13) the mean square deviation of $\delta(N)$ with respect to a $1/N^3$ law, when we follow the transition line determined through DMRG in the OYZ plane. The results are based on Lanczös diagonalizations on finite systems with N=16, 20 and 24 sites. The critical value thus obtained $\mathcal{S} = 0.15201$ is improved up to five digits. This corresponds to the accuracy on the position of the transition line obtained through DMRG (see Fig. 12).

Taking successive cuts of the transition surface through different vertical planes at $J_{\|} = C^{st.}$ or $\mathcal{S} = C^{st.}$ and using the same methods, we have determined the whole tricritical line $\mathrm{T_1TY}$ representing the boundary of the critical part of the transition surface (heavy white line in Fig. 11). Near Y, this line is close to the boundary of the surface of transition in the XYZ plane. We do know however that it does not intersect it: exact field-theoretical results[11] prove that in the XYZ plane the transition is first order near Y.

5 Conclusion

We have presented an overall view on the three-dimensional phase diagram of the frustrated two-leg ladder model. This model contains, as particular cases, the spin-1 Heisenberg chain with first and second nearest-neighbor interactions, the spin-$\frac{1}{2}$ "Zig-Zag chain" with dimerization and frustration, the usual ladder and a variety of previously studied ladder models that we have briefly reviewed.

Only three different phases occur: the ferromagnetic phase (which is considered in our base article [1]) and two gapped Haldane phases with different symmetry and different long-range topological order. In a three dimensional phase diagram, these three phases are separated

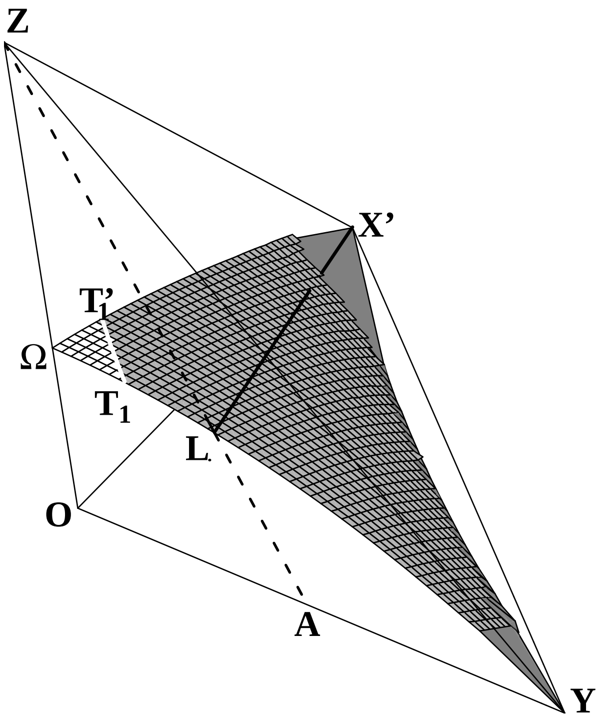

Figure 11: The transition surface between the two Haldane phases with different topological order in the $[-,+,+]$ sector. The dark part of the surface corresponds to the zero-gap critical part. All points of the mesh are DMRG results. Our DMRG result are not accurate in the vicinity of ferromagnetic (X') or antiferromagnetic (Y) decoupled chains. The grey areas are simple extrapolations. The critical part of the surface is bounded through the tricritical T_1T'$_1$ white line. On T_1T'$_1\Omega$ the transition is first order with two degenerate ground-states and finite gap. The cut by the plane ALZX' corresponds to the "Zig-Zag chain" model (cf. Fig. 6 in section IV). The straight line LX' lies in the critical surface.

through transition surfaces. Some parts of the transition surface separating the two gapped Haldane phases are critical with vanishing gap and string-order parameters; the complementary parts corresponds to a first order transition with finite gap and two degenerate S=0 ground-states. In the two dimensional plane cuts of the three-dimensional phase-diagram which correspond to the "Zig-Zag chain model", there is a one to one mapping of these two Haldane phases through a simple symmetry which exchanges rung and diagonal interactions. Consequently, although the topology of these two Haldane phases differ, the critical behavior concerning their thermodynamic properties near the critical surface are the same and they belong to the same class of universality.

Most calculations were performed on the Cray T3E from CEA-Grenoble. Numerical data concerning the transition surfaces are available upon request (from roger@drecam.saclay.cea.fr).

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Low dimensional integrable models and their applications in field theory and statistical physics
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Raković-Chu integrable systems in conformal variables

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Abstract

The study of integrable systems by Raković and Chu related to the hydrogen atom in crossed electric and magnetic field is continued. As the most suitable local variables in phase space the 15 generators of SO(4,2) dynamical group of the Kepler problem are taken. Three integrals of motion are expressed in conformal variables and appeared to be polynomials of 3d or 4th order in the new variables. Limiting cases and possible Lax matrices are discussed.

1 Introduction

Considerable progress in the theory of classical and quantum dynamical integrable systems achieved in last thirty years may be mainly attributed to *the inverse problem of integrability*. Namely we start from some general scheme like the Lax representation or R-matrix algebra which at some stage *generates* an integrable system. The scheme gives not only constants of motion but also indicates a way how to integrate classical equations of motion or to find the spectrum of quantum integrals of motion. Actually this approach can be treated as developing C.Jacobi's idea from his lectures of 1842 to generate integrable systems from sophisticated sets of separated variables [1] (pp. 198–199):

"Die Hauptschwierigkeit bei der Integration gegebener Differentialgleichungen scheint in der Einführung der richtigen Variablen zu bestehen, zu deren Auffindung es keine allgemeine Regel gibt. Man muss daher das umgekehrte Verfahren einschlagen und nach erlangter Kenntniss einer merkwürdigen Substitution die Probleme aufsuchen, bei welchen dieselbe mit Glück zu brauchen ist."

At the same time when we got a Hamiltonian appeared from physics it is not clear whether the system is integrable or not. If we are lucky to prove that our system is integrable, *the direct problem of integrability* arises. We should find the way how to deal with the given system, i.e. to integrate its classical equations of motion or to find the spectra of its quantum integrals. Very little is known about the direct problem. Actually we solve it by checking slightly modified trial sets of known integrable systems, i.e. by doing again inverse problem of integrability. If using specific features of our system we have managed to incorporate it into one of the known schemes, we may switch on powerful machinery of the inverse problem of integrability. So the crucial point is to find relation of the given system with the known schemes generating integrable systems.

Below we study an integrable system appeared from physics and rewrite it in new local coordinates on the same phase manifold in such a way that the Hamiltonian becomes a polynomial in generators of $o(4,2)$ Lie algebra. Lax matrices related with $o(p,q)$ algebras were considered by Reyman and Semenov-Tian-Shansky [2]. We hope that their scheme after some modification will be helpful to find the Lax representation in our case.

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Recently Raković and Chu [3] have shown that the three-dimensional dynamical system with the Hamiltonian function:

$$\mathbf{H}^{(\lambda)} = \frac{1}{2}\mathbf{p}^2 - \frac{Z}{r} - \omega l_3 + f x_1 + \frac{\omega^2}{18}(r^2 + (\lambda^2 - 1)x_3^2) \tag{1}$$

is integrable in classical mechanics at $\lambda = 1, 2$. Here l_i stands for components of angular momentum. If we take the coefficient of oscillator $1/16$ instead of $1/18$ and put $\lambda = 0$, the Hamiltonian (1) becomes nonintegrable and will describe the hydrogen atom in crossed electric and magnetic fields.

Raković and Chu studied in detail phase manifold of the system (1) at $\lambda = 1, 2$ and discussed its application to the hydrogen atom in circularly polarized microwave field [3], [4], [6].

Two additional independent integrals of motion read:

$\lambda = 1$:

$$I^{(1)} = \frac{4}{3}\omega l_3(\mathbf{H} - f x_1) + \frac{2}{9}\omega^2(l_1^2 + l_2^2 + 4l_3^2) + f(l_2 p_3 - l_3 p_2 + \frac{Z x_1}{r})$$
$$+\frac{f^2}{2}(r^2 - x_1^2) + \frac{1}{3}\omega f(3x_1 l_3 - p_2 r^2 + x_3 l_1) - \frac{1}{9}\omega^2 f x_1 r^2, \tag{2}$$

$$J^{(1)} = \omega^2(l_1^2 + l_2^2)(\mathbf{H} - f x_1) + \frac{2}{3}\omega^3 l_3(l_1^2 + l_2^2) + \frac{9}{8}f^2 l_1^2 - \frac{3}{2}\omega f l_1(l_1 p_2 - l_2 p_1 + \frac{Z x_3}{r})$$
$$+\frac{1}{2}\omega^2 f x_3(l_2 \mathbf{r}\mathbf{p} - 2l_1 l_3) + \frac{1}{6}\omega^3 f l_1 x_3 r^2 + \frac{3}{4}\omega f^2 x_1 x_3 l_1 + \frac{1}{8}\omega^2 f^2 x_3^2 r^2; \tag{3}$$

$\lambda = 2$:

$$I^{(2)} = \frac{4}{3}\omega l_3(\mathbf{H} - f x_1) + \frac{8}{9}\omega^2 l_3^2 + f(l_2 p_3 - l_3 p_2 + \frac{Z x_1}{r})$$
$$+\frac{f^2}{2}(r^2 - x_1^2) + \frac{1}{3}\omega f(3x_1 l_3 - p_2 r^2 + x_3 l_1) - \frac{1}{9}\omega^2 f x_1(r^2 + x_3^2), \tag{4}$$

$$J^{(2)} = \omega(l_1 p_2 - l_2 p_1 + \frac{Z x_3}{r}) + \frac{1}{9}\omega^3 x_3(x_1^2 + x_2^2) - \frac{3}{2}f l_1 - \frac{1}{2}\omega f x_1 x_3. \tag{5}$$

Proportional to ω^2 term in the Hamiltonian (1) corresponds to an axially symmetric isotropic oscillator at $\lambda = 1$ or Fermi oscillator with frequencies related as $\omega_1 : \omega_2 : \omega_3 = 1 : 1 : 2$ at $\lambda = 2$. Fermi oscillator is a degenerated system. It admits separation of variables both in Cartesian and parabolic coordinates. Because additional integrals of motion are of the third and forth order in momenta, spatial separation of variables (SoV) doesn't work in this case.

The Hamiltonians of dynamical systems (1) depend on parameter Z and can be written as $\mathbf{H}^{(\lambda)} = \mathbf{H}_0^{(\lambda)} - ZV$, where $V = 1/r$. At the level of fixed energy $\mathbf{H}^{(\lambda)} = E$ we arrive at the curve

$$P(Z, E) = \mathbf{H}_0^{(\lambda)} - ZV - E = 0. \tag{6}$$

This curve can be treated in two dual ways:

- E is a value of the Hamiltonian $\mathbf{H}^{(\lambda)}$ and Z is a free parameter
- Z is a value of new Hamiltonian $\mathbf{Z}^{(\lambda)}$ and E is a free parameter

In our case for $V = 1/r$ $\mathbf{Z}^{(\lambda)}$ is defined by

$$\mathbf{Z}^{(\lambda)} = r(\mathbf{H}_0^{(\lambda)} - E). \tag{7}$$

In quantum mechanics it corresponds to well known Sturmian transformation and we will continue to use this name in classical mechanics.

To get Sturmian counterparts of the integrals of motion one should substitute Z by $\mathbf{Z}^{(\lambda)}$ in (2), (3), (4), (5). It is easy to check that transformed quantities will mutually Poisson commute (see also [8]).

For perturbed Kepler problem one may introduce $o(4,2)$ variables [9], [10], [11] that allow to express Sturmian Hamiltonian $\mathbf{Z}^{(\lambda)}$ as a polynomial in term of its generators and hence to regularize equations of motions.

Let us introduce 15 generators $L_{\alpha\beta} = -L_{\beta\alpha}$ of conformal algebra $o(4,2)$ obeying the following Poisson brackets

$$\{L_{\alpha\beta}, L_{\gamma\delta}\} = -(\delta_{\alpha\gamma} g_{\gamma\mu} L_{\beta\delta} + \delta_{\beta\delta} g_{\delta\mu} L_{\alpha\gamma} + \delta_{\beta\gamma} g_{\gamma\mu} L_{\delta\alpha} + \delta_{\alpha\delta} g_{\delta\mu} L_{\gamma\beta}),$$

$$\alpha, \beta, \gamma, \delta, \mu = 1, 2, \ldots, 6 \tag{8}$$

with diagonal metrics $g_{\alpha\alpha} = (1,1,1,1,-1,-1)$. Following (for instance, [9]) one can take their realization in terms of coordinates and momenta of the Kepler problem in quantum case

$$\begin{aligned}
L_{ij} &= x_i p_j - x_j p_i = \varepsilon_{ijk} l_k, && (9)\\
L_{i4} &= \frac{1}{2}(x_i \mathbf{p}^2 - 2\mathbf{x}\mathbf{p} p_i - x_i), && (10)\\
L_{i5} &= \frac{1}{2}(x_i \mathbf{p}^2 - 2\mathbf{x}\mathbf{p} p_i + x_i), && (11)\\
L_{45} &= \mathbf{x}\mathbf{p}, && (12)\\
L_{46} &= \frac{1}{2}(r\mathbf{p}^2 - r), && (13)\\
L_{i6} &= r p_i, && (14)\\
L_{56} &= \frac{1}{2}(r\mathbf{p}^2 + r), && (15)
\end{aligned}$$

where $i, j = 1, 2, 3$. Then the generators $L_{\alpha\beta}$ satisfy the well known identities

$$L_{\alpha\gamma} g_{\gamma\sigma} L_{\sigma\beta} + L_{\beta\gamma} g_{\gamma\sigma} L_{\sigma\alpha} = 2\delta_{\alpha\beta}, \tag{16}$$

$$\varepsilon_{\rho\mu\alpha\beta\gamma\delta} g_{\alpha\tau} g_{\gamma\sigma} L_{\tau\beta} L_{\sigma\delta} = 0, \tag{17}$$

which fix the representation of $o(4,2)$ up to parity.

Transition from initial 6 Darboux variables x_i and p_i to 15 generators $L_{\alpha\beta}$ of conformal algebra constrained by relations (16), (17) means introduction of new local coordinates on the same phase manifold.

In what follows we need some inverse equalities,

$$\begin{aligned}
r\mathbf{p}^2 &= L_{45} + L_{56}, && (18)\\
r &= -L_{46} + L_{56}, && (19)\\
x_i &= -L_{i4} + L_{i5}. && (20)
\end{aligned}$$

Let us construct $\mathbf{Z}$ by (7). Using explicit form of (1), one gets

$$\mathbf{Z} = r\big(\frac{1}{2}\mathbf{p}^2 - E - \omega l_3 + f x_1 + \frac{\omega^2}{18}(r^2 + (\lambda^2 - 1)x_3^2)\big)\,. \tag{21}$$

First two items in the RHS of (21) can be written as

$$r(\frac{1}{2}\mathbf{p}^2 - E) = \alpha e^{-iL_{45}\theta} L_{56} e^{iL_{45}\theta}, \tag{22}$$

where parameter of the finite pseudo rotation depends on energy

$$e^{-\theta} = \alpha = \sqrt{-2E}. \tag{23}$$

Other terms in $\mathbf{Z}$ also can be expressed as the same similarity transform due to

$$\begin{aligned} r &= \frac{1}{\alpha} e^{-iL_{45}\theta} r e^{iL_{45}\theta}, && (24)\\ r_i &= \frac{1}{\alpha} e^{-iL_{45}\theta} r_i e^{iL_{45}\theta}, && (25)\\ l_i &= e^{-iL_{45}\theta} l_i e^{iL_{45}\theta}. && (26) \end{aligned}$$

Finally one arrives at the Sturmian counterpart of the initial Hamiltonian (1)

$$\begin{aligned} \frac{1}{\alpha}\mathbf{Z}^{(\lambda)}(f,\omega) &= L_{56} + \frac{\omega}{\alpha^2}(L_{56}-L_{46})L_{12} + \frac{f}{\alpha^3}(L_{56}-L_{46})(L_{15}-L_{14}) \\ &+ \frac{\omega^2}{18\alpha^4}(L_{56}-L_{46})\big((L_{56}-L_{46})^2 + (\lambda^2-1)(L_{35}-L_{34})^2\big). \end{aligned} \tag{27}$$

As well as in the initial Hamiltonian (1) factors in all summands of new Hamiltonian $\mathbf{Z}^{(\lambda)}(f,\omega)$ are Poisson commuting, hence the $\mathbf{Z}^{(\lambda)}(f,\omega)$ looks similar both in quantum and classical cases. The RHS of (27) is a cubic polynomial in generators of $o(4,2)$ and a quartic one in parameter $\frac{1}{\alpha}$ (23). The corresponding equations of motion are polynomials in conformal variables no higher then the third order. One can hope that these equations can be rewritten in the Lax form.

Sturmian operator $\mathbf{Z}^{(\lambda)}$ is responsible for dynamics in new time τ. It is connected to the original Schrödinger dynamics due to well-known relation

$$\frac{d\tau}{dt} = r. \tag{28}$$

Getting back the original time is quite similar to [10]. Dynamics conjugated to $\mathbf{Z}(E)$ will be integrable.

In order to find Sturmian counterparts of other integrals of motion one should substitute in (2),(3),(4),(5) the function $\mathbf{Z}(E)$ of phase variables from (21) instead the value of charge Z. Direct computation prove that the resulting quantities will Poisson commute in initial p_i, x_k variables so that dynamics in new time remains integrable. As well as the charge Hamiltonian (21) they can also be expressed in conformal variables. After tedious but straightforward calculations one arrives at two Sturmian integrable integrals of motion in conformal variables

$\lambda = 1$:

$$\begin{aligned} I_{1Z} = &-\frac{2}{3}L_{12}\omega\alpha^2 - L_{14}f\alpha + \frac{2}{9}(L_{23}^2 + L_{31}^2 + 4L_{12}^2)\omega^2 \\ &-\big(4(-L_{14}+L_{15})L_{12} + (-L_{46}+L_{56})L_{26} - (-L_{34}+L_{35})L_{23}\big)\frac{\omega f}{3\alpha} \\ &+\big((-L_{46}+L_{56})^2 + (-L_{14}+L_{15})^2\big)\big)\frac{f^2}{2\alpha^2} - (-L_{14}+L_{15})(-L_{46}+L_{56})^2\frac{\omega^2 f}{18\alpha^3}, \end{aligned} \tag{29}$$

$$
\begin{aligned}
J_{1Z} = &-\frac{1}{2}(L_{23}^2+L_{31}^2)\,\omega^2\alpha^2+\frac{3}{2}L_{23}L_{34}\,f\omega\alpha+\frac{9}{8}L_{23}^2\,f^2+\frac{2}{3}L_{12}(L_{23}^2+L_{31}^2)\,\omega^3\\
&+((L_{23}^2+L_{31}^2)(L_{14}-L_{15})+\frac{1}{2}(-L_{34}+L_{35})(L_{31}L_{45}-2L_{23}L_{12})\\
&+\frac{3}{2}L_{23}L_{12}(-L_{34}+L_{35}))\,\frac{\omega^2 f}{\alpha}+\frac{3}{2}(-L_{14}+L_{15})(-L_{34}+L_{35})L_{23}\,\frac{\omega f^2}{\alpha^2}\\
&+L_{23}(-L_{34}+L_{35})(-L_{46}+L_{56})^2\,\frac{\omega^3 f}{12\alpha^3}+(-L_{34}+L_{35})^2(-L_{46}+L_{56})^2\,\frac{\omega^2 f^2}{8\alpha^4}
\end{aligned}
\tag{30}
$$

$\lambda = 2$:

$$
\begin{aligned}
I_{2Z} = &-\frac{2}{3}L_{12}\,\omega\alpha^2-L_{14}fa+\frac{8}{9}L_{12}^2\,\omega^2\\
&-(4(-L_{14}+L_{15})L_{12}+(-L_{46}+L_{56})L_{26}-(-L_{34}+L_{35})L_{23})\,\frac{\omega f}{3\alpha}\\
&+\frac{1}{2}((-L_{46}+L_{56})^2+(-L_{14}+L_{15})^2)\,\frac{f^2}{\alpha^2}\\
&-(-L_{14}+L_{15})(-L_{46}+L_{56}+L_{34}-L_{35})(-L_{46}+L_{56}-L_{34}+L_{35})\,\frac{\omega^2 f}{18\alpha^3}
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
J_{2Z} = &-\frac{3}{2}L_{23}f-L_{34}\,\alpha\omega-L_{12}(-L_{34}+L_{35})\,\frac{\omega^2}{\alpha}+(-L_{14}+L_{15})(-L_{34}+L_{35})\,\frac{\omega f}{2\alpha^2}\\
&+(-L_{34}+L_{35})((-L_{34}+L_{35})^2+3(-L_{46}+L_{56})^2)\,\frac{\omega^3}{18\alpha^3}
\end{aligned}
\tag{32}
$$

To this end we arrived at two integrable systems (27), (29), (30) at $\lambda = 1$ and (27), (31), (32) at $\lambda = 2$ defined on the orbits of $o(4,2)$ constrained by (16), (17). All integrals of motion are polynomials of third or fourth order in terms of generators of conformal algebra. Because there is no pure quadratic functions among them, the Reyman-Semenov-Tian-Shansky scheme does not work directly.

2 Separation of variables in one field limits at $f = 0$ or $\omega = 0$

We analyze the SoV in limiting cases both in terms of original x_i, p_i and $o(4,2)$ variables.

We first look at the case of $Z = 0$, when the Hamiltonian reads

$$
\mathbf{H}_0^{(\lambda)} = \frac{1}{2}\mathbf{p}^2-\omega l_3+fx_1+\frac{\omega^2}{18}(r^2+(\lambda^2-1)x_3^2). \tag{33}
$$

We got quadratic in coordinates and momenta Hamiltonian that is diagonalized by simplectic rotation with constant matrix. Finally in bosonic creation annihilation variables the $\mathbf{H}_0^{(\lambda)}$ takes diagonal form

$$
\mathbf{H}_0^{(\lambda)} = \frac{\sqrt{10}\omega}{3}(d_1d_1^\dagger+d_2d_2^\dagger)+\frac{\omega\lambda}{3}d_3d_3^\dagger+\frac{9f^2}{16w^2}\,. \tag{34}
$$

The degeneracy of axially symmetric oscillator in 12 plane is removed by switching on perturbation, i.e., the Coulomb potential, in such way that the resulting system remains integrable.

At $f = 0$ the Hamiltonian reads

$$
\mathbf{H}^{(\lambda)}(0,\omega) = \frac{1}{2}\mathbf{p}^2-\frac{1}{r}-\omega l_3+\frac{\omega^2}{18}(r^2+(\lambda^2-1)x_3^2). \tag{35}
$$

At $\lambda = 1$ the system obeys rotational symmetry, its other integrals of motion are reduced to $\mathbf{l}^2$ and l_3 and separation of variables take place in polar spherical coordinates r, θ and ϕ.

The case $\lambda = 2$ is slightly less trivial. At $f = 0$ the second integral of motion drastically simplifies to

$$I_1^{(2)} = l_3, \tag{36}$$

while the third integral of motion reads

$$J_1^{(2)} = \omega(l_1 p_2 - l_2 p_1 + \frac{x_3}{r}) + \frac{1}{9}\omega^3 x_3(x_1^2 + x_2^2)\,. \tag{37}$$

The system admits separation of variables in parabolic coordinates

$$\xi = r + x_3, \quad \eta = r - x_3, \quad \phi = \tan^{-1}(\frac{x_2}{x_1}). \tag{38}$$

We will do it for the Sturmian counterpart of $\mathbf{H}^{(2)}$ taking $\mathbf{Z}$ from (27) at zero electric field $f = 0$

$$\frac{1}{\alpha}\mathbf{Z}^{(2)}(f = 0, \omega) = \frac{1}{\alpha}\mathbf{Z}_w{}^{(2)} =$$

$$= L_{56} + \Omega(L_{56} - L_{46})L_{12} + \frac{\Omega^2}{18}(L_{56} - L_{46})\big((L_{56} - L_{46})^2 + 3(L_{35} - L_{34})^2\big), \tag{39}$$

where $\Omega = \omega/\alpha^2 = \omega/(-2E)$. It has a sense to look at this SoV in terms of subalgebra decompositions of $o(4,2)$. At fixed level of $L_{12} = m$ the modified Hamiltonian $\mathbf{Z}_\omega^{(2)}$ depends only on generators $Q_{\mu\nu} = L_{2+\mu 2+\nu}$, $\mu, \nu{=}1,2,3,4$ of $o(2,2)$ subalgebra of $o(4,2)$. The Casimir operator of $o(2,2)$

$$C = -L_{34}^2 + L_{35}^2 + L_{36}^2 + L_{45}^2 + L_{46}^2 - L_{56}^2 = 1 - L_{12}^2 \tag{40}$$

depends on $m = e.v.L_{12}$ as a result of the identities (16) between the conformal generators in initial representation. Another Casimir operator of $o(2,2)$ vanishes.

