# TOPOLOGICAL QUANTUM FIELD THEORIES IN TERMS OF COLOURED GRAPHS ASSOCIATED TO QUANTUM GROUPS [1]


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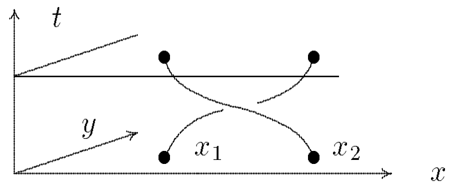
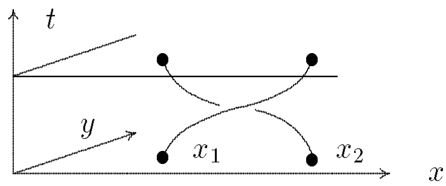
## 1. Introduction

Apart from obvious mathematical applications the investigation is motivated by the problem of braid group statistics in physics. Statistics is one of the central concepts in many body quantum systems. Consider a system of two identical particles located at $x_1$ and $x_2$ in $\mathbf{R}^d$ with a Schrödinger wave function $\psi(x_1, x_2)$. Under the exchange of particles with these coordinates one usually has Bose or Fermi statistics in case $\psi(x_2, x_1) = \pm\psi(x_1, x_2)$. For a quick access to the problem consider the following classical geometric space-time description of the exchange of position for two identical particles, reflecting itself in two quantum mechanical transformation laws




$\psi_+(x_2, x_1) = R\psi(x_1, x_2)$




$\psi_-(x_2, x_1) = R^{-1}\psi(x_1, x_2)$

with $R$ being the so-called statistics operator. The problem arises whether there are situations with $R \neq R^{-1}$. As is well known the question has been answered negatively for space dimensions $d \geq 3$ and the statistics is always given by the permutation group. In $d < 3$ however, the situation is entirely different and one has the possibility that $R \neq \pm 1$ leading to theories with braid group statistics.

At the moment such theories are in a not very good shape, even the theory of "free particles" is not very well understood. This raises the question of finding a good starting point for low dimensional quantum field theories with braid group statistics. A first step in a constructive approach may be provided by topological quantum field theories. In fact, such theories exhibit no dynamics and even no kinematics but they contain all the structures needed to understand the main effects of braid group statistics. Thus one only has to work within the context of linear algebra in finite dimensional spaces.

We briefly review the set-up of topological quantum field theory [1,2] and present our new formulation in terms of coloured graphs [3]. Consider the following assignments

$$\begin{aligned} &\text{2-manifold } \Sigma \to \text{vector space } V^\Sigma \\ &\text{3-manifold } M \to \text{vector} \in V^\Sigma, \quad \text{if } \partial M = \Sigma, \end{aligned} \tag{1}$$

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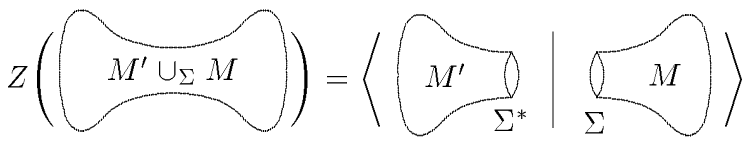
such that one has an invariant of the 3-manifold $M' \cup_\Sigma M$ through the glueing (or surgery) formula given by a pairing




(2)

where the boundaries $\Sigma^* \subset \partial M'$ and $\Sigma \subset \partial M$ have opposite orientation. This pairing has been represented in [3] as a scalar product in $V^\Sigma$ in terms of invariants of 3-manifolds $M$ equipped with a "coloured graph" $G_{\underline{a}} \subset \partial M$. We have derived this representation [4,3] using a generalization of [2] in terms of triangulated manifolds. However, it turns out that one can get rid of the triangulation and derive everything from some simple properties (A0,A1,A2) which are to be considered as axioms in this note.

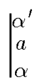
## 2. Coloured graphs on $S^2$

Although the construction presented may be easily generalized we take the data from the q-group $SL_q(2)$ for a root of unity $q = \exp(i\pi/r)$. There are $r-1$ irreps $\pi_a$ ($a \in \{0, 1/2, 1, \dots, r/2-1\}$ = set of colours). A coloured graph consists of lines and vertices embedded in $S^2$ where each line carries a colour.