Semisimple Lie algebra $o(2,2)$ admits decomposition $o(2,2) = su(1,1) \oplus su(1,1)$ with new generators s_i and t_i respectively

$$s_1 = L_{36} + L_{45}\,, \qquad s_2 = L_{35} - L_{46}\,, \qquad s_3 = L_{34} - L_{56}\,, \tag{41}$$

$$t_1 = -L_{36} + L_{45}\,, \qquad t_2 = L_{35} + L_{46}\,, \qquad t_3 = L_{34} + L_{56}\,. \tag{42}$$

They obey the commutators

$$[s_i, s_j] = -2i\epsilon_{ijk} g_{kr} s_r\,, \qquad [t_i, t_j] = -2i\epsilon_{ijk} g_{kr} t_r\,, \tag{43}$$

with metrics $(1,1,-1)$ and the Casimir elements being

$$s_1^2 + s_2^2 - s_3^2 = 1 - L_{12}^2\,, \qquad t_1^2 + t_2^2 - t_3^2 = 1 - L_{12}^2\,. \tag{44}$$

In terms of s_i and t_i generator L_{56} is decomposed as a linear combination of compact generators

$$L_{56} = -\frac{1}{2}(s_3 - t_3) \tag{45}$$

and we arrive at the reduced charge on the orbit of $su(1,1) \oplus su(1,1)$

$$\frac{1}{\alpha}\mathbf{Z}_\omega^{(2)} = \frac{1}{2}(t_3 - s_3) + \frac{\omega}{2\alpha^2}L_{12}(s_2 - s_3 - t_2 + t_3) + \frac{\omega^2}{36\alpha^4}(-(-s_2 + s_3)^3 + (-t_2 + t_3)^3) \tag{46}$$

The separated integrals of motion now read respectively

$$\frac{1}{\alpha}\mathbf{Z}^{(2)}_{\omega s} = -\frac{1}{2}s_3 - \frac{\omega}{2\alpha^2}L_{12}(-s_2+s_3) - \frac{\omega^2}{36\alpha^4}(-s_2+s_3)^3 \tag{47}$$

$$\frac{1}{\alpha}\mathbf{Z}^{(2)}_{\omega t} = \frac{1}{2}t_3 + \frac{\omega}{2\alpha^2}L_{12}(-t_2+t_3) + \frac{\omega^2}{36\alpha^4}(-t_2+t_3)^3 \tag{48}$$

and relation between their eigenvalues

$$\frac{1}{\alpha}Z^{(2)}_{\omega s} + \frac{1}{\alpha}Z^{(2)}_{\omega t} = \frac{Z}{\alpha} \tag{49}$$

allows us to find the eigenenergy E.

In the case $\omega = 0$ the Hamiltonian doesn't depend on λ. The treatment is very similar to $f = 0, \lambda = 2$. Integral J is replaced by $l_1 = L_{23}$ and the SoV is appeared in another parabolic coordinates

$$\xi = r + x_1, \qquad \eta = r - x_1, \qquad \psi = \tan^{-1}(\frac{x_3}{x_2}). \tag{50}$$

with Ox_1 as a symmetry axis. The charge Hamiltonian now doesn't depend on λ and reads

$$\frac{1}{\alpha}\mathbf{Z}(f,\omega=0) = \frac{1}{\alpha}\mathbf{Z}_f = L_{56} + \frac{f}{\alpha^3}(L_{56}-L_{46})(L_{15}-L_{14}). \tag{51}$$

One can repeat the decomposition $o(4,2) \supset so(2,2) = su(1,1) \oplus su(1,1)$ with other choice of subalgebras. Subalgebra $o(2,2)$ is formed by excluding the second and the third row and columns from matrix $L_{\alpha\beta}$. It admits decomposition $so(2,2) = su(1,1) \oplus su(1,1)$ with generators k_i and q_i respectively

$$k_1 = L_{16} + L_{45}\,, \qquad k_2 = L_{15} - L_{46}\,, \qquad k_3 = L_{14} - L_{56}\,, \tag{52}$$

$$q_1 = -L_{16} + L_{45}\,, \qquad q_2 = L_{15} + L_{46}\,, \qquad q_3 = L_{14} + L_{56}\,. \tag{53}$$

This differs from (41), (42) by substitution $L_{1\alpha}$ in s_i, t_i variables by $L_{3\alpha}$. The brackets, values of the subalgebras Casimir operators and definition of parabolic coordinates are similar to (41), (42) and (44). Finally for the reduced charge we arrive at

$$\frac{1}{\alpha}\mathbf{Z}_f = -\frac{1}{2}(k_3 - q_3) + \frac{f}{4\alpha^3}((-k_2+k_3)^2 - (q_2-q_3)^2) \tag{54}$$

with obvious separated operators

$$\frac{1}{\alpha}\mathbf{Z}_{f1} = -\frac{1}{2}k_3 + \frac{f}{4\alpha^3}(-k_2+k_3)^2\,, \tag{55}$$

$$\frac{1}{\alpha}\mathbf{Z}_{f2} = \frac{1}{2}q_3 - \frac{f}{4\alpha^3}(-q_2+q_3)^2\,, \tag{56}$$

$$\mathbf{Z}_f = \mathbf{Z}_{f1} + \mathbf{Z}_{f2}\,. \tag{57}$$

Hence at $\lambda = 2$ initial system generically depends on f and ω in the limits of zero fields it admits the SoV corresponding to different subalgebra extractions from $o(4,2)$. The separated equations in limit cases of $f = 0$ or $\omega = o$ can be naturally treated in the bases of linear combinations of the Cartan elements

$$L_{12}\,, \qquad L_{34}\,, \qquad L_{56}$$

and

$$L_{23}\,, \qquad L_{14}\,, \qquad L_{56}\,,$$

respectively. Transition between them is provided by the element L_{13}. Its commutators with the Cartan elements L_{12}, L_{34} of $o(4)$ subalgebra result in another Cartan elements L_{23}, L_{14}.

Similarly, at $\lambda = 1$, we need to construct interpolation between subgroup basis at $f = 0$

$$L_{12}\,, \qquad C(2) = L_{12}^2 + L_{23}^2 + L_{31}^2\,, \qquad L_{56}$$

and

$$L_{23}\,, \qquad L_{14}\,, \qquad L_{56}$$

at $\omega = 0$, respectively.

The SoV both in spherical and parabolic coordinates does not use specific value of the coefficient $\Omega^2/18$ in the operator $\mathbf{Z}_\omega$ (39).

So the problem arises to find a kind of interpolation scheme matching the two limit cases. A rather promising idea is to apply Reyman–Semenov-Tian-Shansky [2] approach to construction of Lax pairs for the dynamical systems on the orbits of $o(p,q)$ Lie algebras. Their approach should be modified to generate integrals of motion (29), (30), (31), (32). The main feature is that among the integrals of motion in the Raković - Chu case there is no quadratic one.

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Boundaries and symmetry breaking in the nonlinear Schrödinger hierarchy

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Abstract

We describe an algebraic framework for studying the symmetry properties of quantum nonlinear Schrödinger (NLS) equation on the half line. The approach is based on the introduction of a boundary algebra, in the same way the ZF algebra is essential in the quantification of the NLS equation on a line.

1 Introduction

This note is a summary of the paper [1, 2]. It uses the results obtained in [3], and also [4]. We describe an algebraic framework for studying the symmetry properties of quantum nonlinear Schrödinger (NLS) equation on the half line. The approach is based on the introduction of a boundary algebra, in the same way the ZF algebra is essential in the quantification of the NLS equation on a line [5, 6]. It turns out that this algebra both encodes the boundary conditions and generate integrals of motion. We use this direct relationship between boundary conditions and symmetry content to establish the spontaneous breakdown of some internal symmetries, due to the boundary.

The paper is organized as follow: in the first section, we present the Non-Linear Schrödinger equation in 1+1 dimension without boundary, at classical and quantum level. We then study its vectorial version, where the canonical field in an N-vector. In particular, we construct the Yangian symmetry which appears in this case. In section 3, we study the case with boundaries. We classify the possible reflection matrices, construct the boundary condition associated to these matrices. Then, we study the symmetry algebra of the models, and show the spontaneous symmetry breaking induced by the boundary condition.

2 The Non Linear Schrödinger equation on a line

Classically, one defines the Non linear Schrödinger (NLS) equation on a line by

$$\left(i\partial_t + \partial_x^2\right)\Phi(x,t) = 2g\,|\,\Phi(x,t)\,|^2\,\Phi(x,t)\,, \tag{2.1}$$

where $\Phi(x,t)$ is a complex field. Here and below we will take the coupling constant as a positive real number. This equation can be deduce from the following action

$$A[\Phi,\bar\Phi] = \int_{\mathbb{R}} dt \int_{\mathbb{R}} dx \left(i\Phi\partial_t\Phi - |\,\partial_x\Phi\,|^2 - g|\,\Phi\,|^4\right). \tag{2.2}$$

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The solution to (2.1) has been known for a long time [7]. It takes the form of a series in the coupling constant:

$$\Phi(x,t) = \sum_{n=0}^{\infty}(-g)^n\Phi^{(n)}(x,t) \tag{2.3}$$

$$\Phi^{(n)}(x,t) = \int_{\mathbb{R}^{2n+1}}\prod_{\substack{i=1\\j=0}}^{\infty}\frac{dp_i}{2\Pi}\frac{dq_j}{2\Pi}\bar{\lambda}_1\cdots\bar{\lambda}_n\lambda_n\cdots\lambda_0\frac{e^{i\omega_n(x,t)}}{\prod_{i=1}^{\infty}(p_i-q_{i-1})(p_i-q_i)} \tag{2.4}$$

$$\omega_n(x,t) = \sum_{i=1}^{\infty}(p_ix-p_i^2t)+\sum_{j=0}^{\infty}(q_jx-q_j^2t) \tag{2.5}$$

where $\bar{\lambda}_i=\lambda^*(p_i)$ and $\lambda_j=\lambda(q_j)$, λ being an arbitrary complex function, such that the series (2.3) converges (e.g. any Schwarz test function).

A remarkable fact about the above classical solution is that it can be extended to a quantum one [6]. Indeed, the solution (2.3-2.5) defines a canonical field $\Phi(x,t)$,

$$[\Phi(x,t),\Phi(y,t)] = 0 = [\Phi^\dagger(x,t),\Phi^\dagger(y,t)]\,, \qquad [\Phi(x,t),\Phi^\dagger(y,t)] = \delta(x-y)$$

provided one takes $\bar{\lambda}_i = a^\dagger(p_i)$ and $\lambda_j = a(q_j)$, where a and $a^\dagger$ are operators obeying the following exchange relations:

$$a(k_1)a(k_2) = R(k_2-k_1)a(k_2)a(k_1) \tag{2.6}$$

$$a^\dagger(k_1)a^\dagger(k_2) = a^\dagger(k_2)a^\dagger(k_1)R(k_2-k_1) \tag{2.7}$$

$$a(k_1)a^\dagger(k_2) = a^\dagger(k_2)R(k_1-k_2)a(k_1)+\delta(k_1-k_2) \tag{2.8}$$

with

$$R(k) = \frac{k-ig}{k+ig}\,. \tag{2.9}$$

This defines a so-called Zamolodchikov-Faddeev (ZF) algebra [5], which can be viewed as a deformation of an oscillator algebra, g playing the role of the deformation parameter.

Then, one shows that the canonical field Φ obeys a quantum version of the NLS equation:

$$\Big(i\partial_t+\partial_x^2\Big)\Phi(x,t) = 2g\,:\,\Phi(x,t)\bar{\Phi}(x,t)\Phi(x,t)\,: \tag{2.10}$$

where the normal ordering $:\,:$ is done with respect to the operators a and $a^\dagger$.

In the determination of Φ, the ZF algebra has been naturally put in the forefront. Indeed, this algebra plays a central role in the NLS equation. For instance, looking at the NLS-Hamiltonian, one can show that in term of a and $a^\dagger$, it can be rewritten as

$$H_2 = \int_{\mathbb{R}} dk\,k^2a^\dagger(k)a(k)\,. \tag{2.11}$$

The evolution of the canonical field

$$\Phi(x,t) = e^{iH_2t}\,\Phi(x,0)\,e^{iH_2t} \tag{2.12}$$

being compatible with the form (2.3-2.5).

Moreover, one can define an infinite set of commuting operators

$$H_n = \int_{\mathbb{R}} dk\,k^na^\dagger(k)a(k)\,, \qquad [H_n,H_m]=0 \tag{2.13}$$

generating the hierarchy associated to the NLS equation. The canonical field associated to H_m still can be written as (2.3-2.5), with the replacement

$$\omega_n(x,t) \;\to\; \omega_n^{(m)}(x,t) = \sum_{i=1}^{\infty}(p_i x - p_i^m t) + \sum_{j=0}^{\infty}(q_j x - q_j^m t)\,. \tag{2.14}$$

The simple form of the Hamiltonians also grounds the property for the a and $a^\dagger$ operators to be asymptotic states of the models. In fact, constructing the Fock space $\mathcal{F}$ with vacuum Ω, $a(k)\Omega = 0$, one can show that Ω is also cyclic for Φ, and that the Fock space of Φ and $\bar{\Phi}$ is dense into $\mathcal{F}$. For more details, see [3].

The central role plaid by the ZF algebra is confirmed when looking at the "vectorial" version of the NLS equation.

2.1 Generalization to the $gl(N)$-NLS equation and its symmetry

One can extend the above techniques to the case where Φ is an N-vector

$$\Phi(x,t) = \begin{pmatrix} \varphi_1(x,t) \\ \vdots \\ \varphi_n(x,t) \end{pmatrix} \quad \text{and} \quad \Phi^\dagger = (\bar{\varphi}_1, \ldots, \bar{\varphi}_n)\,. \tag{2.15}$$

The $gl(N)$-NLS equation reads

$$\left(i\partial_t + \partial_x^2\right)\Phi(x,t) = 2g\,|\,\Phi(x,t)\,|^2\,\Phi(x,t) \quad \text{with} \quad |\,\Phi\,|^2 = \Phi^\dagger\,\Phi\,. \tag{2.16}$$

Once more, the (quantum) solution takes the form (2.3-2.5), with now

$$\bar{\lambda}_i = a^\dagger_{\alpha_i}(p_i) \quad \text{and} \quad \lambda_j = a_{\alpha_j}(q_j) \tag{2.17}$$

and an implicit summation on the indices $\alpha_i = 1, \ldots, N$. The ZF generators a_i and $a_i^\dagger$, with $i = 1, \ldots, N$ obey

$$\begin{aligned} a_1(k_1)a_2(k_2) &= R_{21}(k_2-k_1)a_2(k_2)a_1(k_1) && (2.18)\\ a_1^\dagger(k_1)a_2^\dagger(k_2) &= a_2^\dagger(k_2)a_1^\dagger(k_1)R_{21}(k_2-k_1) && (2.19)\\ a_1(k_1)a_2^\dagger(k_2) &= a_2^\dagger(k_2)R_{12}(k_1-k_2)a_1(k_1) + \delta_{12}\delta(k_1-k_2)\,. && (2.20) \end{aligned}$$

We have used the notations

$$\begin{aligned} & a_1(k) = \sum_{i=1}^{N} a_i(k)\,e_i\otimes\mathbb{I} \quad , \quad a_2(k_2)\sum_{i=1}^{N} a_i(k)\,\mathbb{I}\otimes e_i \\ & a_1^\dagger(k) = \sum_{i=1}^{N} a_i^\dagger(k)\,e_i^\dagger\otimes\mathbb{I} \quad , \quad a_2^\dagger(k_2)\sum_{i=1}^{N} a_i^\dagger(k)\,\mathbb{I}\otimes e_i^\dagger \\ & \delta_{12} = \sum_{i=1}^{N} e_i\otimes e_i^\dagger \quad , \quad e_i^\dagger = (0,\ldots,0,\overset{i}{1},0,\ldots,0) \quad , \quad e_i^\dagger\cdot e_j = \delta_{ij} \\ & R_{12}(k) = \frac{1}{k+ig}\,(k\,\mathbb{I}_N\otimes\mathbb{I}_N + ig\,P_{12}) \quad , \quad P_{12} = \sum_{i,j=1}^{N} E_{ij}\otimes E_{ji} \end{aligned}$$

where $\mathbb{I}_N$ is the $N \times N$ identity matrix and E_{ij} the $N \times N$ matrix with 1 at position (i,j). With these notations, the exchange relations (2.18-2.20) of the ZF algebra $\mathcal{A}_N$ are understood as equalities in $\mathcal{A}_N \otimes \mathbb{C}^N \otimes \mathbb{C}^N$.

As previously, one defines the Hamiltonians

$$H_m = \sum_{i=1}^{N} \int_{\mathbb{R}} dk\, k^m a_i^\dagger(k) a_i(k) \equiv \int_{\mathbb{R}} dk\, k^m a^\dagger(k) a(k) \tag{2.21}$$

associated to the hierarchy of $gl(N)$-NLS equation, and the Fock space $\mathcal{F}$ of the $\mathcal{A}_N$ algebra provides a vacuum for the canonical fields Φ.

2.2 Yangian symmetry

The R-matrix (2.21) is well-known in the context of quantum groups: it is the R-matrix of the Yangian $Y(N)$ based on $gl(N)$. As an R-matrix, it obeys the Yang-Baxter equation:

$$R_{12}(k_1-k_2)R_{13}(k_1-k_3)R_{23}(k_2-k_3) = R_{23}(k_2-k_3)R_{13}(k_1-k_3)R_{12}(k_1-k_2) \tag{2.22}$$

and one can check it satisfies also the unitary condition

$$R_{12}(k_1-k_2)R_{21}(k_2-k_1) = \mathbb{I}_N \otimes \mathbb{I}_N\,. \tag{2.23}$$

Let us remind that the Yangian $Y(N)$ is a Hopf algebra which can be viewed as a deformation of $\mathbb{C}[t]\otimes gl(N)$, the polynomial algebra based on $gl(N)$. The emergence of $Y(N)$ in the context of $gl(N)$-NLS equation is not surprising, since it has been shown, using QISM formalism, that this algebra is a symmetry of it. It is natural to ask which form takes this symmetry in the framework of $\mathcal{A}_N$ algebra. Indeed, defining

$$Q^a_\alpha = \sum_{n=0}^{\infty} \frac{(-)^n}{n!} Q^{a,(n)}_\alpha,\ \ \alpha=0,1,\ \ a=1,\dots,N^2 \tag{2.24}$$

$$Q^{a,(n)}_\alpha = \int_{\mathbb{R}^n} \prod_{i=1}^{\infty} \frac{dk_i}{2\Pi}\, a^\dagger_{1\dots n} T^{a,(n)}_\alpha a_{n\dots 1} \quad , \quad T^{a,(n)}_\alpha \in \left(\mathbb{C}^{N^2}\right)^{\otimes n}[k_1,\dots,k_n] \tag{2.25}$$

with

$$a^\dagger_{1\dots n} = \sum_{i_1,\dots,i_n=1}^{N} a^\dagger_{i_1}(k_1)\cdots a^\dagger_{i_n}(k_n)\, e^\dagger_{i_1}\otimes\cdots\otimes e^\dagger_{i_n} \tag{2.26}$$

$$a_{n\dots 1} = \sum_{i_1,\dots,i_n=1}^{N} a_{i_n}(k_n)\cdots a_{i_1}(k_1)\, e_{i_1}\otimes\cdots\otimes e_{i_n} \tag{2.27}$$

one can shown that for

$$T^{a,(n)}_0 = \sum_{i=1}^{n} \binom{n-1}{i-1} t^a_i \quad , \quad t^a_i = \underbrace{\mathbb{I}_N\otimes\ldots\otimes\mathbb{I}_N}_{i-1}\otimes t^a\otimes\mathbb{I}_N\otimes\ldots\otimes\mathbb{I}_N \tag{2.28}$$

$$T^{a,(n)}_1 = \sum_{i=1}^{n} \binom{n-1}{i-1}\left(k_i t^a_i + \frac{g}{2}\sum_{j=1}^{i-1} f^a{}_{bc} t^b_i t^c_j\right) \tag{2.29}$$

Q_0^a and Q_1^a satisfy the defining relations (in the Drinfeld presentation) of the Yangian $Y(N)$.

Now, using the simple form of the Hamiltonians in the $\mathcal{A}_N$ framework, it is easy to show that the generators Q_α^a commute with H_n, so that $Y(N)$ generates integral of motions, not only for the $gl(N)$-NLS equation, but for its whole hierarchy.

Summarizing the construction, starting with the classical solution of the NLS equation, we have promoted it to a quantum solution for the $gl(N)$-NLS hierarchy. In such a process, an algebra (the ZF algebra $\mathcal{A}_N$) naturally appears and is used to construct the Fock space, the Hamiltonians and the integral of motions for the hierarchy. It is this strategy of "algebraic quantization" that we want to apply to the case of the NLS equation with boundaries.

3 BNLS: the NLS equation on a half-line

Classically, the BNLS equation is defined through (2.1), with the field Φ submitted to the boundary condition

$$\lim_{x\downarrow 0}(\partial_x - i\eta)\Phi(x,t) = 0 \quad , \quad \eta \geq 0 . \tag{3.30}$$

Note that this condition breaks the translation invariance. This model is integrable, and enters in the class of factorized S-matrix with boundary, introduced by Cherednik [8], and studied lately by Sklyanin [9] in the QISM approach.

The classical solution still takes the form (2.3-2.5), but now the function λ is submitted to the condition

$$\lambda(k) = B(k)\lambda(-k) \quad \text{with a reflection factor} \quad B(k) = \frac{k-i\eta}{k+i\eta} . \tag{3.31}$$

Sticking to our strategy, one tries to quantize the solution keeping its classical form, and seeking for exchange relations for operators a and $a^\dagger$. It can indeed be done, and in [3] it has been shown that the quantum field defined (2.3-2.5) is canonical provided the ZF algebra is replaced by a bigger algebra, the boundary algebra, whose exchange relations are given by:

$$\begin{aligned}
a(k_1)a(k_2) &= R(k_2-k_1)a(k_2)a(k_1) && (3.32)\\
a^\dagger(k_1)a^\dagger(k_2) &= a^\dagger(k_2)a^\dagger(k_1)R(k_2-k_1) && (3.33)\\
a(k_1)a^\dagger(k_2) &= a^\dagger(k_2)R(k_1-k_2)a(k_1) + \frac{1}{2}\delta(k_1-k_2) + \frac{1}{2}b(k_1)\delta(k_1+k_2) && (3.34)\\
a(k_1)b(k_2) &= R(k_2-k_1)b(k_2)R(k_1+k_2)a(k_1) && (3.35)\\
b(k_1)a^\dagger(k_2) &= a^\dagger(k_2)R(k_2-k_1)b(k_1)R(k_1+k_2) && (3.36)\\
b(k_1)b(k_2) &= b(k_2)b(k_1) \quad \text{and} \quad b(k)b(-k) = 1 && (3.37)
\end{aligned}$$

with the same $R(k)$. A new operator appears, which is the quantum analog of the reflection factor. Note that the ZF algebra is <u>not</u> a subalgebra of the boundary algebra, see relation (3.34).

The condition (3.31) turns to be, at the quantum level, an automorphism of the $\mathcal{B}(N)$ algebra:

$$a(k) \to a(-k)b(k) \ , \ a^\dagger(k) \to b(-k)a^\dagger(-k) \ , \ b(k) \to b(k) . \tag{3.38}$$

The Hamiltonian still has the form (2.11). The link with the reflection factor is done when looking at the Fock space representation of the boundary algebra: defining the vacuum by $a(k)\Omega = 0$, it appears that it is an eigenvector of the operator $b(k)$. Thus, there are several Fock spaces $\mathcal{F}_B$, each of them being label by a function $B(k)$ obeying

$$b(k)\Omega = B(k)\Omega \ , \ B(k)B(-k) = 1 . \tag{3.39}$$

The most general solution of the above equation can be written as

$$B(k)=\frac{A(k)+i}{A(k)-i}\quad \text{with } A \text{ odd real function} \tag{3.40}$$

and for $A(k)=\frac{k}{\eta}$, we recover the reflection factor. The other solutions correspond to more complicated boundary conditions (see below).

Before going deeper is the consequences of the algebra, we first define it in the vectorial form of the BNLS equation.

3.1 $gl(N)$-BNLS hierarchy and its symmetry

As in the case without boundary, one generalize the BNLS equation to N-vectors. Once again, the solutions can be written as in (2.3-2.5), with a boundary algebra $\mathcal{B}(N)$ obeying

$$\begin{aligned}
a_1(k_1)a_2(k_2) &= R_{21}(k_2-k_1)a_2(k_2)a_1(k_1) && (3.41)\\
a_1^\dagger(k_1)a_2^\dagger(k_2) &= a_2^\dagger(k_2)a_1^\dagger(k_1)R_{21}(k_2-k_1) && (3.42)\\
a_1(k_1)a_2^\dagger(k_2) &= a_2^\dagger(k_2)R_{12}(k_1-k_2)a_1(k_1)+\frac{\delta_{12}}{2}\delta(k_1-k_2)+\frac{1}{2}b_{12}(k_1)\delta(k_1+k_2) && (3.43)\\
a_1(k_1)b_2(k_2) &= R_{21}(k_2-k_1)b_2(k_2)R_{12}(k_1+k_2)a_1(k_1) && (3.44)\\
b_1(k_1)a_2^\dagger(k_2) &= a_2^\dagger(k_2)R_{21}(k_2-k_1)b_1(k_1)R_{21}(k_1+k_2) && (3.45)\\
b(k)b(-k) &= \mathbb{I} && (3.46)
\end{aligned}$$

$$R(k_1-k_2)\,b_1(k_1)\,R(k_1+k_2)\,b_2(k_2)\;=\;b_2(k_2)\,R(k_1+k_2)\,b_1(k_1)\,R(k_1-k_2)\,. \tag{3.47}$$

The notations are the same as in the previous section, together with

$$b_{12}(k)=\sum_{i,j=1}^{N}b_{ij}(k)\,e_i\otimes e_j\ ,\ b_1(k)=\sum_{i,j=1}^{N}b_{ij}(k)\,E_{ij}\otimes\mathbb{I}_N\ ,\ b_2(k)=\sum_{i,j=1}^{N}b_{ij}(k)\,\mathbb{I}_N\otimes E_{ij}\,.$$

The relation (3.47) shows that the generators $b_{ij}(k)$ form a subalgebra of $\mathcal{B}(N)$, called the reflection algebra.

The same properties appear: $\Phi(x,t)$ is a canonical quantum field, obeying the equation (2.10), and whose evolution operator is H_2.

3.1.1 Classification of the Fock spaces

As in the scalar case, there are several Fock spaces $\mathcal{F}_B$, labeled by $N\times N$ matrices $B(k)$ satisfying

$$\begin{aligned}
&B(k)B(-k) \;=\; \mathbb{I}_N && (3.48)\\
&R(k_1-k_2)\,B_1(k_1)\,R(k_1+k_2)\,B_2(k_2)\;=\;B_2(k_2)\,R(k_1+k_2)\,B_1(k_1)\,R(k_1-k_2)\,.
\end{aligned}$$

We have classified these reflection matrices (and thus the Fock spaces of the boundary algebra). The general solution, up to a conjugation by a constant $gl(N)$ matrix, takes the form:

$$B(k)=\beta(k)\,\frac{i\eta\,\mathbb{I}+k\,\mathbb{E}}{i\eta+k} \tag{3.49}$$

$$B(k)=\beta(k)\,(\mathbb{I}+ak\mathbb{J}) \tag{3.50}$$

where β is a function such that $\beta(k)\beta(-k) = 1$, $\mathbb{E}$ is a diagonal matrix with eigenvalues ± 1 and

$$\mathbb{J} = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_r \end{pmatrix} \quad \text{with} \quad J_i = (0) \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad 1 \leq i \leq r\,. \tag{3.51}$$

3.1.2 Boundary condition for Φ

In each Fock space $\mathcal{F}_B$, the field $\Phi(x,t)$ should obey a boundary condition associated to $B(k)$. We have constructed this boundary condition for the matrices (3.49). It takes the form

$$\lim_{x\downarrow 0} \Big(\mathcal{D}_B \Phi(x,t) \Big) = 0\,, \tag{3.52}$$

where $\mathcal{D}_B$ is a matrix differential operator. Rewriting $B(k)$ as

$$B(k) = \frac{A(k)+i}{A(k)-i}\,\frac{\mathbb{I}_N+\mathbb{E}}{2} + \frac{\widetilde{A}(k)+i}{\widetilde{A}(k)-i}\,\frac{\mathbb{I}_N-\mathbb{E}}{2}\,, \tag{3.53}$$

where $A(k)$ and $\widetilde{A}(k)$ are real odd functions, one has

$$\mathcal{D}_B = \frac{\mathbb{I}_N+\mathbb{E}}{2}\,(D_A+i) + \frac{\mathbb{I}_N-\mathbb{E}}{2}\,(D_{\widetilde{A}}+i)\,. \tag{3.54}$$

For any odd function A, one defines the differential operator D_A by

$$(D_A f)(y) = \int dx\, \hat{A}(x) f(x+y) \quad \text{with} \quad \hat{A}(x) \text{ inverse Fourier transform of } A(k)$$

3.1.3 Hierarchy associated to BNLS

Looking now at the hierarchy associated to the BNLS equation, one starts with the Hamiltonians H_m defined in (2.21). However, these Hamiltonians do not commute anymore:

$$[H_m, H_n] = \Big((-1)^m - (-1)^n\Big) \int dp p^{m+n} a^\dagger(-p) b(-p) a(p)\,. \tag{3.55}$$

Moreover, in the Fock spaces, the Hamiltonians H_{2m+1} vanish identically:

$$H_{2m+1} = 0 \quad \text{in} \quad \mathcal{F}_B\,. \tag{3.56}$$

Thus the hierarchy associated to the BNLS equation is formed by the commuting Hamiltonians H_{2m}.

3.1.4 Integrals of motion

remarking that we have the property

$$[b_{ij}(k), \sum_{\alpha=1}^{N} a_\alpha^\dagger(p) a_\alpha(p)] = 0 \quad , \quad \forall\, i,j = 1,\ldots,N,\ \forall p \tag{3.57}$$

we deduce:

The reflection algebra generated by the $b_{ij}(k)$'s commutes with all the Hamiltonians, and whence is a symmetry of the whole hierarchy.

Note that, contrarily to the case without boundary, the symmetry algebra comes for free with the boundary algebra: we consider now this reflection subalgebra from an algebraic point of view.

3.2 The reflection algebra $\mathcal{S}(N, \ell)$

We consider here the reflection as an algebra defined by the exchange relation

$$R(k_1 - k_2)\, b_1(k_1)\, R(k_1 + k_2)\, b_2(k_2) \;=\; b_2(k_2)\, R(k_1 + k_2)\, b_1(k_1)\, R(k_1 - k_2)\,. \tag{3.58}$$

This relation was introduced in [8], and enters in the class of $ABCD$-algebras, defined in [10].

The consistency of such relations leads to the following constraint

$$b(k)b(-k) = b(-k)b(k) = \rho(k)\mathbb{I},\ \ \rho(k) \text{ central and even in } k\,. \tag{3.59}$$

Here and below we will focus on $\rho(k) = 1$.

We adopt here a mode expansion

$$b(k) = b^{(0)} + \sum_{n>0} k^{-n} b^{(n)} \tag{3.60}$$

where $b^{(0)}$ is a constant matrix. Due to the choice of mode expansion, the solution (3.50) for the reflection matrices has to be excluded.

Note that because of the constraint (3.59), $b^{(0)}$ can be chosen diagonal with eigenvalues ± 1, <u>but</u> the number of + and - indeed plays a role in the commutation relations. This fact is not present when considering the Yangian $Y(N)$. We will denote $\mathcal{S}(N, \ell)$ the corresponding algebra, where ℓ is the number of -1. Up to a multiplication by $-\mathbb{I}$, we can choose $\ell \leq \frac{N}{2}$.

Remark that, depending on the choice of $b^{(0)}$, not all the reflection matrices will be allowed: the solution must fit this choice.

The Lie subalgebra in $\mathcal{S}(N, \ell)$ is a $gl(\ell) \oplus gl(N - \ell)$ one, and the other generators are combined into $(\ell, \overline{N - \ell}) \oplus (\overline{\ell}, N - \ell)$ fundamental representations for the $b^{(2n)}$ generators, and $(\ell^2, 0) \oplus (0, (N - \ell)^2)$ adjoint representations for the $b^{(2n+1)}$ generators.

The commutation relations can be rewritten as:

$$\begin{aligned}
[b_{ij}(k_1), b_{kl}(k_2)] \;&=\; \frac{ig}{k_1 - k_2}\Big(b_{kj}(k_2)b_{il}(k_1) - b_{kj}(k_1)b_{il}(k_2)\Big) \\
&+\; \frac{ig}{k_1 + k_2}\sum_{a=1}^{N}\Big(\delta_{il}\, b_{ka}(k_2)b_{aj}(k_1) - \delta_{kj}\, b_{ia}(k_1)b_{al}(k_2)\Big) \\
&-\; \frac{g^2}{k_1^2 - k_2^2}\,\delta_{ij}\sum_{a=1}^{N}\Big(b_{ka}(k_2)b_{al}(k_1) - b_{ka}(k_1)b_{al}(k_2)\Big)\,.
\end{aligned} \tag{3.61}$$

This structure is very similar to the one encountered in the case of twisted Yangians, and it can be shown that the $\mathcal{S}(N, \ell)$ algebra is a subalgebra of the Yangian $Y(N)$ through the embedding

$$b(k) \;\hookrightarrow\; T(k)b^{(0)}T^{-1}(-k)\,. \tag{3.62}$$

This embedding is natural from the physical point of vie, since it shows that the symmetry algebra of BNLS is a subalgebra of the symmetry algebra of NLS (without boundary).

The finite dimensional irreducible representations of $\mathcal{S}(N, \ell)$ have been classified in [4].

3.3 Spontaneous symmetry breaking

Once the Fock space is fixed, we are in position to determine which of the generators are broken by the vacuum. Indeed, it is exactly the reflection matrix which shows them, since $b(k)\Omega = B(k)\Omega$. Note that since $B(k)$ completely determines the boundary condition satisfied by Φ, it is this boundary condition which generates the spontaneous symmetry breaking.

One can gather these broken generators into a generating function $t_B(k)$ such that

$$t_b(k)\Omega = B(k)\Omega \quad \text{and} \quad \Big(b(k) - t_B(k)\Big)\Omega = 0\,. \tag{3.63}$$

Rewriting the reflection matrix as

$$B(k) = \frac{\beta(k)}{1+a}(\mathbb{E} + ak^{-1}\mathbb{I}) \quad \text{with} \quad \beta(k) = \sum_{p\in J} \beta_p k^{-p},\ J \subset \mathbb{Z}_+ \tag{3.64}$$

the generating function is given by

$$t_B(u) = tr\ d_B\, b(k) \quad \text{with} \quad d_B = \frac{1}{1+a}\Big(\mathbb{E}\, f(u^{-1}\partial) + \mathbb{I}\, a\widetilde{f}(u^{-1}\partial)\Big) \quad ; \quad \partial = \frac{\partial}{\partial k^{-1}}$$

where

$$f(x) = \sum_{p\in J} \frac{1}{p!}x^p \quad \text{and} \quad \widetilde{f}(x) = \sum_{p\in J} \frac{1}{(p+1)!}x^{p+1} \tag{3.65}$$

and tr is the trace in the auxiliary space.

t_B generates a commutative subalgebra of $\mathcal{S}(N,\ell)$

$$[t_B(k_1), t_B(k_2)] = 0\,. \tag{3.66}$$

A special feature appears when $B(k) = b^{(0)} \equiv \mathbb{E}$. In that case, none of the generators is broken, and the symmetry algebra in $\mathcal{F}_{\mathbb{E}}$ is the full reflection algebra $\mathcal{S}(N,\ell)$. Note however that although the reflection factor is “trivial”, the boundary affects the symmetry which is reduced from the Yangian $Y(N)$ to the reflection algebra $\mathcal{S}(N,\ell)$.

3.3.1 Example: the case $N = 2$

We can perform a change of basis in $\mathcal{S}(N,\ell)$, using the Pauli matrices σ_a:

$$t(k) = tr\, b(k) \quad \text{and} \quad t_a(k) = tr\, \sigma_a b(k),\ a = 1,2,3\,. \tag{3.67}$$

Then, in this new basis, one can show that $t(k)$ is a central generator.

The matrix $b^{(0)}$ can take two values: $\mathbb{I}$ or σ_3, and the corresponding possible values for the reflection matrix are

$$b^{(0)} = \mathbb{I} \Rightarrow B(k) = \frac{A(k)+i}{A(k)-i}\begin{pmatrix} 1 & 0 \\ 0 & \frac{i\eta-k}{i\eta+k} \end{pmatrix} \tag{3.68}$$

$$b^{(0)} = \sigma_3 \Rightarrow B(k) = \frac{A(k)+i}{A(k)-i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.69}$$

Looking at the Lie subalgebra, one starts with a $gl(2)$ algebra for $\mathcal{S}(2,0)$ and a $gl(1)\oplus gl(1)$ algebra for $\mathcal{S}(2,1)$. The spontaneous symmetry breaking reduces the $gl(2)$ algebra down to an $sl(2)$ one, and the $gl(1)\oplus gl(1)$ algebra down to a $gl(1)$ one.

We gather below the results for $N = 2$, including the case without boundary, which can be viewed as a $b(k) = 0$ limit.

Model	$b^{(0)}$	Symmetry algebra	Lie subalg.	Reflection matrix	Sym. break. (Lie subalg.)
NLS	0	$Y(2)$	$gl(2)$	-	-
BNLS	$\mathbb{I}$	$\mathcal{S}(2,0)$	$gl(2)$	$\mathbb{I}$	$gl(2)$
				$B(k)$	$sl(2)$
BNLS	σ_3	$\mathcal{S}(2,1)$	$2gl(1)$	σ_3	$2gl(1)$
				$\beta(k)\sigma_3$	$gl(1)$

4 Conclusion

We have shown that in the same way the ZF algebra allows to reconstruct all the informations about the NLS hierarchy, there is an analogous algebra, the boundary algebra $\mathcal{B}(N)$ in the case of BNLS hierarchy. It can be used to reconstruct the canonical field Φ, the Hamiltonians of the hierarchy and the symmetry algebra. Looking at the Fock space representation of $\mathcal{B}(N)$, we have shown that they are classified by reflection matrices $B(k)$, which turn to provide the boundary condition obeyed by Φ. The boundary algebra also contains the integrals of motion which form a subalgebra, the reflection algebra. Through the Fock space determination, one selects which of the symmetry generators will be spontaneously broken.

Note that such a $\mathcal{B}_R$ algebra can be constructed for any R-matrix which obey the Yang-Baxter equation

$$R_{12}(k_1-k_2)R_{13}(k_1-k_3)R_{23}(k_2-k_3) = R_{23}(k_2-k_3)R_{13}(k_1-k_3)R_{12}(k_1-k_2) \tag{4.70}$$

and the unitary condition

$$R_{12}(k_1-k_2)R_{21}(k_2-k_1) = \mathbb{I}\,. \tag{4.71}$$

In this context, the $\mathcal{B}_R$ algebra can be used for the study of a wide class of quantum integrable systems with boundary.

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Staggered anisotropy parameter modification of the Perk-Schultz model

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Abstract

The Perk–Schultz model (anisotropic t-J model) with staggered disposition of the anisotropy parameter along a chain is considered and the corresponding ladder type integrable model is constructed. This is a generalization to spin-1 case of the staggered XXZ spin-1/2 model considered earlier. The corresponding Hamiltonian is calculated and, since it contains next to nearest neighbor interaction terms, can be written in a zig-zag form. The Algebraic Bethe Ansatz technique is applied and the eigenstates, along with eigenvalues of the transfer matrix of the model are found.

1 Introduction

In recent articles [3, 4] the authors have proposed a construction of some integrable chain models with Z_2 grading of the states along the chain, as well as along the time directions. Hence, a way of constructing an integrable models with staggered inhomogeneity was proposed, motivated by the problems of three dimensional Ising model [6] and the Hall effect [7]. The inhomogeneous anisotropic XXZ Heisenberg chain with staggered anisotropy parameter $\pm\Delta$ was constructed in [3], while the isotropic $t-J$ model was considered in [4].

The XXZ and $t-J$ models with only inhomogeneous shift of the spectral parameter was previously considered in a chain of articles [8, 9, 10, 11], but these authors have not analyzed the possibility of the inhomogeneity of the anisotropy parameter Δ (or other model parameters). As a result they have the same intertwining R-matrix as for ordinary homogeneous integrable XXZ and $t-J$ models correspondingly and, therefore, the same quantum group structure behind. Contrary to this, in the construction presented in the articles [3] and [5] the integrable inhomogeneity appeared not only in shifts of the spectral parameter, but also in some structural changes. As a result we got modified Yang–Baxter equations (YBE), leading to a quantum group algebraic structure different from the usual $sl_q(2)$ [5].

As it is shown in the mentioned above articles, due to periodic shift of the spectral parameters the Hamiltonians of all this models contain at most next to nearest neighbor (NNN) interaction terms and therefore can be represented as integrable two leg ladder (zig-zag) models. The interaction between the legs of the ladder is represented in the Hamiltonian by a topological type terms written as the product of three neighbor spins and the anisotropic antisymmetric tensor as

$$\hat{\epsilon}^{abc}\sigma_i^a\sigma_{i+1}^b\sigma_{i+2}^c. \tag{1}$$

As it was shown in [7], the model of free fermions hopping with inhomogeneous parameters, in a case when the rotational invariance is preserved, determines a Peierls type mechanism of mass generation, which is based on the breaking of the translational invariance for the translations of one lattice spacing. Perhaps the above mentioned 3-spin interaction terms are

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responsible for a gaped phase of the system, as it is discussed in the articles [12] in connection with high temperature superconductivity problems.

In this article, by use of the solution of the staggered YBE for the general $Sl_q(n)$ quantum group presented in [5], we pass from spin-1/2 XXZ model to spin-1 case and consider the Perk-Schultz model [13, 14, 15] (or anisotropic $t-J$ model) with staggered sign of the anisotropy parameter. In Section 2 we present the solution of the staggered YBE for the $Sl_q(1,2)$ case and calculate the Hamiltonian of the model in a ladder form. In Section 3 we apply the technique of the Algebraic Bethe Ansatz (ABA) [1, 2] and find the eigenvalues and eigenstates of our Hamiltonian. The Bethe equations (BE) for the allowed values of the spectral parameters (momentums) are written down here.

2 The Yang–Baxter equations, their solution and the Hamiltonian of the model

In this Section we follow the technique developed in the articles [3, 4]. We modify the spin-1 model of Perk and Schultz [13, 14, 15] , which can be regarded as anisotropic supersymmetric $t-J$ model, in order to construct an integrable model with staggered disposition of the sign of the anisotropy parameter Δ.

The principal ingredient of integrable models via Bethe Ansatz technique is the R-matrix. The R_{aj}-matrix acts as an intertwining operator on the direct product of the auxiliary $V_a(v)$ and quantum $V_j(u)$ spaces

$$R_{aj}(u,v) : V_a(u) \otimes V_j(v) \longrightarrow V_j(v) \otimes V_a(u) \ . \tag{2}$$

For the Perk-Schultz model the quantum and auxiliary spaces are three dimensional and corresponding R-matrix can be defined by the following formula

$$R(z) = qzR_q - q^{-1}z^{-1}R_q^{-1}, \tag{3}$$

where we have introduced a new multiplicative spectral parameter $z = e^{iu}$. The parameter q defines the anisotropy of the model and, as it is shown in [15], the so called constant R-matrix R_q is equal to (here $\lambda = q - q^{-1}$)

$$R_q \;=\; \lim_{z\to\infty}\frac{R(z)}{z} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -q^{-1}\lambda & 0 & -q^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 & 0 & 0 \\ 0 & 0 & -q^{-1}\lambda & 0 & 0 & 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -q^{-1}\lambda & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-2} \end{pmatrix}. \tag{4}$$

It is very convenient for this model to consider its fermionized version by use of the technique developed in the article [16, 17]. Let us introduce two Fermi fields $c_{i,\sigma}$, $c^+_{i,\sigma}$ with spin up and down states $\sigma =\uparrow,\downarrow$ at each site i of the chain and three basic vectors of the corresponding spaces V_i as follows

$$\mid -\rangle \equiv\mid 0,\downarrow\rangle, \qquad \mid +\rangle \equiv\mid\uparrow,0\rangle, \qquad \mid 0\rangle \equiv\mid 0,0\rangle, \tag{5}$$

numerated as $|\ 1\rangle, |\ 2\rangle, |\ 3\rangle$ respectively. Therefore we have Z_2 graded quantum $V_j(u)$ and auxiliary $V_a(v)$ spaces with the following parities

$$p(|\ +\rangle) = p(|\ -\rangle) = 1, \qquad p(|\ 0\rangle) = 0. \tag{6}$$

Consider now Hubbard operators $X^{a_1}_{a_2} =|\ a_2\rangle\langle a_1\ |$, $X^{j_1}_{j_2} =|\ j_2\rangle\langle j_1\ |$, with $|\ a\rangle, |\ j\rangle$ defined as in formulas (5) and

$$X^k_m = \begin{pmatrix} |-\rangle\langle-| & |-\rangle\langle+| & |-\rangle\langle 0| \\ |+\rangle\langle-| & |+\rangle\langle+| & |+\rangle\langle 0| \\ |0\rangle\langle-| & |0\rangle\langle+| & |0\rangle\langle 0| \end{pmatrix} = \begin{pmatrix} (1-n_\uparrow)n_\downarrow & c^+_\downarrow c_\uparrow & (1-n_\uparrow)c^+_\downarrow \\ c^+_\uparrow c_\downarrow & n_\uparrow(1-n_\downarrow) & c^+_\uparrow(1-n_\downarrow) \\ (1-n_\uparrow)c_\downarrow & c_\uparrow(1-n_\downarrow) & (1-n_\uparrow)(1-n_\downarrow) \end{pmatrix}, \tag{7}$$

where $n_\sigma = c^+_\sigma c_\sigma$ is the particle number operator. The demand that the trace of this operator acts on auxiliary and quantum spaces as the identity operator means that the double occupancy of the sites by fermions is excluded

$$\Delta\ |\ l\rangle = X^m_m\ |\ l\rangle = (1 - n_\uparrow n_\downarrow)\ |\ l\rangle =|\ l\rangle, \qquad m, l = 1, 2, 3. \tag{8}$$

Now we can write down the formula connecting the fermionic R_{aj}-operator of the Perk-Schultz model with the R_{aj}-matrix defined by the expression (4)

$$\begin{aligned} R_{aj} &= R_{aj}\ |\ j_1\rangle\ |\ a_1\rangle\langle a_1\ |\ \langle j_1\ |= (R_{aj})^{a_2 j_2}_{a_1 j_1}\ |\ a_2\rangle\ |\ j_2\rangle\langle a_1\ |\ \langle j_1\ | \\ &= (-1)^{p(a_1)p(j_2)} (R_{aj})^{a_2 j_2}_{a_1 j_1} X^{a_1}_{a_2} X^{j_1}_{j_2}. \end{aligned} \tag{9}$$

It is straightforward to check that this fermionic R-operator satisfies the YBE in the operator form [17].

Having the R-matrix of the integrable model we should construct the monodromy matrix (operator), which is usually a homogeneous product of the R-matrices along the chain. But now let us instead consider two different monodromy matrices (operators) $M_0(u)$ and $M_1(u)$ defined by staggered inhomogeneous product of the R-operators

$$M_1(u) = R^{\iota_1\iota_2}_{01}(u) R^{\iota_1}_{02}(\bar{u}) R^{\iota_1\iota_2}_{03}(u)\cdots, \qquad M_0(u) = R^{\iota_2}_{01}(\bar{u}) R_{02}(u) R^{\iota_2}_{03}(\bar{u})\cdots. \tag{10}$$

Here the operations ι_1 and ι_2 transform the R-operator of the Perk-Schultz model to $R^{\iota_1}(u)$, $R^{\iota_2}(u)$ with the property $\iota_1^2 = \iota_2^2 = 1$. These operations will be defined below.