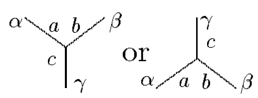
**Definition**: We define

$$Z(G_{\underline{a}}) = \sum_{\underline{\alpha}} \prod_{\text{lines}} w(l) \prod_{\text{vertices}} w(v) \tag{3}$$

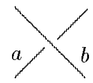
as the value of a closed coloured graph $G_{\underline{a}} \subset \mathbf{R}^2$, where the sum is over all "magnetic quantum numbers" (counting the basis vectors of the representation spaces). The weights are given for

a line 

by : $w(l) = \delta_{\alpha'\alpha}, \quad (\alpha = -a, -a+1, \dots, a)$,

a 3-vertex 

by : $w(v_3) = \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix}$ Clebsh-Gordan coefficient, (4)

a 4-vertex 

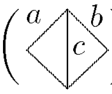
by : $w(v_4)$ = R-martix (tensor basis).

Note that there are no horizontal lines, these are obtained (up to a normalization) from the 3-vertex for $c = 0$ yielding the "Markov trace". The explicit expressions can be found e.g. in [5]. With suitable normalizations one finds as a result the following

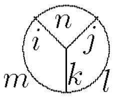
**Theorem**: *$Z(G_{\underline{a}})$ is invariant under deformation of the graph $G_{\underline{a}} \subset S^2$.*

**Examples**:

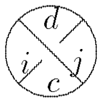
$$Z(\bigcirc^a) = w_a^2 = (2a+1)_{-q} = (-1)^{2a} \text{ "q-dimension of } \pi_a\text{"} \quad (x_q = \frac{\sin\frac{\pi}{r}x}{\sin\frac{\pi}{r}}), \tag{5a}$$

$Z\Big($ 

$\Big) = N_{ab}^c = \left\{\begin{matrix} 1 & \text{if } \pi_c \subset \pi_a \times \pi_b \\ 0 & \text{else}\end{matrix}\right\} =$ "fusion matrix", (5$b$)

$Z\Big($ 

$\Big) = \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} =$ "Racah-Wigner 6j-symbol", (5$c$)

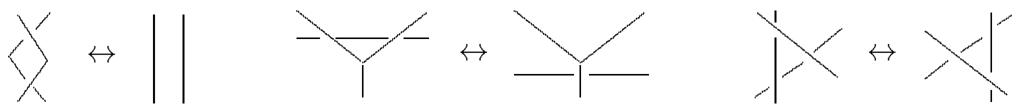
$Z\Big(a$ 

$b\Big) = (R_{ij}^{ab})_c^d = \frac{q_a q_b}{q_c q_d}\begin{vmatrix} i & a & c \\ j & b & d \end{vmatrix} = \begin{cases} \text{"R-matrix" (path basis)} \\ \big(q_a = (-1)^{-a} q^{a(a+1)}\big). \end{cases}$ (5$d$)

There are additional invariances of $Z(G_{\underline{a}})$ under changes of graphs, due to algebraic identities for Clebsh-Gordan coefficients and the R-matrix. We have inside closed graphs the

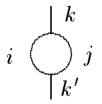
**Relations**:

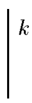
"unitarity" "fusion" "Yang-Baxter"



(6)

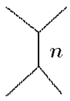
"orthogonality"

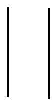


$\leftrightarrow$ 

$N_{ij}^k \frac{\delta_{kk'}}{w_k^2}$

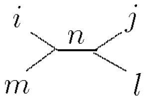
"completeness"

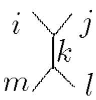
$\sum_n w_n^2$ 

$\leftrightarrow$ 

(7)

"Biedenharn-Eliott's"

$\sum_n w_n^2 \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}$ 

$\leftrightarrow$ 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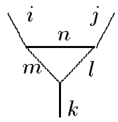
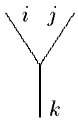
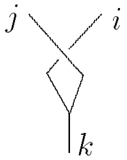


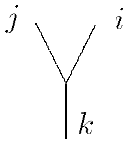
$\leftrightarrow$ 

$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}$

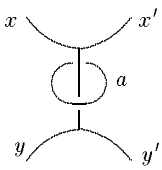
(8)

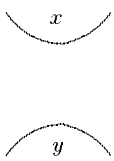
"Racah's"



$\leftrightarrow \frac{q_k}{q_i q_j}$ 


"uniqueness of the vacuum"