The full monodromy operator $M(u)$ of the model is defined as a product of these two monodromy operators

$$M(u) = M_0(u) M_1(u), \tag{11}$$

while the transfer matrix is the trace of $M(u)$ over auxiliary spaces 0 and $\bar{0}$

$$\tau(u) = Tr_0 Tr_{\bar{0}} M(u) = \tau_0(u)\tau_1(u). \tag{12}$$

It is known that the Yang–Baxter equations (YBE) insure the commutativity of the transfer matrices $\tau(u)$ and $\tau(v)$ for different values of the spectral parameters $u \neq v$. As it was shown in the article [3] for our inhomogeneous case we will need two sets of YBE defined as

$$\check{R}_{12}(u,v)\check{R}^{\iota_1}_{23}(\bar{u})\check{R}_{12}(v) = \check{R}^{\iota_1}_{23}(v)\check{R}_{12}(\bar{u})\check{\check{R}}_{23}(u,v)\,, \tag{13}$$

$$\check{\check{R}}_{12}(u,v)\check{R}^{\iota_2\iota_1}_{23}(u)\check{R}^{\iota_1}_{12}(\bar{v}) = \check{R}^{\iota_2\iota_1}_{23}(\bar{v})\check{R}^{\iota_2}_{12}(u)\check{R}_{23}(u,v)\,, \tag{14}$$

where $\check{R}_{ij} = P_{ij}R_{ij}$ is the R-matrix in the braid formalism and P_{ij} is the permutation operator.

In order to have a integrable model this YBE-s should have a solution of the equations (13, 14) for some nontrivial ι_1 and ι_2 operations. The solution of the first set of equations can be found in the article [5] by extending $Sl_q(1,2)$ case from the $Sl_q(3)$ solutions. It is

$$\check{R}^{\iota_1}(z) = qz\check{R}_q^{\iota_1} - q^{-1}z^{-1}(\check{R}_q^{\iota_1})^{-1}, \tag{15}$$

where

$$\check{R}_q^{\iota_1} = \tilde{R}_q = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -q^{-1}\lambda & 0 & -q^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -q^{-1}\lambda & 0 & 0 & 0 & -q^{-1} & 0 & 0 \\ 0 & -q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -q^{-1} & 0 \\ 0 & 0 & -q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -q^{-1} & 0 & -q^{-1}\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-2} \end{pmatrix} \tag{16}$$

and the bar operation is a multiplicative shift of the multiplicative spectral parameter $z = e^{iu}$, $\check{R}(\bar{z}) = \check{R}(e^{i\bar{u}}) = \check{R}(hz^{-1})$.

If we define the operation ι_2 on $\check{R}(z)$ as

$$\check{R}^{\iota_2}(z) = \check{R}^{\iota_1}(z^{-1}), \tag{17}$$

then it is easy to see that second set of YBEs will coincide with the first set of YBE-s.

Therefore we have found an integrable inhomogeneous model and our aim now is to write down the expression for the Hamiltonian. But in order to have a local Hamiltonian it is necessary to have a point u_0, where

$$\check{R}_{ij}(u_0) = \hat{I}_{ij}, \tag{18}$$

with $\hat{I}_{ij}$ being an identity operator.

From the form of $\check{R}$-matrix (15) one can easily see that the value $u_0 = 0$ (or $z_0 = e^{iu_0} = 1$) is just the needed one. Let us notice that though $\check{R}_{ij}(u)|_{u=0} = \hat{I}_{ij}$, we have $\check{R}_{ij}(\bar{u})|_{u=0} \neq \hat{I}_{ij}$. Because of this, and as future calculations will show, the Hamiltonian of our model contains, together with nearest-neighbor interaction terms, also next-to nearest-neighbor interaction terms.

So being provided by such value, our Hamiltonian is defined as logarithmic derivative of the transfer matrix at that point

$$H = -\frac{\partial \ln \tau(u)}{\partial u}|_{u=0}.$$

For the calculation of this logarithmic derivative we need the following linear expressions of the operators $\check{R}(z), \check{R}^{\iota_1\iota_2}(z), \check{R}^{\iota_1}(\bar{z}), \check{R}^{\iota_2}(\bar{z})$ around the point $u = 0$

$$\begin{aligned} \check{R}(u) &= -\lambda + iuH, \qquad H = q\check{R}_q + q^{-1}\check{R}_q^{-1}, \qquad \check{R}^{\iota_1\iota_2}(u) = -\lambda - iuH, \\ \check{R}(\bar{u}) &= \check{R}(h) + iu\bar{H}, \qquad \bar{H} = -qh\check{R}_q - q^{-1}h^{-1}\check{R}_q^{-1}, \\ \check{R}^{\iota_1}(\bar{u}) &= \check{R}^{\iota_1}(h) + iu\bar{H}^{\iota_1}, \qquad \bar{H}^{\iota_1} = -qh\check{R}_q^{\iota_1} - q^{-1}h^{-1}(\check{R}_q^{\iota_1})^{-1}, \\ \check{R}^{\iota_2}(\bar{u}) &= \check{R}^{\iota_1}(h^{-1}) + iu\bar{H}^{\iota_2}, \qquad \bar{H}^{\iota_2} = qh^{-1}\check{R}_q^{\iota_1} + q^{-1}h(\check{R}_q^{\iota_1})^{-1}, \end{aligned} \tag{19}$$

and we also have that

$$\check{R}^{\iota_1}(h)\check{R}^{\iota_1}(h^{-1}) = (\lambda^2 + 4\cos^2\theta)\hat{I},$$

with $\theta = -i \log h$, as an additive shift of the spectral parameter.

The next step consists in inserting expressions(19) in expressions (11) and (10) of the monodromy matrix (in the braid formalism)

$$M(u) = \check{R}^{\iota_2}_{01}(\bar{u})\check{R}_{12}(u)\check{R}^{\iota_2}_{23}(\bar{u})\cdots\check{R}^{\iota_1}_{01}(\bar{u})\check{R}^{\iota_1\iota_2}_{12}(u)\check{R}^{\iota_1}_{23}(\bar{u})\cdots$$

and to extract the linear in u terms from the product. Finally we obtain

$$\begin{aligned}
- &\; i\lambda(\lambda^2 + 4\sin^2\theta)H_{j-1,j,j+1} = \\
= &\; (-1)^{j-1}\sum_{\sigma=\uparrow,\downarrow}\Big\{c^+_{j-1,\sigma}c_{j+1,\sigma}\Big[(1-n_{j,\bar{\sigma}})f_4(q,h^{(-1)^j}) + n_{j,\bar{\sigma}}f_0(q,h^{(-1)^j})\Big] \\
+ &\; c^+_{j+1,\sigma}c_{j-1,\sigma}\Big[(1-n_{j,\bar{\sigma}})f_4(q,h^{-(-1)^j}) + n_{j,\bar{\sigma}}f_0(q,h^{-(-1)^j})\Big]\Big\} \\
- &\; (-1)^{j-1}\Big\{S^\mu_{j-1}S^\mu_{j+1}\Big[(1-n_j)f_0(q,h) - n_j\frac{f_4(q,h)+f_4(q,h^{-1})}{2}\Big] \\
- &\; \Big[S^3_{j-1}S^3_{j+1} + \frac{1}{4}n_{j-1}n_{j+1}\Big](2n_j-1)f_6(q,h)\Big\} + (-1)^{j-1}\sum_{\sigma=\uparrow,\downarrow}\Big\{c^+_{j,\sigma}c_{j+1,\sigma} \\
\times &\; \big[(1-n_{j+2,\bar{\sigma}})f_{j(mod2)+1}(q,h) + (-1)^j(1-n_{j-1,\bar{\sigma}})f_{j(mod2)+1}(q,h^{-1})\big] \\
+ &\; c^+_{j+1,\sigma}c_{j,\sigma}\big[(1-n_{j-1,\bar{\sigma}})f_{j(mod2)+1}(q,h) + (-1)^j(1-n_{j+2,\bar{\sigma}}f_{j(mod2)+1}(q,h^{-1})\big]\Big\} \\
+ &\; \Big[S^\mu_{2j-1}S^\mu_{2j}\frac{f_2(q,h)+f_2(q,h^{-1})}{2} + f_6(q,h)n_{2j-1}n_{2j}\Big](n_{2j-2}-n_{2j+1}) \\
+ &\; 2S^a_{j-1}S^b_jS^c_{j+1}\hat{\epsilon}^{abc} + (-1)^{j+1}\sum_{\{klm\}=\{j-1,j,j+1\}}S^\mu_kS^\mu_{lm}\hat{\epsilon}^{klm}f_5(q,h) \\
+ &\; f_3(q,h)(n_{j-1}n_j + n_jn_{j+1})
\end{aligned} \tag{20}$$

where $\mu = 1, 2$.

In this expression for the Hamiltonian the spin variables are as in usual $t-J$ model

$$S^+ = c^+_\uparrow c_\downarrow, \qquad S^- = c^+_\downarrow c_\uparrow, \qquad S^3 = \frac{n_\uparrow - n_\downarrow}{2}, \tag{21}$$

the anisotropic antisymmetric tensors $\hat{\epsilon}^{abc}$, $\quad a,b,c = 1,2,3$ and $\hat{\epsilon}^{klm}$, $k,l,m, = j-1, j, j+1$ are defined as follows

$$\begin{array}{ll} \hat{\epsilon}^{a3b} = f_1(q^{1/2},h^2)\epsilon^{a3b} \\ \hat{\epsilon}^{3ab} = \hat{\epsilon}^{ab3} = f_1(q,h)\epsilon^{3ab} \end{array} \qquad \hat{\epsilon}^{klm} = \begin{cases} -\hat{\epsilon}^{lkm} & \text{if} \quad |k-l| = 2, \\ \hat{\epsilon}^{lkm} & \text{if} \quad |k-l| = 1 \end{cases} \tag{22}$$

where ϵ^{abc} is ordinary antisymmetric tensor, and for functions $f_r(q,h)$ we got

$$\begin{aligned}
&f_0(q,z) = 2(h-h^{-1})^2\,, \qquad f_1(q,z) = 2(h-h^{-1})(q^2-q^{-2})\,, \\
&f_3(q,z) = 4(h^{-1}-h)(3qh + qh^{-1} - q^{-1}h - 3q^{-1}h^{-1})\,, \\
&f_4(q,z) = 2(q^{1/2}h^{-1} - q^{-1/2}h)^2 - 2(q^{1/2}-q^{-1/2})^2, \\
&f_5(q,z) = 2(q-q^{-1})(h-h^{-1})\,, \qquad f_6(q,z) = 2(q+q^{-1})(h-h^{-1})^2.
\end{aligned} \tag{23}$$

It is convenient now to write the Hamiltonian (20) in a ladder form. Let us consider the even $(2j)$ and the odd $(2j+1)$ points of the chain as a sites (j) of two different chains labeled by $s = 0$ and 1 correspondingly. The Fermi fields will be labeled now as $c_{j,s} = c_{2j+s}$, $s = 0, 1$.

With these notations it is straightforward to obtain from the expression (20) the following ladder Hamiltonian

$$
\begin{aligned}
- \quad & i\lambda(\lambda^2+4\sin^2\theta)H_{j,s} = H^c_{j,s} + H^r_{j,s} + H^{3S}_{j,s} + H^{hS}_{j,s} \\
= \quad & (-1)^s \sum_{\sigma=\uparrow,\downarrow} \Big\{ c^+_{j,s,\sigma} c_{j+1,s,\sigma} \Big[(1-n_{j+s,s+1,\bar\sigma}) f_4(q,h^{(-1)^{s+1}}) + n_{j+s,s+1,\bar\sigma} f_0(q,h^{(-1)^{s+1}}) \Big] \\
+ \quad & c^+_{j+1,s,\sigma} c_{j,s,\sigma} \Big[(1-n_{j+s,s+1,\bar\sigma}) f_4(q,h^{-(-1)^s}) + n_{j+s,s+1,\bar\sigma} f_0(q,h^{-(-1)^s}) \Big] \Big\} \\
- \quad & (-1)^s \Big\{ S^\mu_{j,s} S^\mu_{j+1,s} \Big[(1-n_{j+s,s+1}) f_0(q,h) - n_{j+s,s+1} \frac{f_4(q,h)+f_4(q,h^{-1})}{2} \Big] \\
- \quad & \Big[S^3_{j,s} S^3_{j+1,s} + \frac{1}{4} n_{j,s} n_{j+1,s} \Big] (2n_{j+s,s}-1) f_6(q,h) \Big\} + (-1)^{j-1} \sum_{\sigma=\uparrow,\downarrow} \Big\{ c^+_{j,s,\sigma} c_{j+s,s+1,\sigma} \\
\times \quad & \Big[(1-n_{j+1,s,\bar\sigma}) f_{s+1}(q,h^{(-1)^s}) + (-1)^s (1-n_{j-1+s,s+1,,\bar\sigma}) f_{s+1}(q,h^{(-1)^{s+1}}) \Big] \\
+ \quad & c^+_{j+s,s+1,\sigma} c_{j,s,\sigma} \Big[(1-n_{j+1,s,\bar\sigma}) f_{s+1}(q,h^{(-1)^{s+1}}) + (-1)^s (1-n_{j-1+s,s+1,\bar\sigma} f_{s+1}(q,h^{(-1)^s}) \Big] \Big\} \\
+ \quad & \Big[S^\mu_{j,1} S^\mu_{j+1,0} \frac{f_2(q,h)+f_2(q,h^{-1})}{2} + f_6(q,h) n_{j,1} n_{j+1,0} \Big] (n_{j,0} - n_{j+1,1}) \\
+ \quad & 2 S^a_{j,s} S^b_{j+s,s+1} S^c_{j+1,s} \hat\epsilon^{abc} + (-1)^s \sum_{\{klm\}=\{(j,s),(j+s,s+1),(j+1,s)\}} S^\mu_k S^\mu_{lm} \hat\epsilon^{klm} f_5(q,h) \\
+ \quad & f_3(q,h) n_{j,s} n_{j+s,s+1}
\end{aligned} \tag{24}
$$

Though this expression is big but one can recognize four clear terms there. The first one is the Hamiltonian of anisotropic $t-J$ model for the each of chains (marked as H^c_{js}). The second is anisotropic $t-J$ term for the rungs (H^r_{ja}). The third term (H^{3S}_{js}), which was discussed in the introduction, is written for the each triangle of the zig-zag, represents the interaction between chains and has a topological form. The last term contains spin-spin Heisenberg interaction together with hopping of fermions and can be thought as spin-orbit interaction.

At the end of this Section let us write the $q \to 1$ limit of this Hamiltonian , which corresponds to isotropic case. In this limit it is necessary to rescale the additive spectral parameter u and introduce $w = \log z/\log q$ together with new additional model parameter $\theta = \log z/\log q$, which should be kept finite. Then we will obtain

$$\tfrac{(-1)^{s+1}}{2} H_{js} = \textstyle\sum_{\sigma=\uparrow,\downarrow} \big(c^+_{j,s,\sigma} c_{j+1.s,\sigma} + c^+_{j+1,s,\sigma} c_{j.s,\sigma} \big) - n_{j,s} n_{j+s,s+1} n_{j+1.s} \tag{25}$$

$$-(-1)^s 5 n_{j,s} n_{j+s,s+1} + \Big[S^\mu_{j,s} S^\mu_{j+1,s} + S^3_{j,s} S^3_{j+1,s} + \tfrac{1}{4} n_{j,s} n_{j+1,s} \Big] (2n_{j+s,s+1} - 1), \tag{26}$$

which looks like an ordinary supersymmetric $t-J$ model for each of chains. But the presence of the factor $2n-1$ in front of spin-spin interaction means that it is ferromagnetic or antiferromagnetic regarding whether the corresponding site in the other chain is occupied by fermion or not.

3 *ABA* solution of the model

The *ABA* solution for the homogeneous Perk-Schultz model was carried out in [14] and in a chain of articles [18] for a model with open boundaries. In this section we will use the technique of Algebraic Bethe Ansatz ([1]-[2]) in order to find the eigenstates and eigenvalues of the Hamiltonian (24) for staggered inhomogeneity.

At the beginning we need the definition of the L-operator

$$L_{i,j}(z)_m^{m'} = \langle m \mid R_{i,j}(z) \mid m' \rangle = (-1)^{p(m')p(n')}(R_{ij}(z))_{mn}^{m'n'} X_{n'}^n \tag{27}$$

which is a 3×3 matrix in a horizontal auxiliary space, with the matrix elements acting in quantum space. It has the following expression

$$(L_{ij})_m^{m'}(z) = \tag{28}$$
$$\begin{pmatrix} \begin{matrix}(qz - q^{-1}z^{-1})(1-n_\uparrow)n_\downarrow \\ +(z-z^{-1})(1-n_\downarrow)\end{matrix} & \lambda z^{-1}c_\downarrow^+ c_\uparrow & -\lambda z^{-1}(1-n_\uparrow)c_\downarrow^+ \\ \lambda z c_\uparrow^+ c_\downarrow & \begin{matrix}(z-z^{-1})(1-n_\uparrow)n_\downarrow \\ -(-qz+q^{-1}z^{-1})n_\uparrow(1-n_\downarrow) \\ +(z-z^{-1})(1-n_\uparrow)(1-n_\downarrow)\end{matrix} & -\lambda z^{-1}c_\uparrow^+(1-n_\downarrow) \\ -\lambda z(1-n_\uparrow)c_\downarrow & -\lambda z c_\uparrow(1-n_\downarrow) & \begin{matrix}(z-z^{-1})(1-n_\uparrow)n_\downarrow \\ +(z-z^{-1})n_\uparrow(1-n_\downarrow)+ \\ (q^{-1}z-qz^{-1})(1-n_\uparrow)(1-n_\downarrow)\end{matrix} \end{pmatrix}.$$

The matrix elements of the monodromy operator (11) between auxiliary states $\mid k'\rangle$ and $\langle k \mid$ can be expressed as a product of L_{ij} matrices

$$\begin{aligned} M_0(z)_{k'}^k &= \langle k \mid M_0(z) \mid k'\rangle = L_{01}^{\iota_2}(\bar{z})_{k_1}^k L_{02}(z)_{k_2}^{k_1} \cdots L_{0N}(z)_{k'}^{k_{N-1}}, \\ M_1(z)_{k'}^k &= \langle k \mid M_1(z) \mid k'\rangle = L_{01}^{\iota_1\iota_2}(z)_{k_1}^k L_{02}^{\iota_1}(\bar{z})_{k_2}^{k_1} \cdots L_{0N}^{\iota_1}(\bar{z})_{k'}^{k_{N-1}} \end{aligned} \tag{29}$$

which in matrix form looks like

$$M_s(z)_{k'}^k = \begin{pmatrix} A_{s,11}(z) & A_{s,12}(z) & B_{s,1}(z) \\ A_{s,12}(z) & A_{s,22}(z) & B_{s,2}(z) \\ C_{s,1}(z) & C_{s,2}(z) & D_s(z) \end{pmatrix}, \quad s = 0, 1, \tag{30}$$

where $A_{s,ab}$, $B_{s,a}$, $C_{s,a}$, D_s; $(a, b = 1, 2)$ are operators in the the quantum space. The graded property of the model is the origin of the following form of the transfer matrix (12)

$$\tau_s(z) = -A_{s,11}(z) - A_{s,22}(z) + D_s(z), \quad s = 0, 1. \tag{31}$$

Now we would like to take the empty fermionic state

$$\mid \Omega\rangle_s = \mid 0, 0, \ldots, 0\rangle_s = \mid 0\rangle_{1s} \mid 0\rangle_{2s} \cdots \mid 0\rangle_{Ns}, \quad s = 0, 1, \tag{32}$$

as a "test" vacuum and let us demonstrate that the "vacuum" $\mid \Omega\rangle$ is indeed eigenstate of the transfer matrix (31)

$$\tau_s(z) \mid \Omega\rangle_s = \nu_s^{(0)} \mid \Omega\rangle_{1-s}. \tag{33}$$

The action of $L_{0k}(z)$ and $L_{0k}^{\iota_2}(\bar{z})$ on the k-th empty state $\mid 0\rangle_k$ is easy to calculate

$$L_{0k}(z)_m^{m'} \mid 0\rangle = \begin{pmatrix} z - z^{-1} & 0 & -\lambda z^{-1}c_\downarrow^+ \\ 0 & z - z^{-1} & -\lambda z^{-1}c_\uparrow^+ \\ 0 & 0 & q^{-1}z - qz^{-1} \end{pmatrix} \mid 0\rangle_k, \tag{34}$$

$$L_{0k}^{\iota_2}(\bar{z})_m^{m'} \mid 0\rangle = \begin{pmatrix} -h^{-1}z + hz^{-1} & 0 & \lambda hz^{-1}c_\downarrow^+ \\ 0 & -h^{-1}z + hz^{-1} & \lambda hz^{-1}c_\uparrow^+ \\ 0 & 0 & h^{-1}q^{-1}z - hqz^{-1} \end{pmatrix} \mid 0\rangle_k, \tag{35}$$

and as we see it has an upper-triangular form. Because the upper triangular matrices form a semigroup the action of the monodromy matrix $M_0(z)^k_{k'}$ on $\mid \Omega\rangle_0$ also have an upper-triangular form directly following from the expression (29)

$$M_0(z)^{m'}_m \mid \Omega\rangle_0 = \tag{36}$$
$$\begin{pmatrix} \begin{matrix}(-h^{-1}z+hz^{-1})^{\frac{N}{2}}\\ (z-z^{-1})^{\frac{N}{2}}\end{matrix} & 0 & B_{0,1} \\ 0 & (-h^{-1}z+hz^{-1})^{\frac{N}{2}}(z-z^{-1})^{\frac{N}{2}} & B_{0,2} \\ 0 & 0 & \begin{matrix}(h^{-1}q^{-1}z-hqz^{-1})^{\frac{N}{2}}\\ (q^{-1}z-qz^{-1})^{\frac{N}{2}}\end{matrix} \end{pmatrix} \mid \Omega\rangle_0 .$$

Following the definitions (16-17) of the ι_1 and ι_2 operations one can easily find the actions of $L^{\iota_1\iota_2}_{0k}(z)$ and $L^{\iota_1}_{0k}(\bar z)$ on $\mid 0\rangle_k$ by use of formulas

$$L^{\iota_1\iota_2}(z) = L(z^{-1}), \qquad L^{\iota_1}(\bar z) = L^{\iota_2}(\bar z^{-1}), \tag{37}$$

and correspondingly the action of $M_1(z)^k_{k'}$ on $\mid \Omega\rangle_1$ will be

$$M_1(z)^{m'}_m \mid \Omega\rangle_1 = \tag{38}$$
$$\begin{pmatrix} \begin{matrix}(h^{-1}z-hz^{-1})^{\frac{N}{2}}\\ (-z+z^{-1})^{\frac{N}{2}}\end{matrix} & 0 & B_{1,1} \\ 0 & (h^{-1}z-hz^{-1})^{\frac{N}{2}}(-z+z^{-1})^{\frac{N}{2}} & B_{1,2} \\ 0 & 0 & \begin{matrix}(hq^{-1}z^{-1}-h^{-1}qz)^{\frac{N}{2}}\\ (q^{-1}z^{-1}-qz)^{\frac{N}{2}}\end{matrix} \end{pmatrix} \mid \Omega\rangle_1 .$$

Now we see that $\mid \Omega\rangle$ is the eigenstate of $\tau(u)$ with the eigenvalue $\nu^{(0)}(z)$

$$\begin{aligned} \nu^{(0)}(z) &= \nu^{(0)}_0(z)\nu^{(0)}_1(z), \\ \nu^{(0)}_0(z) &= -2(-h^{-1}z+hz^{-1})^{\frac{N}{2}}(z-z^{-1})^{\frac{N}{2}} + (h^{-1}q^{-1}z-hqz^{-1})^{\frac{N}{2}}(q^{-1}z-qz^{-1})^{\frac{N}{2}}, \\ \nu^{(0)}_1(z) &= -2(h^{-1}z-hz^{-1})^{\frac{N}{2}}(-z+z^{-1})^{\frac{N}{2}} + (hq^{-1}z^{-1}-h^{-1}qz)^{\frac{N}{2}}(q^{-1}z^{-1}-qz)^{\frac{N}{2}}, \end{aligned} \tag{39}$$

which follows from the expressions (12), (31), (33), (36). Simultaneously we see that the $B_{s,1}(z)$ and $B_{s,2}(z)$ $(s=0,1)$ operators are creation operators, while the operators $C_s(z)$ acts on $\mid \Omega\rangle$ as the annihilation operators. This motivates us to consider the states

$$\mid v_1, v_2, \dots v_n \mid F\rangle_0 = F^{a_n \dots a_1} B_{0,a_1}(v_1) B_{1,a_2}(v_2) \cdots B_{0,a_n}(v_n) \mid \Omega\rangle_0, \quad a_i = 1,2; \tag{40}$$

as n-particle eigenstates of the transfer matrix $\tau(u)$. $F^{a_n \dots a_1}$ is a function of the spectral parameters v_j, which should be found later.

In order to proceed further we need to rewrite the YBEs (13) and (14) in terms of the matrix elements of the monodromy matrix

$$\begin{aligned} &(-1)^{p(k'')(p(m')+p(m''))} \check{R}^{km}_{k'm'}(\bar u, v) M_1{}^{m'}_{m''}(u) M_0{}^{k'}_{k''}(v) \\ &\qquad = (-1)^{p(k')(p(m)+p(m'))} M_1{}^{m}_{m'}(v) M_0{}^{k}_{k'}(u) \check{R}^{k'm'}_{k''m''}(\bar u, v), \\ &(-1)^{p(k'')(p(m')+p(m''))} (\check{\bar R}^{\iota_1})^{km}_{k'm'}(\bar u, v) M_0{}^{m'}_{m''}(u) M_1{}^{k'}_{k''}(v) \\ &\qquad = (-1)^{p(k')(p(m)+p(m'))} M_0{}^{m}_{m'}(v) M_1{}^{k}_{k'}(u) (\check{\bar R}^{\iota_1})^{k'm'}_{k''m''}(\bar u, v). \end{aligned} \tag{41}$$

For our purpose, the interesting components of the equations (41) are the following commutation relations between the operators $A_{bc}(u)$ and $D(u)$ with the $B_c(v)$

$$
\begin{aligned}
A_{1,ba}(u)B_{0,c}(v) &= \frac{r_{bc}^{b'c'}(u,v)}{b(u,v)}B_{1,c'}(v)A_{0,b'a}(u) + \frac{a_a(u,v)}{b(u,v)}B_{1,b}(u)A_{0,ca}(v), && (42)\\
A_{0,ba}(u)B_{1,c}(v) &= -\frac{r_{bc}^{b'c'}(u,v)}{b(u,v)}B_{0,c'}(v)A_{1,b'a}(u) - \frac{a_a(u,v)}{b(u,v)}B_{0,b}(u)A_{1,ca}(v), && (43)\\
D_1(u)B_{0,a}(v) &= \frac{1}{b'(u,v)}B_{1,a}(v)D_0(u) - \frac{{a'}_a(v,u)}{b'(v,u)}B_{1,a}(u)D_0(v), && (44)\\
D_0(u)B_{1,a}(v) &= -\frac{1}{b'(u,v)}B_{0,a}(v)D_1(u) + \frac{{a'}_a(v,u)}{b'(v,u)}B_{0,a}(u)D_1(v), && (45)\\
B_{1,a}(u)B_{0,b}(v) &= r_{ab}^{b'a'}(u,v)B_{1,a'}B_{0,b'}(u), && (46)\\
B_{0,a}(u)B_{1,b}(v) &= r_{ab}^{b'a'}(u,v)B_{0,a'}B_{1,b'}(u), && (47)
\end{aligned}
$$

where we have defined $b(u,v),\ a_a(u,v),\ b'(u,v),\ a'_a(v,u)$ as follows

$$
\begin{aligned}
b(u,v) &= z(u,v) - z^{-1}(u,v),\\
a_a(u,v) &= \begin{cases} -\lambda z(u,v), & \text{for} \quad a=1,\\ -\lambda z^{-1}(u,v), & \text{for} \quad a=2,\end{cases}\\
b'(u,v) &= -\frac{z(u,v)-z^{-1}(u,v)}{q^{-1}z^{-1}(u,v)-qz(u,v)},\\
a'_a(v,u) &= \begin{cases} \frac{\lambda z(u,v)}{q^{-1}z(u,v)-qz^{-1}(u,v)}, & \text{for} \quad a=1,\\ \frac{\lambda z^{-1}(u,v)}{q^{-1}z(u,v)-qz^{-1}(u,v)}, & \text{for} \quad a=2\end{cases} && (48)
\end{aligned}
$$

and

$$
r_{bc}^{b'c'}(u,v) = \tag{49}
$$
$$
\begin{pmatrix}
qz(u,v)-q^{-1}z^{-1}(u,v) & 0 & 0 & 0\\
0 & z(u,v)-z^{-1}(u,v) & \lambda z^{-1}(u,v) & 0\\
0 & \lambda z(u,v) & z(u,v)-z^{-1}(u,v) & 0\\
0 & 0 & 0 & qz(u,v)-q^{-1}z^{-1}(u,v)
\end{pmatrix}.
$$

Here we have used the notation $z(u,v) = \exp\left(i[u-v]\right) = \frac{z(u)}{z(v)}$.

The first terms in the commutation relations (42)-(45) are so called "wanted" terms. As we will see they are contributing to the eigenvalues of the transfer matrix $\tau_s(u),\ s=0,1$. The second terms in the (42)-(45) are so called "unwanted" terms and their contribution should be canceled in order to have an eigenstate.

Using (42)-(47) we can represent the action of the diagonal elements of the monodromy matrix as follows

$$
\begin{aligned}
&D_1(u) \mid v_1,\ldots,v_n \mid F\rangle_1 =\\
&(-1)^{\frac{n}{2}}\prod_{i=1}^{n}\frac{1}{b'(\bar u,v_i)}[q^{-1}z^{-1}(u)-qz(u)]^{\frac{N}{2}}[hq^{-1}z^{-1}(u)-h^{-1}qz(u)]^{\frac{N}{2}} \mid v_1,\ldots,v_n \mid F\rangle_0\\
&+\sum_{k=1}^{n}(\tilde\Lambda_k)_{a_1\ldots a_n}^{b_1\ldots b_n}F^{a_n\ldots a_1}B_{1,b_k}(\bar u)\prod_{j=1,j\neq k}^{n}B_{b_j}(v_j) \mid \Omega\rangle_1, && (50)
\end{aligned}
$$

and

$$\begin{aligned}
&[A_{1,11}(u) + A_{1,22}(u)] \mid v_1, \ldots, v_n \mid F\rangle_1 = \\
= \; & (-1)^{\frac{n}{2}+1} \prod_{i=1}^{n} \frac{1}{b(\bar{u}, v_i)} [-z(u) + z^{-1}(u)]^{\frac{N}{2}} [h^{-1} z(u) - h z^{-1}(u)]^{\frac{N}{2}} \\
& \times \tau^{(1)}{}^{a'_1 \ldots a'_n}_{a_1 \ldots a_n} (\bar{u}, v_1, \ldots, v_n) F^{a_n \ldots a_1} \prod_{i=1}^{n} B_{a'_i}(v_i) \mid \Omega\rangle_1 + \\
& + \sum_{k=1}^{n} (\Lambda_k)^{b_1 \ldots b_n}_{a_1 \ldots a_n} F^{a_n \ldots a_1} B_{1,b_k}(\bar{u}) \prod_{i=1, j \neq k}^{n} B_{b_j}(v_j) \mid \Omega\rangle_1,
\end{aligned} \tag{51}$$

where

$$\tau^{(1)}{}^{a'_1 \ldots a'_n}_{a_1 \ldots a_n} (\bar{u}, v_1, \ldots, v_n) = -r^{b_1 a'_1}_{c a_1} (\bar{u}, v_1) r^{b_2 a'_2}_{b_1 a_2} (\bar{u}, v_2) \cdots r^{c a'_n}_{b_{n-1} a_n} (u, v_n). \tag{52}$$

In the equations (50) and (51) the first terms appeared due to the "wanted" terms of the commutation relations (42)-(45) while the second terms result from the "unwanted" terms.

The expression (52) for $\tau^{(1)}{}^{a'_1 \ldots a'_n}_{a_1 \ldots a_n}$ can be rewritten as $str[l_n(\bar{u}, v_n) l_{(n-1)}(\bar{u}, v_{n-1}) \ldots l_1(\bar{u}, v_1)]$, with

$$\begin{aligned}
[l_k(\bar{u}, v_k)]^{b_k}_{b_{k-1}} &= r^{b_k a'_k}_{b_{k-1} a_k} (\bar{u}, v_k) \\
&= \begin{pmatrix} \begin{array}{c} [q z(\bar{u}, v_k) - q^{-1} z^{-1}(\bar{u}, v_k)] e^1_1 \\ + (z(\bar{u}, v_k) - z^{-1}(\bar{u}, v_k)) e^2_2 \end{array} & \lambda z^{-1}(\bar{u}, v_k) e^2_1 \\ \lambda z^(\bar{u}, v_k) e^1_2 & \begin{array}{c} [q z(\bar{u}, v_k) - q^{-1} z^{-1}(\bar{u}, v_k)] e^2_2 \\ + (z(\bar{u}, v_k) - z^{-1}(\bar{u}, v_k)) e^1_1 \end{array} \end{pmatrix}.
\end{aligned} \tag{53}$$

Here the quantum operators e^b_a act on the k-th space and have $(e^b_a)^\alpha_\beta = \delta_a{}^\alpha \delta_\beta{}^b$ matrix representation.

As it follows from the equations(50),(51) and (31), in order $\mid v_1, \ldots, v_n \mid F\rangle$ to be an eigenstate of $\tau_s(u)$

$$\tau_s(u) \mid v_1, \ldots, v_n \mid F\rangle = \nu_s(u; v_1, \ldots, v_n,) \mid v_1, \ldots, v_n \mid F\rangle \tag{54}$$

we should demand:

$i)$ the cancellation of the unwanted terms

$$[(\tilde{\Lambda}_k)^{b_1 \ldots b_n}_{a_1 \ldots a_n} - (\Lambda_k)^{b_1 \ldots b_n}_{a_1 \ldots a_n}] F^{a_n \ldots a_1} = 0, \tag{55}$$

$ii)$ F should be an eigenvector of the small (nested) transfer matrix $\tau^{(1)}(u)$

$$\tau^{(1)}{}^{a'_1 \ldots a'_n}_{a_1 \ldots a_n} (u; v_1, \ldots, v_n) F^{a_n \ldots a_1} = \nu^{(1)}(u; v_1, \ldots, v_n) F^{a'_n \ldots a'_1}. \tag{56}$$

Once this conditions are fulfilled, the expressions for the eigenvalues of $\tau_s(u)$ become

$$\begin{aligned}
&\nu_1(u; v_1, \ldots, v_n) = \\
= \; & (-1)^{\frac{n}{2}} \prod_{i=1}^{n} \frac{1}{b'(\bar{u}, v_i)} [q^{-1} z^{-1}(u) - q z(u)]^{\frac{N}{2}} [h q^{-1} z^{-1}(u) - h^{-1} q z(u)]^{\frac{N}{2}} \\
- \; & (-1)^{\frac{n}{2}} \prod_{i=1}^{n} \frac{1}{b(\bar{u}, v_i)} [-z(u) + z^{-1}(u)]^{\frac{N}{2}} [h^{-1} z(u) - h z^{-1}(u)]^{\frac{N}{2}} \nu^{(1)}(u; v_1, \ldots, v_n),
\end{aligned} \tag{57}$$

with

$$\nu_0(u) = \nu_1(\bar{u}). \tag{58}$$

Therefore, as it is obvious from the preceding analysis, in order to know exactly the eigenvalue we need to solve an other eigenvalue-eigenstate problem (56) for the transfer matrix $\tau^{(1)}$ with the reduced amount of degrees of freedom. That is why the all procedure is called Nested Bethe Ansatz (NBA).

In the article [19] the computation of the quantities $\tilde{\Lambda}_k$ and Λ_k for the ordinary $t-J$ model was demonstrated and the condition of cancellation of the unwanted terms was reduced to some equations, which are defining the first set of Bethe equations. It is not necessary to repeat this calculations here, but let us mention the main difference, namely

$$\begin{aligned}
(A_{1,11} + A_{1,22}) \mid \Omega\rangle &= \left[-z(u) + z^{-1}(u)\right]^{\frac{N}{2}} \left[h^{-1}z(u) - hz^{-1}(u)\right]^{\frac{N}{2}} \mid \Omega\rangle, \\
D_1 \mid \Omega\rangle &= \left[-qz(u) + q^{-1}z^{-1}(u)\right]^{\frac{N}{2}} \left[-h^{-1}z(u)q + hz^{-1}(u)q^{-1}\right]^{\frac{N}{2}} \mid \Omega\rangle,
\end{aligned} \tag{59}$$

which leads to the condition

$$\begin{aligned}
\tau^{(1)}{}^{b'_1 \ldots b'_n}_{b_1 \ldots b_n}(v_k; v_1, \ldots, v_n) F^{b_n \ldots b_1} &= -\lambda \prod_{i=1; i \neq k}^{n} \left[qz(v_k, v_i) - q^{-1}z^{-1}(v_k, v_i)\right] \times \\
\times \frac{[q^{-1}z^{-1}(\bar{v}_k) - qz(\bar{v}_k)]^{\frac{N}{2}} [hq^{-1}z^{-1}(\bar{v}_k) - h^{-1}qz(\bar{v}_k)]^{\frac{N}{2}}}{[-z(\bar{v}_k) + z^{-1}(\bar{v}_k)]^{\frac{N}{2}} [h^{-1}z(\bar{v}_k) - hz^{-1}(\bar{v}_k)]^{\frac{N}{2}}} F^{b'_n \ldots b'_1}.
\end{aligned} \tag{60}$$

The next step of NABA is the diagonalization of the small transfer matrix $\tau^{(1)}(u)$ for a chain of length n in order for F to become an eigenvector of it. Following the ABA technique we write the YBE

$$\check{r}^{km}_{k'm'}(u,v) M^{(1)}{}^{m'}_{m''}(u) M^{(1)}{}^{k'}_{k''}(v) = M^{(1)}{}^{m}_{m'}(v) M^{(1)}{}^{k}_{k'}(u) \check{r}^{k'm'}_{k''m''}(u,v), \tag{61}$$

where $M^{(1)}{}^{a'}_{a}$ is a monodromy matrix for the "nested" problem, and

$$\begin{aligned}
&\check{r}^{b'c'}_{bc}(u,v) \qquad\qquad (62) \\
&= \begin{pmatrix} qz(u,v) - q^{-1}z^{-1}(u,v) & 0 & 0 & 0 \\ 0 & \lambda z(u,v) & z(u,v) - z^{-1}(u,v) & 0 \\ 0 & z(u,v) - z^{-1}(u,v) & \lambda z^{-1}(u,v) & 0 \\ 0 & 0 & 0 & qz(u,v) - q^{-1}z^{-1}(u,v) \end{pmatrix}.
\end{aligned}$$

is the r-matrix from (49) in a braid formalism.

Let us now take

$$M^{(1)}(u) = \begin{pmatrix} A^{(1)}(u) & B^{(1)}(u) \\ C^{(1)}(u) & D^{(1)}(u) \end{pmatrix}, \tag{63}$$

and the corresponding trace

$$\tau^{(1)}(u) = -A^{(1)}(u) - D^{(1)}(u). \tag{64}$$

From the formulas (61) one can choose following algebraic relations

$$
\begin{aligned}
A^{(1)}(u)B^{(1)}(v) &= \frac{qz(u,v)-q^{-1}z^{-1}(u,v)}{z(u,v)-z^{-1}(u,v)}B^{(1)}(v)A^{(1)}(u) \\
&\quad -\frac{\lambda z(u,v)}{z(u,v)-z^{-1}(u,v)}B^{(1)}(u)A^{(1)}(v), \\
D^{(1)}(u)B^{(1)}(v) &= \frac{q^{-1}z(u,v)-qz^{-1}(u,v)}{z(u,v)-z^{-1}(u,v)}B^{(1)}(v)D^{(1)}(u) \\
&\quad +\frac{\lambda z(u,v)}{z(u,v)-z^{-1}(u,v)}B^{(1)}(u)D^{(1)}(v), \\
B^{(1)}(u)B^{(1)}(v) &= B^{(1)}(v)B^{(1)}(u),
\end{aligned} \tag{65}
$$

which are the YBE for the XXZ model with staggered inhomogeneity, defined in the article [3]. The first (second) term in the expression (65) called "wanted" (correspondingly "unwanted") term.

Let us take

$$
\mid 0\rangle_k^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mid \Omega\rangle^{(1)} = \mid 0\rangle_1^{(1)} \cdots \mid 0\rangle_n^{(1)} = \bigotimes_{k=1}^{n} \mid 0\rangle_k^{(1)}. \tag{66}
$$

as a reference state. In order to find the action of nested monodromy matrix $M^{(1)}(u)$ on the reference state one should act by $l_k(u)$ from (53) on $\mid 0\rangle_k^{(1)}$.

After simple calculations we obtain

$$
\begin{aligned}
A^{(1)}(u) \mid \Omega\rangle^{(1)} &= \prod_{i=1}^{n}[qz(u,v_i)-q^{-1}z^{-1}(u,v_i)] \mid \Omega\rangle^{(1)}, \\
D^{(1)}(u) \mid \Omega\rangle^{(1)} &= \prod_{i=1}^{n}[z(u,v_i)-z^{-1}(u,v_i)] \mid \Omega\rangle^{(1)}.
\end{aligned} \tag{67}
$$

Now let us make the following Ansatz for the eigenstates of $\tau^{(1)}(u)$

$$
\mid v_1^{(1)},\ldots,v_m^{(1)}\rangle = B^{(1)}(v_1^{(1)})B^{(1)}(v_2^{(1)})\ldots B^{(1)}(v_m^{(1)}) \mid \Omega\rangle^{(1)}. \tag{68}
$$

By use of (65) the actions of $A^{(1)}(u)$ and $D^{(1)}(u)$ on the states (68) is given by

$$
\begin{aligned}
&D^{(1)}(u) \mid v_1^{(1)},\ldots,v_m^{(1)}\rangle \\
&= \prod_{j=1}^{m}\frac{q^{-1}z(u,v_i^{(1)})-qz^{-1}(u,v_i^{(1)})}{z(u,v_i^{(1)})-z^{-1}(u,v_i^{(1)})}\prod_{j=1}^{n}[z(u,v_j)-z^{-1}(u,v_j)] \mid v_1^{(1)},\ldots,v_m^{(1)}\rangle \\
&\quad +\sum_{k=1}^{m}\Lambda_k^{(1)}B^{(1)}(u)\prod_{j=1,j\neq k}^{m}B^{(1)}(v_j^{(1)}) \mid \Omega\rangle^{(1)}, \\
&A^{(1)}(u) \mid v_1^{(1)},\ldots,v_m^{(1)}\rangle \\
&= \prod_{i=1}^{m}\frac{qz(u,v_i^{(1)})-q^{-1}z^{-1}(u,v_i^{(1)})}{z(u,v_i^{(1)})-z^{-1}(u,v_i^{(1)})}\prod_{j=1}^{n}[qz(u,v_i)-q^{-1}z^{-1}(u,v_i)] \mid v_1^{(1)},\ldots,v_m^{(1)}\rangle \\
&\quad +\sum_{k=1}^{m}\tilde{\Lambda}_k^{(1)}B^{(1)}(u)\prod_{j=1,j\neq k}^{m}B^{(1)}(v_j^{(1)}) \mid \Omega\rangle^{(1)}.
\end{aligned} \tag{69}
$$

The first terms in this expressions are the "wanted" terms, while the second terms (with $\Lambda_k^{(1)}$, $\tilde{\Lambda}_k^{(1)}$) are the "unwanted" terms of the nested problem and should cancel each other in the eigenvalue equation (56). Hence we can write the eigenvalues $\nu^{(1)(u;v_1,\ldots,v_n;v_1^{(1)},\ldots,v_m^{(1)})}$ of $\tau^{(1)}(u)$

$$\begin{aligned}
&\tau^{(1)}(u)\mid v_1^{(1)},\ldots,v_m^{(1)}\rangle = [-A^{(1)}(u)-D^{(1)}(u)]\mid v_1^{(1)},\ldots,v_m^{(1)}\rangle \\
&= -\left[\prod_{i=1}^{m}\frac{qz(u,v_i^{(1)})-q^{-1}z^{-1}(u,v_i^{(1)})}{z(u,v_i^{(1)})-z^{-1}(u,v_i^{(1)})}\prod_{j=1}^{n}[qz(u,v_i)-q^{-1}z^{-1}(u,v_i)]\right. \\
&\left.+\prod_{j=1}^{m}\frac{q^{-1}z(u,v_i^{(1)})-qz^{-1}(u,v_i^{(1)})}{z(u,v_i^{(1)})-z^{-1}(u,v_i^{(1)})}\prod_{j=1}^{n}[z(u,v_j)-z^{-1}(u,v_j)]\right]\mid v_1^{(1)},\ldots,v_m^{(1)}\rangle. \qquad (70)
\end{aligned}$$

Setting now $u=v_k$ in (70) and comparing it with the condition (60) we obtain the first set of Bethe equations

$$\begin{aligned}
\prod_{i=1}^{m}\frac{qz(v_k,v_i^{(1)})-q^{-1}z^{-1}(v_k,v_i^{(1)})}{z(v_k,v_i^{(1)})-z^{-1}(v_k,v_i^{(1)})} &= \frac{[h^{-1}q^{-1}z(v_k)-hqz^{-1}(v_k)]^{\frac{N}{2}}[q^{-1}z(v_k)-qz^{-1}(v_k)]^{\frac{N}{2}}}{[-h^{-1}z(v_k)+hz^{-1}(v_k)]^{\frac{N}{2}}[z(v_k)-z^{-1}(v_k)]^{\frac{N}{2}}}, \\
k &= 1,\ldots,n. \qquad (71)
\end{aligned}$$

The equations which are obtained from the cancellation of the unwanted terms $\Lambda_k^{(1)}$ and $\tilde{\Lambda}_k^{(1)}$

$$\prod_{i=1,i\neq k}^{m}\frac{q^{-1}z(v_k^{(1)},v_i^{(1)})-qz^{-1}(v_k^{(1)},v_i^{(1)})}{qz(v_k^{(1)},v_i^{(1)})-q^{-1}z^{-1}(v_k^{(1)},v_i^{(1)})}=\prod_{j=1}^{n}\frac{qz(v_k^{(1)},v_j)-q^{-1}z^{-1}(v_k^{(1)},v_j)}{z(v_k^{(1)},v_j)-z^{-1}(v_k^{(1)},v_j)}, \quad k=1\ldots,m \qquad (72)$$

are the second set of Bethe equations.

Finally we obtain the eigenvalues of $\tau_1(u)$ and $\tau_0(u)$ from (57) and (58) as

$$\begin{aligned}
\nu_1(u;\{v\};\{v^{(1)}\}) &= (-1)^{\frac{n}{2}}\prod_{i=1}^{n}\frac{1}{b'(\bar{u},v_i)}[q^{-1}z^{-1}(u)-qz(u)]^{\frac{N}{2}}[hq^{-1}z^{-1}(u)-h^{-1}qz(u)]^{\frac{N}{2}} \\
&+(-1)^{\frac{n}{2}}\prod_{i=1}^{n}\frac{1}{b(\bar{u},v_i)}[-z(u)+z^{-1}(u)]^{\frac{N}{2}}[h^{-1}z(u)-hz^{-1}(u)]^{\frac{N}{2}}\times \\
&\times\left[\prod_{i=1}^{m}\frac{qz(u,v_i^{(1)})-q^{-1}z^{-1}(u,v_i^{(1)})}{z(u,v_i^{(1)})-z^{-1}(u,v_i^{(1)})}\prod_{j=1}^{n}[qz(u,v_i)-q^{-1}z^{-1}(u,v_i)]\right. \\
&\left.+\prod_{j=1}^{m}\frac{q^{-1}z(u,v_i^{(1)})-qz^{-1}(u,v_i^{(1)})}{z(u,v_i^{(1)})-z^{-1}(u,v_i^{(1)})}\prod_{j=1}^{n}[z(u,v_j)-z^{-1}(u,v_j)]\right],
\end{aligned} \qquad (73)$$

and

$$\nu_0(u;\{v\};\{v^{(1)}\})=\nu_1(\bar{u};\{v\};\{v^{(1)}\}). \qquad (74)$$

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Consistency of the bootstrap for Liouville field theory

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Abstract

I review the (re-)construction of Liouville field theory based on the quantization of the classical correspondence with free field theory. The main task is to construct general exponential field operators and to verify their mutual locality. The main technical ingredient for the latter is the derivation of braid relations for a certain class of covariant chiral operators, which is outlined in this talk.

Motivation

- Applications: Noncritical Strings, models of low-dim. gravity, D-brane physics.
- Relations to other integrable models (e.g. sinh-Gordon).

General perspective

There are many interesting 2D CFT besides Liouville theory which are expected to have continuous spectrum of representations of conformal symmetry (noncompact WZNW, Sine-Liouville etc.). One needs new techniques to handle the difficulties associated with the continuous spectrum (no null-vectors!). Liouville theory seems to be the natural prototypical example to start the theoretical study of such CFT, just as

- The solution of the minimal models (BPZ) was the starting point for RCFT.
- The studies of Lorentz group / $SL(2,\mathbb{R})$ by Gelfand, Naimark and Bargmann were the starting point for the harmonic analysis on noncompact groups.

Liouville theory may be seen as a CFT that exhibits strong analogies to the harmonic analysis of noncompact groups.

The exposition of the results presented below is based on the work [1]. Further references can be found there.