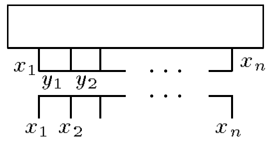
$\sum_a w_a^2$ 

$\leftrightarrow \frac{\delta_{xx'}\delta_{yy'}}{w_x w_y}$ 


(9)

"Wigner-Eckart's"

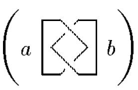
$$= \sum_{\underline{y}} \frac{\delta_{x_1 y_1}\delta_{x_n y_{n-1}}}{w_{x_1}^2 w_{x_n}^2} \prod_{i=1}^{n-1} w_{y_i}^2$$

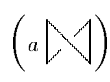


$$(n \geq 2). \tag{10}$$

The modular matrices $S$ and $T$ fulfil $SS^* = TT^* = (ST)^3 = 1$, where

$$S_{ab} = \frac{1}{w} Z\left(\ \right), \quad T_{ab} = \frac{\rho\delta_{ab}}{w_a^2} Z\left(\ \right) \tag{11}$$





and $w^2 = \sum_c w_c^4 = -2r(q-q^{-1})^{-2}$, $\rho = (\sum_c w_c^4 q_c^2/w)^{1/3} = q^{1/2}e^{-i\pi/4}$.

**3. Invariants of 3-manifolds $M$ equipped with coloured graphs $G_{\underline{a}} \subset \partial M$**

There exist various formulations of invariants of 3-manifolds:

i) Witten's [6] Chern-Simons functional integral $Z_{CS}(M)$,

ii) Reshetikhin and Turaev's [7] construction via ribbon graphs,

iii) Turaev and Viro's [2] state sum $Z(M)$ via a triangulation of $M$.

We present here a new formulation via coloured graphs on the boundary of 3-manifolds. In [4,3] we generalized iii) and constructed a topological quantum field theory in terms of invariants $Z(M, G_{\underline{a}})$ of 3-manifolds equipped with coloured graphs on the boundaries. However, it turns out that we can define $Z(M, G_{\underline{a}})$ without using a triangulation but the following simple properties (A0,A1,A2) which are to be considered as axioms.

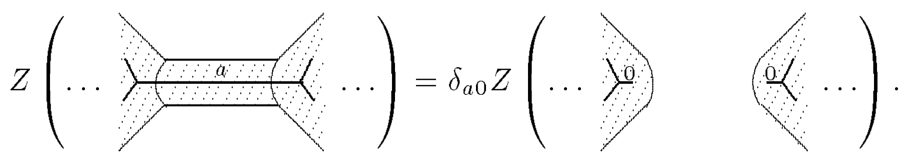
**A0 Factorization:**

a) $Z(M \cup M', G_{\underline{a}} \cup G'_{\underline{a}'}) = Z(M, G_{\underline{a}})Z(M', G'_{\underline{a}'})$, if $M \cap M' = \emptyset$ and $G^{(\prime)}_{\underline{a}^{(\prime)}} \subset \partial M^{(\prime)}$,

b) $Z(M \setminus D^3, G_{\underline{a}}) = \text{const.} Z(M)Z(G_{\underline{a}})$, if $G_{\underline{a}} \subset \partial D^3 = S^2$, $Z(G_{\underline{a}})$ as defined above.

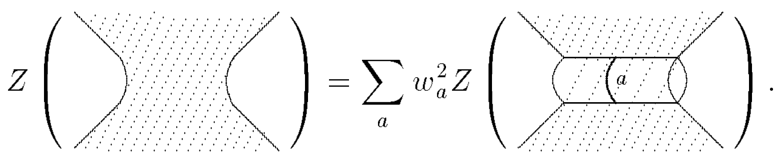
**A1 Cutting "full cylinders":** If the 3-manifold $M$ contains a cylinder $D^2 \times [0,1]$ such that $\partial D^2 \times [0,1] \subset \partial M$ and (for a point $P \in \partial D^2$) $P \times [0,1]$ is part of a line (of colour $a$) of a graph $G_{\underline{a}}$, then the cylinder may be removed using the formula

$$Z\left(\dots \quad \dots\right) = \delta_{a0} Z\left(\dots \qquad \dots\right).$$



**A2 Introducing "empty tubes":** If the 3-manifold $M$ contains a cylinder $D^2 \times [0,1]$ such that $D^2 \times \{0,1\} \subset \partial M$, then the cylinder may be removed (i.e. a tube may be introduced) using the formula

$$Z\left(\ \right) = \sum_a w_a^2 Z\left(\ \right).$$



The Heegaard decomposition of 3-manifolds and the surgery formula below gives the

**Theorem:** *Let $Z(M, G_{\underline{a}})$ be an invariant of the 3-manifold $M$ with a coloured graph on its boundary, then it is uniquely determined by A0, A1 and A2.*

**Examples:** $Z(D^3) = 1,\ Z(S^2 \times S^1) = 1,\ Z(S^3) = w^{-2},\ Z(S^1 \times S^1 \times S^1) = (r-1)^2$, $Z(M \setminus D^3) = w^2 Z(M)$.

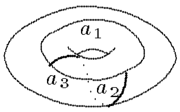
**4. Surgery formula**

Using the properties A0, A1 and A2 of the invariant $Z(M, G_{\underline{a}})$ we derived in [3] for $\Sigma \subset \partial M$ and $\Sigma^* \subset \partial M'$ the surgery formula

$$Z(M' \cup_\Sigma M) = \langle M'|M\rangle = \sum_{\underline{a}} W_{\underline{a}}^\Sigma Z(M', G_{\underline{a}}^{\Sigma^*}) Z(M, G_{\underline{a}}^\Sigma) \tag{12}$$

where $W_{\underline{a}}^\Sigma = w^{-2} \prod_{i=1}^{6g-3} w_{a_i}^2$ is a weight factor and $G_{\underline{a}}^\Sigma$ is what we call a canonical graph for $\Sigma$. For example for the torus it is given as

$$G_{a_1,a_2,a_3}^{S^1 \times S^1} \quad \leftrightarrow \tag{13}$$




The sum in eq. (12) is finite over a set of colours $a_1, \ldots$. This exhibits the replacement of classical groups (here SU(2)) by quantum groups as a regularization procedure. Equation (12) defines a topological quantum field theory. As a function of $\underline{a}$ the invariant $\langle \underline{a}|M\rangle = Z(M, G_{\underline{a}}^\Sigma)$ may be considered as a vector [associated to $M$ by (1)] in the space $V^\Sigma$ associated to $\Sigma$. Moreover the coloured graphs $G_{\underline{a}}^\Sigma$ may be considered as vectors in the dual space $(V^\Sigma)^*$ These vectors are in general not linear independent. However, in [3] we also constructed a vector basis. We mention that the spaces $V^\Sigma$ and $V^{\Sigma^*}$ (where $\Sigma^*$ has orientation opposite to $\Sigma$) are dual to each other. Also there is an antilinear map $\tau : V^\Sigma \to V^{\Sigma^*}$ such that i) the induced hermitian scalar product on $V^\Sigma$ is positive definite and ii) $\tau v(M) = v(M^*)$ for $\partial M = \Sigma$, where $v(M) \in V^\Sigma$ is the state corresponding to $M$.

The representation of vectors in $V^\Sigma$ introduced above implies a matrix representation of the **mapping class** group. A homoeomorhpism $F : \Sigma^g \to \Sigma^g$ of the Riemann surface $\Sigma^g$ induces a map

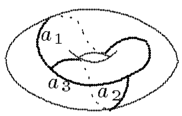
$$F: \ G_{\underline{a}}^{\Sigma^g} \mapsto G'_{\underline{a}}. \tag{14}$$

Therefore

$$\langle \underline{a}|F|\underline{a}'\rangle = Z(\Sigma^g \times [0,1], G_{\underline{a}}^{\Sigma^g} \cup G'_{\underline{a}'}) \quad \left(G_{\underline{a}}^{\Sigma^g} \subset \Sigma^g \times \{0\},\ G'_{\underline{a}'} \subset \Sigma^g \times \{1\}\right) \tag{15}$$

defines a matrix representation of the mapping class group of $\Sigma^g$. For example for the case of $g = 1$ a "Dehn-twist" maps the canonical graph (13) as follows

$$G_{a_1,a_2,a_3}^{S^1 \times S^1} \mapsto G'_{a_1,a_2,a_3} \quad \leftrightarrow \tag{16}$$

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**Heegaard's Theorem** now implies that for any 3-manifold $M$ there exist a 2-manifold $\Sigma$ and a homoeomorphism $F : \Sigma \to \Sigma$ such that the invariant of $M$ is given by

$$Z(M) = \langle M_\Sigma | F | M_\Sigma \rangle \tag{17}$$

where $M_\Sigma$ is the handle body with $\partial M_\Sigma = \Sigma$.