1 Classical solution

Claim: *The following statements are equivalent:*
(i) φ satisfies the classical Liouville equation of motion

$$\partial_+\partial_-\varphi = -2\pi\mu_c e^{\varphi}$$

(ii) $e^{-\frac{1}{2}\varphi}$ satisfies

$$\begin{aligned} \partial_+^2 e^{-\frac{1}{2}\varphi(x_+,x_-)} &= T_{++}(x_+)e^{-\frac{1}{2}\varphi(x_+,x_-)} \\ \partial_-^2 e^{-\frac{1}{2}\varphi(x_+,x_-)} &= T_{--}(x_-)e^{-\frac{1}{2}\varphi(x_+,x_-)}, \end{aligned} \tag{1}$$

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where T_{++} (resp. T_{--}) depend on x_+ (resp. x_-) only.
Proof: (i)⇒(ii): Any $\varphi(x_+,x_-)$ will satisfy (1) with

$$T_{\pm\pm}(x_+,x_-) \;\equiv\; \tfrac{1}{4}(\partial_\pm\varphi)^2 - \tfrac{1}{2}\partial_\pm^2\varphi.$$

However, direct calculation shows that $\partial_\mp T_{\pm\pm} = 0$ if the Liouville equation of motion is satisfied.
(ii)⇒(i): Equation (1) implies that $e^{-\frac{1}{2}\varphi} = \sum_{i,j=1}^2 \; f_i^+(x_+)C_{ij}f_j^-(x_-)$, where $f_i^\pm(x_\pm)$, $i=1,2$ are two linearly independent real solutions of $\partial_\pm^2 f_i^\pm = T_{\pm\pm}f_i^\pm$, which can and will be normalized such that

$$f_2^\pm\partial_\pm f_1^\pm - f_1^\pm\partial_\pm f_2^\pm = 1. \tag{2}$$

W.l.o.g.: $C_{ij} = C\delta_{ij}$ (by forming linear combinations of the $f_i^\pm$). By direct calculation using (2) one may then verify that

$$\varphi(x_+,x_-) \;=\; -2\log\Big(\sqrt{2\pi\mu_c}\sum_{i=1}^{2}\; f_i^+(x_+)f_i^-(x_-)\Big) \tag{3}$$

satisfies the Liouville equation of motion.

Remarks

- This means that the solution of the nonlinear Liouville equation of motion can be reduced to the integration of the linear equations $\partial_\pm^2 f_i^\pm = T_{\pm\pm}f_i^\pm$, where $T_{\pm\pm}(x_\pm)$ can be considered as data, given in terms of either initial or asymptotic conditions. From these one may then reconstruct $e^{-\frac{1}{2}\varphi(x_+,x_-)}$ by solving *linear* differential equations, and finally φ itself by taking the logarithm. Any solution of the Liouville equation is obtained in this way, and the data $T_{\pm\pm}(x_\pm)$ characterize a solution $\varphi(x_+,x_-)$ uniquely up to an *additive constant* (`zero mode`).

- We are interested in the case where $\varphi(\sigma,t)$ is periodic, $\varphi(\sigma+2\pi,t)=\varphi(\sigma,t)$. It follows that $T_{\pm\pm}(x_\pm)$ must also be periodic. However, the solutions $f_i^\pm$ will only be *quasi*-periodic

$$f^\pm(x_\pm \pm 2\pi) \;=\; M^\pm\cdot f^\pm(x_\pm). \tag{4}$$

  The Liouville field only depends on the conjugacy class of the matrix M. It can be shown (Pogrebkov/Polivanov) that the Liouville field is regular if and only if Tr$M>2$, corresponding to the so-called *hyperbolic* conjugacy classes of $SL(2,\mathbb{R})$ One may therefore assume that $M=\mathrm{diag}(e^{\frac{p}{2}},e^{-\frac{p}{2}})$.

2 Map to free field

Introduce $\varphi_\pm^{\mathrm{F}}(x_\pm)$ by means of

$$f_1^\pm(x_\pm) \;\equiv\; e^{-\frac{1}{2}\varphi_\pm^{\mathrm{F}}(x_\pm)}.$$

$\varphi^{\mathrm{F}}(x_+,x_-)\equiv\varphi_+^{\mathrm{F}}(x_+)+\varphi_-^{\mathrm{F}}(x_-)$ is a solution of the free wave equation $\partial_+\partial_-\varphi^{\mathrm{F}}=0$ on the *cylinder*. The second solution of $\partial_\pm^2 f_i^\pm = T_{\pm\pm}f_i^\pm$ can also be expressed in terms of $\varphi_\pm^{\mathrm{F}}(x_\pm)$:

$$f_2^\pm(x_\pm) \;=\; e^{-\frac{1}{2}\varphi_\pm^{\mathrm{F}}(x_\pm)}\frac{1}{e^{\pm p}-1}\int_0^{2\pi} dy_\pm\; e^{\varphi_\pm(y_\pm+x_\pm)}.$$

It follows that the space $\mathcal{P}^{\mathrm{F}}$ of solutions to the free field equations provides a parameterization of the space $\mathcal{P}$ of solutions to the Liouville equation. Direct, but tedious calculation shows: The canonical relations $\{\Pi_{\varphi}^{\mathrm{F}}(\sigma), \varphi^{\mathrm{F}}(\sigma')\} = \delta(\sigma-\sigma')$ imply $\{\Pi_{\varphi}(\sigma), \varphi(\sigma')\} = \delta(\sigma-\sigma')$. But:

$$\boxed{\text{The parametrization of } \mathcal{P} \text{ by free fields is ambiguous!}}$$

To see this, introduce a second free field $\tilde{\varphi}^{\mathrm{F}}$ by

$$f_2^{\pm}(x_\pm) \equiv e^{-\frac{1}{2}\tilde{\varphi}_{\pm}^{\mathrm{F}}(x_\pm)}.$$

Since the exchanging $\tilde{\varphi}^{\mathrm{F}} \leftrightarrow \varphi^{\mathrm{F}}$ only amounts to $f_1^{\pm} \leftrightarrow f_2^{\pm}$ it follows that

$$\varphi[\tilde{\varphi}^{\mathrm{F}}] = \varphi[\varphi^{\mathrm{F}}].$$

Define an operator $S : \mathcal{P}^{\mathrm{F}} \to \mathcal{P}^{\mathrm{F}}$ by $S \cdot \varphi^{\mathrm{F}} = \tilde{\varphi}^{\mathrm{F}}$. Then $\mathcal{P} \simeq \mathcal{P}^{\mathrm{F}}/\sim$, where $\varphi_1^{\mathrm{F}} \sim \varphi_2^{\mathrm{F}}$ iff $S \cdot \varphi_1 = \varphi_2$.

3 Quantization of the free field theory

Let us introduce the free field $\phi^{\mathrm{F}}(\sigma, t)$ with CCR

$$[\phi^{\mathrm{F}}(\sigma), \partial_t \phi^{\mathrm{F}}(\sigma)] = 2\pi i \delta(\sigma - \sigma') \tag{5}$$

at time $t = 0$. Modes are introduced via

$$\phi^{\mathrm{F}}(\sigma, t) \;=\; \mathsf{q}^{\mathrm{F}} + 2\mathsf{p}t + \sum_{n \neq 0} \frac{1}{n}\big(\mathsf{a}_n^{\mathrm{F}} e^{-inx_+} + \mathsf{b}_n^{\mathrm{F}} e^{-inx_-}\big), \tag{6}$$

and the Hilbert-space will be defined as

$$\mathcal{H}^{\mathrm{F}} \;\equiv\; L^2(\mathbb{R}) \otimes \mathcal{F} \otimes \mathcal{F}, \tag{7}$$

where $\mathcal{F} \otimes \mathcal{F}$ is the Fock-space generated by action of the modes a_n, b_n, $n < 0$ on the Fock-vacuum v that satisfies $\mathsf{a}_n v = 0$ and $\mathsf{b}_n v = 0$, $n > 0$. We will assume that p is diagonal.

3.1 Conformal symmetry:

Conformal symmetry is realized on $\mathcal{H}^{\mathrm{F}}$ by means of

$$\begin{aligned}
\mathsf{T}_{++}(x_+) \;&=\; \sum_{n\in\mathbb{Z}} \mathsf{L}_n^{\mathrm{F}} e^{-inx_+}, \qquad & \mathsf{L}_n^{\mathrm{F}} &= (2\mathsf{p} + inQ)\mathsf{a}_n + \sum_{k\neq 0,n} \mathsf{a}_k \mathsf{a}_{n-k}, \\
\mathsf{T}_{--}(x_-) \;&=\; \sum_{n\in\mathbb{Z}} \bar{L}_n^{\mathrm{F}} e^{-inx_-}, \qquad & \bar{\mathsf{L}}_n^{\mathrm{F}} &= (2\mathsf{p} + inQ)\mathsf{b}_n + \sum_{k\neq 0,n} \mathsf{b}_k \mathsf{b}_{n-k}.
\end{aligned}$$

This means that $\mathcal{H}^{\mathrm{F}}$ decomposes as the direct integral of Fock-space representations of the Virasoro algebra:

$$\mathcal{H}^{\mathrm{F}} \;\simeq\; \int_{-\infty}^{\infty} dP\; \mathcal{F}_P \otimes \bar{\mathcal{F}}_P, \tag{8}$$

where $\mathcal{F}_P$ (resp. $\bar{\mathcal{F}}_P$) denote the Virasoro representations defined on $\mathcal{F}$ by means of the generators $\mathsf{L}_n^{\mathrm{F}}$ (resp. $\bar{L}_n^{\mathrm{F}}$).

3.2 Building blocks:

The basic building blocks of all constructions will be:
NORMAL ORDERED EXPONENTIALS

$$\begin{aligned}\mathsf{E}^{\alpha}(x_+) &\equiv e^{\alpha \mathsf{q}^{\mathrm{F}}} \exp\Big(\sum_{n<0} \frac{2\alpha}{n} \mathsf{a}_n^{\mathrm{F}} e^{-inx_+}\Big) e^{2\alpha x + \mathsf{p}} \exp\Big(\sum_{n>0} \frac{2\alpha}{n} \mathsf{a}_n^{\mathrm{F}} e^{-inx_+}\Big) e^{\alpha \mathsf{q}^{\mathrm{F}}} \\ \bar{\mathsf{E}}^{\alpha}(x_-) &\equiv e^{\alpha \mathsf{q}^{\mathrm{F}}} \exp\Big(\sum_{n<0} \frac{2\alpha}{n} \mathsf{b}_n^{\mathrm{F}} e^{-inx_-}\Big) e^{2\alpha x - \mathsf{p}} \exp\Big(\sum_{n>0} \frac{2\alpha}{n} \mathsf{b}_n^{\mathrm{F}} e^{-inx_-}\Big) e^{\alpha \mathsf{q}^{\mathrm{F}}}.\end{aligned}$$

SCREENING CHARGES

$$\mathsf{Q}(x_+) \equiv \int_0^{2\pi} dy_+ \, \mathsf{E}^b(x_+ + y_+), \quad \bar{\mathsf{Q}}(x_-) \equiv \int_0^{2\pi} dy_+ \, \bar{\mathsf{E}}^b(x_- + y_-).$$

The normal ordered exponentials can be understood as true (unbounded) operators if $\Im(x_\pm) > 0$ (negative euclidean time), or as operator valued distributions for real values of $x_\pm$.

The screening charges on the contrary represent densely defined unbounded operators even for $x_\pm \in \mathbb{R}$. They can be seen to have a canonical self-adjoint extension due to their property of positivity.

$$\boxed{\text{This defines } \big(\mathsf{Q}(x_+)\big)^\lambda \text{ and } \big(\bar{\mathsf{Q}}(x_+)\big)^\lambda \text{ for } \lambda \in \mathbb{C}!}$$

The behavior of these operators under conformal transformations can be summarized by

$$\begin{gathered}\big[\mathsf{L}_n, \mathsf{E}^\alpha(x_+)\big] = e^{inx_+}\big(-i\partial_+ + n\Delta_\alpha\big)\, \mathsf{E}^\alpha(x_+), \quad \big[\bar{\mathsf{L}}_n, \bar{\mathsf{E}}^\alpha(x_-)\big] = e^{inx_-}\big(-i\partial_- + n\Delta_\alpha\big)\, \bar{\mathsf{E}}^\alpha(x_-), \\ \big[\mathsf{L}_n, \mathsf{Q}(x_+)\big] = -ie^{inx_+}\partial_+ \mathsf{Q}(x_+), \quad \big[\bar{\mathsf{L}}_n, \bar{\mathsf{Q}}(x_-)\big] = -ie^{inx_-}\partial_- \bar{\mathsf{Q}}(x_-),\end{gathered}$$

3.3 Quantum counterparts of f_i, $\bar{f}_i$:

Let us define the following set of operators:

$$\begin{aligned} \mathsf{f}_1(x_+) &= \mathsf{E}^{-\frac{b}{2}}(x_+), & \mathsf{f}_2(x_+) &= \frac{e^{-2\pi b(\mathsf{p}+i\frac{b}{2})}}{\sin(\pi b(Q-2i\mathsf{p}))} \mathsf{Q}(x_+)\mathsf{f}_1(x_+), \\ \bar{\mathsf{f}}_1(x_-) &= \bar{\mathsf{E}}^{-\frac{b}{2}}(x_-), & \bar{\mathsf{f}}_2(x_-) &= \bar{\mathsf{f}}_1(x_-)\bar{\mathsf{Q}}(x_-)\frac{e^{-2\pi b(\mathsf{p}-i\frac{b}{2})}}{2\sin(\pi b(Q+2i\mathsf{p}))}. \end{aligned}$$

The correspondence with their classical counterparts is quite obvious (The shifts of p in the p-dependent pre-factors could be absorbed by choosing a different ordering).

It is encouraging to note that the operators f_i $i = 1, 2$ satisfy a second order differential equation of a very similar form as their classical counterparts:

$$\partial_+^2 \, \mathsf{f}_i = -b^2 : \mathsf{T} \, \mathsf{f}_i :, \quad i = 1, 2, \tag{9}$$

where the normal ordering of the expression on the right hand side is defined as follows:

$$: \mathsf{T} \, \mathsf{f}_i : = \sum_{n<0} \mathsf{L}_n e^{-inx_+} \, \mathsf{f}_i + \sum_{n>0} \mathsf{f}_1 \, e^{-inx_+} \mathsf{L}_n + \frac{1}{2}\big(\mathsf{L}_0 \mathsf{f}_i^+ + \mathsf{f}_i^+ \mathsf{L}_0\big) - \left(\frac{c-1}{24} - \frac{b^2}{16}\right) \mathsf{f}_i.$$

The operators $\bar{\mathsf{f}}_i$ $i = 1, 2$ satisfy a second order differential equation that is obtained by obvious replacements.

3.4 Quantum counterpart of $e^{-\frac{1}{2}\varphi}$:

Let us consider the following operator:

$$\mathsf{V}(\sigma,t) \;=\; \mathsf{f}_1(x_+)\, e^{b\mathsf{q}}\, \bar{\mathsf{f}}_1(x_-) + \mu_{\rm e}\, \mathsf{f}_2(x_+)\, e^{-b\mathsf{q}}\, \bar{\mathsf{f}}_2(x_-).$$

It can be shown to satisfy the following properties:

a) Conformal covariance

$$\begin{aligned}\left[\mathsf{L}_n, \mathsf{V}(\sigma,t)\right] &= e^{+inx_+}\left(-i\partial_+ + n\Delta_b\right)\mathsf{V}(\sigma,t)\,,\\ \left[\bar{L}_n, \mathsf{V}(\sigma,t)\right] &= e^{-inx_-}\left(-i\partial_- + n\Delta_b\right)\mathsf{V}(\sigma,t)\,,\end{aligned}$$

b) Equation of motion

$$\partial_+^2\,\mathsf{V} \;=\; -b^2 : \mathsf{T}(x_+)\,\mathsf{V} :, \qquad \partial_-^2\,\mathsf{V} \;=\; -b^2 : \bar{\mathsf{T}}(x_-)\,\mathsf{V} :,$$

c) Locality

$$\langle\psi_2|\,[\;\mathsf{V}(f)\;,\;\mathsf{V}(f')\;]\,|\psi_1\rangle \;=\; 0\,,$$

d) Positivity

$$\langle\psi|\,\mathsf{V}(f)\,|\psi\rangle \;>\; 0\,,$$

where $|\psi\rangle$, $|\psi_1\rangle$, $|\psi_2\rangle$ are taken from a dense subset of $\mathcal{H}^{\rm F}$ and the *smeared* operator $\mathsf{V}(f)$ is defined as $\mathsf{V}(f) \;\equiv\; \int_{S^1} d\sigma \;\; f(\sigma)\,\mathsf{V}(\sigma)$. Properties a) and b) are obvious. Locality is proved by using the exchange relations of the operators f_i, $\bar{\mathsf{f}}_i$ that follow those discovered by Gervais/Neveu.

These properties strongly support the identification:

$$\mathsf{V}(\sigma) \;\equiv\; [e^{-b\phi}]_b(\sigma),$$

which identifies $\mathsf{V}(\sigma)$ as quantum counterpart of $e^{-\frac{1}{2}\varphi}$ taking into account $\varphi = 2b\phi$.

4 Reflection operator

Recall: Liouville phase space was parameterized by the free field phase space in an ambiguous way. An unambiguous parameterization is obtained by identifying configurations related by the operator S: $\varphi_1^{\rm F} \sim \varphi_2^{\rm F}$ iff $S[\varphi_1] = \varphi_2$.

The quantum analogue of S should be an operator S on $\mathcal{H}^{\rm F}$ that maps p to $-\mathsf{p}$ but leaves $[e^{-b\phi}]_b$ unchanged,

$$\mathsf{S}^{-1}\cdot[e^{-b\phi}]_b\cdot\mathsf{S} = [e^{-b\phi}]_b. \tag{10}$$

Then

$$\mathcal{H} \;=\; \{|\psi\rangle \in \mathcal{H}^{\rm F}; (1-\mathsf{S})\Psi = 0\}. \tag{11}$$

Compatibility with conformal symmetry:

$$\mathsf{S}\cdot\mathsf{L}_n^{\rm F}(\mathsf{p}) \;=\; \mathsf{L}_n^{\rm F}(-\mathsf{p})\cdot\mathsf{S}. \tag{12}$$

Since the Fock-space representations $\mathcal{F}_P$ and $\mathcal{F}_{-P}$ are unitarily equivalent, one has a unique operator $\mathsf{S}(P) : \mathcal{F}\otimes\mathcal{F} \to \mathcal{F}\otimes\mathcal{F}$ that satisfies (12) and $\mathsf{S}(P)v = v$. Therefore

$$\mathsf{S} \;=\; \mathsf{P}\cdot R(\mathsf{p})\mathsf{S}(\mathsf{p})\,, \tag{13}$$

where P denotes the parity operation in $L^2(\mathbb{R}) \otimes \mathcal{F}$, $\mathsf{P}\psi(P) = \psi(-P)$. One may then show that there exists a unique choice of the function $R(P)$ that was introduced in (13) such that (10) is satisfied as well:

$$R(P) = -\big(\mu_{\rm e}\Gamma^2(b^2)\big)^{-\frac{2iP}{b}} \frac{\Gamma(1+2ibP)\Gamma(1+2ib^{-1}P)}{\Gamma(1-2ibP)\Gamma(1-2ib^{-1}P)}\,. \tag{14}$$

5 General exponential operators. Abstract definition

We would like to define $[e^{2\alpha\phi}]_b(\sigma)$ as something like $[e^{-b\phi}]_b \equiv \mathsf{V}$ raised to the power -2α. This is of course a nontrivial thing to do in the case of local operators. However, the usual exponential function is the unique solution to the functional equation $\exp(x)\exp(y) = \exp(x+y)$. This motivates:

DEFINITION. *Let $\mathsf{V}_\alpha(\sigma)$ be a family of operators that has the properties*

i) There exists a complex number ρ such that

$$\big(\mathsf{V}_\alpha \star \mathsf{V}_\beta\big)(\sigma) \;\equiv\; \lim_{\sigma'\to\sigma} \big|\sigma'-\sigma\big|^\rho \mathsf{V}_\alpha(\sigma')\mathsf{V}_\beta(\sigma)$$

exists and satisfies

$$\big(\mathsf{V}_\alpha \star \mathsf{V}_\beta\big)(\sigma) \;=\; \mathsf{V}_{\alpha+\beta}(\sigma).$$

ii) Each two operators $\mathsf{V}_\alpha(\sigma)$ and $\mathsf{V}_\beta(\sigma)$ are mutually local:

$$\big[\,\mathsf{V}_\alpha(\sigma')\,,\,\mathsf{V}_\beta(\sigma)\,\big] = 0.$$

iii) $\mathsf{V}_\alpha(\sigma)$ reduces to $\mathsf{V}(\sigma)$ for $\alpha = -\frac{b}{2}$.

We then call $\mathsf{V}_\alpha(\sigma)$ a renormalized power function of V, in symbols $\mathsf{V}_\alpha(\sigma) \equiv \mathsf{V}^{-2\alpha}(\sigma)$.
It is then natural to identify

$$\mathsf{V}_\alpha(\sigma) \;\equiv\; [e^{2\alpha\phi}]_b(\sigma),$$

as will be substantiated in the following.

6 General exponential operators. Concrete realization

Let us now consider the operator

$$\boxed{\mathsf{V}_\alpha(t,\sigma) \;\equiv\; \int_{i\mathbb{R}} ds\; \mu_{\rm e}^s\; B_b(\alpha,s)\; \mathsf{f}_s^\alpha(x_+)\; \mathsf{T}_s^\alpha\; \bar{\mathsf{f}}_s^\alpha(x_-),}$$

where

- $\mathsf{T}_s^\alpha \equiv e^{-2(\alpha+bs)\mathsf{q}}$,
- the b-binomial coefficients $B_b(\alpha,s)$ are given by

$$B_b(\alpha,s) \;=\; \frac{S_b(-bs)S_b(2\alpha+bs)}{S_b(2\alpha)},$$

  where the special function $S_b(x)$ is defined as

$$\log\big(S_b(x)e^{\frac{\pi i}{2}(x^2-Qx+\frac{1}{6}(b^2+3+b^{-2}))}\big) = \int_{\mathbb{R}+i0} \frac{dt}{t}\frac{e^x t}{(1-e^{bt})(1-e^{t/b})}.$$

- The operators $\mathsf{f}_s^\alpha(x_+)$, $\bar{\mathsf{f}}_s^\alpha(x_-)$ can be represented explicitly as follows:

$$\begin{aligned}
\mathsf{f}_s^\alpha(x_+) &\equiv e^{-2\pi b s(\mathsf{p}+i\frac{b}{2})}\frac{S_b(Q-2i\mathsf{p})}{S_b(Q-2i\mathsf{p}+bs)}\big(\mathsf{Q}(x_+)\big)^s\mathsf{E}^\alpha(x_+)\,,\\
\bar{\mathsf{f}}_s^\alpha(x_-) &\equiv \bar{\mathsf{E}}^\alpha(x_-)\big(\bar{\mathsf{Q}}(x_-)\big)^s\frac{S_b(Q+2i\mathsf{p})}{S_b(Q+2i\mathsf{p}+bs)}e^{-2\pi b s(\mathsf{p}-i\frac{b}{2})}\,.
\end{aligned}$$

REMARK: S_b is known under various names: "Double Sine function" (Barnes/Shintani), "Quantum Dilogarithm" (Faddeev/Kashaev), "Quantum Exponential function" (Woronowicz). It seems that these people have discovered this remarkable function independently.

7 Comments

7.1 Correspondence to classical expression

$$e^{\lambda\varphi(\sigma,t)} \;=\; \big(f_1\bar{f}_1+2\pi\mu_c f_2\bar{f}_2\big)^{-2\lambda}. \tag{15}$$

Let us note that at least for $\lambda>0$ one has a useful representation of (15) as sum of *imaginary* powers of $f_i\bar{f}_i$:

$$e^{\lambda\varphi(\sigma,t)} \;=\; \frac{i}{2\pi}\int\limits_{i\mathbb{R}} ds\;(2\pi\mu_c)^s\,\frac{\Gamma(s+2\lambda)\Gamma(-s)}{\Gamma(2\lambda)}\,f_s^\lambda(x_+)\bar{f}_s^\lambda(x_-),$$

where $f_s^\lambda=f_2^s f_1^{-2\lambda-s}$, $\bar{f}_s^\lambda=\bar{f}_2^s\bar{f}_1^{-2\lambda-s}$. This expansion is related to the binomial expansion by writing the integral as sum over residues. However, it has advantages for quantization:

The essential point is captured by the following example: Consider the operator v_λ of multiplication by the function $h_\lambda(q)\equiv(e^{\frac{q}{2}}+e^{-\frac{q}{2}})^{-2\lambda}$ on $L^2(\mathbb{R})$ in the case that $\Re(\lambda)>0$. v_λ is a nice, bounded operator on $L^2(\mathbb{R})$ since $h_\lambda(q)$ behaves asymptotically as $h_\lambda(q)\sim e^{-\lambda|q|}$. Now let us try to represent this operator by using the usual binomial expansion:

$$\mathsf{v}_\lambda \;=\; e^{\lambda\mathsf{q}}\sum_{n=0}^{\infty}\frac{(-)^n\Gamma(n+2\lambda)}{\Gamma(2\lambda)\Gamma(n+1)}e^{n\mathsf{q}} \tag{16}$$

which represents the operator on wave-functions with support on the negative half-axis only. The domain in which (16) serves to represent the operator v_λ is not even dense in $L^2(\mathbb{R})$!

7.2 Relation with canonical quantization

Recover Liouville field and its canonical momentum as

$$\phi(\sigma) \;=\; \tfrac{1}{2}\partial_\alpha\mathsf{V}_\alpha(\sigma)|_{\alpha=0},\quad \Pi_\phi(\sigma) \;=\; \tfrac{1}{4\pi}\partial_t\big(\partial_\alpha\mathsf{V}_\alpha(t,\sigma)|_{\alpha=0}\big)_{t=0}.$$

It can be shown to satisfy weakly (within matrix elements)

- Liouville equation of motion

$$(\partial_t^2-\partial_\sigma^2)\phi=-4\pi\mu b\,[e^{2b\phi}]_b.$$

- Canonical commutation relations

$$\big[\phi(\sigma)\;,\;\Pi_\phi(\sigma')\big]=i\delta(\sigma-\sigma').$$

7.3 Relation with DOZZ-proposal

Dorn/Otto and the brothers Zamolodchikov [2, 3] proposed a formula for the VEV of three operators V_α. This is equivalent to the knowledge of matrix elements of an operators V_α between highest weight states of the Virasoro algebra:

$$\langle P_2|\mathsf{V}_\alpha(\sigma)|P_1\rangle \;=\; C\big(\tfrac{Q}{2}-iP_2,\alpha,\tfrac{Q}{2}+iP_1\big). \tag{17}$$

The formula for $C(\alpha_3,\alpha_2,\alpha_1)$ that was proposed by DOZZ reads:

$$\begin{aligned} C(\alpha_1,\alpha_2,\alpha_3) = \Big[\pi\mu\gamma(b^2)b^{2-2b^2}\Big]^{(Q-\sum_{i=1}^3\alpha_i)/b}\times \\ \times\frac{\Upsilon_0\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon(\alpha_1+\alpha_2-\alpha_3)\Upsilon(\alpha_2+\alpha_3-\alpha_1)\Upsilon(\alpha_3+\alpha_1-\alpha_2)}. \end{aligned}$$

Our previous definition for V_α gives precisely (17).

$$\boxed{\Rightarrow \text{Proof of the DOZZ-proposal!}}$$

8 Covariant chiral operators

Introduce the operators

$$\mathsf{g}^\alpha_s(x_+) = \mathsf{E}^\alpha(x_+)\big(\mathsf{Q}(x_+)\big)^s.$$

The following two results are the main technical ingredients to our proof of the DOZZ-proposal.
Normalization: The matrix elements of operators g^α_s between primary states for the Virasoro algebra are given as

$$\langle P'|\mathsf{g}^\alpha_s(x_+)|P\rangle \;=\; \delta\big(P^\alpha_s-P'\big)\,e^{ix_+(\Delta(P')-\Delta(P))}\,G^\alpha_s(P),$$

where $P^\alpha_s\equiv P-i\alpha-ibs$ and

$$\begin{aligned} G^\alpha_s(P) \;=\; \Big(\Gamma(1+b^2)b^{-1-b^2}\Big)^s e^{2\pi bsP}e^{-\pi ib^2s^2}\times \\ \times\frac{\Gamma_b(Q-2iP-2\alpha-sb)\Gamma_b(Q+2iP+sb)\Gamma_b(Q-2\alpha-sb)\Gamma_b(Q+sb)}{\Gamma_b(Q-2iP-2\alpha-2sb)\Gamma_b(Q+2iP)\Gamma_b(Q-2\alpha)\Gamma_b(Q)}. \end{aligned}$$

Braid relations: The operators g^α_s satisfy braid relations of the form

$$\mathsf{g}^{\alpha_2}_{s_2}(\sigma_2)\mathsf{g}^{\alpha_1}_{s_1}(\sigma_1) \;=\; \frac{1}{4}\int_{\mathbb{T}} dt_2dt_1\;\mathsf{g}^{\alpha_1}_{t_1}(\sigma_1)\mathsf{g}^{\alpha_2}_{t_2}(\sigma_2)\;B_\epsilon(\alpha_2,\alpha_1|\mathsf{p})^{s_2s_1}_{t_2t_1},$$

where $\mathbb{T}\equiv-\frac{Q}{2}+i\mathbb{R}$ and $\epsilon\equiv\operatorname{sgn}(\sigma_2-\sigma_1)$. The distributional kernel $B_\epsilon(\alpha_2,\alpha_1|\mathsf{p})^{s_2s_1}_{t_2t_1}$ has support for $t_2+t_1=s_2+s_1$ only.

9 Derivation of braid relations

Consider case $\alpha_i\in\frac{Q}{2}+i\mathbb{R}$, $s_i\in-\frac{Q}{2}+i\mathbb{R}$, $t_i\in-\frac{Q}{2}+i\mathbb{R}$, $i=1,2$. The general case can be obtained by analytic continuation.

9.1 Step 1: Reformulation via Weyl-type operators

Nice trick introduced by Gervais/Schnittger: Consider the case $\sigma_2 < \sigma_1 \leftrightarrow \epsilon = -1$. Let us split $\mathsf{Q}(\sigma) = \mathsf{Q}_I^c + \mathsf{Q}_I$, $\mathsf{Q}(\sigma') = \mathsf{Q}_I^c + \mathsf{Q}'_I$, where

$$\mathsf{Q}_I^c \equiv \int_{\sigma'}^{\sigma+2\pi} d\varphi\, \mathsf{E}^b(\varphi), \quad \mathsf{Q}_I \equiv \int_{\sigma}^{\sigma'} d\varphi\, \mathsf{E}^b(\varphi), \quad \mathsf{Q}'_I \equiv \int_{\sigma+2\pi}^{\sigma'+2\pi} d\varphi\, \mathsf{E}^b(\varphi).$$

Exchange relations

$$\mathsf{E}^{\alpha_2}(\sigma_2)\mathsf{E}^{\alpha_1}(\sigma_1) = e^{-2\pi i\alpha_2\alpha_1\epsilon(\sigma_2-\sigma_1)}\mathsf{E}^{\alpha}(\sigma_1)\mathsf{E}^{b}(\sigma_2),$$

$\Rightarrow$ braid relations are equivalent to the following identity:

$$\begin{aligned} & e^{4\pi i\alpha_2\alpha_1}\big(e^{-2\pi ib\alpha_1}\mathsf{Q}_I^c + e^{2\pi ib\alpha_1}\mathsf{Q}_I\big)^{s_2}\big(\mathsf{Q}_I^c + \mathsf{Q}'_I\big)^{s_1} = \\ & = \int_{\mathbb{T}} dt_2dt_1\, \big(e^{-2\pi ib\alpha_2}\mathsf{Q}_I^c + e^{-6\pi ib\alpha_2}\mathsf{Q}'_I\big)^{t_1}\big(\mathsf{Q}_I^c + \mathsf{Q}_I\big)^{t_2} B_-(\alpha_2,\alpha_1|\mathsf{p})_{t_2t_1}^{s_2s_1}. \end{aligned}$$

The operators Q_I, Q'_I, Q_I^c satisfy the following algebra:

$$\mathsf{Q}_I^c\mathsf{Q}_I = q^{-2}\mathsf{Q}_I\mathsf{Q}_I^c, \qquad \mathsf{Q}_I^c\mathsf{Q}'_I = q^{+2}\mathsf{Q}'_I\mathsf{Q}_I^c, \qquad \mathsf{Q}_I\mathsf{Q}'_I = q^4\mathsf{Q}'_I\mathsf{Q}_I,$$

where $q = e^{\pi ib^2}$. It follows that these operators can be realized as

$$\mathsf{Q}_I^c = e^{2b\mathsf{x}}e^{-\pi ibt}, \quad \mathsf{Q}_I = e^{b\mathsf{x}}e^{-2\pi b\mathsf{p}}e^{b\mathsf{x}}e^{\pi ibt}, \quad \mathsf{Q}'_I = e^{b\mathsf{x}}e^{2\pi b\mathsf{p}}e^{b\mathsf{x}}e^{\pi ibt},$$

where x and p satisfy $[\mathsf{x},\mathsf{p}] = \frac{i}{2}$ and $t \in i\mathbb{R}$ commutes with x,p.

9.2 Step2: Ordering the operators

Translate braid-relations into a relation that only contains commuting quantities: This becomes possible thanks to identities such as

$$\big(e^{-2\pi ib\alpha_1}\mathsf{Q}_I^c + e^{2\pi ib\alpha_1}\mathsf{Q}_I\big)^{s_2} = e^{2bs_2\mathsf{q}}e^{-\pi bs_2(\mathsf{p}-is_2\frac{b}{2})}\frac{S_b\big(\frac{Q}{2}+2\alpha_1+i\mathsf{p}+s_2b+t\big)}{S_b\big(\frac{Q}{2}+2\alpha_1+i\mathsf{p}+t\big)},$$

which follow quite easily from $S_b(x+b) = 2\sin(\pi bx)S_b(x)$. These identities allow one to collect all x-dependent factors to the left of the terms containing p. The desired identity will therefore follow once one has shown that

$$\begin{aligned} & e^{\frac{\pi i}{2}b^2(s_2^2-s_1^2)}e^{\pi ib^2s_1s_2}e^{\pi b(s_1-s_2)\mathsf{p}}\;\frac{S_b\big(\frac{Q}{2}+i\mathsf{p}+2\alpha_1+(s_2+s_1)b+t\big)S_b\big(\frac{Q}{2}-i\mathsf{p}+t\big)}{S_b\big(\frac{Q}{2}+i\mathsf{p}+2\alpha_1+bs_1+t\big)S_b\big(\frac{Q}{2}-i\mathsf{p}-bs_1+t\big)} = \\ & = \int_{\mathbb{T}} dt_2dt_1\, B_-(\alpha_2,\alpha_1|\mathsf{p})_{t_2t_1}^{s_2s_1}\, e^{-4\pi i(\alpha_1+bt_1)\alpha_2}e^{\frac{\pi i}{2}b^2(t_2^2-t_1^2)}e^{-\pi ib^2t_1t_2}e^{\pi b(t_1-t_2)\mathsf{p}}\times \\ & \qquad \times\frac{S_b\big(\frac{Q}{2}+i\mathsf{p}+2\alpha_2+(t_2+t_1)b-t\big)S_b\big(\frac{Q}{2}-i\mathsf{p}-t\big)}{S_b\big(\frac{Q}{2}+i\mathsf{p}+2\alpha_2+bt_2-t\big)S_b\big(\frac{Q}{2}-i\mathsf{p}-bt_2-t\big)}. \end{aligned}$$

It is furthermore convenient to take the Fourier-transformation w.r.t. the variable t of this identity. Let us introduce

$$\Phi_\lambda(A,B,C;y) = \int_{\mathbb{R}} d\tau\; e^{2\pi i\tau y}\,\tilde{\Phi}_\lambda(A,B,C;\tau),$$

where

$$\tilde{\Phi}_\lambda(A,B,C;\tau) \;\equiv\; \frac{S_b\big(\frac{Q}{2}+i(\tau+A)\big)S_b\big(\frac{Q}{2}+i(\tau+B)\big)}{S_b\big(Q+i(\tau-C+\lambda+i0)\big)S_b\big(Q+i(\tau-C-\lambda+i0)\big)},$$

with real parameters A,B,C . The desired identity thereby takes the form

$$\begin{aligned} & e^{4\pi i(\alpha_1+bs)\alpha_2}e^{\pi i b^2(s^2-s_1^2)}e^{2\pi b(s_1-s)\mathsf{p}}\Phi_\lambda(A_1,B_1;C_1;y) = \\ & \quad = \int\limits_{\mathbb{T}} dt_2 dt_1 \;\; B_-(\alpha_2,\alpha_1|\mathsf{p})_{t_2t_1}^{s_2s_1}\, e^{4\pi i b t_2\alpha_2}e^{\pi i b^2 t_2^2}e^{-2\pi b t_2 \mathsf{p}}\,\Phi_\lambda(A_2,B_2;C_2;-y), \end{aligned}$$

with $s=s_1+s_2=t_1+t_2$.

9.3 Step 3: Exploiting the completeness of the Φ_λ

In [4] it is found that for any choice of $A,B,C\in\mathbb{R}$ the set $\{\Phi_\lambda(A,B;C;y);\lambda\in\mathbb{R}^+\}$ forms a basis for $L^2(\mathbb{R})$ with

$$\int_{\mathbb{R}} dy\;\big(\Phi_\lambda(A,B;C;y)\big)^*\Phi_{\lambda'}(A,B;C;y) = \frac{\delta(\lambda-\lambda')}{|S_b(Q+2i\lambda)|^2}.$$

This means that in the case $\alpha_i\in\frac{Q}{2}+i\mathbb{R}$, $s_i\in-\frac{Q}{2}+i\mathbb{R}$, $t_i\in-\frac{Q}{2}+i\mathbb{R}$, $i=1,2$, the sought-for relation is nothing but a CHANGE OF BASIS:

$$\text{from}\qquad \{\Phi_\lambda(A_1,B_1;C_1;y);\lambda\in\mathbb{R}^+\}\qquad \text{to}\qquad \{\Phi_\lambda(A_2,B_2;C_2;-y);\lambda\in\mathbb{R}^+\},$$

with kernel $B_\epsilon(\alpha_2,\alpha_1|\mathsf{p})_{t_2t_1}^{s_2s_1}$ given in terms of the overlap

$$\int_{\mathbb{R}} dy\;\big(\Phi_\lambda(A_1,B_1;C_1;y)\big)^*\Phi_{\lambda'}(A_2,B_2;C_2;-y).$$

The rest of the calculation is straightforward. The result can be written as

$$\boxed{B_\epsilon(\alpha_2,\alpha_1|P)_{t_2t_1}^{s_2s_1} \;=\; \delta_{st}\,\frac{e^{\epsilon A}e^{B}}{|S_b(2C)|^2}\int_{i\mathbb{R}} ds\,\prod_{i=1}^{4}\frac{S_b(s+R_i)}{S_b(s+S_i)},}$$

where $\delta_{st}\equiv\delta(s_1+s_2-t_1+t_2)$, A, B, and C are given as

$$\begin{aligned} A &= \pi i b^2(s_1^2+t_2^2-s^2)-2\pi bP(s_1+t_2-s)-2\pi i b\big((s-t_2)\alpha_2+(s-s_1)\alpha_1\big), \\ B &= 2\pi i b\big((s-t_2)\alpha_2-(s-s_1)\alpha_1\big), \qquad C \;=\; -iP-\alpha_2-bt_2, \end{aligned}$$

and the coefficients R_i and S_i, $i=1,\dots,4$ are defined by

$$\begin{aligned} R_1 &= \tfrac{Q}{2}-2\alpha_2-iP-b(t_1+t_2), & S_1 &= \tfrac{3Q}{2}-2\alpha_2-iP-bt_2, \\ R_2 &= \tfrac{Q}{2}+2\alpha_1+iP+b(s_1+s_2), & S_2 &= \tfrac{Q}{2}+2\alpha_1+iP+bs_1, \\ R_3 &= \tfrac{Q}{2}+iP, & S_3 &= \tfrac{3Q}{2}+iP+bt_2, \\ R_4 &= \tfrac{Q}{2}-iP, & S_4 &= \tfrac{Q}{2}-iP-bs_1. \end{aligned}$$

10 Locality

To complete the proof of locality one may proceed via the following steps:

- Express the braid-coefficients $B_\epsilon(\alpha_2, \alpha_1|P)^{s_2 s_1}_{t_2 t_1}$ in terms of the b-Racah-Wigner symbols [4]
$$\left\{ \begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \bar{\alpha}_4 \end{smallmatrix} \middle| \begin{smallmatrix} \alpha_s \\ \alpha_t \end{smallmatrix} \right\}_b .$$
The b-Racah-Wigner symbols are analogs of the usual Racah-Wigner symbols where the role of the rotation group is played by the quantum group $\mathcal{U}_q(sl(2,\mathbb{R}))$ (Faddeev [5]: "modular double").

$$\boxed{\Rightarrow \text{"Modular double": Internal symmetry of Liouville theory!}}$$

- Use (part of) tetrahedral symmetry
$$\left\{ \begin{smallmatrix} \alpha_2 & \alpha_1 \\ \bar{\alpha}_4 & \bar{\alpha}_3 \end{smallmatrix} \middle| \begin{smallmatrix} \alpha_s \\ \bar{\alpha}_t \end{smallmatrix} \right\}_b = \left\{ \begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{smallmatrix} \middle| \begin{smallmatrix} \alpha_s \\ \alpha_t \end{smallmatrix} \right\}_b .$$
(Follows from representation theory of $\mathcal{U}_q(sl(2,\mathbb{R}))$.
- Locality becomes equivalent to the following unitarity relation:
$$\int_{\mathbb{S}} d\alpha_s \; M_b(\alpha_s) \; \left\{ \begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{smallmatrix} \middle| \begin{smallmatrix} \alpha_s \\ \alpha_t \end{smallmatrix} \right\}_b \left\{ \begin{smallmatrix} \bar{\alpha}_1 & \bar{\alpha}_2 \\ \bar{\alpha}_3 & \bar{\alpha}_4 \end{smallmatrix} \middle| \begin{smallmatrix} \bar{\alpha}_s \\ \bar{\alpha}'_t \end{smallmatrix} \right\}_b = M_b(\alpha_t) \; \delta(\alpha_t - \alpha'_t),$$
where $\bar{\alpha}_i = Q - \alpha_i$, $\alpha_t, \beta_t \in \mathbb{S}$, and the measure $M_b(\alpha)$ is given by
$$M_b(\alpha) = |S_b(2\alpha)|^2 = -4\sin(\pi b(2\alpha - Q))\sin(\pi b^{-1}(2\alpha - Q)).$$

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Separation of variables for integrable systems on the Poisson manifolds

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Abstract

A systematic way for realizing integrable hamiltonian system on coadjoint orbits of Lie algebras is provided by the Classical Inverse Scattering Method [6], which is a working machine to produce examples of classical integrable systems together with their solutions. One of the the key points of this method is duality between the integrals of motion and the Casimir elements in the underlying hidden symmetry algebra.

The aim of this paper is to show that the same Casimir elements not only give rise integrals of motion but they can generate the separated variables for this family of integrals.

1 Introduction

Let us consider some $2n$-dimensional symplectic manifold $(\mathcal{M},\Omega)$ endowed with a symplectic form Ω. The Hamilton-Jacoby equation

$$\mathcal{Q}^{2n-1}: \qquad H-E=0\,, \qquad E\in\mathbb{R},$$

is a hyperplane $\mathcal{Q}^{2n-1}$ in $\mathcal{M}$ [10, 1]. Solution of this equation is an n-dimensional Lagrangian submanifold F lying on hyperplane $\mathcal{Q}^{2n-1}$. By definition, a Lagrangian submanifold is one where the symplectic form Ω vanishes when restricted to it, i.e. $\Omega|_{\mathcal{C}}=0$.

Integrable system on $\mathcal{M}$ is defined by n functionally independent integrals of motion I_j. The inverse images $\cap I_j^{-1}(\alpha_j)$ of the corresponding moment map is Lagrangian submanifold F. Thus, any integrable system is associated with the Lagrangian fibration $\rho_i:\mathcal{M}\to\mathcal{A}^n$ whose fibers F are Lagrangian submanifolds depending at least on the n arbitrary parameters α_j, which are identified with values of integrals of motion $I_j=\alpha_j$ [10, 1].

Let us consider the Poisson manifold $(\mathcal{M},\mathcal{P})$ with degenerate Poisson form $\mathcal{P}$. Any Poisson manifold turns out to be foliated in symplectic leaves or minimal Poisson submanifolds S_a for which the induced Poisson structure is nondegenerate. This foliation $\rho_c:\mathcal{M}\to\mathcal{A}^m$ may be described by the Casimir functions whose restrictions to symplectic leaves S_a are constants. So, fixing the values of Casimirs provides a rough classification of symplectic leaves, although it is not true in general that different symplectic leaves are separated by the Casimirs.

Thus, for an integrable system on the Poisson manifolds we can consider composition of foliations $\rho=\rho_i\circ\rho_c:\mathcal{M}\to\mathcal{A}^{n+m}$ whose fibers F depend on the n values of integrals of motion α_j and m values a_i of the Casimirs. Using some additional assumptions on foliation ρ we can define canonical symplectic connection ∇, which gives rise parallel translations T_γ of fibers F under some smooth curve γ in base $\mathcal{A}^{n+m}$.

Let a given integrable system on $\mathcal{M}$ and the corresponding separated variables are invariant with respect to translations T_γ, i.e. separated variables are independent on the values of integrals α_j and values of the Casimirs a_i. In this case ∇N, where ∇ is symplectic connection

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on ρ and N is the Nijenhuis tensor, which eigenvalues are the separated variables [2]. Thus, we suppose that the separated variables may be explicitly defined using these translations T_γ. In this paper we proof this proposition for some examples. The general geometric construction will be studied in forthcoming publication.

For the separable integrable systems on the Poisson manifold the Lagrangian submanifolds

$$F \simeq \mathcal{C}^{(n)}: \qquad \mathcal{C}_1 \times \mathcal{C}_2 \times \cdots \times \mathcal{C}_n \ \times \mathcal{A}^m \subset \mathbb{R}^m \tag{1.1}$$

are product of n plane curves $\mathcal{C}_j$, which may be defined explicitly

$$\mathcal{C}_j: \qquad \Phi_j\,(\mu_j, \lambda_j, \alpha_1, \ldots, \alpha_n, a_1, \ldots, a_m) = 0\,. \qquad j = 1, \ldots, n\,. \tag{1.2}$$

Here (μ_j, λ_j) are coordinates on the j-th plane. Below we shall drop these indexes in notations.

In the inverse scattering method we usually study the Lax form of the equations of motion

$$\{H, L\} = [L, A]\,,$$

where L and A are matrix valued functions on $\mathcal{M}$. The coefficients of the characteristic polynomial of Lax matrix L are integrals of motion and Casimir functions. If the given Lax matrix $L(\lambda)$ depends on the second spectral parameter λ then the characteristic equation

$$\det(\mu - L(\lambda)\,) = \Phi(\mu, \lambda, \alpha_1 \ldots, \alpha_n) = 0 \tag{1.3}$$

defines only one plane curve $\mathcal{C}$. The corresponding Lagrangian submanifold

$$\mathcal{C}^{(n)}: \qquad Sym^n(\mathcal{C}) \times \mathcal{A}^m \subset \mathbb{R}^m$$

is a symmetric product of this spectral curve of the Lax matrix. It allows us to identify Lagrangian foliation with affine variety of $\mathcal{C}$ and to switch on the powerful algebraic-geometric machinery [5, 3, 9]. In this case, phase space M may be regarded as a fiber bundle $\mathcal{M} \to \mathcal{A}^{n+m}$. The fibers are (generalized) Jacobians of these curves [3, 9].

2 Separation of variables

Let $\{p_i, q_i\}_{i=1}^n$ are some local coordinates on the phase space $\mathcal{M}$. In the separation of variables method [1, 7], we are looking for a special canonical transformation $(p, q) \mapsto (P, Q)$ such that separated variables have to satisfy to the separated equations

$$\Phi_j\,\Big(P_j, Q_j, I_1(P, Q), \ldots, I_n(P, Q)\Big) = 0\,, \qquad j = 1, \ldots, n\,. \tag{2.1}$$

These equations may be obtained from the equations of the plane curves (1.2) after substitution of the integrals of motion $I_j(P, Q)$ as functions of the separated variables instead of their values α_j and separated variables (P, Q) instead of variables (μ, λ).

We have to emphasize that variables μ and λ are complex variables related by numeric equation (1.2). The separated variables P_i and Q_i are canonical coordinates of $\mathcal{M}$. Constructing the separated equations (2.1) we have to understand that numerical parameters α_j and plane coordinates $\mu = \phi(P_i, Q_i)$ and $\lambda = \psi(P_i, Q_i)$ become some functions on $\mathcal{M}$.