In [3] we also introduced graphs $\mathcal{G}_{\underline{a}} = \big(c(\mathcal{G}), \mathcal{T}(c(\mathcal{G})), G_{\underline{a}}\big) \subset M$ embedded in the internal of a 3-manifold $M$ consisting of lines and vertices. Starting from the "core" $c(\mathcal{G})$ of $\mathcal{G}_{\underline{a}}$ as a finite 1-dimensional simplicial complex, we cut out a tubular neighbourhood $\mathcal{T}(c(\mathcal{G}))$ of the core. A graph $G_{\underline{a}}$ of the type as defined above is then introduced on the boundary of this neighbourhood. It turns out that any line may be decomposed into a "left-" and a "right handed" one, as more fundamental objects. These oriented lines carry "meridians", "under-" or "over-crossing" the lines, respectively. The interesting property of the oriented lines is that they do not "interact". There is no branching of left and right handed lines. Moreover the relative braiding of left and right handed lines is trivial. (For more details see Ref. [3])

The surgery formula also holds in the case that a graph $\mathcal{G}_{\underline{c},\underline{\bar{d}}}$ is embedded in the 3-manifold $M$, leading to punctured surfaces $\Sigma_{\underline{c},\underline{\bar{d}}}$, where $\underline{c}, \underline{\bar{d}}$ correspond to the intersection points with left handed lines (colours $c_i$) and right handed lines (colours $\bar{d}_i$).

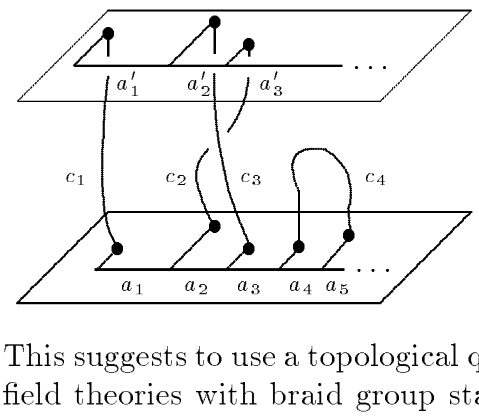
The dimension of the vector spaces $V^\Sigma$ is given by $Z(\Sigma \times S^1)$ [1]. For the punctured case we find that it is a product of two generalized Verlinde formulas [8]

$$\dim V^{\Sigma^g_{\underline{c},\underline{\bar{d}}}} = \mathrm{tr}\big(N^{c_1} \ldots N^{c_n} (\textstyle\sum_a N^a N^a)^{g-1}\big)\mathrm{tr}\big(N^{\bar{d}_1} \ldots N^{\bar{d}_{\bar{n}}} (\textstyle\sum_a N^a N^a)^{g-1}\big). \tag{18}$$

For the special case of a link $\mathcal{L}_{\underline{c}_0}$ of left handed lines we have a braid group representations by

$$\big\langle \underline{a}' | \mathcal{L}_{\underline{c}_0} | \underline{a} \big\rangle = Z(\mathcal{L}_{\underline{c}_0} \cap \Sigma \times [0,1], G_{\underline{a}'}^{\Sigma_{\underline{c}'}} \cup G_{\underline{a}}^{\Sigma^*_{\underline{c}}}). \tag{19}$$

For example for $\Sigma = S^2$ we have




$$\to \big\langle \underline{a}' | \mathcal{L}_{\underline{c}_0} | \underline{a} \big\rangle = \delta_{a_1 c_1} \delta^{a'_1}_{a_1} (R^{a_1 a_3}_{c_2 c_3})^{a'_2}_{a_2} \delta^{a'_3}_{a_3} N^{c_4}_{a_3 a_4} \delta_{a_3 a_5} \ldots$$

This suggests to use a topological quantum field theory as a starting point for quantum field theories with braid group statistics. As open problems remain 1) to introduce kinematics to obtain "free particles" with braid group statistics (what ever this may mean) and 2) to introduce dynamics to describe interacting particles.

We believe that such quantum field theories may apply to 2-dimensional phenomena in solid state physics as high-$T_c$ superconducting, fractional quantum Hall effect etc.

**Acknowledgement:**

This talk is based on a collaboration with R. Schrader.

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