Thus, in the separation of variables method two related problems might be distinguished. At first we must determine decomposition of the Lagrangian submanifold $\mathcal{C}^{(n)}$ (1.1) on a product of some plane curves. At second we have to introduce canonical separated variables $\{P_i, Q_i\}_{i=1}^n$ satisfying to the corresponding separated equations (2.1).

The aim of this note is to discuss how to solve the second problem if we know solution of the first problem. Below we shall consider Lagrangian submanifolds depending on the $n+m$ arbitrary parameters. As above the n constants $\alpha_1,\ldots,\alpha_n$ are identified with the values of integrals of motion $I_j=\alpha_j$ [10]. These integrals I_j depend on the remaining constants a_k (charges or coupling constants)

$$I_j(p,q,a)=I_j(p_1,\ldots,p_n,q_1,\ldots,q_n,a_1,\ldots,a_m)\,,\qquad a_k\in\mathbb{R}. \tag{2.2}$$

Let us substitute functions $I_j(p,q)$ instead of their values α_j into the known equations of the plane curves (1.2)

$$\Phi_j(\mu,\lambda,I_1(p,q,a),\ldots,I_n(p,q,a),a_1\ldots,a_m)=0\,,\qquad j=1,\ldots,n\,. \tag{2.3}$$

We can consider these equations by different values of parameters, for instance by a_k=0

$$\Phi_j(\mu,\lambda,I_1(p,q,a),\ldots,I_n(p,q,a),a_1\ldots,a_m)|_{\,a=0}=0\,, \tag{2.4}$$

either differentiate these equations by parameters a_k

$$\frac{\partial}{\partial a_k}\Phi_j\Big(\,\mu,\lambda,\,I_1(p,q,a),\,\ldots,\,I_n(p,q,a),a_1\ldots,a_m\,\Big)=0\,,\qquad k=1,\ldots,m\,. \tag{2.5}$$

The enlarged system of equations (2.3-2.5) may be useful for explicit construction of the separated variables.

Proposition 1 *If the desired canonical transformation of variables $(p,q)\mapsto(P,Q)$ is independent on parameters a_k, then the solutions of equations (2.3-2.5) are separated variables*

$$Q_i=f_i(p,q)\,,\qquad P_i=g_i(p,q)\,,\qquad i=1,\ldots,n, \tag{2.6}$$

as functions of initial variables (p,q).

Here we suppose that the generating function of the unknown canonical transformation $(p,q)\mapsto(P,Q)$ is independent on parameters a_k. Clearly, if we can calculate solutions of equations (2.3-2.5) then we can check that these solutions are independent on parameters and that they are canonically conjugated.

As an example let us consider two-dimensional Kepler problem. The corresponding Lagrangian manifold $\mathcal{C}^{(2)}=\mathcal{C}_1\times\mathcal{C}_2$ depends on two values of integrals of motion

$$I_1=p_x^2+p_y^2+\frac{Z}{\sqrt{x^2+y^2}}\,,\qquad I_2=2p_x(p_x\,y-p_y\,x)+\frac{Zy}{\sqrt{x^2+y^2}}\,.$$

and the arbitrary value of the charge Z. We can substitute functions $I_k(p_x,x,p_y,y)$ instead their values α_k into the equations of the corresponding plane curves

$$\mathcal{C}_{1,2}:\qquad \Phi_{1,2}(\mu,\lambda)=\Big(\mu^2+\alpha_1\lambda+Z\pm\alpha_2\Big)=0\,,\qquad I_1=\alpha_1\quad I_2=\alpha_2$$

and differentiate these equations by Z. Solutions of the resulting system of algebraic equations are usual parabolic coordinates

$$\frac{\partial}{\partial Z}\Phi_{1,2}(\,\mu,\lambda)=0\,,\quad\Leftrightarrow\quad \lambda_{1,2}=y\pm\sqrt{x^2+y^2}\,,$$

which are independent on the charge Z.

In the proposed construction of separated variables the main problem is related with existence of the additional arbitrary parameters a_k. For integrable hamiltonian system on coadjoint orbits of Lie algebras values of the Casimir operators may be considered as these necessary parameters. Recall, that the Casimir operators give rise a stratification of the underlying phase space $\mathcal{M}$ on usually equivalent symplectic leaves. Since, we can suppose that the desired separated variables are independent on the "numbers" of symplectic leaves.

3 The Neumann system

Let us consider motion of a particle on the $(n-1)$-dimensional sphere $\mathcal{S}^{n-1}$ under the action of a quadratic potential (see [6] and references within). If x, p are canonical variables on $T^*\mathcal{S}^{n-1}$ then the corresponding Lax representation is given by

$$L(\lambda) = A + J\lambda + X\lambda^2\,, \qquad J = p \wedge x\,, \quad X = x \otimes x\,. \tag{3.1}$$

Here $A = \mathrm{diag}(a_1, \ldots, a_n)$ is arbitrary numerical diagonal matrix and the Hamilton function reads as

$$H = \frac{1}{2}\sum p_j^2 - \sum a_j x_j^2\,.$$

When $n = 3$ this system bears the name Neumann system.

The corresponding phase space may be identified with coadjoint orbits of Euclidean Lie algebras [1]. Let two vectors $J \in so(3) \simeq \mathbb{R}^3$ and $x \in \mathbb{R}^3$ be coordinates in the dual space $e^*(3)$ equipped with a natural Lie-Poisson brackets

$$\{J_i\,, J_j\} = \varepsilon_{ijk}\, J_k\,, \qquad \{J_i\,, x_j\} = \varepsilon_{ijk}\, x_k\,, \qquad \{x_i\,, x_j\} = 0\,, \qquad i,j,k = 1,2,3,. \tag{3.2}$$

Here ε_{ijk} is the standard totally skew-symmetric tensor.

The Poisson structure is degenerate and the initial phase space $e^*(3)$ decomposes into minimal Poisson submanifolds for which the induced Poisson structure is non-degenerate. The generic coadjoint orbits of $E(3)$ in $e^*(3)$ are four dimensional symplectic leaves specified by the two second order Casimir elements

$$I_1 = (x, x) = x_i x_i; \qquad I_2 = (J, x) = J_i x_i\,. \tag{3.3}$$

Here (x, y) means inner product in $\mathbb{R}^3$. Thus the dual space $e^*(3)$ decomposes into the coadjoint orbits

$$\mathcal{O}_{b,\,c} = \Big\{\{J, x\} \in \mathbb{R}^6 \;:\; I_1 = b\,,\; I_2 = c\Big\} \tag{3.4}$$

which are invariant with respect to the usual Euler-Poisson equations in $e^*(3)$ [1, 6]. The Neumann system is a completely integrable system on the one-parameter subset of orbits $\mathcal{O}_b$ $(I_1 = b\,,\; I_2 = 0)$ in $e(3)^*$.

For the Neumann system we can consider motion on the different spheres $\mathcal{S}_b^2$ with radius $I_1 = b$, i.e. we can consider different orbits labeled by one parameter b. To construct equations (2.3-2.5) we have to determine restriction of integrals of motion to symplectic leaves. Instead of it we shall use scaling transformation $x \to \sqrt{b}\,x$ which relates different orbits $\mathcal{O}_b$, changes integrals of motion

$$\begin{aligned}
I_3 &= J_1^2 + J_2^2 + J_3^2 + b\Big((a_3 + a_2)x_1^2 + (a_1 + a_3)x_2^2 + (a_1 + a_2)x_3^2\Big)\,,\\
I_4 &= a_1 J_1^2 + a_2 J_2^2 + a_3 J_3^2 + b\left(a_2 a_3 x_1^2 + a_1 a_3 x_2^2 + a_1 a_2 x_3^2\right),
\end{aligned}$$

transforms the corresponding Lax representation

$$L(\lambda) \to L(\lambda, b) = A + J\lambda + b\,X\lambda^2\,,$$

and modifies the spectral curve

$$\mathcal{C}\;:\qquad \Phi(\mu,\lambda) = \frac{(\mu - a_1)(\mu - a_2)(\mu - a_3)}{\lambda^2} + \mu\, I_3 - I_4 - b\Big(\mu^2\, I_1 + \lambda^2\, I_2^2\Big) = 0\,.$$

Such modification of the spectral curve allows us to consider this transformation as a scaling transformation of the Casimir functions $I_1 \to b\, I_1$ and $I_2 \to \sqrt{b}\; I_2$.

Proposition 2 *Solutions of the following system of equations (2.3-2.5)*

$$\det\Big(L(\lambda,b)-\mu\Big)=0\,,\qquad \frac{\partial\det\Big(L(\lambda,b)-\mu\Big)}{\partial b}=0\,. \tag{3.5}$$

are separated variables for the Neumann system.

The last equation in the system (3.5) looks like

$$\prod_{j=1}^{3}(\mu-a_j)\left(\frac{x_1^2}{\mu-a_1}+\frac{x_2^2}{\mu-a_2}+\frac{x_3^2}{\mu-a_3}\right)=0\,.$$

It is well-known definition of the elliptical spherical coordinates. These coordinates are separated variables for the Neumann system [5].

Notice that third additional equation (2.4) is equal to

$$\det\Big(L(\lambda,b=0)-\mu\Big)=\det\Big(L(\lambda,b)-\mu\Big)-b\,\frac{\partial\det\Big(L(\lambda,b)-\mu\Big)}{\partial b}=0 \tag{3.6}$$

and the Lax matrix $L(\lambda,b=0)$ may be associated with the Euler top.

4 On the stationary flow of the KdV hierarchy

The separation of variables for constrained flows of soliton equation has been studied, for instance in [4, 11]. Below we shell consider the KdV hierarchy only.

Let $\{\mathbf{e},\mathbf{f},\mathbf{h}\}$ be generators of the Lie algebra $sl(2)$

$$[\mathbf{h},\mathbf{e}]=\mathbf{e}\,,\qquad [\mathbf{h},\mathbf{f}]=-\mathbf{f}\,,\qquad [\mathbf{e},\mathbf{f}]=2\mathbf{h}\,, \tag{4.1}$$

and the element

$$\Delta=\mathbf{h}^2+\frac{1}{2}(\mathbf{ef}+\mathbf{fe}) \tag{4.2}$$

of the universal enveloping algebra be Laplace operator in $SL(2)$. Let us consider infinite-dimensional irreducible representation $\mathcal{W}_j$ of the Lie algebra $sl(2)$ in the linear space V_j such that

$$\mathcal{W}_j:\ \{\mathbf{e},\mathbf{f},\mathbf{h}\}\to\{e_j,f_j,h_j\}\in\mathrm{End}(V_j)\,.$$

Using these representations and an appropriate completion of the direct sum of $n+1$ copies of the loop algebra $sl(2,\lambda)$ we can construct the following multi-pole Lax matrix

$$L_0(\lambda)=\begin{pmatrix}0&1\\0&0\end{pmatrix}+\sum_{j=1}^{n}\frac{1}{2(\lambda-d_j)}\begin{pmatrix}h_j&e_j\\f_j&-h_j\end{pmatrix} \tag{4.3}$$

in framework of the classical r-matrix theory [6].

Below we shall consider infinite-dimensional representations of $sl(2)$ in which spectrums of all the Casimir elements are continuous. In this case the corresponding $2n$-dimensional Lagrangian foliation is given by

$$\mathcal{C}^{(2n)}:\qquad \overbrace{\mathcal{C}\times\cdots\times\mathcal{C}}^{n\text{ times}}\times\overbrace{\mathbb{R}\times\cdots\times\mathbb{R}}^{n\text{ times}},$$

where $\mathcal{C}$ is a spectral curve of the Lax matrix

$$\mathcal{C}\ :\qquad \mu^2+h(\lambda)^2+e(\lambda)\,f(\lambda)=\mu^2+P(\lambda,I_1,\ldots,I_n)+\sum_{j=1}^{n}\frac{\Delta_k}{(\lambda-d_k)^2}=0\,.$$

Here $h(\lambda)$, $f(\lambda)$ and $e(\lambda)$ are entries of the Lax matrices $L_0(\lambda)$ (4.3) and explicit expressions for the function P and integrals $I_1,\ldots,I_n$ may be founded in [8, 11].

This general construction is an ample source of examples of integrable systems. Any concrete realization of the $sl(2)$ is related with the family of integrable systems. For instance, the following realization of the infinite-dimensional representation of $sl(2)$

$$h_i=x_ip_i\,,\quad e_i=x_i^2\,,\quad f_i=-p_i^2+\frac{a_i}{x_i^2}\,,\quad a_i\in\mathbb{R} \tag{4.4}$$

may be associated to standard constrained flows and with the geodesic motion on the Riemann manifolds with the Stäckel metrics [8].

To study hight-order binary constrained flows we have to use so-called m-bosonic realization of the infinite-dimensional representation of $sl(2)$ instead of the one-bosonic realization (4.4). As an example, two-bosonic realization looks like

$$h_i=\frac{1}{2}(x_ip_{x,i}-y_ip_{y,i})\,,\quad e_i=x_i\,p_{y,i}\,,\quad f_i=y_i\,p_{x,i}\,. \tag{4.5}$$

The first realization of $sl(2)$ (4.4) is embedding of the symplectic manifold $\mathbb{R}^{2n}$ into the Poisson manifold $\mathcal{M}=\oplus sl(2)^*$ and, therefore, the Casimir elements Δ_i (4.2) describe trivial dynamics. Moreover, all the integrals of motion depend on values a_i of these Casimir elements Δ_i. Using parameters a_i we can construct the following system of equations (2.5)

$$\frac{\partial}{\partial a_k}\Phi\Big(\mu,\lambda,\,\alpha,a\Big)=0\qquad\Leftrightarrow\qquad\frac{\partial f(\lambda)}{\partial a_k}\,e(\lambda)=0\,.$$

Zeroes of the great common divisor $e(\lambda)$ of these equations coincide with poles of the corresponding Baker-Akhiezer function normalized on vector $\vec{\alpha}=(1,0)$. Here we have explicitly n poles and we can introduce $2n$ separated variables.

The two-bosonic realization of $sl(2)$ (4.5) describes immersion of the Poisson manifold $\mathcal{M}=\oplus sl(2)^*$ into the symplectic manifold $\mathbb{R}^{4n}$. Since, the first n integrals of motion I_j describe potential motion on the Riemann manifold with the Stäckel metrics, whereas remaining n integrals

$$I_{n+j}=\frac{(x_jp_{x_j}+y_jp_{y,j})^2}{4}\,,\qquad j=1,\ldots,n$$

are Casimir elements Δ_j (4.2). In contrast with the previous example these functions I_{n+j} describe non-trivial dynamics on the $4n$-dimensional symplectic manifold $\mathbb{R}^{4n}$.

The initial Poisson structure on $\mathcal{M}$ is degenerate. A possible way to analyze integrability is to eliminate the Casimir elements of the Poisson tensor by fixing the values of its Casimirs. On a level surface S_b functions $I_{n+j}=b_j$ the Poisson structure becomes non-degenerate.

Let us consider restriction of initial phase space to S_b. On the each leaf the n-dimensional Lagrangian foliation depends on n values of integrals I_j and n parameters b_j. It allows us to apply the proposed method for construction of the separated variables.

Proposition 3 *Solutions of the following system of equations*

$$\Phi\Big(\mu,\lambda,\,\widehat{I}_1,\,\ldots,\,\widehat{I}_n,b\Big)=0\,,\qquad\frac{\partial}{\partial b_k}\Phi\Big(\mu,\lambda,\,\widehat{I}_1,\,\ldots,\,\widehat{I}_n,b\Big)=0 \tag{4.6}$$

are separated variables. Here $\widehat{I}_j$ are restrictions of the integrals I_j to generic leaf S_b.

To prove this proposition we can start with the following equations [11]

$$\{I_{n+j}, e(\lambda)\} = \{I_{n+j}, h(\lambda)\} = 0\,,$$

which are preserved by reduction procedure. Since, using reduction of the Lax matrix $L(\lambda)$ to S_b, we can rewrite equations (2.5) at the following form

$$\frac{\partial f(\lambda)}{\partial b_k}\, e(\lambda) = 0 \qquad \Leftrightarrow \qquad e(\lambda) = 0\,.$$

As above, zeroes of the great common divisor $e(\lambda)$ of these equations coincide with poles of the corresponding Baker-Akhiezer function normalized on vector $\vec{\alpha} = (1, 0)$.

Let us turn back to the initial $4n$-dimensional manifold $\mathbb{R}^{4n}$. By definition of the first $2n$ separated variables (4.6) the Poisson brackets between Casimir elements I_{n+j} and these separated variables are equal to zero. So, to construct a complete family of separated variables on the initial $4n$-dimensional phase space we can join $2n$ solutions of (4.6) with n Casimir elements I_{n+j} and n conjugated to them variables. For this additional variables the separated equations are equal to $P_{n+j} = I_{n+j}$.

5 Stationary flow of the Boussinesq hierarchy

This example was considered within a general framework for the analysis of stationary flows of n-Gel'fand-Dickey hierarchies [2]. According to this work we introduce 10-dimensional Poisson manifold $\mathcal{M}$ with coordinates $h_1, k_1, h_2, k_2, \ldots, h_5, k_5$ and with the following Poisson tensor

$$P_0 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 & & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
 & & & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0 & -h_1 & -2h_1 & -k_1 - h_2 \\
 & & & & & 0 & -h_1 & -2k_1 & -k_1 - h_2 & -2k_2 \\
 & & & & & & 0 & 0 & -2h_2 & -k_2 \\
 & & & & & & & 0 & -k_2 & 0 \\
 & & & & & & & & 0 & 0 \\
 & & & & & & & & & 0
\end{pmatrix}.$$

This Poisson structure is degenerate and two Casimir elements

$$I_1 = 3k_5 - 3k_1k_2 - 3k_2h_2\,, \qquad I_3 = -3h_1k_2 + 3k_4 - 3k_1^2\,. \tag{5.1}$$

describe trivial dynamics $\{I_{1,3}, h_j\} = \{I_{1,3}, k_j\} = 0$. On this phase space we consider integrable system with the following Lax matrices

$$\begin{aligned}
L(\lambda) = \;& \lambda^2 \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0 & 1 \\ h_2 & h_1 & 0 \\ -h_3 + k_2 & k_1 - h_2 & -h_1 \end{pmatrix} \\
&+ \begin{pmatrix} -k_3 & -k_2 & -k_1 \\ L_0^{21} & -h_2h_1 + h_4 & -h_1^2 + h_3 \\ L_0^{31} & L_0^{32} & h2h_1 - h_4 + k_3 \end{pmatrix},
\end{aligned} \tag{5.2}$$

where

$$L_0^{21} = -h_1k_2 + k_4 - k_1^2 - h_1h_3 + h_5\,, \qquad L_0^{32} = -k_1h_2 + h_2^2 - h_5 + 2k_4 - k_1^2\,,$$

$$L_0^{31} = h_1k_3 - 2h_1h_4 + 3k_5 + h_2h_1^2 - 2k_1k_2 - 2k_1h_3 + h_1^2k_1 - 2k_2h_2 + h_2h_3\,.$$

The second matrix of the Lax pair may be founded in [2]. This Lax representation may be obtained using the Lax formulation for the Boussinesq hierarchy or applying general r-matrix theory to the loop algebra $sl(3,\lambda)$ [2].

The spectral curve of this Lax matrix (5.2)

$$\Phi(\mu,\lambda) = \mu^3 - \mu\,(I_1\lambda + I_2) - (\lambda^5 + I_3\lambda^3 + I_4\lambda^2 + I_5\lambda + I_6) = 0 \tag{5.3}$$

depends on two Casimir elements and on four integrals of motion. The explicit expression for the Hamilton function is

$$\begin{aligned} I_2 = {} & k_3^2 + 2h_1h_4k_1 - h_1k_3k_1 + h_3k_1^2 - 2h_3k_1h_2 + 3k_1^2k_2 + h_1k_2^2 + h_2h_1k_3 - 2h_2h_1h_4 + h_4^2 \\ & -k_4k_2 + h_1h_3k_2 + h_1^2h_5 + 2k_2h_2k_1 - 3k_5k_1 + 2h_3k_4 + h_3h_2^2 - 2h_1^2k_4 - h_4k_3 - h_5k_2 - h_3h_5 \end{aligned}$$

whereas other integrals may be founded in [2].

In the initial ten-dimensional phase space $\mathcal{M}$ the five-dimensional Lagrangian foliation

$$\mathcal{C}^{(5)}: \qquad \mathcal{C}\times\mathcal{C}\times\mathcal{C}\times\mathcal{C}\times\mathbb{R}^2$$

is non-compact. Let us introduce a level surface S_{ab} of the Casimir elements $I_1 = a$ and $I_3 = b$. It is an eight-dimensional symplectic leaf, which may be identified with the coadjoint orbit of the corresponding Lie algebra [2].

On the leaf S_{ab} the corresponding four-dimensional Lagrangian submanifold depends on four values of integrals of motion $\alpha_j = \widehat{I}_j$ and on two free parameters a and b.

Proposition 4 *The separated variables are solutions of the following system of equations*

$$\Phi(\mu,\lambda,\widehat{I},a,b) = 0\,, \qquad \frac{\partial}{\partial a}\,\Phi(\mu,\lambda,\widehat{I},a,b) = 0\,, \qquad \frac{\partial}{\partial b}\,\Phi(\mu,\lambda,\widehat{I},a,b) = 0\,,$$

where $\Phi(\mu,\lambda)$ determines the spectral curve of the Lax matrix $\mathcal{C}$ (5.3) and $\widehat{I}_j$ are restrictions of the original integrals I_j to S_{ab}.

So, to construct the separated variables we have to compute integrals $\widehat{I}_j$ on a generic symplectic leaf S_{ab}. This problem was solved in [2]. Substituting integrals $\widehat{I}_j$ into the equations (5.4) we can directly check that solutions of these equations are independent on parameters a and b. Of course, these separated variables coincide with Darboux-Nijenhuis coordinates [2].

In order to avoid calculations of the restrictions of integrals of motion on the symplectic leaf S_{ab} we can consider mapping $S_{ab} \to S_{cd}$, which relates different symplectic leaves. To determine such mapping we can apply canonical transformations of the coordinates (h,k) or the following shift of the Lax matrix

$$\widetilde{L}(\lambda) = L(\lambda) + \begin{pmatrix} 0 & 0 & 0 \\ b_1\,I_3 + b_2 & 0 & 0 \\ a_1\,I_1 + a_2 & c_1\,I_3 + c_2 & 0 \end{pmatrix}, \qquad a_k, b_k \in \mathbb{R}\,. \tag{5.5}$$

This transformation changes coefficients of the spectral curve

$$\Phi(\mu,\lambda) = \mu^3 - \mu\left(\widetilde{I}_1\lambda + \widetilde{I}_2\right) - (\lambda^5 + \widetilde{I}_3\lambda^3 + \widetilde{I}_4\lambda^2 + \widetilde{I}_5\lambda + \widetilde{I}_6) = 0$$

and yields scaling and shift of the Casimir elements

$$\widetilde{I}_1 = (a_1+1)\, I_1 + a_2\,, \qquad \widetilde{I}_3 = (b_1+c_1+1)\, I_3 + b_2 + c_2\,.$$

The explicit expressions for other integrals may be simple calculated from definition of the Lax matrix (5.5). For instance, the new Hamilton function is given by

$$\widetilde{I}_2 = I_2 - k_1(a_1 I_1 - a_2) - k_2(b_1 I_3 - b_2) - (h_1^2 - h_3)(c_1 I_3 - c_2)\,.$$

The corresponding Lagrangian foliation depends on six arbitrary parameters. However, pairs of parameters x_1 and x_2, where $x = a, b$ or c give rise to equivalent equations

$$\frac{\partial}{\partial x_1}\,\Phi(\mu,\lambda) = 0\,, \quad \Leftrightarrow \quad \frac{\partial}{\partial x_2}\,\Phi(\mu,\lambda) = 0\,, \qquad x = a, b, c\,.$$

So, differentiating by parameters a, b or c one gets three equations in addition to the initial equation of the plane curve.

Proposition 5 *Solutions of two different systems of equations*

$$\Phi\left(\mu,\lambda,\widetilde{I},a,b,c\right) = 0\,, \qquad \frac{\partial}{\partial a_k}\Phi(\mu,\lambda) = 0\,, \qquad \frac{\partial}{\partial b_k}\Phi(\mu,\lambda) = 0 \tag{5.6}$$

and

$$\Phi\left(\mu,\lambda,\widetilde{I},a,b,c\right) = 0\,, \qquad \frac{\partial}{\partial a_k}\Phi(\mu,\lambda) = 0\,, \qquad \frac{\partial}{\partial c_k}\Phi(\mu,\lambda) = 0 \tag{5.7}$$

are separated variables depending on parameters $c_{1,2}$ or $b_{1,2}$, respectively.

These two family of variables are related by canonical transformation of initial phase space $\mathcal{M}$ associated with the left and right translations on the group $SL(3)$.

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